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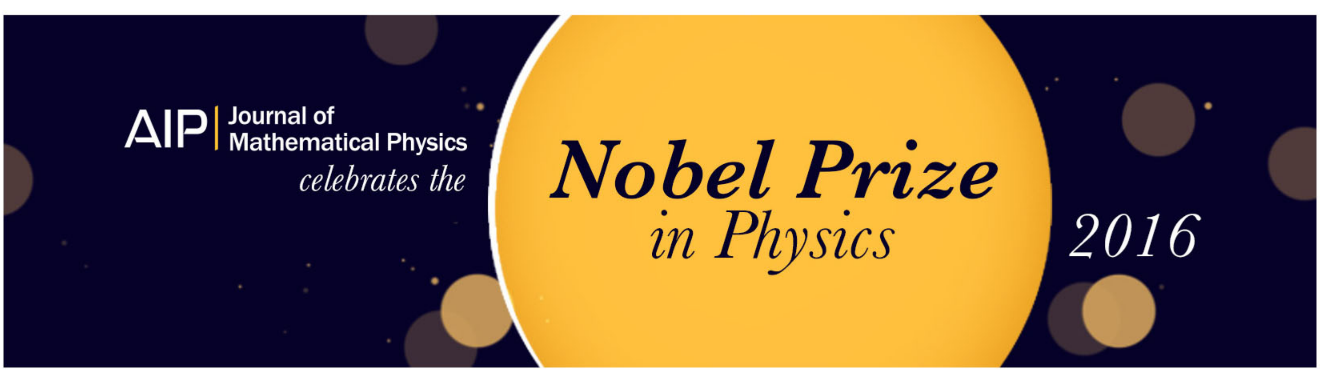

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Mapping the geometry of the E_6 group

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In this paper, we present a construction for the compact form of the exceptional Lie group E_6 by exponentiating the corresponding Lie algebra $\mathfrak{e}_6$, which we realize as the sum of $\mathfrak{f}_4$, the derivations of the exceptional Jordan algebra J_3 of dimension 3 with octonionic entries, and the right multiplication by the elements of J_3 with vanishing trace. Our parametrization is a generalization of the Euler angles for SU(2) and it is based on the fibration of E_6 via an F_4 subgroup as the fiber. It makes use of a similar construction we have performed in a previous article for F_4. An interesting first application of these results lies in the fact that we are able to determine an explicit expression for the Haar invariant measure on the E_6 group manifold. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2830522]

I. INTRODUCTION

The standard model (SM) provides a very good description of elementary particle physics. However, despite its success, there are reasons to go beyond it, for example, the recent discovery of neutrino oscillations, the fine tuning of the mixing matrices, the hierarchy problem, the difficulty in including gravity, and so on.

A starting point could be the fact that the renormalization flow of the coupling constants suggests the unification of gauge interactions at energies of the order of 10^{15} GeV, which can be improved (fine tuned) by supersymmetry. It is natural to expect the gauge group G of the grand unification theory (GUT) theory to be a simple group. Obviously, at low energies, the GUT model must reproduce the SM physics, so that not only G should contain $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ (the SM gauge group) but it should also predict the correct spectra after spontaneous symmetry breaking. The increasing accuracy in the analysis of particle spectra imposes even more restrictions in the possible choices for G. For example, it is already known that the current estimate for the lower bound of the proton lifetime rules out some of the GUT model candidates. Recently, the particular structure of the neutrino mixing matrix seems to suggest that a good candidate for a GUT could be based on the semidirect product between the exceptional group E_6^4 and the discrete group S_4.[3,6]

In general, while the local properties of group G, which for a Lie group are all encoded in the corresponding Lie algebra, are enough to perform a perturbative analysis, in order to obtain nonperturbative results, such as instantonic calculations or lattice simulations, a knowledge of the

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entire group structure is required, in particular, of the invariant measure on the group in a suitable global parametrization. There are many ways to give such an explicit expression for the Haar measure on a Lie group. However, for large dimensions, it becomes quite hard to find a realization, which is able at the same time to provide both a reasonably simple form for the measure and an explicit determination for the range of the angles. In this paper, we solve this problem for the exceptional Lie group E_6, by constructing a generalization of the Euler parametrization, with a technique we have introduced in Ref. 2 and fully developed in Ref. 1. In Sec. II, we explain how the $\mathfrak{e}_6$ algebra can be represented using a theorem due to Chevalley and Schafer. The construction of the group and the determination of the corresponding Haar measure is made in Sec. III.

II. THE CONSTRUCTION OF THE $\mathfrak{e}_6$ ALGEBRA

Our starting point for the construction of the exceptional algebra $\mathfrak{e}_6$ is a theorem due to Chevalley and Schafer[5,7] which we rewrite here for convenience.

Theorem 2.1: *The exceptional simple Lie algebra* $\mathfrak{f}_4$ *of dimension* 52 *and rank* 4 *over* K *is the derivation algebra* $\mathfrak{D}$ *of the exceptional Jordan algebra* $\mathfrak{J}$ *of dimension* 27 *over* K. *The exceptional simple Lie algebra* $\mathfrak{e}_6$ *of dimension* 78 *and rank* 6 *over* K *is the Lie algebra*

$$\mathfrak{D} + \{R_Y\}, \quad \mathrm{Tr}\, Y = 0, \tag{2.1}$$

spanned by the derivations of $\mathfrak{J}$ *and the right multiplications of elements* Y *of trace 0.*

We will refer to Ref. 8 for the notations. Then, the exceptional Jordan algebra is the algebra J_3. For our purposes, K will be $\mathbb{R}$ or $\mathbb{C}$. The right multiplication is $R_Y(X) = Y \circ X$ and the trace is the sum of diagonal elements. The exceptional Jordan algebra is 27 dimensional, just like the principal fundamental representation of $\mathfrak{e}_6$.

In our previous paper,[1] we determined the algebra of derivations $\mathfrak{D}$ and used it to obtain an irreducible representation of the exceptional algebra $\mathfrak{f}_4$. To obtain a representation of $\mathfrak{e}_6$, we only need to determine the matrix representation of R_Y. This is a very simple task and we solved it in $\mathbb{R}^{27}$ with the product inherited from J_3 by means of the linear isomorphism

$$\Phi: J_3 \to \mathbb{R}^{27}, \quad A \mapsto \Phi(A),$$

$$\Phi(A) := \begin{pmatrix} a_1 \\ \rho(o_1) \\ \rho(o_2) \\ a_2 \\ \rho(o_3) \\ a_3 \end{pmatrix}, \tag{2.2}$$

where A is the Jordan matrix,

$$A = \begin{pmatrix} a_1 & o_1 & o_2 \\ \overset{*}{o}_1 & a_2 & o_3 \\ \overset{*}{o}_2 & \overset{*}{o}_3 & a_3 \end{pmatrix}, \tag{2.3}$$

and ρ is the linear isomorphism between the octonions $\mathbb{O}$ and $\mathbb{R}^8$ given by[1]

$$\rho: \mathbb{O} \to \mathbb{R}, \quad o = o^0 + \sum_{i=1}^{7} o^i i_i \mapsto \rho(o),$$

[1]See Ref. 1 for more details.

$$\rho(o) := \begin{pmatrix} o^0 \\ o^1 \\ o^2 \\ o^3 \\ o^4 \\ o^5 \\ o^6 \\ o^7 \end{pmatrix}. \tag{2.4}$$

After choosing a 26 dimensional base of traceless Jordan matrices, we used the MATHEMATICA program in Appendix A to find the matrices which complete the base for $\mathfrak{f}_4$ to a base for the whole $\mathfrak{e}_6$ algebra. In particular, if we work in the real case, we obtain the split form of $\mathfrak{e}_6$, with signature (52, 26). However, multiplying the 26 added generators by i, we find that the algebra remains real, but the Killing form becomes the compact one.

III. THE CONSTRUCTION OF THE GROUP E_6

A. The generalized Euler parametrization for E_6

To give an Euler parametrization for E_6, we start by choosing its maximal subgroup. It is $H=F_4$, the group generated by the first 52 matrices. Let $\mathcal{P}$ be the linear complement of $\mathfrak{f}_4$ in $\mathfrak{e}_6$. We then search for a minimal linear subset V of $\mathcal{P}$, which, under the action of $\mathrm{Ad}(F_4)$, generates the whole $\mathcal{P}$. Looking at the structure constants, we see that V can be chosen as the linear space generated by c_{53},c_{70}. Note that they commute.

