SEPARABLE LINDENSTRAUSS SPACES WHOSE DUALS LACK THE WEAK* FIXED POINT PROPERTY FOR NONEXPANSIVE MAPPINGS

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Abstract. In this paper we study the w^*-fixed point property for nonexpansive mappings. First we show that the dual space X^* lacks the w^*-fixed point property whenever X contains an isometric copy of the space c. Then, the main result of our paper provides several characterizations of weak-star topologies that fail the fixed point property for nonexpansive mappings in ℓ_1 space. This result allows us to obtain a characterization of all separable Lindenstrauss spaces X inducing the failure of the w^*-fixed point property in X^*.

1. Introduction

Let X be an infinite dimensional real Banach space and let us denote by B_X its closed unit ball. A nonempty bounded closed and convex subset C of X has the fixed point property (shortly, FPP) if each nonexpansive mapping (i.e., a mapping $T : C \to C$ such that $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$) has a fixed point. The space X^* is said to have the $\sigma(X^*, X)$-fixed point property ($\sigma(X^*, X)$-FPP) if every nonempty, convex, $\sigma(X^*, X)$-compact subset C of X^* has the FPP. The study of the $\sigma(X^*, X)$-FPP proves to be of special interest whenever a dual space has different preduals. Indeed, the behaviour with respect to the $\sigma(X^*, X)$-FPP of a given dual space can be completely different if we consider two different preduals. For instance, this situation occurs when we consider the space ℓ_1 and its preduals c_0 and c where it is well-known (see [7]) that ℓ_1 has the $\sigma(\ell_1, c_0)$-FPP whereas it lacks the $\sigma(\ell_1, c)$-FPP.

The main aim of this paper is to study some structural features of a separable space X linked to $\sigma(X^*, X)$-FPP on its dual.

At the beginning of Section 3, we state a sufficient condition for the failure of the $\sigma(X^*, X)$-FPP. Indeed, Theorem 3.2 shows that the presence of an isometric copy of c in a separable space X implies the failure of the $\sigma(X^*, X)$-FPP. This theorem extends a result of Smyth (Theorem 1 in [10]) to a broader class of spaces. Moreover, it allows us to show that every separable Lindenstrauss space X (i.e., a space such that its dual is a space $L_1(\mu)$ for some measure μ), whose dual is nonseparable, lack the $\sigma(X^*, X)$-FPP. Taking into account these last facts, it seems to be natural to investigate if the presence of an isometric copy of c in the space X is also a necessary condition for the failure of the $\sigma(X^*, X)$-FPP. The simple example where $X = \ell_1$ shows that the answer is negative in a general framework. Moreover, by considering a suitable class of hyperplanes of c, we are able to

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show that the answer remains negative even if we add the assumption that X is a separable Lindenstrauss space. This class of hyperplanes of c with duals isometric to ℓ_1 and failing the w^*-FPP, will play an important role in this paper and subsequently members of the class will be referred to as *"bad"* W_f (see Section 2 for a detailed description of these spaces). The first interesting result involving this class of spaces is Theorem 3.7, where we prove that if a separable space X contains a *"bad"* W_f then X^* still fails the $\sigma(X^*, X)$-FPP. A simple but relevant consequence of this theorem is Remark 3.8 where it is stated that the dual space X^* lacks the $\sigma(X^*, X)$-FPP whenever there is a quotient of X that contains an isometric copy of a "bad" W_f. The last section is devoted to the characterization of the preduals of ℓ_1 such that ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP. Theorem 4.1, which is the main theorem of this paper, lists several properties of a predual X of ℓ_1 that are all equivalent to the lack of the $\sigma(\ell_1, X)$-FPP for ℓ_1. Among them, one property is exactly the structural condition that appears in Remark 3.8. Another property listed in this result (see condition (4) in Theorem 4.1) seems to be meaningful. It is related to the w^*-cluster points of the standard basis of ℓ_1 and it allows us to extend Theorem 8 in [6] in the case of w^*-topologies. Indeed, we can prove this theorem without assuming the strong assumption on w^*-convergence of the standard basis of ℓ_1 that was made in [6].

Throughout the paper we will follow standard terminology and notations. In particular, it is well-known that c^* can be isometrically identified with ℓ_1 in the following way. For every $x^* \in c^*$ there exists a unique element $f = (f(1), f(2), \dots) \in \ell_1$ such that

$$x^*(x) = \sum_{n=0}^{\infty} f(n+1)x(n) = f(x)$$

with $x = (x(1), x(2), \dots) \in c$ and $x(0) = \lim x(n)$.

2. A class of hyperplanes in the space of convergent sequences

This section is devoted to recalling some properties of a class of hyperplanes of c that will play a crucial role in the remainder of our paper. For the convenience of the reader we repeat some materials from [4] without proofs, thus making our exposition self-contained. Moreover, we prove some additional properties of these hyperplanes directly related to the topic studied in the present paper.

