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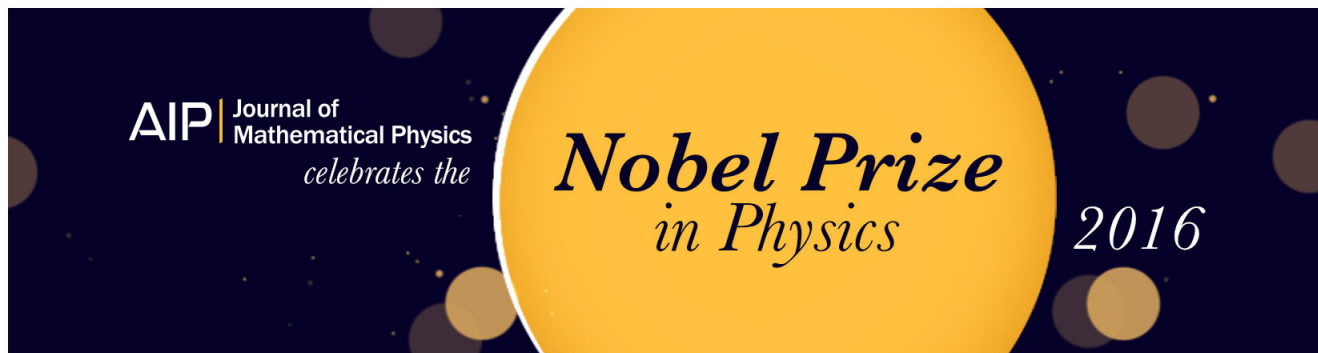Am. J. Phys. **40**, 1036 (1972); 10.1119/1.1986739

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On the Euler angles for SU(*N*)

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In this paper we reconsider the problem of the Euler parametrization for the unitary groups. After constructing the generic group element in terms of generalized angles, we compute the invariant measure on SU(N) and then we determine the full range of the parameters, using both topological and geometrical methods. In particular, we show that the given parametrization realizes the group SU($N+1$) as a fibration of U(N) over the complex projective space $\mathbb{CP}^n$. This justifies the interpretation of the parameters as generalized Euler angles. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2190898]

I. INTRODUCTION

The importance of group theory in all branches of physics is a well-known fact. Explicit realizations of group representations are often necessary technical tools. Often it is finite dimensional and compact Lie groups and then the knowledge of the associated algebra, which describes the group in a neighborhood of the identity, is enough for this purpose.

There are however cases where an explicit expression of the full global group structure is needed, as, for example, when nonperturbative computations come into play. In most of these cases, the main objectives are two: First, one would like to find a relative simple parametrization, making all the computations manageable. Second, one needs to determine the full range of the parameters, in order to be able to handle global questions.

If both such points can seem unnecessary at an abstract level, they become essential at a most concrete level, e.g., in instantonic calculus or in nonperturbative lattice gauge theory computations. The necessary computer memory for simulations is in fact drastically diminished.

The case of SU(N) was first considered and solved by Tilma and Sudarshan, in Ref. 1. There, they provide a parametrization, in terms of angular parameters, for the unitary groups. In particular, in the first paper they consider special groups, SU(N), together with some applications to qubit and qutrit configurations. In the second paper, they give an extension to U(N) groups, using the fibration structure of SU($N+1$) as U(N) fiber over the complex projective space $\mathbb{CP}^n$.

In this paper we reconsider the problem of finding a generalized Euler parametrization for special unitary groups. The intent is to provide a fully explicit and elementary (which does not mean short) proof of the beautiful results of Ref. 1. Our motivation is that the determination of the range of the parameters is a quite difficult task, so that disagreements are present in the literature

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even for SU(3) (for example, in Ref. 2). Therefore, we think that a careful deduction is necessary in order to corroborate the results of Tilma and Sudarshan. Also, all our proofs based essentially on inductive procedures, and they are explicit, in order to be easily accessible to anyone who needs them.

Our construction is quite different from Ref. 1, and as a result our parametrization differs slightly from theirs. However, this does not affect the final expression of the invariant measure.

To illustrate the spirit of our construction, let us start by taking a look at the Euler parametrization for SU(2).

Starting from the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.1}$$

it is known that the generic element of SU(2) can be written as

$$g = e^{i\phi\sigma_3} e^{i\theta\sigma_2} e^{i\psi\sigma_3}. \tag{1.2}$$

Here $\phi \in [0,\pi]$, $\theta \in [0,\pi/2]$, $\psi \in [0,2\pi]$ are the so-called Euler angles for SU(2). They are related to the well-known Euler angles traditionally used in classical mechanics to describe the motion of a spin. From the point of view of the structure of the representation, (1.2) is obtained starting from a one parameter subgroup $\exp(i\theta\sigma_2)$ and then acting on it both from the left and from the right with a maximal subgroup of SU(2) which does not contain the first subgroup. We can rewrite it in the schematic form $g = U(1)\exp(i\theta\sigma_2)U(1)$. On the other hand, the group SU(2) is topologically equivalent to the three-sphere S^3, and admits a Hopf fibration structure with fiber S^1 over the base $S^2 \simeq \mathbb{CP}^1$.

To recognize this fibration structure in (1.2), we can apply the methods used in Refs. 3 and 4. After introducing the metric $\langle A|B\rangle = \frac{1}{2}\mathrm{Tr}(AB)$ on the algebra, the metric on the group can be computed as $ds^2 = \frac{1}{2}\mathrm{Tr}\, J \otimes J$, where $J = -ig^{-1}\,dg$ are the left-invariant currents. Following Ref. 3, it is possible to separate the fiber from the base by writing $g = hU(1)$, where $h = e^{i\phi\sigma_3}e^{i\theta\sigma_2}$ and $U(1) = e^{i\psi\sigma_3}$. To find the metric on the fiber, let us fix the point on the base and compute the currents along the fiber, $J_F = -iU(1)^{-1}dU(1) = d\psi\sigma_3$. The metric on the fiber is then simply given by $ds_F^2 = d\psi^2$. To determine the metric on the base, we first must project out from the current $J_B = -ih^{-1}\,dh$ the component along the fiber, in order to be left with the reduced current on the basis $\tilde{J}_B = d\psi\sigma_2 + \sin(2\psi)d\psi\,\sigma_1$, which then in turn provides the metric

$$ds_B^2 = \tfrac{1}{4}[d(2\psi)^2 + \sin^2(2\psi)d(2\phi)^2]. \tag{1.3}$$

This corresponds in fact with the metric of a sphere of radius $\frac{1}{2}$. It is easy to see that, introducing the complex coordinates $z = \tan\psi e^{i\phi}$ and their complex conjugates, the metric ds_B^2 reduces to the standard Fubini-Study metric for $\mathbb{CP}^1$.