If we then write the general element g of E_6 in the form

$$g=\exp(\tilde{h})\exp(v)\exp(h), \quad h,\tilde{h} \in \mathfrak{f}_4, \quad v \in V, \tag{3.1}$$

we have a redundancy of dimension 28. As argued in Ref. 1, we expect to find a 28 dimensional subgroup of F_4, which, acting by adjunction, defines an automorphism of V. This can be done by noticing that V commutes with the first 28 matrices c_i, $i=1,\ldots,28$, which generate an SO(8) subgroup of F_4.

We found convenient to introduce the change of base

$$\tilde{c}_{53} := \tfrac{1}{2}c_{53} + \tfrac{\sqrt{3}}{2}c_{70}, \tag{3.2}$$

$$\tilde{c}_{70} := -\tfrac{\sqrt{3}}{2}c_{53} + \tfrac{1}{2}c_{70}. \tag{3.3}$$

Thus, we have

$$g[x_1,\ldots,x_{78}] = B[x_1,\ldots,x_{24}]e^{x_{25}\tilde{c}_{53}}e^{x_{26}\tilde{c}_{70}}F_4[x_{27},\ldots,x_{78}],$$

where $B=F_4/\mathrm{SO}(8)$. We chose for F_4 the Euler parametrization given in Ref. 1 so that we found for B,

$$B[x_1,\ldots,x_{24}] = B_{F_4}[x_1,\ldots,x_{16}]B_9[x_{17},\ldots,x_{23}]e^{x_{24}c_{45}}, \tag{3.4}$$

where

$$B_{F_4}[x_1,\ldots,x_{15}] = B_9[x_1,\ldots,x_7]e^{x_8c_{45}}B_8[x_9,\ldots,x_{14}]e^{x_{15}\tilde{c}_{30}}e^{x_{16}c_{22}}, \tag{3.5}$$

$$B_9[x_1,\ldots,x_7] = e^{x_1\tilde{c}_3}e^{x_2\tilde{c}_{16}}e^{x_3\tilde{c}_{15}}e^{x_4\tilde{c}_{35}}e^{x_5\tilde{c}_5}e^{x_6\tilde{c}_1}e^{x_7\tilde{c}_{30}}, \tag{3.6}$$

$$B_8[x_1,\ldots,x_6] = e^{x_1\tilde{c}_3}e^{x_2\tilde{c}_{16}}e^{x_3\tilde{c}_{15}}e^{x_4\tilde{c}_{35}}e^{x_5\tilde{c}_5}e^{x_6\tilde{c}_1}, \tag{3.7}$$

and the tilded matrices are the ones introduced in Ref. 1.

B. Determination of the range for the parameters

To determine the range of the parameters, we will use the topological method we have developed in Ref. 2. Let us first determine the volume of E_6 by means of the Macdonald formula.

1. The volume of E_6

The rational homology of the exceptional Lie group E_6 is that of a product of odd dimensional spheres,[10] $H*(E_6)=H*(\Pi_{i=1}^{6}S^{d_i})$, with[4]

$$d_1 = 3, \quad d_2 = 9, \quad d_3 = 11, \quad d_4 = 15, \quad d_5 = 17, \quad d_6 = 23. \tag{3.8}$$

The simple roots of E_6 are

$$r_1 = L_1 + L_2, \tag{3.9}$$

$$r_2 = L_2 - L_1, \tag{3.10}$$

$$r_3 = L_3 - L_2, \tag{3.11}$$

$$r_4 = L_4 - L_3, \tag{3.12}$$

$$r_5 = L_5 - L_4, \tag{3.13}$$

$$r_6 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2}, \tag{3.14}$$

where L_i, $i=1,\ldots,6$, is an orthonormal base for the Cartan algebra. The volume of the fundamental region is then

$$\mathrm{Vol}(f_{E_6}) = \frac{2}{L}. \tag{3.15}$$

Indeed, we computed the 36 positive roots, all having length $\sqrt{2}$. They have exactly the structure given in Ref. 8, with $L_i=e_i$, the canonical base of $\mathbb{R}^6$. The Macdonald formula[9,11] gives for the volume of the compact form of E_6,

$$\mathrm{Vol}(E_6) = \frac{\sqrt{3}\cdot 2^{17}\cdot \pi^{42}}{3^{10}\cdot 5^5\cdot 7^3\cdot 11}. \tag{3.16}$$

2. The invariant measure on E_6

With the chosen generalized Euler parametrization, the invariant measure on E_6 decomposes into the product of the measure of F_4 and the one on $M=E_6/F_4$. The invariant measure on F_4 was computed in Ref. 1 so that we need to compute here only the induced measure on M.

Using the notation of Ref. 1, let us define

$$J_M \coloneqq \pi_{\mathcal{P}}(e^{-x_{26}\tilde{c}_{70}}e^{-x_{25}\tilde{c}_{53}}B[x_1,\ldots,x_{24}]^{-1}d(B[x_1,\ldots,x_{24}]e^{x_{25}\tilde{c}_{53}}e^{x_{26}\tilde{c}_{70}})), \tag{3.17}$$

where $\pi_{\mathcal{P}}$ is the projection on the subspace generated by c_j, $j=53,\ldots,78$. The metric induced on M by the bi-invariant metric on E_6 is then

$$ds_M^2 = -\frac{1}{6}\mathrm{Tr}(J_M \otimes J_M), \tag{3.18}$$

and the invariant measure on E_6 is then

$$d\mu_{E_6} = |\det(J_{Mi}^j)| d\mu_{F_4} \prod_{l=1}^{26} dx_l, \tag{3.19}$$

where J_{Mi}^j is the 26×26 matrix defined by

$$J_M = \sum_{i,j=1}^{26} J_{Mi}^j c_{i_j} dx^i, \tag{3.20}$$

with c_{i_j} a base $\{\tilde{c}_{53}, \tilde{c}_{70}, c_{54}, \ldots, c_{69}, c_{71}, \ldots, c_{78}\}$ of $\mathcal{P}$.

In order to compute $|\det(J_{Mi}^j)|$, it is convenient to introduce the notations

$$\omega[x,y,z] = e^{xc_{45}} e^{y\tilde{c}_{53}} e^{z\tilde{c}_{70}}, \tag{3.21}$$

$$J_\omega \coloneqq \omega^{-1} d\omega, \tag{3.22}$$

$$J_9 \coloneqq B_9^{-1} dB_9, \tag{3.23}$$

$$J_{F_4} \coloneqq B_{F_4}^{-1} dB_{F_4}, \tag{3.24}$$

so that

$$J_M[x_1, \ldots, x_{26}] = \omega[x_{24}, x_{25}, x_{26}]^{-1} B_9[x_{17}, \ldots, x_{23}]^{-1} J_{F_4}[x_1, \ldots, x_{16}] B_9[x_{17}, \ldots, x_{23}] \omega[x_{24}, x_{25}, x_{26}] + \omega[x_{24}, x_{25}, x_{26}]^{-1} J_9[x_{17}, \ldots, x_{23}] \omega[x_{24}, x_{25}, x_{26}] + J_\omega[x_{24}, x_{25}, x_{26}]. \tag{3.25}$$

Some remarks are in order now.

(1) The following relations are true:

$$e^{-\alpha\tilde{c}_{53}} c_L e^{\alpha\tilde{c}_{53}} = \cos\alpha c_L + \sin\alpha c_{L+26}, \tag{3.26}$$

if $L=45, \ldots, 52$. Moreover, $\tilde{c}_{70}$ commutes with c_I, $I=45, \ldots, 52, 71, \ldots, 78$, and with $\tilde{c}_{53}$.

(2) $\tilde{c}_{53}$ and $\tilde{c}_{70}$ commute with the so(8) algebra generated by the matrices c_I, $I=1, \ldots, 21, 30, \ldots, 36$.

(3) The adjoint action of $e^{x_{24}c_{45}}$ on the above so(8) algebra generates in addition the matrices c_J, $J=46, \ldots, 52$.

(4) From $J_{F_4} \in \mathfrak{f}_4$ and $B_9 \subset F_4$, it follows that

$$\tilde{J}_{F_4} \coloneqq B_9^{-1} J_{F_4} B_9 \in \mathfrak{f}_4.$$

(5) The adjoint action of ω on the $\mathfrak{f}_4$ matrices generates all the remaining matrices of $\mathfrak{e}_6$. In particular, the projection of $\omega^{-1} c_L \omega$, $L=1, \ldots, 52$, on c_J, $J=54, \ldots, 69$, is different from zero only if $L=22, \ldots, 29, 37, \ldots, 44$.