Let $f \in \ell_1 = c^*$ be such that $\|f\| = 1$. We consider the hyperplane of c defined by

$$W_f = \{x \in c : f(x) = 0\}.$$

In [4], the following results are proved:

(I) there exists $j_0 \geq 1$ such that $|f(j_0)| \geq 1/2$ if and only if W_f^* is isometric to ℓ_1.

(II) there exists $j_0 \geq 2$ such that $|f(j_0)| \geq 1/2$ if and only if W_f is isometric to c.

For our aims, an important case to be considered is when $|f(1)| \geq 1/2$ and $|f(j)| < 1/2$ for every $j \geq 2$. Under these additional assumptions, Theorem 4.3 in

[4] identifies W_f^* with ℓ_1 by giving the following dual action: for every $x^* \in W_f^*$ there exists a unique element $g \in \ell_1$ such that

$$x^*(x) = \sum_{n=1}^{\infty} g(n)x(n) = g(x) \tag{2.1}$$

where $x = (x(1), x(2), \dots) \in W_f$. We conclude this section by proving some additional properties of the spaces W_f that will be useful in the sequel. The first proposition gives a necessary and sufficient condition for the existence of a subspace of W_f isometric to c.

Proposition 2.1. Let $f \in \ell_1 = c^*$ be such that $\|f\| = 1$ and $|f(1)| \geq \frac{1}{2}$. Then the following statements are equivalent.

(1) W_f contains a subspace isometric to c.
(2) $|f(1)| = \frac{1}{2}$, $\{n \in \mathbb{N} : f(1)f(n+1) > 0\}$ is a finite set and $\{n \in \mathbb{N} : f(n+1) = 0\}$ is an infinite set.

Proof. (2)$\Longrightarrow$(1). Let $\{n \in \mathbb{N} : f(n+1) = 0\} = \{n_k\}_{k=1}^{+\infty}$ and let us consider the mapping $T : c \to W_f$ defined for every $x = (x(1), x(2), \dots) \in c$ by $T(x) = ((T(x))(1), (T(x))(2), \dots) \in W_f$, where

$$(T(x))(i) = \begin{cases} x(k) & \text{if} \quad i = n_k, \\ -\operatorname{sgn}(f(1)f(i+1)) \cdot \lim_j x(j) & \text{if} \quad i \in \mathbb{N} \setminus \{n_k\}. \end{cases}$$

It is easy to see that T is a linear isometry of c into W_f.

(1)$\Longrightarrow$(2). If W_f is isometric to c, then the assertion follows immediately from the result recalled in item (II) at the beginning of this section. Suppose that W_f is not isometric to c. Let $(e_n^*)_{n\geq 1}$ be the standard basis of $\ell_1 = c^*$. For every $n \geq 2$ we take a norm-one extension of e_n^* to the whole space W_f and we denote it by g_n^*. Consider a $\sigma(\ell_1, W_f)$-convergent subsequence $(g_{n_k}^*)_{k\geq 2}$ of $(g_n^*)_{n\geq 2}$ and denote its limit by $g_{n_1}^*$. Obviously, $g_{n_1}^*$ is a norm-one extension of e_1^* to the whole W_f. It is easy to see that $\left\|g_{n_k}^* \pm g_{n_l}^*\right\| = 2$ for all $k, l \in \mathbb{N}$, $k \neq l$. Consequently,

$$\operatorname{supp} g_{n_k}^* \cap \operatorname{supp} g_{n_l}^* = \emptyset \tag{2.2}$$

for all $k, l \in \mathbb{N}$, $k \neq l$, where $\operatorname{supp} g_n^* := \{i \in \mathbb{N} : g_n^*(i) \neq 0\}$. Hence, by using the argument presented at the beginning of the proof of Theorem 8 in [6] and Theorem 4.3 in [4], we obtain

$$g_{n_1}^* = \pm\left(\frac{f(2)}{f(1)}, \frac{f(3)}{f(1)}, \frac{f(4)}{f(1)}, \dots\right).$$

Therefore we have that $|f(1)| = \frac{1}{2}$ and $\{n \in \mathbb{N} : f(n+1) = 0\}$ is an infinite set. Since there exists $x \in c \subset W_f$ such that $\|x\| = 1$ and $e_n^*(x) = 1$ for every n, we get

$$e_{n_k}^*(x) = g_{n_k}^*(x) = g_{n_1}^*(x) = 1$$

for every $k \geq 2$. From the above relation and the standard duality of W_f (see (2.1) above) we have

$$x(i) = \operatorname{sgn}(g_{n_k}^*(i)) \tag{2.3}$$

for every $i \in \operatorname{supp} g^*_{n_k}$ and for every $k \in \mathbb{N}$. Taking into account (2.2) and (2.3) we conclude that there exists i_0 such that either $x(i) = 1$ or $x(i) = -1$ for infinitely many $i \geq i_0$. Therefore $\{n \in \mathbb{N} : f(1)f(n+1) > 0\}$ is a finite set. □

The last proposition of this section characterizes a class of spaces W_f such that ℓ_1 enjoys the $\sigma(\ell_1, W_f)$-FPP.