This shows that the Euler parametrization captures the Hopf fibration structure of SU(2), which is the starting point for our construction. Mimicking what we said about SU(2), let us write the generic element of SU(N+1) as $g = U(N)e(\theta)U(N)$, where $e(\theta)$ is a one parameter subgroup not contained in U(N). The first difficulty we must face here is that this expression for a generic SU(N+1) group has redundancies, which must be eliminated. After this problem is solved, we then must show that the parametrization respects the Hopf fibration structure of SU(N+1).

II. THE SU(N) ALGEBRA

The generators of su(N) are all the $N \times N$ traceless Hermitian matrices. A convenient choice for a base are the generalized Gell-Mann matrices as explained in Ref. 1. Let us remind how they can be constructed using an inductive procedure. Let $\{\lambda_i\}_{i=1}^{N^2-1}$ be the Gell-Mann base for su(N): They are $N \times N$ matrices which can be embedded in su(N+1) adding a null column and a null row

$$\widetilde{\lambda}_i = \begin{pmatrix} \lambda_i & \vec{0} \\ \vec{0} & 0 \end{pmatrix}. \tag{2.1}$$

We will omit the tilde from now on. The dimension of SU(N) being $(N+1)^2-1$, we must add $2N+1$ matrices to obtain a Gell-Mann base for su(N+1). This can be done as follows: Set

$$\{\lambda_{N^2+2a-2}\}_{\alpha\beta} = \delta_{\alpha,a}\delta_{\beta,N+1} + \delta_{\alpha,N+1}\delta_{\beta,a},$$

$$\{\lambda_{N^2+2a-1}\}_{\alpha\beta} = i(-\delta_{\alpha,a}\delta_{\beta,N+1} + \delta_{\alpha,N+1}\delta_{\beta,a}), \tag{2.2}$$

for $a=1,\ldots,N$. The last matrix we need is diagonal and traceless so that we can take $\lambda_{(N+1)^2-1} = \epsilon_{N+1}\,\mathrm{diag}\{1,\ldots,1,-N\}$.

One can easily verify that the base of matrices $\{\lambda_I\}_{I=1}^{(N+1)^2-1}$ so obtained satisfies the normalization condition $\mathrm{Trace}\{\lambda_I\lambda_J\}=2\delta_{IJ}$ if we choose $\epsilon_{N+1}=\sqrt{2/[(N+1)^2-(N+1)]}$.

These are exactly the matrices we need to generate the group elements.

III. THE EULER PARAMETRIZATION FOR SU(N+1): INDUCTIVE CONSTRUCTION

It is a well-known fact that special unitary groups SU(N+1) can be geometrically understood as U(N) fibration over the complex projective space $\mathbb{CP}^N$. Now U(N) is generated by the first N^2-1 generalized Gell-Mann matrices plus the last one $\lambda_{(N+1)^2-1}$. Using the fact that all the remaining generators of SU(N+1) can be obtained from the commutators of these matrices with λ_{N^2+1}, one is tempted to write the general element of SU(N+1) in the form

$$\mathrm{SU}(N+1) = \mathrm{U}(N)e^{ix\lambda_{N^2+1}}\mathrm{U}(N). \tag{3.1}$$

However to describe SU(N+1) we need $(N+1)^2-1$ parameters, while in the rhs they are $2N^2+1$: There are $(N-1)^2$ redundancies. Inspired at first by dimensional arguments, we propose that an U(N−1) subgroup can be subtracted from the left U(N) in the following way. Let us write U(N) in the form $\mathrm{U}(N)=\mathrm{SU}(N)e^{i\psi\lambda_{(N+1)^2-1}}$. Inductively, we can think that also SU(N) can be recovered from $\mathrm{U}(N-1)e^{i\phi\lambda_{(N-1)^2+1}}\mathrm{U}(N-1)$ eliminating the redundant parameters, so that it will have the form $\mathrm{SU}(N)=he^{i\phi\lambda_{(N-1)^2+1}}\mathrm{SU}(N-1)e^{i\theta\lambda_{N^2-1}}$. We then choose to eliminate the appearing SU(N−1) together with the phase $e^{i\psi\lambda_{(N+1)^2-1}}$. In this way the SU(N+1) group element can be written in the form $\mathrm{SU}(N+1)=he^{i\phi\lambda_{(N-1)^2+1}}e^{i\theta\lambda_{N^2-1}}e^{ix\lambda_{N^2+1}}\mathrm{U}(N)$. By induction, assuming $N\geq 2$ we arrive to the final form of our Ansatz about the parametrization of the general element $g \in \mathrm{SU}(N+1)$,

$$g = e^{i\theta_1\lambda_3}e^{i\phi_1\lambda_2}\prod_{a=2}^{N}[e^{i(\theta_a/\epsilon_a)\lambda_{a^2-1}}e^{i\phi_a\lambda_{a^2+1}}]\mathrm{U}(N)[\alpha_1,\ldots,\alpha_{N^2}], \tag{3.2}$$

where $\mathrm{U}(N)[\alpha_1,\ldots,\alpha_{N^2}]$ is a parametrization of U(N) which in turn can be obtained inductively using the fact