(6) The adjoint action of the SO(8) group, corresponding to the above so(8) algebra, gives a rotation both on the indices $I=\{22, \ldots, 29\}$ and $J=\{37, \ldots, 44\}$. More precisely,

$$\mathrm{SO}(8)^{-1} c_I \mathrm{SO}(8) = R_I^L c_L, \tag{3.27}$$

$$\mathrm{SO}(8)^{-1}c_J\mathrm{SO}(8)=\tilde{R}_J^K c_K, \tag{3.28}$$

where L runs from 22 to 29 and K runs from 37 to 44, and $R,\tilde{R}$ are both orthogonal matrices.

To verify these, it suffices to note that

$$e^{-xc_A}c_I e^{xc_A}=\cos\frac{x}{2}c_I\pm\sin\frac{x}{2}c_{I_A}, \tag{3.29}$$

$$e^{-xc_A}c_J e^{xc_A}=\cos\frac{x}{2}c_J\pm\sin\frac{x}{2}c_{J_A}, \tag{3.30}$$

where $A=1,\ldots,21,30,\ldots,36$, $I,I_A\in\{22,\ldots,29\}$, $J,J_A\in\{37,\ldots,44\}$. In particular, $\det(R\otimes\tilde{R})=1$.

From the first three points, we see that the matrix J_M takes the form

$$M=\begin{pmatrix} A & 0 & 0\\ * & C & 0\\ * & * & D\end{pmatrix}, \tag{3.31}$$

where on the rows we indicate the coefficients of dx_I with $I=1,\ldots,26$ starting from the bottom and on the columns the projections on $\tilde{c}_A$, c_L, following the order $A=53,70$, $L=71,\ldots,78,54,\ldots,69$. The asterisks indicate the elements which do not contribute to the determinant,

$$\det J_M=\det A\det C\det D.$$

In particular, A is a 3×3 block obtained by projecting J_ω on $\tilde{c}_{53},\tilde{c}_{70},c_{71}$. Point 1 implies

$$A=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \sin x_{25}\end{pmatrix}, \tag{3.32}$$

so that

$$\det A=\sin x_{25}. \tag{3.33}$$

The C block is a 7×7 matrix obtained by projecting $\omega^{-1}J_9\omega$ on c_K, $K=72,\ldots,78$. From point 1, it follows that it can be obtained by multiplying the projection of $e^{-x_{24}c_{45}}J_9e^{x_{24}c_{45}}$ on c_L, $L=46,\ldots,52$, by $\sin x_{25}$. If we call $\tilde{C}$ the matrix corresponding to such a projection, we find $\det C=\sin^7 x_{25}\det\tilde{C}$. The determinant of $\tilde{C}$ can be computed directly and gives

$$\det C=\sin x_{20}\cos x_{21}\cos x_{22}\sin^2 x_{22}\sin^2 x_{23}\cos^4 x_{23}\sin^7 x_{24}\sin^7 x_{25}. \tag{3.34}$$

The D block requires some further discussion. It is a 16×16 matrix obtained by projecting $\omega^{-1}\tilde{J}_{B_{F_4}}\omega$ on c_I, $I=54,\ldots,69$. First, from points (4) and (5) of our remarks, we see that only the c_L with $L=22,\ldots,29,37,\ldots,44$ in $\tilde{J}_{F_4}$ contribute to the determinant. Let us define the 16×16 matrix U with

$$U_A^B:=-\tfrac{1}{6}\mathrm{Tr}(\omega^{-1}c_A\omega c_B), \tag{3.35}$$

$$A=22,\ldots,29,37,\ldots,44, \tag{3.36}$$

$$B = 54, \ldots, 69. \tag{3.37}$$

Moreover, let $\widetilde{D}$ be the matrix obtained by projecting J_{F_4} on c_L with $L=22,\ldots,29,37,\ldots,44$, and

$$Q := \begin{pmatrix} R & 0 \\ 0 & \widetilde{R} \end{pmatrix}, \tag{3.38}$$

where R and $\widetilde{R}$ are the rotation matrices defined at point (6). Then, we can deduce from points (4)–(6) that

$$\det D = \det(U \cdot Q \cdot \widetilde{D}) = \det U \det \widetilde{D}. \tag{3.39}$$

The matrix U and its determinant are easily computed. Indeed, we have

$$\omega^{-1} c_{22} \omega = \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{61} + \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{69} + \cdots,$$

$$\omega^{-1} c_{23} \omega = -\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{57} - \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{65} + \cdots,$$

$$\omega^{-1} c_{24} \omega = -\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{60} - \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{68} + \cdots,$$

$$\omega^{-1} c_{25} \omega = \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{55} + \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{63} + \cdots,$$

$$\omega^{-1} c_{26} \omega = -\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{59} - \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{67} + \cdots,$$

$$\omega^{-1} c_{27} \omega = \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{58} + \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{66} + \cdots,$$

$$\omega^{-1} c_{28} \omega = \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{56} + \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{64} + \cdots,$$

$$\omega^{-1} c_{29} \omega = -\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{54} - \sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{62} + \cdots,$$

$$\omega^{-1} c_{37} \omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{54} + \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{62} + \cdots,$$

$$\omega^{-1} c_{38} \omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{55} + \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{63} + \cdots,$$

$$\omega^{-1} c_{39} \omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{56} + \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{64} + \cdots,$$

$$\omega^{-1}c_{40}\omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)c_{57}+\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right)c_{65}+\cdots,$$

$$\omega^{-1}c_{41}\omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)c_{58}+\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right)c_{66}+\cdots,$$

$$\omega^{-1}c_{42}\omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)c_{59}+\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right)c_{67}+\cdots,$$

$$\omega^{-1}c_{43}\omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)c_{60}+\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right)c_{68}+\cdots,$$

$$\omega^{-1}c_{44}\omega = -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)c_{61}+\cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right)c_{69}+\cdots,$$

where ellipses stay for terms which vanish when projected on c_B, $B=54,\dots,61,70,\dots,78$. From these, it follows

$$\det U = \sin^8\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)\sin^8\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right). \tag{3.40}$$

At this point, we need to compute $\det \tilde{D}$. This can be done by noticing that J_{F_4} coincides exactly with the current J_M of F_4 in Ref. 1. We then find

$$\det \tilde{D} = 2^7 \sin^{15}\frac{x_{16}}{2}\cos^7\frac{x_{16}}{2}\sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8$$
$$\times \sin x_{12}\cos x_{13}\cos x_{14}\sin^2 x_{14}\cos^2 x_{15}\sin^4 x_{15}. \tag{3.41}$$

We can finally write down the invariant measure on the base space,

$$d\mu_M = 2^7 \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8$$
$$\times \sin x_{12}\cos x_{13}\cos x_{14}\sin^2 x_{14}\cos^2 x_{15}\sin^4 x_{15}\sin^{15}\frac{x_{16}}{2}\cos^7\frac{x_{16}}{2}$$
$$\times \sin x_{20}\cos x_{21}\cos x_{22}\sin^2 x_{22}\sin^2 x_{23}\cos^4 x_{23}\sin^7 x_{24}$$
$$\times \sin^8 x_{25}\sin^8\left(\frac{\sqrt{3}}{2}x_{26}+\frac{x_{25}}{2}\right)\sin^8\left(\frac{\sqrt{3}}{2}x_{26}-\frac{x_{25}}{2}\right). \tag{3.42}$$

Note that the periods of the variables are 4π so that one should take the range $x_i=[0,4\pi]$ for $i=1,2,3,,9,10,11,17,18,19$. However, as in Ref. 1, it is easy to show directly from the parametrization that they can all be restricted to $[0,2\pi]$. The range of x_i is then

$$x_1 \in [0,2\pi], \quad x_2 \in [0,2\pi], \quad x_3 \in [0,2\pi], \quad x_4 \in [0,\pi],$$

$$x_5 \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \quad x_6 \in \left[0,\frac{\pi}{2}\right], \quad x_7 \in \left[0,\frac{\pi}{2}\right], \quad x_8 \in [0,\pi],$$

$$x_9 \in [0,2\pi], \quad x_{10} \in [0,2\pi], \quad x_{11} \in [0,2\pi], \quad x_{12} \in [0,\pi],$$

$$x_{13} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \quad x_{14} \in \left[0,\frac{\pi}{2}\right], \quad x_{15} \in \left[0,\frac{\pi}{2}\right], \quad x_{16} \in [0,\pi], \tag{3.43}$$

$$x_{17} \in [0,2\pi], \quad x_{18} \in [0,2\pi], \quad x_{19} \in [0,2\pi], \quad x_{20} \in [0,\pi],$$

$$x_{21} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \quad x_{22} \in \left[0,\frac{\pi}{2}\right], \quad x_{23} \in \left[0,\frac{\pi}{2}\right], \quad x_{24} \in [0,\pi],$$

$$x_{25} \in \left[0,\frac{\pi}{2}\right], \quad -\frac{x_{25}}{\sqrt{3}} \leqslant x_{26} \leqslant \frac{x_{25}}{\sqrt{3}}.$$