Proposition 2.2. Let $f \in \ell_1 = c^*$ be such that $\|f\| = 1$, $\frac{1}{2} \leq |f(1)| < 1$ and $|f(j)| < \frac{1}{2}$ for every $j \geq 2$. The space ℓ_1 has the $\sigma(\ell_1, W_f)$-FPP if and only if one of the following conditions holds

(1) $|f(1)| > \frac{1}{2}$
(2) $|f(1)| = \frac{1}{2}$ and the set $N^+ = \{n \in \mathbb{N} : f(1)f(n+1) \leq 0\}$ is finite.

Proof. We have that $W^*_f = \ell_1$ as recalled at the beginning of this section (see item (I) above). Now, Theorem 4.3 in [4] shows that

$$e^*_n \overset{\sigma(\ell_1, W_f)}{\longrightarrow} e^*,$$

where $e^* = (-f(2)/f(1), -f(3)/f(1), \dots)$. The conclusion follows immediately from Theorem 8 in [6]. □

Proposition 2.2 and item (II) lead us to introducing the following definition.

Definition 2.3. A space W_f is called "bad with respect to $\sigma(\ell_1, W_f)$-FPP" (shortly "bad") if $f \in \ell_1$ is such that $\|f\| = 1$, $|f(1)| = \frac{1}{2}$ and the set $N^+ = \{n \in \mathbb{N} : f(1)f(n+1) \leq 0\}$ is infinite.

We underline that, by combining Propositions 2.1 and 2.2, we find an example of a ℓ_1-predual space X such that ℓ_1 fails the $\sigma(\ell_1, X)$-FPP but X does not contain an isometric copy of c.

Example 2.4. Let us consider the space W_f where

$$f = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right) \in \ell_1.$$

We have that

- $W^*_f = \ell_1$;
- W_f does not contain an isometric copy of c (by Proposition 2.1);
- ℓ_1 lacks the $\sigma(\ell_1, W_f)$-FPP (by Proposition 2.2).

We point out another feature of this space that will be useful in the last section. The space W_f does not have a quotient that contains an isometric copy of c. Indeed, let us suppose $c \subseteq W_f/Y$. Then, by following the reasoning from the proof of Proposition 2.1, we obtain a sequence $(x^*_n)_{n\geq 1} \subset (W_f/Y)^*$ such that

- $x^*_n \overset{\sigma((W_f/Y)^*, W_f/Y)}{\longrightarrow} x^*_1$,
- $\left\|x^*_n\right\| = 1$ for every $n \in \mathbb{N}$,
- $\left\|x^*_n \pm x^*_m\right\| = 2$ for all $m, n \in \mathbb{N}$, $m \neq n$.

Now, for each $u \in v + Y$, $v \in W_f$, we put $y^*_n(u) = x^*_n(v + Y)$. Consequently, the sequence $(y^*_n)_{n\geq 1} \subset W^*_f$ is equivalent to the standard basis in ℓ_1 and $y^*_n \overset{\sigma(\ell_1, W_f)}{\longrightarrow} y^*_1$.

Again, by following the argument developed at the beginning of the proof of Theorem 8 in [6], Theorem 4.3 in [4] yields

$$y_1^* = \pm\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right).$$

The last equality gives a contradiction.

Two remarks conclude this section by relating Proposition 2.2 to some results appeared in the literature.

Remark 2.5. If we restrict our attention to w^*-topologies on ℓ_1, the assumptions of Theorem 8 in [6] are equivalent to those of Proposition 2.2. Indeed, if X is a predual of ℓ_1 such that the standard basis of ℓ_1 is a $\sigma(\ell_1, X)$-convergent sequence, then there exists a suitable W_f isometric to X (see Corollary 4.4 in [4]).

Remark 2.6. In the case of a particular family of sets in ℓ_1, a characterization of the fixed point property for nonexpansive mappings was established in [5]. For example, for every $\varepsilon \in (0, 1)$ we define the set $C_\varepsilon \subset \ell_1$ by

$$C_\varepsilon = \left\{\alpha_1(1-\varepsilon)e_1^* + \sum_{i=2}^{\infty} \alpha_i e_i^* : \alpha_i \geq 0, \sum_{i=1}^{\infty} \alpha_i = 1\right\}.$$

The set C_ε is convex, bounded and closed. Moreover, it has the FPP (see [5]). Obviously C_ε is neither $\sigma(\ell_1, c)$-compact nor $\sigma(\ell_1, c_0)$-compact.