$$\mathrm{U}(N) = [\mathrm{SU}(N)\times\mathrm{U}(1)]/\mathbb{Z}_N. \tag{3.3}$$

The Ansatz (3.2) contains the correct number of parameters. However, we need to show that it is a good Ansatz, meaning that at least locally it must generate the whole tangent space to the identity. Using the Backer-Campbell-Hausdorff formula and some properties of the Gell-Mann matrices, (essentially the fact that the commutators of $\lambda_{(k-1)^2+1}$ with the first $(k-1)^2-1$ matrices generate all the remaining matrices of the su(k) algebra but the last one) it is easy to show that

$$e^{i\theta_1\lambda_1}e^{i\phi_1\lambda_3}\prod_{a=2}^{N}\left[e^{i(\theta_a/\epsilon_a)\lambda_{a^2-1}}e^{i\phi_a\lambda_{a^2+1}}\right] = e^{i\sum_{j=1}^{(N+1)^2-2}a_j\lambda_j}, \tag{3.4}$$

where a_j are all nonvanishing functions of the $2N$ parameters θ_a, ϕ_a. Thus in a change of coordinates (from the θ_a, ϕ_a to the a_j) only $2N$ of the a_j can be chosen as independent parameters. We could choose the last ones, corresponding to the coefficients of the matrices $\{\lambda_k\}_{k=N^2}^{N^2+2N-1}$. In this way, the N^2 free parameters for the remaining matrices come out exactly from the U(N) factors in (3.2).

We have not entered into details here because a second simple proof of the validity of this parametrization will be given by constructing a nonsingular invariant measure from our Ansatz.

IV. INVARIANT MEASURE AND THE RANGE OF THE PARAMETERS

A. The invariant measure

To construct the invariant measure for the group starting from (3.2), we will adopt the same method used in Ref. 3, with $U := \mathrm{U}(N)$ as the fiber group. Let us then write (3.2) as

$$g = h \cdot U. \tag{4.1}$$

Starting from the computation of the left invariant currents $j_h = -ih^{-1}\,dh$, we can define the one forms

$$e^l := \tfrac{1}{2}\mathrm{Tr}[j_h \cdot \lambda_{N^2+l-1}], \quad l = 1, \dots, 2N, \tag{4.2}$$

which turns out to give the Vielbein one forms of the base space of the fibration. If $\underline{e}$ denotes the corresponding Vielbein matrix, the invariant measure for SU($N+1$) will then take the form

$$d\mu_{\mathrm{SU}(N+1)} = \det \underline{e} \cdot d\mu_{\mathrm{U}(N)}, \tag{4.3}$$

$d\mu_{\mathrm{U}(N)}$ being the invariant measure for U(N). Using (3.3) with $\mathrm{U}(1) = e^{i(\omega/\epsilon_{N+1})\lambda_{(N+1)^2-1}}$ we obtain the recursion relation (note that here ω is allowed to vary in the range $[0, 2\pi/N]$)

$$d\mu_{\mathrm{SU}(N+1)} = \det \underline{e} \cdot d\mu_{\mathrm{SU}(N)}\frac{d\omega}{\epsilon_{N+1}}. \tag{4.4}$$

Then we will concentrate on the det $\underline{e}$ term. To this end let us write (3.2) in the form

$$g = h_{N+1}[\theta_a, \phi_a] \cdot U[\alpha_i]. \tag{4.5}$$

Here we will consider $N \geq 3$ so that the relation

$$h_{N+1} = h_N e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}e^{i\phi_N\lambda_{N^2+1}}, \tag{4.6}$$

is true. If we introduce the right currents $J_{h_{N+1}} = -ih_{N+1}^{-1}\,dh_{N+1}$ then the Vielbein (4.2) takes the form

$$e_l^{\{N\}} = \frac{1}{2}\mathrm{Tr}\{J_{h_{N+1}}\lambda_l\} = d\phi_N\delta_{l,N^2+1} + \frac{1}{2\epsilon_N}d\theta_N\,\mathrm{Tr}\{e^{-i\phi_N\lambda_{N^2+1}}\lambda_{N^2-1}e^{i\phi_N\lambda_{N^2+1}}\lambda_{N^2+l}\}$$

$$+ \frac{1}{2}\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}e^{i\phi_N\lambda_{N^2+1}}\lambda_{N^2+l}e^{-i\phi_N\lambda_{N^2+1}}\}, \tag{4.7}$$

and using the relations in Appendix A we find

$$
\underline{e}^{\{N\}} = \begin{pmatrix} d\phi_N & 0 & 0 \\ 0 & \sin\phi_N\cos\phi_N\,d\theta_N & \sin\phi_N\cos\phi_N\dfrac{1}{2}\sum_{a=2}^{N}\mathrm{Tr}\left[\dfrac{1}{\alpha_a}J_{h_N}\lambda_{a^2-1}\right] \\ 0 & 0 & \dfrac{1}{2}\sin\phi_N\,\mathrm{Tr}[e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\underline{M}] \end{pmatrix}, \qquad (4.8)
$$

where we introduced the $\{N\}$ index to remember that this is a $2N\times 2N$ matrix associated to the group SU($N+1$). Here $\underline{M}$ is a column of matrices, $M_{2j-1}=\lambda_{j^2+1}$, $M_{2j}=\lambda_{j^2}$, $j=1,2,\ldots,N-1$. Formula (4.8) then reads as follows: J_{h_N} is a 1-form with components $J_{h_N,c}$, $c=1,2,\ldots,2n-2$, with respects to the coordinates X^c, defined as $X^{2j-1}=\theta_j$, $X^{2j}=\phi_j$. To find the component (r,c) of (4.8) one must then take the cth component of $J_{h_N,c}$ and the rth component of $\underline{M}$ before to compute the trace.