The remaining parameters x_j, $j=27,\ldots,78$, will run over the range for F_4, as given in Ref. 1. The volume of the whole closed cycle V so obtained is then

$$\mathrm{Vol}(V) = \mathrm{Vol}(F_4)\int_R d\mu_M = \frac{\sqrt{3}\cdot 2^{17}\cdot \pi^{42}}{3^{10}\cdot 5^5\cdot 7^3\cdot 11}, \tag{3.44}$$

where R is the range of parameters x_i, $i=1,\ldots,26$. This is the volume of E_6, so that we cover the group exactly once.[2]

IV. CONCLUSIONS

In this paper, we have performed an explicit construction for the compact form of the simple Lie group E_6. This is particularly interesting because recently it has been argued that this group could be the most promising for unification in GUT theories.[3,6] To parameterize the group, we have used the generalized Euler angle method, a technique we introduced in Refs. 2 and 1 to give the most simple expression for the invariant measure on the group, while at the same time still being able to provide an explicit expression for the range of the parameters. Both these requirements are necessary in order to minimize the computation power needed for computer simulations, for example, of lattice models.

Our results can be easily extended to the GUT group $E_6^4 \rtimes S_4$. Also, a modified Euler parametrization could be applied to evidentiate the subgroups related to the correct symmetry breaking, see Ref. 3. Finally, our parametrization could be used for a straightforward geometrical analysis of the exceptional Lie group E_6 and its quotients.

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APPENDIX A: THE $\mathfrak{e}_6$ MATRICES

The matrices we found using MATHEMATICA, and orthonormalized with respect to the scalar product $\langle a,b\rangle := -\frac{1}{6}\mathrm{Tr}(ab)$, was computed by means of the followings programs.

The MATHEMATICA program

% % % % the octonionic products

[2]Obviously, there is a subset of vanishing measure which is multiply covered.

QQ[1, 1] = e[1]; QQ[1, 2] = e[2]; QQ[1, 3] = e[3];

QQ[1, 4] = e[4]; QQ[1, 5] = e[5]; QQ[1, 6] = e[6];

QQ[1, 7] = e[7]; QQ[1, 8] = e[8]; QQ[2, 1] = e[2];

QQ[2, 2] = − e[1]; QQ[2, 3] = e[5]; QQ[2, 4] = e[8];

QQ[2, 5] = − e[3]; QQ[2, 6] = e[7]; QQ[2, 7] = − e[6];

QQ[2, 8] = − e[4]; QQ[3, 1] = e[3]; QQ[3, 2] = − e[5];

QQ[3, 3] = − e[1]; QQ[3, 4] = e[6]; QQ[3, 5] = e[2];

QQ[3, 6] = − e[4]; QQ[3, 7] = e[8]; QQ[3, 8] = − e[7];

QQ[4, 1] = e[4]; QQ[4, 2] = − e[8]; QQ[4, 3] = − e[6];

QQ[4, 4] = − e[1]; QQ[4, 5] = e[7]; QQ[4, 6] = e[3];

QQ[4, 7] = − e[5]; QQ[4, 8] = e[2]; QQ[5, 1] = e[5];

QQ[5, 2] = e[3]; QQ[5, 3] = − e[2]; QQ[5, 4] = − e[7];

QQ[5, 5] = − e[1]; QQ[5, 6] = e[8]; QQ[5, 7] = e[4];

QQ[5, 8] = − e[6]; QQ[6, 1] = e[6]; QQ[6, 2] = − e[7];

QQ[6, 3] = e[4]; QQ[6, 4] = − e[3]; QQ[6, 5] = − e[8];

QQ[6, 6] = − e[1]; QQ[6, 7] = e[2]; QQ[6, 8] = e[5];

QQ[7, 1] = e[7]; QQ[7, 2] = e[6]; QQ[7, 3] = − e[8];

QQ[7, 4] = e[5]; QQ[7, 5] = − e[4]; QQ[7, 6] = − e[2];

QQ[7, 7] = − e[1]; QQ[7, 8] = e[3]; QQ[8, 1] = e[8];

QQ[8, 2] = e[4]; QQ[8, 3] = e[7]; QQ[8, 4] = − e[2];

QQ[8, 5] = e[6]; QQ[8, 6] = − e[5]; QQ[8, 7] = − e[3];

QQ[8, 8] = − e[1];