Let $f = \left(\frac{1}{2-\varepsilon}, -\frac{1-\varepsilon}{2-\varepsilon}, 0, 0, \dots\right)$, from Theorem 4.3 in [4] we know that $W_f^* = \ell_1$ and

$$e_n^* \xrightarrow{\sigma(\ell_1, W_f)} (1-\varepsilon)e_1^*.$$

Hence, Corollary 2 in [6] implies that C_ε is $\sigma(\ell_1, W_f)$-compact. By Proposition 2.2, ℓ_1 has the $\sigma(\ell_1, W_f)$-FPP.

3. Sufficient conditions for the lack of weak* fixed point property in the dual of a separable Banach space

This section is devoted to proving some sufficient conditions for the lack of the $\sigma(X^*, X)$-FPP where X is a separable space. The first step is suggested by the well-known example of $X = c$. Indeed, we start by showing that the presence in X of a copy of c implies the failure of the $\sigma(X^*, X)$-FPP. In order to prove this theorem we use an auxiliary result about the existence of a 1-complemented copy of c.

Proposition 3.1. Let X be a separable Banach space that contains an isometric copy of c. Then there is a subspace Y of X such that Y is isometric to c and 1-complemented in X.

Proof. Let $(e_n^*)_{n\geq 1}$ be the standard basis of $c^* = \ell_1$. For each $n \in \mathbb{N}$, we consider a norm preserving extension of e_n^* to the whole X, we denote it by x_n^*. Then, there exists a subsequence $\left(x_{n_j}^*\right)$ such that $n_1 > 1$ and

$$x_{n_j}^* \xrightarrow{\sigma(X^*, X)} \overline{x}^*.$$

Let us consider the subspace

$$Y = \left\{ y \in c : \lim y(n) = y(0) = y(s) \text{ for each } s \in \mathbb{N} \setminus \left\{ n_j - 1 \right\} \right\}$$

and the mapping $P : X \to Y$ defined by

$$P(x) = \overline{x}^*(x)e_0 + \sum_{j=1}^{\infty} \left(x^*_{n_j} - \overline{x}^* \right)(x) e_{n_j - 1},$$

where $e_0 = (1, 1, \ldots, 1, \ldots)$. It is easy to see that Y is isometric to c and P is a norm-one projection onto Y. □

Theorem 3.2. Let X be a separable Banach space that contains a subspace isometric to c. Then X^* fails the $\sigma(X^*, X)$-FPP.

Proof. By Proposition 3.1 we may assume that c is 1-complemented in X. So, there is a projection P of X onto c with $\|P\| = 1$. Then, $P^* : c^* \to X^*$ is a w^*-continuous isometry. Since c^* fails to have the $\sigma(c^*, c)$-FPP, there exists a $\sigma(c^*, c)$-compact convex set C that lacks the FPP. Therefore $P^*(C)$ is a convex, $\sigma(X^*, X)$-compact set in X^* which lacks the FPP. □

Remark 3.3. It is easy to find a $\sigma(c^*, c)$-compact and convex set $C \subset c^*$ which fails the FPP for isometries. Moreover, Lennard (see Ex. 3.2-3.3, pp. 41-43 in [9]) found an example of a convex, $\sigma(c^*, c)$-compact set $C \subset c^*$ that fails the FPP for affine (as well as for non affine) contractive mappings (i.e., a mapping $T : C \to C$ such that $\|T(x) - T(y)\| < \|x - y\|$ for all $x, y \in C, x \neq y$). Therefore, under the same assumptions of the previous theorem, X^* fails the $\sigma(X^*, X)$-FPP for isometries and affine contractive mappings.

A consequence of Theorem 3.2 shows that every separable Lindenstrauss space X with a nonseparable dual is such that X^* lacks the $\sigma(X^*, X)$-FPP.

Corollary 3.4. Let X be a separable Lindenstrauss space such that X^* is a nonseparable space. Then X^* lacks the $\sigma(X^*, X)$-FPP.

Proof. Theorem 2.3 in [8] proves that a separable Lindenstrauss space X with nonseparable dual contains a subspace isometric to the space $C(\Delta)$ where Δ is the Cantor set. Since $C(\Delta)$ contains an isometric copy of c, the conclusion follows directly from Theorem 3.2. □

A simple extension of Theorem 3.2 can be easily obtained by considering a quotient of X instead of a subspace.

Remark 3.5. Let X be a separable Banach space and let us suppose that there exists a quotient X/Y of X isometric to c. Theorem 3.2 shows that $Y^\perp$ fails the $\sigma(Y^\perp, X/Y)$-FPP and it follows easily that also X^* fails the $\sigma(X^*, X)$-FPP.

The following example shows that to consider a quotient of X is a true extension of Theorem 3.2.