The invariant measure is then

$$
\det\underline{e}^{\{N\}} = d\phi_N\,d\theta_N\cos\theta_N\sin^{2N-1}\phi_N\det\left(\frac{1}{2}\mathrm{Tr}[e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\underline{M}]\right). \qquad (4.9)
$$

We now use the recurrence relation

$$
J_{h_N} = \lambda_{(N-1)^2+1}\,d\phi_{N-1} + \frac{1}{\epsilon_{N-1}}e^{-i\phi_{N-1}\lambda_{(N-1)^2+1}}\lambda_{(N-1)^2-1}e^{i\phi_{N-1}\lambda_{(N-1)^2+1}}\,d\theta_{N-1}
$$
$$
+ e^{-i\phi_{N-1}\lambda_{(N-1)^2+1}}e^{-i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}J_{h_{N-1}}e^{i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}e^{i\phi_{N-1}\lambda_{(N-1)^2+1}}. \qquad (4.10)
$$

Computing the traces different cases arise depending on whether $j=N-1$ or $j<N-1$; using again the relations in Appendix A it is not too difficult to show that the last determinant is equal to

$$
\det\begin{pmatrix} d\phi_{N-1}\cos(N\theta_N) & -\dfrac{1}{2}\sin(N\theta_N)\sin(2\phi_{N-1})d\theta_{N-1} \\ d\phi_{N-1}\sin(N\theta_N) & \dfrac{1}{2}\cos(N\theta_N)\sin(2\phi_{N-1})d\theta_{N-1} \end{pmatrix}
$$
$$
\times\det\left(\frac{1}{2}\cos\phi_{N-1}\mathrm{Tr}[e^{-i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}J_{h_{N-1}}e^{i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}\underline{M}]\right),
$$

which set into (4.9) in turn yields the recurrence relation

$$
\det\underline{e}^{\{N\}} = d\phi_N\,d\theta_N\frac{\sin^{2N-1}\phi_N}{\tan^{2N-4}\phi_{N-1}}\det\underline{e}^{\{N-1\}}, \qquad (4.11)
$$

which can be solved to give

$$
\det\underline{e}^{\{N\}} = 2\,d\theta_N\,d\phi_N\cos\phi_N\sin^{2N-1}\phi_N\prod_{a=1}^{N-1}[\sin\phi_a\cos^{2a-1}\phi_a\,d\theta_a\,d\phi_a]. \qquad (4.12)
$$

This is the same result as found in Ref. 1.

B. The range of the parameters

At this point we are able to determine the range of the parameters in such a way as to cover the whole group. We will do this only for the base space: The remaining ranges for the fiber can be determined recursively, as discussed above, remembering in particular that the U(1) phase in U(k) can be taken in $[0,2\pi/k]$.

We then proceed as in Ref. 3. We first choose the ranges so as to generate a closed $[(N+1)^2-1]$-dimensional closed manifold which then must wrap around the group manifold of

$SU(N+1)$ an integer number of times. This can be done by looking at the measure (4.12) on the base manifold and noticing that it is nonsingular when $0 < \phi_a < \pi/2$, whereas θ_a can take all the period values $\theta_a \in [0, 2\pi]$, for all $a = 1, \ldots, N$. However, note that the angles $\theta_1, \phi_1, \theta_2$ generate the whole $SU(2)$ group when $0 \leq \theta_1 \leq \pi$, $0 < \phi_a < \pi/2$ and $0 \leq \theta_2 \leq 2\pi$. We can then restrict $\theta_1 \in [0, \pi]$. The rest of the variety is generated by the remaining $U(N)$ part.

   If we call V_{N+1} the manifold obtained this way we then find

$$\text{Vol}(V_{N+1}/U(N)) = \int_0^\pi d\theta \prod_{a=2}^N \int_0^{2\pi} d\theta_a \prod_{b=1}^N \int_0^{\pi/2} d\phi_b \left\{ \cos \phi_N \sin^{2N-1} \phi_N \prod_{c=1}^{N-1} [\sin \phi_c \cos^{2c-1} \phi_c] \right\}$$

$$= \frac{\pi^N}{N!},$$ (4.13)

or equivalently

$$\text{Vol}(V_{N+1}) = \text{Vol}(U(N)) \frac{\pi^N}{N!}.$$ (4.14)

This is exactly the recursion relation found in Appendix B. Therefore, it is the correct range of the parameters for every $N \geq 2$, if we have $V_3 = SU(3)$, as can be easily checked directly or by comparison with the results given in Appendix A of Ref. 2 (see also Appendix B of Ref. 4). The next step is to determine the parametrization of $SU(N+1)$ for every value of N. It is given by (3.2) with

$$0 \leq \theta_1 \leq \pi, \quad 0 \leq \theta_a \leq 2\pi, a = 2, \ldots, N,$$

$$0 \leq \omega \leq \frac{2\pi}{N}, \quad 0 \leq \phi_a \leq \frac{\pi}{2}, a = 1, \ldots, N,$$ (4.15)

and the remaining parameters which cover $SU(N)$ (determined inductively).

   To prove that our parametrization is well defined we can do more: We are in fact able to show that the induced metric on the base manifold is exactly the Fubini-Study metric over $\mathbb{CP}^N$.

V. THE GEOMETRIC ANALYSIS OF THE FIBRATION

   We will now show that the metric induced on the base space takes exactly the form of the Fubini-Study metric in trigonometric coordinates as given in Appendix C. To do so we will again use inductive arguments.

   The metric on the base is $ds_B^2 = [\underline{e}^{\{N\}}]^T \otimes \underline{e}^{\{N\}}$, where T indicates transposition and $\underline{e}^{\{N\}}$ is given in (4.8). Using the relations in Appendix A and defining

$$X_N = \frac{1}{2} \sum_{a=2}^N \text{Tr}[J_{h_N} \epsilon_a \lambda_{a^2-1}]$$ (5.1)

the metric takes the form

$$ds_B^2 = d^2\phi_N + \sin^2 \phi_N \left\{ [d\theta_N + X_N]^2 + \sum_{j=1}^{N-1} \left[\frac{1}{2} \text{Tr}(e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}} J_{h_N} e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}) \lambda_{j^2} \right]^2 \right.$$

$$\left. + \sum_{j=1}^{N-1} \left[\frac{1}{2} \text{Tr}(e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}} J_{h_N} e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}) \lambda_{j^2+1} \right]^2 \right\} - \sin^4 \phi_N [d\theta_N + X_N]^2.$$ (5.2)