% % % % the Jordan algebra product

```
Qm[x_, y_] := Sum[Sum[x[[i]]y[[j]]QQ[i, j], {i, 8}], {j, 8}]

QP[x_, y_] := {Coefficient[Qm[x, y], e[1]], Coefficient[Qm[x, y], e[2]],
    Coefficient[Qm[x, y], e[3]], Coefficient[Qm[x, y], e[4]],
    Coefficient[Qm[x, y], e[5]], [Qm[x, y], e[6]], Coefficient[Qm[x, y], e[7]],
    Coefficient[Qm[x, y], e[8]]}

OctP[a_, b_] := {{Sum[QP[Part[Part[a, 1], i], Part[Part[b, i], 1]], {i, 3}],
    Sum[QP[Part[Part[a, 1], i], Part[Part[b, i], 2]], {i, 3}],
    Sum[QP[Part[Part[a, 1], i], Part[Part[b, i], 3]], {i, 3}]},
    {Sum[QP[Part[Part[a, 2], i], Part[Part[b, i], 1]], {i, 3}],
    Sum[QP[Part[Part[a, 2], i], Part[Part[b, i], 2]], {i, 3}],
    Sum[QP[Part[Part[a, 2], i], Part[Part[b, i], 3]], {i, 3}]},
    {Sum[QP[Part[Part[a, 3], i], Part[Part[b, i], 1]], {i, 3}],
    Sum[QP[Part[Part[a, 3], i], Part[Part[b, i], 2]], {i, 3}],
    Sum[QP[Part[Part[a, 3], i], Part[Part[b, i], 3]], {i, 3}]}}

OctPS[a_, b_] := 1/2(OctP[a, b] + OctP[b, a])

% % % % correspondence between the Jordan algebra and ℝ^27

FF[AA_] := {Part[{Part[Part[Part[AA, 1], 1], 1]}, 1], Part[{Part[Part[Part[AA, 1], 2], 1]}, 1],
    Part[{Part[Part[Part[AA, 1], 2], 2]}, 1], Part[{Part[Part[Part[AA, 1], 2], 3]}, 1],
    Part[{Part[Part[Part[AA, 1], 2], 4]}, 1], Part[{Part[Part[Part[AA, 1], 2], 5]}, 1],
    Part[{Part[Part[Part[AA, 1], 2], 6]}, 1], Part[{Part[Part[Part[AA, 1], 2], 7]}, 1],
    Part[{Part[Part[Part[AA, 1], 2], 8]}, 1], Part[{Part[Part[Part[AA, 1], 3], 1]}, 1],
    Part[{Part[Part[Part[AA, 1], 3], 2]}, 1], Part[{Part[Part[Part[AA, 1], 3], 3]}, 1],
    Part[{Part[Part[Part[AA, 1], 3], 4]}, 1], Part[{Part[Part[Part[AA, 1], 3], 5]}, 1],
    Part[{Part[Part[Part[AA, 1], 3], 6]}, 1], Part[{Part[Part[Part[AA, 1], 3], 7]}, 1],
    Part[{Part[Part[Part[AA, 1], 3], 8]}, 1], Part[{Part[Part[Part[AA, 2], 2], 1]}, 1],
    Part[{Part[Part[Part[AA, 2], 3], 1]}, 1], Part[{Part[Part[Part[AA, 2], 3], 2]}, 1],
    Part[{Part[Part[Part[AA, 2], 3], 3]}, 1], Part[{Part[Part[Part[AA, 2], 3], 4]}, 1],
    Part[{Part[Part[Part[AA, 2], 3], 5]}, 1], Part[{Part[Part[Part[AA, 2], 3], 6]}, 1],
    Part[{Part[Part[Part[AA, 2], 3], 7]}, 1], Part[{Part[Part[Part[AA, 2], 3], 8]}, 1],
    Part[{Part[Part[Part[AA, 3], 3], 1]}, 1]}
```

```
FFi[vv_] := {{{Part[vv, 1], 0, 0, 0, 0, 0, 0, 0}, {Part[vv, 2], Part[vv, 3], Part[vv, 4],
        Part[vv, 5], Part[vv, 6], Part[vv, 7], Part[vv, 8], Part[vv, 9]},
        {Part[vv, 10], Part[vv, 11], Part[vv, 12], Part[vv, 13], Part[vv, 14], Part[vv, 15],
        Part[vv, 16], Part[vv, 17]}}, {{Part[vv, 2], - Part[vv, 3], - Part[vv, 4],
        - Part[vv, 5], - Part[vv, 6], - Part[vv, 7], - Part[vv, 8],
        - Part[vv, 9]}, {Part[vv, 18], 0, 0, 0, 0, 0, 0, 0}, {Part[vv, 19], Part[vv, 20],
        Part[vv, 21], Part[vv, 22], Part[vv, 23], Part[vv, 24], Part[vv, 25], Part[vv, 26]}},
        {{Part[vv, 10], - Part[vv, 11], - Part[vv, 12], - Part[vv, 13], - Part[vv, 14],
        - Part[vv, 15], - Part[vv, 16], - Part[vv, 17]}, {Part[vv, 19], - Part[vv, 20],
        - Part[vv, 21], - Part[vv, 22], - Part[vv, 23], - Part[vv, 24], - Part[vv, 25],
        - Part[vv, 26]}, {Part[vv, 27], 0, 0, 0, 0, 0, 0, 0}}}

%%%% construction of the matrices

MT = {{{mt[1], 0, 0, 0, 0, 0, 0, 0}, {mt[2], mt[3], mt[4], mt[5], mt[6], mt[7], mt[8], mt[9]},
    {mt[10], mt[11], mt[12], mt[13], mt[14], mt[15], mt[16], mt[17]}}, {{mt[2], - mt[3],
    - mt[4], - mt[5], - mt[6], - mt[7], - mt[8], - mt[9]},
    {mt[18], 0, 0, 0, 0, 0, 0, 0}, {mt[19], mt[20], mt[21], mt[22], mt[23], mt[24],
    mt[25], mt[26]}}, {{mt[10], - mt[11], - mt[12], - mt[13], - mt[14], - mt[15],
    - mt[16], - mt[17]}, {mt[19], - mt[20], - mt[21], - mt[22], - mt[23], - mt[24],
    - mt[25], - mt[26]}, {- mt[1] - mt[18], 0, 0, 0, 0, 0, 0, 0}}};

h = Array[hh, 27];

M = Array[mm, {27, 27}];

imm = FF[OctPS[MT, FFi[h]]];

Do[Do[mm[i, j] = Coefficient[Part[imm, i], Part[h, j]], {i, 27}], {j, 27}]

Do[econi[j] = D[M, mt[j]], {j, 26}];

Mno[1] = econi[1];

Do[Do[AA[j, i] = 0, {i, 26}], {j, 26}]

Do[Do[AA[j, i] = - Tr[econi[j] . Mno[i]]/Tr[Mno[i] . Mno[i]], {i, j - 1}];Mno[j] = econi[j]
    + Sum[AA[j, i]Mno[i], {i, j - 1}], {j, 2, 26}]

Do[Mnno[i] = - √6Mno[i]/Sqrt[Tr[Mno[i] . Mno[i]]], {i, 26}];
```

```
% % % % rotation to give the irreducible 26 representations

XXX = {{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
    − 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}/Sqrt[2],
    {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,  − 2}/Sqrt[6],
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
    {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1}/Sqrt[3]};

Do[cc[i + 52] = XXX . Mnno[i] . Transpose[XXX], {i, 26}];
```

The matrices cc[i], i=53, …, 78, so obtained, must be added to the 52 matrices cc[i] determined in Ref. 1 to give a base of the $\mathfrak{e}_6$ algebra.

APPENDIX B: STRUCTURE CONSTANTS

The structure constants s_{IJ}^{K} are defined by $[C_I, C_J]=\Sigma_{K=1}^{78} s_{IJ}^{K} C_K$. We checked that the coefficients $s_{IJK} := s_{IJ}^{K}$ are completely antisymmetric in the indices. We write only the nonvanishing terms, up to symmetries, which are not yet written in Ref. 1,

$$s_{1,54,55}=\tfrac{1}{2},\quad s_{1,56,58}=-\tfrac{1}{2},\quad s_{1,57,61}=-\tfrac{1}{2},\quad s_{1,59,60}=-\tfrac{1}{2},$$

$$s_{1,62,63}=-\tfrac{1}{2},\quad s_{1,64,66}=\tfrac{1}{2},\quad s_{1,65,69}=\tfrac{1}{2},\quad s_{1,67,68}=\tfrac{1}{2},$$

$$s_{1,71,72}=-1,\quad s_{2,54,56}=\tfrac{1}{2},\quad s_{2,55,58}=\tfrac{1}{2},\quad s_{2,57,59}=-\tfrac{1}{2},$$

$$s_{2,60,61}=-\tfrac{1}{2},\quad s_{2,62,64}=-\tfrac{1}{2},\quad s_{2,63,66}=-\tfrac{1}{2},\quad s_{2,65,67}=\tfrac{1}{2},$$

$$s_{2,68,69}=\tfrac{1}{2},\quad s_{2,71,73}=-1,\quad s_{3,54,58}=\tfrac{1}{2},\quad s_{3,55,56}=-\tfrac{1}{2},$$

$$s_{3,57,60}=-\tfrac{1}{2},\quad s_{3,59,61}=\tfrac{1}{2},\quad s_{3,62,66}=\tfrac{1}{2},\quad s_{3,63,64}=-\tfrac{1}{2},$$

$$s_{3,65,68}=-\tfrac{1}{2},\quad s_{3,67,69}=\tfrac{1}{2},\quad s_{3,72,73}=-1,\quad s_{4,54,57}=\tfrac{1}{2},$$

$$s_{4,55,61}=\tfrac{1}{2},\quad s_{4,56,59}=\tfrac{1}{2},\quad s_{4,58,60}=-\tfrac{1}{2},\quad s_{4,62,65}=-\tfrac{1}{2},$$

$$s_{4,63,69}=-\tfrac{1}{2},\quad s_{4,64,67}=-\tfrac{1}{2},\quad s_{4,66,68}=\tfrac{1}{2},\quad s_{4,71,74}=-1,$$

$$s_{5,54,61}=\tfrac{1}{2},\quad s_{5,55,57}=-\tfrac{1}{2},\quad s_{5,56,60}=\tfrac{1}{2},\quad s_{5,58,59}=\tfrac{1}{2},$$

$$s_{5,62,69}=\tfrac{1}{2},\quad s_{5,63,65}=-\tfrac{1}{2},\quad s_{5,64,68}=\tfrac{1}{2},\quad s_{5,66,67}=\tfrac{1}{2},$$

$$s_{5,72,74}=-1,\quad s_{6,54,59}=\tfrac{1}{2},\quad s_{6,55,60}=-\tfrac{1}{2},\quad s_{6,56,57}=-\tfrac{1}{2},$$

$$s_{6,58,61}=-\tfrac{1}{2},\quad s_{6,62,67}=\tfrac{1}{2},\quad s_{6,63,68}=-\tfrac{1}{2},\quad s_{6,64,65}=-\tfrac{1}{2},$$

$$s_{6,66,69}=-\tfrac{1}{2},\quad s_{6,73,74}=-1,\quad s_{7,54,58}=\tfrac{1}{2},\quad s_{7,55,56}=-\tfrac{1}{2},$$

$$s_{7,57,60}=\tfrac{1}{2},\quad s_{7,59,61}=-\tfrac{1}{2},\quad s_{7,62,66}=-\tfrac{1}{2},\quad s_{7,63,64}=\tfrac{1}{2},$$

$$s_{7,65,68}=-\tfrac{1}{2},\quad s_{7,67,69}=\tfrac{1}{2},\quad s_{7,71,75}=-1,\quad s_{8,54,56}=-\tfrac{1}{2},$$

$$s_{8,55,58}=-\tfrac{1}{2},\quad s_{8,57,59}=-\tfrac{1}{2},\quad s_{8,60,61}=-\tfrac{1}{2},\quad s_{8,62,64}=-\tfrac{1}{2},$$