Example 3.6. Let us consider the space W_f where

$$f = \left(-\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{8}, 0, \frac{1}{16}, 0, \ldots \right) \in \ell_1.$$

We have that

- $W_f^* = \ell_1$;
- W_f does not contain an isometric copy of c (by Proposition 2.1);
- ℓ_1 lacks the $\sigma(\ell_1, W_f)$-FPP (by Proposition 2.2).

Moreover, there exists a quotient of W_f isometric to c. Indeed, let us consider the subspace

$$Y = \left\{ y \in W_f : y(2k) = 0 \text{ for all } k \in \mathbb{N} \right\}$$

and the map $T : c \longrightarrow W_f / Y$ defined by

$$T(x) = \left(\frac{7}{3} x(0), x(1), x(0), x(2), x(0), \dots \right) + Y$$

for every $x \in c$. The map T is easily seen to be a surjective isometry.

It is easy to observe that the lack of the $\sigma(X^*, X)$-FPP does not imply that $c \subset X$ when X is a generic separable Banach space. Indeed, the well-known example by Alspach [2] shows that ℓ_∞ fails the $\sigma(\ell_\infty, \ell_1)$-FPP, whereas its only predual does not contain an isometric copy of c. Moreover, Example 2.4 shows that also a Lindenstrauss space exhibits the same behaviour. The same example proves that also the lack of a quotient of X containing an isometric copy of c is not a necessary condition for the lack of $\sigma(X^*, X)$-FPP.

The next result extends Theorem 3.2. Indeed, the space c can be regarded as a special member of the family of "bad" W_f by taking $f = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \dots\right)$ (see Section 2).

Theorem 3.7. Let X be a separable Banach space. If X contains a subspace isometric to a "bad" W_f, then X^* fails the $\sigma(X^*, X)$-FPP.

Proof. Let $x \in W_f$ and (e_n^*) be a sequence of elements of W_f^* defined by

$$e_n^*(x) = x(n)$$

for every $n \in \mathbb{N}$. From Theorem 4.3 in [4] we have that

$$e_n^* \xrightarrow{\sigma(\ell_1, W_f)} e^*$$

where $e^* = \left(-\frac{f(2)}{f(1)}, -\frac{f(3)}{f(1)}, -\frac{f(4)}{f(1)}, \dots \right)$ (observe that the same relation holds when $|f(j)| = \frac{1}{2}$ for some $j \geq 2$). We denote by x_n^* the equal norm extensions to the whole space X of the functionals e_n^*. By the assumption about W_f we have that the set $N^+ = \{n \in \mathbb{N} : f(1)f(n+1) \leq 0\}$ has infinitely many elements. Therefore we can choose an increasing sequence $\left(n_j\right) \subset N^+$ such that

$$x_{n_j}^* \xrightarrow{\sigma(X^*, X)} x^*$$

and $w_0 = e^* - u_0 \neq 0$ where $u_0 = \sum_{j=1}^{+\infty} e^*(n_j) e_{n_j}^*$. Now we consider the extension of u_0 to the whole space X defined by $\widetilde{u_0} = \sum_{j=1}^{+\infty} e^*(n_j) x_{n_j}^*$ and the elements $\widetilde{w_0} = x^* - \widetilde{u_0}$ and $\widetilde{w} = \frac{\widetilde{w_0}}{\|w_0\|}$. Now, by adapting to our framework the approach developed in the last part of the proof of Theorem 8 in [6], we show that the $\sigma(X^*, X)$-compact, convex set

$$C = \left\{ \mu_1 x^* + \mu_2 \widetilde{w} + \sum_{j=1}^{+\infty} \mu_{j+2} x_{n_j}^* : \sum_{k=1}^{+\infty} \mu_k = 1, \, \mu_k \geq 0, \, k = 1, 2, \dots \right\}$$

can be rewritten as

$$C = \left\{ \lambda_1 \widetilde{w} + \sum_{j=1}^{+\infty} \lambda_{j+1} x_{n_j}^* : \sum_{k=1}^{+\infty} \lambda_k = 1,\ \lambda_k \geq 0,\ k = 1, 2, \dots \right\}.$$

Now, we consider the map $T : C \to C$ defined by:

$$T\left(\lambda_1 \widetilde{w} + \sum_{j=1}^{+\infty} \lambda_{j+1} x_{n_j}^* \right) = \sum_{j=1}^{+\infty} \lambda_j x_{n_j}^*.$$

Since $x = \lambda_1 \widetilde{w} + \sum_{j=1}^{+\infty} \lambda_{j+1} x_{n_j}^* \in C$ has a unique representation the map T is well defined. Moreover it is a nonexpansive map. Indeed, for every $\alpha_j \in \mathbb{R}$, $j = 1, 2, \dots$ it holds

$$\begin{aligned} \left\| \alpha_1 \widetilde{w} + \sum_{j=1}^{+\infty} \alpha_{j+1} x_{n_j}^* \right\| &\geq \left\| \alpha_1 \frac{w_0}{\|w_0\|} + \sum_{j=1}^{+\infty} \alpha_{j+1} e_{n_j}^* \right\| = \sum_{j=1}^{+\infty} \left| \alpha_j \right| \\ &= \sum_{j=1}^{+\infty} \left| \alpha_j \right| \left\| x_{n_j}^* \right\| \geq \left\| \sum_{j=1}^{+\infty} \alpha_j x_{n_j}^* \right\|. \end{aligned}$$

Finally, it is easy to see that T has not a fixed point in C. □

As already pointed out with respect to Theorem 3.2 (see Remark 3.5), we can extend Theorem 3.7 by assuming a property of the quotients of X.