This is an encouraging form, which upon comparison with (C3) suggests the identification $\xi = \phi_N$. With this identification in mind, let us first remark that the following recursion relation holds:

$$X_N = \cos^2 \phi_{N-1}(\mathrm{d}\theta_{N-1} + X_{N-1}), \tag{5.3}$$

which can be shown by inserting (4.10) in (5.1) and then applying (A6) and (A11). A direct computation yields

$$X_3 = \cos^2 \phi_2(\mathrm{d}\theta_2 + \cos(2\phi_1)\mathrm{d}\theta_1), \tag{5.4}$$

from which, through repeated application of the recurrence relation (5.3), we obtain

$$X_N = \sum_{k=1}^{N-3}\left[\prod_{i=1}^{k} \cos^2 \phi_{N-i}\right]\mathrm{d}\theta_{N-k} + \left[\prod_{i=1}^{N-2} \cos^2 \phi_{N-i}\right](\mathrm{d}\theta_2 + \cos(2\phi_1)\mathrm{d}\theta_1). \tag{5.5}$$

At this point we must to compare $\mathrm{d}\theta_N + X_N$ with the coefficient of $\sin^4 \xi$ in (C3). In fact, to bring $\mathrm{d}\theta_N + X_N$ to the desired form $\Sigma_{i=1}^{N} (\widetilde{R}^i)^2 \,\mathrm{d}\psi_i$, one is tempted to just set $\theta_i = \psi_i$ and $\phi_\mu = \omega_\mu$. However, this cannot be the case because the $\widetilde{R}^i$ does not satisfy the condition $\Sigma(\widetilde{R}^i)^2 = 1$.

These observations, together with explicit calculations for the case $N=4$ and $N=5$, suggest that we should simply take some linear combination $\psi_i = \psi_i(\theta_j)$. This can be done as follows: Let us introduce new variables $\widetilde{\theta}_K$, $k=1,\dots,N$, such that

$$\widetilde{\theta}_N = \theta_N, \quad \theta_{N-k} = \widetilde{\theta}_{N-k} - \widetilde{\theta}_{N-k+1}, \quad k = 1, \dots, N-3,$$

$$\theta_1 + \theta_2 = \widetilde{\theta}_1 - \widetilde{\theta}_3, \quad \theta_1 - \theta_2 = \widetilde{\theta}_3 - \widetilde{\theta}_2. \tag{5.6}$$

In this way $\mathrm{d}\theta_N + X_N$ takes the desired form

$$\mathrm{d}\theta_N + X_N = \sum_{i=1}^{N} (R^i(\omega_\mu))^2 \,\mathrm{d}\psi_i \tag{5.7}$$

with $\omega_\mu = \phi_\mu$, $\mu = 1, \dots N-1$, $\psi_i = \widetilde{\theta}_{N-i+1}$, $i = 1, \dots, N$ and

$$R_1 = \sin \phi_{N-1}, \quad R_k = \sin \phi_{N-k}\prod_{i=1}^{k-1} \cos \phi_{N-i}, \quad k = 2, \dots, N-1,$$

$$R_N = \prod_{i=1}^{N-1} \cos \phi_{N-i}. \tag{5.8}$$

These formulas agree with the expressions in Appendix C. As the last step, in Appendix D we finally show that, after performing the change of variables described above, the coefficients of $\sin^2 \xi$ and $\sin^2 \phi_N$ also agree. This proves that the metric induced on the base $\mathbb{CP}^N$ of the U(N) fibration by the invariant metric on SU($N+1$) is nothing else but the natural Fubini-Study metric in trigonometric coordinates.

We can now use this result as a different method to fix the range of the parameters. In fact, $(R^1, \dots, R^N)$ parametrize the positive orthant of a sphere, if $0 < \phi_i < \pi/2$, $i = 1, \dots, N-1$. Moreover, the identification of ϕ_N with ξ yields $\phi \in [0, \pi/2]$. Finally, it is easy to show that the conditions $\widetilde{\theta}_i \in [0, 2\pi]$ are equivalent to $\theta_1 \in [0, \pi]$ and $\theta_i \in [0, 2\pi]$, $i = 2, \dots, N$. These are the same results obtained in (4.15).

VI. CONCLUSIONS

In this paper, we have reconsidered the problem of constructing a generalized Euler parametrization for SU(N). The parametrization we find differs slightly from the one described by Tilma and Sudarshan. In fact, comparing our results with the expression (18) in Ref. 1, it is possible to

see that we have chosen $\lambda_{(k-1)^2-1}$ instead of λ_3. Furthermore, we have computed the corresponding invariant measure, which turns out to coincide with the result in Ref. 1, despite the slight differences in the choice of the parametrization.

To determine the range of the parameters, we have used two distinct methods, both yielding the same result. To better motivate the name "Euler angles," we have carefully shown that the parametrization captures the Hopf fibration structure of the SU(N) groups. In particular the change of coordinate we found to evidentiate the fibrations, gives an explicit map between the Euler coordinates introduced starting from the generalized Gell-Mann matrices, and the ones introduced in Ref. 5 using geometrical considerations.

We have given a quite explicit proof of every assertion. Apart from corroborating the results of Tilma and Sudarshan, we think that our work is providing a complete toolbox of computation techniques useful in applied theoretical physics as well as for experimental physicists.

ACKNOWLEDGMENTS

APPENDIX A: SOME COMMUTATORS

Using the explicit form of the generalized Gell-Mann matrices constructed with the conventions of Sec. II, we find the useful commutators

$$[\lambda_{N^2+1},\lambda_{N^2+2j}]=-i\lambda_{j^2},$$

$$[\lambda_{N^2+1},\lambda_{N^2+2j+1}]=i\lambda_{j^2+1}, \tag{A1}$$

when $j=1,\ldots,N-1$.