$$s_{8,63,66}=-\tfrac{1}{2},\quad s_{8,65,67}=-\tfrac{1}{2},\quad s_{8,68,69}=-\tfrac{1}{2},\quad s_{8,72,75}=-1,$$

$$s_{9,54,55}=\tfrac{1}{2},\quad s_{9,56,58}=-\tfrac{1}{2},\quad s_{9,57,61}=\tfrac{1}{2},\quad s_{9,59,60}=\tfrac{1}{2},$$

$$s_{9,62,63}=\tfrac{1}{2},\quad s_{9,64,66}=-\tfrac{1}{2},\quad s_{9,65,69}=\tfrac{1}{2},\quad s_{9,67,68}=\tfrac{1}{2},$$

$$s_{9,73,75}=-1,\quad s_{10,54,60}=\tfrac{1}{2},\quad s_{10,55,59}=\tfrac{1}{2},\quad s_{10,56,61}=-\tfrac{1}{2},$$

$$s_{10,57,58}=-\tfrac{1}{2},\quad s_{10,62,68}=\tfrac{1}{2},\quad s_{10,63,67}=\tfrac{1}{2},\quad s_{10,64,69}=-\tfrac{1}{2},$$

$$s_{10,65,66} = -\tfrac{1}{2}, \quad s_{10,74,75} = -1, \quad s_{11,54,59} = \tfrac{1}{2}, \quad s_{11,55,60} = \tfrac{1}{2},$$

$$s_{11,56,57} = -\tfrac{1}{2}, \quad s_{11,58,61} = \tfrac{1}{2}, \quad s_{11,62,67} = -\tfrac{1}{2}, \quad s_{11,63,68} = -\tfrac{1}{2},$$

$$s_{11,64,65} = \tfrac{1}{2}, \quad s_{11,66,69} = -\tfrac{1}{2}, \quad s_{11,71,76} = -1, \quad s_{12,54,60} = \tfrac{1}{2},$$

$$s_{12,55,59} = -\tfrac{1}{2}, \quad s_{12,56,61} = -\tfrac{1}{2}, \quad s_{12,57,58} = \tfrac{1}{2}, \quad s_{12,62,68} = \tfrac{1}{2},$$

$$s_{12,63,67} = -\tfrac{1}{2}, \quad s_{12,64,69} = -\tfrac{1}{2}, \quad s_{12,65,66} = \tfrac{1}{2}, \quad s_{12,72,76} = -1,$$

$$s_{13,54,57} = -\tfrac{1}{2}, \quad s_{13,55,61} = \tfrac{1}{2}, \quad s_{13,56,59} = -\tfrac{1}{2}, \quad s_{13,58,60} = -\tfrac{1}{2},$$

$$s_{13,62,65} = -\tfrac{1}{2}, \quad s_{13,63,69} = \tfrac{1}{2}, \quad s_{13,64,67} = -\tfrac{1}{2}, \quad s_{13,66,68} = \tfrac{1}{2},$$

$$s_{13,73,76} = -1, \quad s_{14,54,56} = \tfrac{1}{2}, \quad s_{14,55,58} = -\tfrac{1}{2}, \quad s_{14,57,59} = -\tfrac{1}{2},$$

$$s_{14,60,61} = \tfrac{1}{2}, \quad s_{14,62,64} = \tfrac{1}{2}, \quad s_{14,63,66} = -\tfrac{1}{2}, \quad s_{14,65,67} = -\tfrac{1}{2},$$

$$s_{14,68,69} = \tfrac{1}{2}, \quad s_{14,74,76} = -1, \quad s_{15,54,61} = \tfrac{1}{2}, \quad s_{15,55,57} = \tfrac{1}{2},$$

$$s_{15,56,60} = \tfrac{1}{2}, \quad s_{15,58,59} = -\tfrac{1}{2}, \quad s_{15,62,69} = \tfrac{1}{2}, \quad s_{15,63,65} = \tfrac{1}{2},$$

$$s_{15,64,68} = \tfrac{1}{2}, \quad s_{15,66,67} = -\tfrac{1}{2}, \quad s_{15,75,76} = -1, \quad s_{16,54,60} = \tfrac{1}{2},$$

$$s_{16,55,59} = -\tfrac{1}{2}, \quad s_{16,56,61} = \tfrac{1}{2}, \quad s_{16,57,58} = -\tfrac{1}{2}, \quad s_{16,62,68} = -\tfrac{1}{2},$$

$$s_{16,63,67} = \tfrac{1}{2}, \quad s_{16,64,69} = -\tfrac{1}{2}, \quad s_{16,65,66} = \tfrac{1}{2}, \quad s_{16,71,77} = -1,$$

$$s_{17,54,59} = -\tfrac{1}{2}, \quad s_{17,55,60} = -\tfrac{1}{2}, \quad s_{17,56,57} = -\tfrac{1}{2}, \quad s_{17,58,61} = \tfrac{1}{2},$$

$$s_{17,62,67} = -\tfrac{1}{2}, \quad s_{17,63,68} = -\tfrac{1}{2}, \quad s_{17,64,65} = -\tfrac{1}{2}, \quad s_{17,66,69} = \tfrac{1}{2},$$

$$s_{17,72,77} = -1, \quad s_{18,54,61} = \tfrac{1}{2}, \quad s_{18,55,57} = \tfrac{1}{2}, \quad s_{18,56,60} = -\tfrac{1}{2},$$

$$s_{18,58,59} = \tfrac{1}{2}, \quad s_{18,62,69} = \tfrac{1}{2}, \quad s_{18,63,65} = \tfrac{1}{2}, \quad s_{18,64,68} = -\tfrac{1}{2},$$

$$s_{18,66,67} = \tfrac{1}{2}, \quad s_{18,73,77} = -1, \quad s_{19,54,58} = -\tfrac{1}{2}, \quad s_{19,55,56} = -\tfrac{1}{2},$$

$$s_{19,57,60} = -\tfrac{1}{2}, \quad s_{19,59,61} = -\tfrac{1}{2}, \quad s_{19,62,66} = -\tfrac{1}{2}, \quad s_{19,63,64} = -\tfrac{1}{2},$$

$$s_{19,65,68} = -\tfrac{1}{2}, \quad s_{19,67,69} = -\tfrac{1}{2}, \quad s_{19,74,77} = -1, \quad s_{20,54,57} = \tfrac{1}{2},$$

$$s_{20,55,61} = -\tfrac{1}{2}, \quad s_{20,56,59} = -\tfrac{1}{2}, \quad s_{20,58,60} = -\tfrac{1}{2}, \quad s_{20,62,65} = \tfrac{1}{2},$$

$$s_{20,63,69}=-\tfrac{1}{2},\quad s_{20,64,67}=-\tfrac{1}{2},\quad s_{20,66,68}=-\tfrac{1}{2},\quad s_{20,75,77}=-1,$$

$$s_{21,54,55}=\tfrac{1}{2},\quad s_{21,56,58}=\tfrac{1}{2},\quad s_{21,57,61}=\tfrac{1}{2},\quad s_{21,59,60}=-\tfrac{1}{2},$$

$$s_{21,62,63}=\tfrac{1}{2},\quad s_{21,64,66}=\tfrac{1}{2},\quad s_{21,65,69}=\tfrac{1}{2},\quad s_{21,67,68}=-\tfrac{1}{2},$$

$$s_{21,76,77}=-1,\quad s_{22,53,61}=\tfrac{1}{2},\quad s_{22,61,70}=-\tfrac{\sqrt{3}}{2},\quad s_{22,62,78}=-\tfrac{1}{2},$$

$$s_{22,63,74}=-\tfrac{1}{2},\quad s_{22,64,77}=-\tfrac{1}{2},\quad s_{22,65,72}=\tfrac{1}{2},\quad s_{22,66,76}=-\tfrac{1}{2},$$

$$s_{22,67,75}=\tfrac{1}{2},\quad s_{22,68,73}=\tfrac{1}{2},\quad s_{22,69,71}=\tfrac{1}{2},\quad s_{23,53,57}=-\tfrac{1}{2},$$

$$s_{23,57,70}=-\tfrac{\sqrt{3}}{2},\quad s_{23,62,74}=\tfrac{1}{2},\quad s_{23,63,78}=-\tfrac{1}{2},\quad s_{23,64,76}=-\tfrac{1}{2},$$

$$s_{23,65,71}=-\tfrac{1}{2},\quad s_{23,66,77}=\tfrac{1}{2},\quad s_{23,67,73}=\tfrac{1}{2},\quad s_{23,68,75}=-\tfrac{1}{2},$$