Remark 3.8. Let X be a separable Banach space and let us suppose that a "bad" W_f is a subspace of a quotient X/Y of X. Theorem 3.7 shows that $Y^\perp$ fails the $\sigma(Y^\perp, X/Y)$-FPP. It is straightforward to see that also X^* fails the $\sigma(X^*, X)$-FPP.

In the next section we will see that the property stated in previous remark becomes a necessary condition if we additionaly assume that X is a separable Lindenstrauss space.

4. The case of separable Lindenstrauss spaces

This section is devoted to the main result of our paper. We characterize the separable Lindenstrauss spaces X such that X^* fails the $\sigma(X^*, X)$-FPP.

By taking in account Corollary 3.4, we can limit ourselves to study the Lindenstrauss spaces whose dual is isometric to ℓ_1.

It is worth pointing out that the sufficient condition for the failure of the $\sigma(X^*, X)$-FPP stated in Remark 3.8, turn out to be also necessary. This fact emphasizes the crucial role played in the study of the $\sigma(X^*, X)$-FPP by the spaces "bad" W_f. Moreover, we are able to find also a condition involving the limit of a $\sigma(\ell_1, X)$-convergent subsequence of the standard basis of ℓ_1 that is equivalent to the failure of the $\sigma(\ell_1, X)$-FPP. This property allows us to give a characterization of the $\sigma(\ell_1, X)$-FPP in ℓ_1 by removing the restrictive assumption about the convergence of the standard basis of ℓ_1 used in Theorem 8 in [6].

Theorem 4.1. Let X be a predual of ℓ_1. Then the following are equivalent.

(1) ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP for nonexpansive mappings.
(2) ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP for isometries.
(3) ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP for contractive mappings.

(4) There is a subsequence $(e^*_{n_k})_{k\in\mathbb{N}}$ of the standard basis $(e^*_n)_{n\in\mathbb{N}}$ in ℓ_1 which is $\sigma(\ell_1, X)$-convergent to a norm-one element $e^* \in \ell_1$ with $e^*(n_k) \geq 0$ for all $k \in \mathbb{N}$.
(5) There is a quotient of X isometric to a "bad" W_f.
(6) There is a quotient of X that contains a subspace isometric to a "bad" W_g.

Proof. We divide the proof of this theorem in several parts. First of all we remark that some implications are straightforward to prove. Indeed, it is easy to check that (2) $\Longrightarrow$ (1), (3) $\Longrightarrow$ (1) and (5) $\Longrightarrow$ (6). The implication (6) $\Longrightarrow$ (1) follows immediately from Remark 3.8.

(4) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (3). By adapting to our setting the method developed in the last part of Theorem 8 in [6], we obtain a $\sigma(\ell_1, X)$-compact and convex set $C \subset \ell_1$ and an isometry $T : C \to C$ fixed point free. Moreover, by following the idea of [3], we consider the mapping $S : C \to C$ defined as

$$S(x) = \sum_{j=0}^{\infty} \frac{T^j(x)}{2^{j+1}},$$

where T is as above. It is easy to prove that the mapping S is a fixed point free contractive mapping.

(4) $\Longrightarrow$ (5). By choosing a subsequence we may assume that $u^* = e^* - \sum_{k=2}^{\infty} e^*(n_k)e^*_{n_k} \neq 0$. Put $x^*_1 = \frac{u^*}{\|u^*\|}$ and $x^*_k = e^*_{n_k}$ for $k \geq 2$. It is easy to see that $(x^*_k)_{k\in\mathbb{N}}$ is normalized sequence which is equivalent to the standard basis in ℓ_1. Let us denote by $Y = \left[\left\{x^*_k : k \in \mathbb{N}\right\}\right]$ the closed linear span of $\left\{x^*_k : k \in \mathbb{N}\right\}$. Since $\overline{\left\{x^*_k : k \in \mathbb{N}\right\}}^{w^*} = \left\{x^*_k : k \in \mathbb{N}\right\} \cup \{e^*\} \subset Y$, Lemma 1 in [1] guarantees that $\overline{\left[\left\{x^*_k : k \in \mathbb{N}\right\}\right]}^{w^*} = Y$. Let us consider $W_f \subset c$ where

$$f = \left(-\frac{1}{2}, \frac{1}{2}\left(1 - \sum_{k=2}^{\infty} e^*(n_k)\right), \frac{1}{2}e^*(n_2), \frac{1}{2}e^*(n_3), \frac{1}{2}e^*(n_4), \dots\right).$$