Others interesting relations easy to check are

$$[\lambda_{N^2+1},\lambda_{N^2}]=-i(N+1)\epsilon_{N+1}\lambda_{(N+1)^2-1}-i\sum_{a=2}^{N}\epsilon_a\lambda_{a^2-1},$$

$$[\lambda_{N^2+1},\lambda_{a^2-1}]=i\epsilon_a\lambda_{N^2}, \tag{A2}$$

$$[\lambda_{N^2+1},\lambda_{(N+1)^2-1}]=i(N+1)\epsilon_{N+1}\lambda_{N^2},$$

where $a=1,\ldots,N$, from which remembering that $\epsilon_k=\sqrt{2/k(k-1)}$, one also finds

$$[\lambda_{N^2+1},[\lambda_{N^2+1},\lambda_{N^2}]]=4\lambda_{N^2}. \tag{A3}$$

From the first two commutators we find the very useful relations

$$e^{ix\lambda_{N^2+1}}\lambda_{j^2+1}e^{-ix\lambda_{N^2+1}}=\frac{1}{\sin x}\lambda_{N^2+2j+1}-\frac{1}{\tan x}e^{ix\lambda_{N^2+1}}\lambda_{N^2+2j+1}e^{-ix\lambda_{N^2+1}},$$

$$e^{ix\lambda_{N^2+1}}\lambda_{j^2}e^{-ix\lambda_{N^2+1}} = -\frac{1}{\sin x}\lambda_{N^2+2j} + \frac{1}{\tan x}e^{ix\lambda_{N^2+1}}\lambda_{N^2+2j}e^{-ix\lambda_{N^2+1}}, \tag{A4}$$

when $j = 1, \ldots, N-1$.

Other useful relations easy to prove using the previous relations are

$$\sum_{a=2}^{N}\epsilon_a^2 + (N+1)^2\epsilon_{N+1}^2 = 4, \tag{A5}$$

$$\sum_{a=2}^{N}\epsilon_a^2 + (N+1)\epsilon_{N+1}^2 = 2, \tag{A6}$$

$$\mathrm{Tr}[e^{-ix\lambda_{N^2+1}}\lambda_{N^2-1}e^{ix\lambda_{N^2+1}}\lambda_{N^2+I}] = \epsilon_N\delta_{I0}\sin(2x), \tag{A7}$$

$$\frac{1}{2}\mathrm{Tr}[\lambda_a e^{ix\lambda_{N^2+1}}\lambda_{N^2+2i}e^{-ix\lambda_{N^2+1}}] = \delta_{a,i^2}\sin x, \ a \leq N^2-1, \ i = 1, \ldots, N-1, \tag{A8}$$

$$\frac{1}{2}\mathrm{Tr}[\lambda_a e^{ix\lambda_{N^2+1}}\lambda_{N^2+2i-1}e^{-ix\lambda_{N^2+1}}] = -\delta_{a,i^2+1}\sin x, \ a \leq N^2-1, \ i = 1, \ldots, N-1, \tag{A9}$$

$$\frac{1}{2}\mathrm{Tr}\left[e^{ix\lambda_{N^2+1}}\lambda_{N^2}e^{-ix\lambda_{N^2+1}}\sum_{a=1}^{N^2-1}C^a\lambda_a\right] = \sin(2x)\frac{1}{2}\sum_{b=2}^{N}\mathrm{Tr}[C^{b^2-1}\epsilon_{b^2-1}], \tag{A10}$$

$$\sum_{a=2}^{N}\mathrm{Tr}[Ae^{ix\lambda_{(N-1)^2+1}}\lambda_{a^2-1}e^{-ix\lambda_{(N-1)^2+1}}] = \cos^2 x\sum_{a=2}^{N}\mathrm{Tr}[A\epsilon_a\lambda_{a^2-1}], \tag{A11}$$

$$e^{ix\lambda_{N^2-1}}\lambda_{(N-1)^2}e^{-ix\lambda_{N^2-1}} = \cos(N\epsilon_N x)\lambda_{(N-1)^2} - \sin(N\epsilon_N x)\lambda_{(N-1)^2+1}, \tag{A12}$$

$$e^{ix\lambda_{N^2-1}}\lambda_{(N-1)^2+1}e^{-ix\lambda_{N^2-1}} = \cos(N\epsilon_N x)\lambda_{(N-1)^2+1} + \sin(N\epsilon_N x)\lambda_{(N-1)^2}, \tag{A13}$$

where we used $A := \sum_{i=1}^{(N-1)^2-1}A^i\lambda_i$.

APPENDIX B: THE TOTAL VOLUME OF SU(k)

The total volume for the groups $SU(k)$ can be found following Macdonald in Ref. 6. First remember that, in the sense of rational cohomology, $SU(k)$ is equivalent to the product of odd dimensional spheres

$$SU(k+1) \sim \prod_{j=1}^{k}S^{2i+1}, \tag{B1}$$

where we choosen $k+1$ to obtain recursive relations. The total volume of the group is then uniquely determined when a metric is established on the Lie algebra. We chose the metric induced by the scalar product $(A|B) = \frac{1}{2}\mathrm{Tr}(AB)$, for $A, B \in su(k+1)$. In this way the Gell-Mann generators are orthonormal. The formula for the total volume is (Ref. 6)

$$\text{Vol}(\text{SU}(k+1)) = \prod_{j=1}^{k} \text{Vol}(S^{2i+1}) \cdot \text{Vol}(\mathbb{T}_k) \prod_{\alpha>0} |\alpha^{\vee}|^2, \tag{B2}$$

where $\alpha^{\vee}$ are the coroots associated to positive roots and $\text{Vol}(\mathbb{T}_k)$ is the volume of the torus generated by the simple coroots.