$$s_{23,69,72}=\tfrac{1}{2},\quad s_{24,53,60}=-\tfrac{1}{2},\quad s_{24,60,70}=-\tfrac{\sqrt{3}}{2},\quad s_{24,62,77}=\tfrac{1}{2},$$

$$s_{24,63,76}=\tfrac{1}{2},\quad s_{24,64,78}=-\tfrac{1}{2},\quad s_{24,65,75}=\tfrac{1}{2},\quad s_{24,66,74}=-\tfrac{1}{2},$$

$$s_{24,67,72}=-\tfrac{1}{2},\quad s_{24,68,71}=-\tfrac{1}{2},\quad s_{24,69,73}=\tfrac{1}{2},\quad s_{25,53,55}=\tfrac{1}{2},$$

$$s_{25,55,70}=\tfrac{\sqrt{3}}{2},\quad s_{25,62,72}=-\tfrac{1}{2},\quad s_{25,63,71}=\tfrac{1}{2},\quad s_{25,64,75}=-\tfrac{1}{2},$$

$$s_{25,65,78}=-\tfrac{1}{2},\quad s_{25,66,73}=\tfrac{1}{2},\quad s_{25,67,77}=-\tfrac{1}{2},\quad s_{25,68,76}=\tfrac{1}{2},$$

$$s_{25,68,74}=\tfrac{1}{2},\quad s_{26,53,59}=-\tfrac{1}{2},\quad s_{26,59,70}=-\tfrac{\sqrt{3}}{2},\quad s_{26,62,76}=\tfrac{1}{2},$$

$$s_{26,63,77}=-\tfrac{1}{2},\quad s_{26,64,74}=\tfrac{1}{2},\quad s_{26,65,73}=-\tfrac{1}{2},\quad s_{26,66,78}=-\tfrac{1}{2},$$

$$s_{26,67,71}=-\tfrac{1}{2},\quad s_{26,68,72}=\tfrac{1}{2},\quad s_{26,69,75}=\tfrac{1}{2},\quad s_{27,53,58}=\tfrac{1}{2},$$

$$s_{27,58,70}=\tfrac{\sqrt{3}}{2},\quad s_{27,62,75}=-\tfrac{1}{2},\quad s_{27,63,73}=-\tfrac{1}{2},\quad s_{27,64,72}=\tfrac{1}{2},$$

$$s_{27,64,77}=\tfrac{1}{2},\quad s_{27,66,71}=\tfrac{1}{2},\quad s_{27,67,78}=-\tfrac{1}{2},\quad s_{27,68,74}=-\tfrac{1}{2},$$

$$s_{27,69,76}=\tfrac{1}{2},\quad s_{28,53,56}=\tfrac{1}{2},\quad s_{28,56,70}=\tfrac{\sqrt{3}}{2},\quad s_{28,62,73}=-\tfrac{1}{2},$$

$$s_{28,63,75}=\tfrac{1}{2},\quad s_{28,64,71}=\tfrac{1}{2},\quad s_{28,65,76}=-\tfrac{1}{2},\quad s_{28,66,72}=-\tfrac{1}{2},$$

$$s_{28,67,74}=\tfrac{1}{2},\quad s_{28,68,78}=-\tfrac{1}{2},\quad s_{28,69,77}=\tfrac{1}{2},\quad s_{29,53,54}=-\tfrac{1}{2},$$

$$s_{29,54,70}=-\tfrac{\sqrt{3}}{2},\quad s_{29,62,71}=-\tfrac{1}{2},\quad s_{29,63,72}=-\tfrac{1}{2},\quad s_{29,64,73}=-\tfrac{1}{2},$$

$$s_{29,65,74}=-\tfrac{1}{2},\quad s_{29,66,75}=-\tfrac{1}{2},\quad s_{29,67,76}=-\tfrac{1}{2},\quad s_{29,68,77}=-\tfrac{1}{2},$$

$$s_{29,69,78}=-\tfrac{1}{2},\quad s_{30,54,61}=\tfrac{1}{2},\quad s_{30,55,57}=-\tfrac{1}{2},\quad s_{30,56,60}=-\tfrac{1}{2},$$

$$s_{30,58,59}=-\tfrac{1}{2},\quad s_{30,62,69}=-\tfrac{1}{2},\quad s_{30,63,65}=\tfrac{1}{2},\quad s_{30,64,68}=\tfrac{1}{2},$$

$$s_{30,66,67}=\tfrac{1}{2},\quad s_{30,71,78}=-1,\quad s_{31,54,57}=-\tfrac{1}{2},\quad s_{31,55,61}=-\tfrac{1}{2},$$

$$s_{31,56,59}=\tfrac{1}{2},\quad s_{31,58,60}=-\tfrac{1}{2},\quad s_{31,62,65}=-\tfrac{1}{2},\quad s_{31,63,69}=-\tfrac{1}{2},$$

$$s_{31,64,67}=\tfrac{1}{2},\quad s_{31,66,68}=-\tfrac{1}{2},\quad s_{31,72,78}=-1,\quad s_{32,54,60}=-\tfrac{1}{2},$$

$$s_{32,55,59}=-\tfrac{1}{2},\quad s_{32,56,61}=-\tfrac{1}{2},\quad s_{52,57,58}=-\tfrac{1}{2},\quad s_{32,62,68}=-\tfrac{1}{2},$$

$$s_{32,63,67}=-\tfrac{1}{2},\quad s_{32,64,69}=-\tfrac{1}{2},\quad s_{32,65,66}=-\tfrac{1}{2},\quad s_{32,73,78}=-1,$$

$$s_{33,54,55}=\tfrac{1}{2},\quad s_{33,56,58}=\tfrac{1}{2},\quad s_{33,57,61}=-\tfrac{1}{2},\quad s_{33,59,60}=\tfrac{1}{2},$$

$$s_{33,62,63}=\tfrac{1}{2},\quad s_{33,64,66}=\tfrac{1}{2},\quad s_{33,65,69}=-\tfrac{1}{2},\quad s_{33,67,68}=\tfrac{1}{2},$$

$$s_{33,74,78}=-1,\quad s_{34,54,59}=-\tfrac{1}{2},\quad s_{34,55,60}=\tfrac{1}{2},\quad s_{34,56,57}=-\tfrac{1}{2},$$

$$s_{34,58,61}=-\tfrac{1}{2},\quad s_{34,62,67}=-\tfrac{1}{2},\quad s_{34,63,68}=\tfrac{1}{2},\quad s_{34,64,65}=-\tfrac{1}{2},$$

$$s_{34,66,69}=-\tfrac{1}{2},\quad s_{34,75,78}=-1,\quad s_{35,54,58}=\tfrac{1}{2},\quad s_{35,55,56}=\tfrac{1}{2},$$

$$s_{35,57,60}=-\tfrac{1}{2},\quad s_{35,59,61}=-\tfrac{1}{2},\quad s_{35,62,66}=\tfrac{1}{2},\quad s_{35,63,64}=\tfrac{1}{2},$$

$$s_{35,65,68}=-\tfrac{1}{2},\quad s_{35,67,69}=-\tfrac{1}{2},\quad s_{35,76,78}=-1,\quad s_{36,54,56}=\tfrac{1}{2},$$

$$s_{36,55,58}=-\tfrac{1}{2},\quad s_{36,57,59}=\tfrac{1}{2},\quad s_{36,60,61}=-\tfrac{1}{2},\quad s_{36,62,64}=\tfrac{1}{2},$$

$$s_{36,63,66}=-\tfrac{1}{2},\quad s_{36,65,67}=\tfrac{1}{2},\quad s_{36,68,69}=-\tfrac{1}{2},\quad s_{36,77,78}=-1,$$

$$s_{37,53,62}=-1,\quad s_{37,54,71}=-\tfrac{1}{2},\quad s_{37,55,72}=\tfrac{1}{2},\quad s_{37,56,73}=\tfrac{1}{2},$$

$$s_{37,57,74}=\tfrac{1}{2},\quad s_{37,58,75}=\tfrac{1}{2},\quad s_{37,59,76}=\tfrac{1}{2},\quad s_{37,60,77}=\tfrac{1}{2},$$

$$s_{37,61,78}=\tfrac{1}{2},\quad s_{38,53,63}=-1,\quad s_{38,54,72}=-\tfrac{1}{2},\quad s_{38,55,71}=-\tfrac{1}{2},$$