Then, by recalling Definition 2.3, we have that W_f is a "bad" W_f. Let $(y^*_n)_{n\in\mathbb{N}}$ denote the standard basis in $\ell_1 = W^*_f$. We shall consider two cases. Suppose $\sum_{k=2}^{\infty} e^*(n_k) > 0$. Then, applying Theorem 4.3 in [4], we obtain $y^*_n \overset{\sigma(\ell_1, W_f)}{\longrightarrow} y^*$, where $y^* = (1 - \sum_{k=2}^{\infty} e^*(n_k), e^*(n_2), e^*(n_3), e^*(n_4), \dots)$. Let ϕ be the basis to basis map of Y onto $\ell_1 = W^*_f$, $\phi\left(\sum_{k=1}^{\infty} a_k x^*_k\right) = \sum_{k=1}^{\infty} a_k y^*_k$. Then we have

$$\begin{aligned} \phi(e^*) &= \phi\left(u^* + \sum_{k=2}^{\infty} e^*(n_k)e^*_{n_k}\right) = \phi\left(\|u^*\|\, x^*_1 + \sum_{k=2}^{\infty} e^*(n_k)x^*_k\right) \\ &= \|u^*\|\, y^*_1 + \sum_{k=2}^{\infty} e^*(n_k)y^*_k = \left(1 - \sum_{k=2}^{\infty} e^*(n_k), e^*(n_2), e^*(n_3), \dots\right) = y^*. \end{aligned}$$

Consequently, ϕ is a w^*-continuous homeomorphism from $\overline{\left\{x^*_k : k \in \mathbb{N}\right\}}^{w^*}$ onto $\overline{\left\{y^*_k : k \in \mathbb{N}\right\}}^{w^*} = \left\{y^*_k : k \in \mathbb{N}\right\} \cup \{y^*\}$. So, in view of Lemma 2 in [1] we see that ϕ is a w^*-continuous isometry from Y onto $\ell_1 = W^*_f$. This implies that W_f is isometric to $X/{}^{\perp}Y$. Finally, if $\sum_{k=2}^{\infty} e^*(n_k) = 0$ then W_f is isometric to c. By following the same reasoning as above, we easily conclude that c is isometric to a quotient of X.

To conclude the proof it remains to show that (1)$\Longrightarrow$(4). This part is the key point of the whole proof and we split it in several steps for the sake of convenience of the reader.

(1) $\Longrightarrow$ (4).**The Final Step.** Suppose that we have already constructed a sequence $(x_m)_{m\in\mathbb{N}} \subset B_X$, a $\sigma(\ell_1, X)$-convergent subsequence $(e^*_{n_k})_{k\in\mathbb{N}}$ of the standard basis $(e^*_n)_{n\in\mathbb{N}}$ in $\ell_1 = X^*$ and a null sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ in $(0,1)$ such that for all $k, m \in \mathbb{N}$ we have $e^*_{n_k}(x_m) > 1 - \varepsilon_m$. If e^* denotes the $\sigma(\ell_1, X)$-limit of $(e^*_{n_k})_{k\in\mathbb{N}}$, then $\|e^*\| = 1$ and $e^*(n_k) \geq 0$ for all $k \in \mathbb{N}$.

Indeed, let $k_0 \in \mathbb{N}$ be arbitrarily chosen. Since $e^*_{n_k}(x_m) \xrightarrow{k} e^*(x_m)$, we get $e^*(x_m) \geq 1 - \varepsilon_m$. Consequently, for each $m \in \mathbb{N}$, we have

$$e^*_{n_{k_0}}(x_m) + e^*(x_m) > 1 - \varepsilon_m + 1 - \varepsilon_m = 2 - 2\varepsilon_m.$$

Hence, $\left\|e^*_{n_{k_0}} + e^*\right\| \geq 2$, from which our assertion follows at once.

In the sequel we present how to construct sequences $(x_m)_{m\in\mathbb{N}}$, $(e^*_{n_k})_{k\in\mathbb{N}}$ *and* $(\varepsilon_m)_{m\in\mathbb{N}}$ *described above.*

Step 1. *The sequence* $(x^*_n)_{n\in\mathbb{N}\cup\{0\}}$. Assume that ℓ_1 lacks the $\sigma(\ell_1, X)$-FPP. Then, from the proof of Theorem 8 in [6], we know that there is a sequence $(x^*_n)_{n\in\mathbb{N}\cup\{0\}}$ in ℓ_1 with the following properties:

(i) $x^*_n \xrightarrow{\sigma(\ell_1,X)} x^*_0$,

(ii) $(x^*_n)_{n\in\mathbb{N}}$ tends to 0 coordinatewise,

(iii) $\lim_{n\to\infty} \left\|u^* - x^*_n\right\| = 2$ for every $u^* \in \operatorname{conv}\{x^*_n : n \geq 0\}$,

(iv) $\lim_{n\to\infty} \left\|x^*_n\right\| = 1 = \left\|x^*_0\right\|$.