For $su(k+1)$ the simple coroots are $s_i = L_i - L_{i+1}$, $i=1,\ldots,k$ where L_i is the diagonal matrix with the only nonvanishing entry $\{L_i\}_{ii}=1$. After writing s_i in terms of λ_j, as

$$s_i = \sum_{a=1}^{k} \frac{1}{2} \text{Tr}\{s_i \lambda_{(a+1)^2-1}\} \lambda_{(a+1)^2-1}, \tag{B3}$$

it is easy to prove the recursive relation

$$\text{Vol}(\mathbb{T}_k) = \sqrt{\frac{k+1}{2k}} \text{Vol}(\mathbb{T}_{k-1}). \tag{B4}$$

From this we find

$$\text{Vol}(\text{SU}(k+1)) = \text{Vol}(\text{SU}(k)) 2 \frac{\pi^{k+1}}{k!} \sqrt{\frac{k+1}{2k}}, \tag{B5}$$

where we used the fact that all the positive coroots have unitary length. If we note that the phase $e^{i(\theta_k/\epsilon_k)\lambda_{(k+1)^2-1}}$ generates a U(1) group of volume $2\pi\sqrt{k(k+1)/2}$ and that $\text{U}(k)=[\text{SU}(k) \times \text{U}(1)]/Z_k$, we can finally write

$$\text{Vol}(\text{SU}(k+1)) = \text{Vol}(\text{U}(k)) \frac{\pi^k}{k!}, \tag{B6}$$

APPENDIX C: THE FUBINI-STUDY METRIC FOR $\mathbb{CP}^N$

$\mathbb{CP}^N$ is a Kähler manifold of complex dimension N. In a local chart, which uses holomorphic inhomogeneous coordinates $\{z^i\}_{i=1}^{N} \in \mathbb{C}$, the Kähler potential is $K(z^i, \bar{z}^j) = k/2 \log(1+\sum_{i=1}^{N}|z^i|^2)$ with k a constant. The associated Kähler metric $g_{i\bar{j}} = \partial^2 K/\partial z^i \partial \bar{z}^j$ is then

$$ds^2_{\mathbb{CP}^N} = k \left(\frac{\sum_{i=1}^{N} dz^i \, d\bar{z}^i}{1+\sum_{i=1}^{N}|z^i|^2} - \frac{\sum_{i,j=1}^{N} z^i \, d\bar{z}^i \, \bar{z}^j \, dz^j}{(1+\sum_{i=1}^{N}|z^i|^2)^2} \right). \tag{C1}$$

Notice that obviously it is not possible to cover the whole space with a single chart, but the set of points which cannot be covered has vanishing measure. For our purpose it is therefore enough to consider a single chart.

Let us now search for a trigonometric coordinatization. To this aim let us introduce the new real coordinates ξ, ω_μ, ψ_i, $\mu=1,\ldots,N-1$, $i=1,\ldots,N$, such that

$$z^i = \tan \xi R^i(\omega_\mu) e^{i\psi_i}. \tag{C2}$$

Here $R^i(\omega_\mu)$ is a parametrization of the unit sphere S^{n-1}, construced as an immersion in $\mathbb{R}^N$, where $\sum_{i=1}^{N}(R^i)^2=1$ and ω_μ are the angles of the sphere. However, notice that we are restricted to the positive orthant only: $R_i>0$. If ω_μ are the standard angles (starting, for example, with the azimuthal one ω_1), then $\omega_\mu \in [0,\pi/2]$, $\xi \in [0,\pi/2]$, and $\psi_i \in [0,2\pi]$. This choice of coordinates finally gives

$$\mathrm{d}s^2_{\mathbb{C}P^N} = \mathrm{d}\xi^2 + \sin^2 \xi \left[\sum_{i=1}^{N} \mathrm{d}R^i \, \mathrm{d}R^i + \sum_{i=1}^{N} (R^i)^2 d^2\psi_i \right] - \sin^4 \xi \left[\sum_{i=1}^{N} (R^i)^2 \, \mathrm{d}\psi_i \right]^2. \tag{C3}$$

In particular notice that the coefficient of $\sin^2 \xi$ yields a metric for (the positive orthant of) the sphere S^{N-1}.

APPENDIX D: FINAL CHECKS

Here we verify that the change of variables introduced in Sec. V transforms the terms

$$[\mathrm{d}\theta_N + X_N]^2 + \sum_{j=1}^{N-1} \left[\frac{1}{2}\mathrm{Tr}(e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}})\lambda_{j^2} \right]^2$$

$$+ \sum_{j=1}^{N-1} \left[\frac{1}{2}\mathrm{Tr}(e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}})\lambda_{j^2+1} \right]^2, \tag{D1}$$

into the coefficient of $\sin^2 \xi$ in (C3).

First, using (4.10) and the relations in Appendix A, it is possible to show that

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_{j^2}\}$$
$$= \cos \phi_{N-1} \, \mathrm{Tr}\{e^{-i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}J_{h_{N-1}}e^{i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N^2-1)-1}}\lambda_{j^2}\}, \quad j < N-1,$$

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_{j^2+1}\}$$
$$= \cos\phi_{N-1}\mathrm{Tr}\{e^{-i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}J_{h_{N-1}}e^{i(\theta_{N-1}/\epsilon_{N-1})\lambda_{(N-1)^2-1}}\lambda_{j^2+1}\}, \quad j < N-1,$$

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_{(N-1)^2}\} = \sin(2\phi_{N-1})\cos(N\theta_N)[\mathrm{d}\theta_{N-1} + X_{N-1}] - 2\sin(N\theta_N)\mathrm{d}\phi_{N-1},$$

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_{(N-1)^2+1}\} = \sin(2\phi_{N-1})\sin(N\theta_N)[\mathrm{d}\theta_{N-1} + X_{N-1}] + 2\cos(N\theta_N)\mathrm{d}\phi_{N-1}.$$
$$\tag{D2}$$

Note that these are true for $N \geq 3$, if we define $X_2 := \cos(2\phi_1)\mathrm{d}\theta_1$. From these relations we find

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_{j^2}\} = \left[\prod_{k=j+1}^{N-1} \cos \phi_k \right] [\sin(2\phi_j)\cos[(j+1)\theta_{j+1}](\mathrm{d}\theta_j + X_j)$$
$$- 2\sin[(j+1)\theta_{j+1}]\mathrm{d}\phi_j],$$