$$s_{38,56,75}=-\tfrac{1}{2},\quad s_{38,57,78}=-\tfrac{1}{2},\quad s_{38,58,73}=\tfrac{1}{2},\quad s_{38,59,77}=-\tfrac{1}{2},$$

$$s_{38,60,76}=\tfrac{1}{2},\quad s_{38,61,74}=\tfrac{1}{2},\quad s_{39,53,64}=-1,\quad s_{39,54,73}=-\tfrac{1}{2},$$

$$s_{39,55,75}=\tfrac{1}{2},\quad s_{39,56,71}=-\tfrac{1}{2},\quad s_{39,57,76}=-\tfrac{1}{2},\quad s_{39,58,72}=-\tfrac{1}{2},$$

$$s_{39,59,74}=\tfrac{1}{2},\quad s_{39,60,78}=-\tfrac{1}{2},\quad s_{39,61,77}=\tfrac{1}{2},\quad s_{40,53,65}=-1,$$

$$s_{40,54,74}=-\tfrac{1}{2},\quad s_{40,55,78}=\tfrac{1}{2},\quad s_{40,56,76}=\tfrac{1}{2},\quad s_{40,57,71}=-\tfrac{1}{2},$$

$$s_{40,58,77}=-\tfrac{1}{2},\quad s_{40,59,73}=-\tfrac{1}{2},\quad s_{40,60,75}=\tfrac{1}{2},\quad s_{40,61,72}=-\tfrac{1}{2},$$

$$s_{41,53,66}=-1,\quad s_{41,54,75}=-\tfrac{1}{2},\quad s_{41,55,73}=-\tfrac{1}{2},\quad s_{41,56,72}=\tfrac{1}{2},$$

$$s_{41,57,77}=\tfrac{1}{2},\quad s_{41,58,71}=-\tfrac{1}{2},\quad s_{41,59,78}=-\tfrac{1}{2},\quad s_{41,60,74}=-\tfrac{1}{2},$$

$$s_{41,61,76}=\tfrac{1}{2},\quad s_{42,53,67}=-1,\quad s_{42,54,76}=-\tfrac{1}{2},\quad s_{42,55,77}=\tfrac{1}{2},$$

$$s_{42,56,74}=-\tfrac{1}{2},\quad s_{42,57,73}=\tfrac{1}{2},\quad s_{42,58,78}=\tfrac{1}{2},\quad s_{42,59,71}=-\tfrac{1}{2},$$

$$s_{42,60,72}=-\tfrac{1}{2},\quad s_{42,61,75}=-\tfrac{1}{2},\quad s_{43,53,68}=-1,\quad s_{43,54,77}=-\tfrac{1}{2},$$

$$s_{43,55,76}=-\tfrac{1}{2},\quad s_{43,56,78}=\tfrac{1}{2},\quad s_{43,57,75}=-\tfrac{1}{2},\quad s_{43,58,74}=\tfrac{1}{2},$$

$$s_{43,59,72}=\tfrac{1}{2},\quad s_{43,60,71}=-\tfrac{1}{2},\quad s_{43,61,73}=-\tfrac{1}{2},\quad s_{44,53,69}=-1,$$

$$s_{44,54,78}=-\tfrac{1}{2},\quad s_{44,55,74}=-\tfrac{1}{2},\quad s_{44,56,77}=-\tfrac{1}{2},\quad s_{44,57,72}=\tfrac{1}{2},$$

$$s_{44,58,76}=-\tfrac{1}{2},\quad s_{44,59,75}=\tfrac{1}{2},\quad s_{44,60,73}=\tfrac{1}{2},\quad s_{44,61,71}=-\tfrac{1}{2},$$

$$s_{45,53,71}=-\tfrac{1}{2},\quad s_{45,54,62}=-\tfrac{1}{2},\quad s_{45,55,63}=-\tfrac{1}{2},\quad s_{45,56,64}=-\tfrac{1}{2},$$

$$s_{45,57,65}=-\tfrac{1}{2},\quad s_{45,58,66}=-\tfrac{1}{2},\quad s_{45,59,67}=-\tfrac{1}{2},\quad s_{45,60,68}=-\tfrac{1}{2},$$

$$s_{45,61,69}=-\tfrac{1}{2},\quad s_{45,70,71}=-\tfrac{\sqrt{3}}{2},\quad s_{46,53,72}=-\tfrac{1}{2},\quad s_{46,54,63}=-\tfrac{1}{2},$$

$$s_{46,55,62}=\tfrac{1}{2},\quad s_{46,56,66}=\tfrac{1}{2},\quad s_{46,57,69}=\tfrac{1}{2},\quad s_{46,58,64}=-\tfrac{1}{2},$$

$$s_{46,59,68}=\tfrac{1}{2},\quad s_{46,60,67}=-\tfrac{1}{2},\quad s_{46,61,65}=-\tfrac{1}{2},\quad s_{46,70,72}=-\tfrac{\sqrt{3}}{2},$$

$$s_{47,53,73}=-\tfrac{1}{2},\quad s_{47,54,64}=-\tfrac{1}{2},\quad s_{47,55,66}=-\tfrac{1}{2},\quad s_{47,56,62}=\tfrac{1}{2},$$

$$s_{47,57,67}=\tfrac{1}{2},\quad s_{47,58,63}=\tfrac{1}{2},\quad s_{47,59,65}=-\tfrac{1}{2},\quad s_{47,60,69}=\tfrac{1}{2},$$

$$s_{47,61,68}=-\tfrac{1}{2},\quad s_{47,70,73}=-\tfrac{\sqrt{3}}{2},\quad s_{48,53,74}=-\tfrac{1}{2},\quad s_{48,54,65}=-\tfrac{1}{2},$$

$$s_{48,55,69}=-\tfrac{1}{2},\quad s_{48,56,67}=-\tfrac{1}{2},\quad s_{48,57,62}=-\tfrac{1}{2},\quad s_{48,58,68}=\tfrac{1}{2},$$

$$s_{48,59,64}=\frac{1}{2},\quad s_{48,60,66}=-\frac{1}{2},\quad s_{48,61,63}=\frac{1}{2},\quad s_{48,70,74}=-\frac{\sqrt{3}}{2},$$

$$s_{49,53,75}=-\frac{1}{2},\quad s_{49,54,66}=-\frac{1}{2},\quad s_{49,55,64}=\frac{1}{2},\quad s_{49,56,63}=-\frac{1}{2},$$

$$s_{49,57,68}=-\frac{1}{2},\quad s_{49,58,62}=\frac{1}{2},\quad s_{49,59,69}=\frac{1}{2},\quad s_{49,60,65}=\frac{1}{2},$$

$$s_{49,61,67}=-\frac{1}{2},\quad s_{49,70,75}=-\frac{\sqrt{3}}{2},\quad s_{50,53,76}=-\frac{1}{2},\quad s_{50,54,67}=-\frac{1}{2},$$

$$s_{50,55,68}=-\frac{1}{2},\quad s_{50,56,65}=\frac{1}{2},\quad s_{50,57,64}=-\frac{1}{2},\quad s_{50,58,69}=-\frac{1}{2},$$

$$s_{50,59,62}=\frac{1}{2},\quad s_{50,60,63}=\frac{1}{2},\quad s_{50,61,66}=\frac{1}{2},\quad s_{50,70,76}=-\frac{\sqrt{3}}{2},$$

$$s_{51,53,77}=-\frac{1}{2},\quad s_{51,54,68}=-\frac{1}{2},\quad s_{51,55,67}=\frac{1}{2},\quad s_{51,56,69}=-\frac{1}{2},$$

$$s_{51,57,66}=\frac{1}{2},\quad s_{51,58,65}=-\frac{1}{2},\quad s_{51,59,63}=-\frac{1}{2},\quad s_{51,60,62}=\frac{1}{2},$$

$$s_{51,61,64}=\frac{1}{2},\quad s_{51,70,77}=-\frac{\sqrt{3}}{2},\quad s_{52,53,78}=-\frac{1}{2},\quad s_{52,54,69}=-\frac{1}{2},$$

$$s_{52,55,65}=\frac{1}{2},\quad s_{52,56,68}=-\frac{1}{2},\quad s_{52,57,63}=-\frac{1}{2},\quad s_{52,58,67}=\frac{1}{2},$$

$$s_{52,59,66}=-\frac{1}{2},\quad s_{52,60,64}=-\frac{1}{2},\quad s_{52,61,62}=\frac{1}{2},\quad s_{52,70,78}=-\frac{\sqrt{3}}{2}.$$

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