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_{j^2+1}\} = \left[\prod_{k=j+1}^{N-1} \cos \phi_k \right] [\sin(2\phi_j)\sin[(j+1)\theta_{j+1}](\mathrm{d}\theta_j + X_j)$$
$$+ 2\cos[(j+1)\theta_{j+1}]\mathrm{d}\phi_j],$$

with $j = 2, \ldots, N-1$. For $j = 1$,

$$\mathrm{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_1\} = \left[\prod_{k=2}^{N-1} \cos \phi_k \right] [\sin(2\phi_1)\cos(2\theta_2)\mathrm{d}\theta_1 - \sin(2\theta_2)\mathrm{d}\phi_1],$$

$$\text{Tr}\{e^{-i(\theta_N/\epsilon_N)\lambda_{N^2-1}}J_{h_N}e^{i(\theta_N/\epsilon_N)\lambda_{N^2-1}}\lambda_2\} = \left[\prod_{k=2}^{N-1}\cos\phi_k\right][\sin(2\phi_1)\sin(2\theta_2)\mathrm{d}\theta_1 + \cos(2\theta_2)\mathrm{d}\phi_1].$$

Thus we see that (D2) takes the form $S_N + U_N$, where

$$S_N = \mathrm{d}\phi_{N-1}^2 + \sum_{j=1}^{N-2}\left[\prod_{k=j+1}^{N-1}\cos^2\phi_k\right]\mathrm{d}\phi_j^2, \tag{D3}$$

$$U_N = (\mathrm{d}\theta_N + X_N)^2 + \sum_{j=2}^{N-1}\sin^2\phi_j\left[\prod_{k=j}^{N-1}\cos^2\phi_k\right](\mathrm{d}\theta_j + X_j)^2 + \prod_{k=2}^{N-1}\cos^2\phi_k\sin^2(2\phi_1)\mathrm{d}\theta_1^2. \tag{D4}$$

First, we show that

$$S_N = \sum_{j=1}^{N}\mathrm{d}R^j\mathrm{d}R^j, \tag{D5}$$

with R^j as in (5.8). To this aim let us define the N-dimensional vector $\vec{R}_N = (R^1, \ldots, R^N)$. Such a vector has unit length, and satisfies the recurrence relation $\vec{R}_N = (\sin\phi_{N-1}, \cos\phi_{N-1}\vec{R}_{N-1})$, from which we find

$$\mathrm{d}\vec{R}_N \cdot \mathrm{d}\vec{R}_N = \mathrm{d}\phi_{N-1}^2 + \cos^2\phi_{N-1}\,\mathrm{d}\vec{R}_{N-1}\cdot\mathrm{d}\vec{R}_{N-1}. \tag{D6}$$

Here the dot indicates the scalar product in N dimensions. Now, from (D3), we also have

$$S_N = \mathrm{d}\phi_{N-1}^2 + \cos^2\phi_{N-1}S_{N-1}. \tag{D7}$$

As S_N and $\mathrm{d}\vec{R}_N\cdot\mathrm{d}\vec{R}_N$ both satisfy the same recurrence relation, the thesis follows because of $S_2 = \mathrm{d}\vec{R}_2\cdot\mathrm{d}\vec{R}_2$.

The second and last step of our proof consists in showing that after the change of coordinates (5.6) the Eq. (D4) takes the form

$$U_N = \sum_{i=1}^{N}(R^i)^2\mathrm{d}\psi_i^2. \tag{D8}$$

The structure of (D4) suggests that it is convenient to make the change of variables starting from θ_N and θ_{N-1} step by step. Note that X_{N-1} is invariant under this transformation, so that we have

$$(\mathrm{d}\theta_N + X_N)^2 + \sin^2\phi_{N-1}\cos^2\phi_{N-1}(\mathrm{d}\theta_{N-1} + X_{N-1})^2 = \sin^2\phi_{N-1}\mathrm{d}\widetilde{\theta}_N^2 + \cos^2\phi_{N-1}(\mathrm{d}\widetilde{\theta}_{N-1} + X_{N-1})^2. \tag{D9}$$

Here we have used (5.3) to express X_N in terms of X_{N-1}. Then U_N takes the form

$$U_N = \sin^2\phi_{N-1}\mathrm{d}\widetilde{\theta}_N^2 + \cos^2\phi_{N-1}[(\mathrm{d}\widetilde{\theta}_{N-1} + X_{N-1})^2 + \sin^2\phi_{N-2}\cos^2\phi_{N-2}(\mathrm{d}\theta_{N-2} + X_{N-2})^2] + \cdots. \tag{D10}$$

Now it is possible to use (D9) with $N-1$ in place of N in order to write θ_{N-2} in terms of $\widetilde{\theta}_{N-2}$. In fact, this relation can be applied recursively up to $\mathrm{d}\theta_3$, obtaining

$$U_N = \sin^2 \phi_{N-1} \, d\widetilde{\theta}_N^2 + \sum_{j=2}^{N-4} \sin^2 \phi_{N-j} \left[\prod_{l=N-j+1}^{N-1} \cos^2 \phi_l \right] d\widetilde{\theta}_{N-j+1}^2 + \left[\prod_{l=3}^{N-1} \cos^2 \phi_l \right] (d\widetilde{\theta}_3 + X_3)^2$$

$$+ \sin^2 \phi_2 \left[\prod_{k=2}^{N-1} \cos^2 \phi_k \right] (d\theta_2 + \cos(2\phi_1) d\theta_1)^2 + \left[\prod_{k=2}^{N-1} \cos^2 \phi_k \right] \sin^2(2\phi_1) \theta_1^2. \qquad (D11)$$

At this point we can perform the last two changes of coordinates in (5.6), to show that

$$d\widetilde{\theta}_3 + X_3 = \sin^2 \phi_2 \, d\widetilde{\theta}_3 + \cos^2 \phi_2 (\sin^2 \phi_1 \widetilde{\theta}_2 + \cos^2 \phi_1 \, d\widetilde{\theta}_1), \qquad (D12)$$

and this completes the proof.

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