Now, using (ii), (iii) and (iv), one may observe that for every $n \in \mathbb{N}$ we have

$$2 = \lim_{m\to\infty} \left\|x^*_n - x^*_m\right\| = \left\|x^*_n\right\| + \lim_{m\to\infty} \left\|x^*_m\right\| = \left\|x^*_n\right\| + 1$$

and, consequently,

(v) $\left\|x^*_n\right\| = 1$ for all $n \geq 0$.

Again, using (ii), (iii) and (iv), one may notice that for all $m, n \in \mathbb{N} \cup \{0\}$ we obtain

$$2 = \lim_{k\to\infty} \left\|\frac{1}{2}(x^*_n + x^*_m) - x^*_k\right\| = \left\|\frac{1}{2}(x^*_n + x^*_m)\right\| + \lim_{k\to\infty} \left\|x^*_k\right\| = \left\|\frac{1}{2}(x^*_n + x^*_m)\right\| + 1,$$

hence,

(vi) $\left\|x^*_n + x^*_m\right\| = 2$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Taking into account (v) and (vi) we easily conclude that

(vii) $x^*_n(i) \cdot x^*_m(i) \geq 0$ for all $m, n \in \mathbb{N} \cup \{0\}$ and for every $i \in \mathbb{N}$.

From now on we set $\sum_{i\in\emptyset} a_i := 0$.

Step 2. *Grinding of the sequence* $(x^*_n)_{n\in\mathbb{N}\cup\{0\}}$. Let $(x^*_n)_{n\in\mathbb{N}\cup\{0\}}$ be as above. We show that there is a sequence $(y^*_k)_{k\in\mathbb{N}\cup\{0\}}$ in ℓ_1 and numbers $s^+ \in (0,1]$, $s^- \in (-1,0]$ such that

(a) $\left\|y^*_k\right\| = 1$ for every $k \geq 0$,

(b) for every $k \in \mathbb{N}$ the set $\operatorname{supp} y_k^* := \left\{i \in \mathbb{N} : y_k^*(i) \neq 0\right\}$ is finite and $\max \operatorname{supp} y_k^* < \min \operatorname{supp} y_{k+1}^*$,
(c) $y_m^*(i) \cdot y_n^*(i) \geq 0$ for all $m, n \in \mathbb{N} \cup \{0\}$ and for every $i \in \mathbb{N}$,
(d) for every $k \in \mathbb{N}$

$$s^+(y_k^*) := \sum_{i \in \operatorname{supp}_+ y_k^*} y_k^*(i) = s^+ \quad \text{and} \quad s^-(y_k^*) := \sum_{i \in \operatorname{supp}_- y_k^*} y_k^*(i) = s^-,$$

where $\operatorname{supp}_+ y_k^* := \left\{i \in \mathbb{N} : y_k^*(i) > 0\right\}$, $\operatorname{supp}_- y_k^* := \left\{i \in \mathbb{N} : y_k^*(i) < 0\right\}$,
(e) $y_k^* \stackrel{\sigma(\ell_1, X)}{\longrightarrow} y_0^*$.

Indeed, using (ii) and (v), we can choose a subsequence $(x_{n_k}^*)_{k \in \mathbb{N}}$ of $(x_n^*)_{n \in \mathbb{N}}$ and a sequence $(m_k)_{k \in \mathbb{N} \cup \{0\}}$, $m_k \in \mathbb{N} \cup \{0\}$, $0 = m_0 < m_1 < m_2 < \cdots < m_k < m_{k+1} < \ldots$, such that for every $k \in \mathbb{N}$

$$\sum_{i=m_{k-1}+1}^{m_k} \left|x_{n_k}^*(i)\right| > 1 - \frac{1}{2^k}. \tag{4.1}$$

Now, for every $k \in \mathbb{N}$ we put $\widetilde{x_{n_k}^*} = \sum_{i=m_{k-1}+1}^{m_k} x_{n_k}^*(i) e_i^*$ and $\widetilde{\widetilde{x_{n_k}^*}} = \frac{\widetilde{x_{n_k}^*}}{\left\|\widetilde{x_{n_k}^*}\right\|}$. We can assume that the limits $\lim_k s^+\left(\widetilde{\widetilde{x_{n_k}^*}}\right)$ and $\lim_k s^-\left(\widetilde{\widetilde{x_{n_k}^*}}\right)$ exist, and let $s_0^+ := \lim_k s^+\left(\widetilde{\widetilde{x_{n_k}^*}}\right)$ and $s_0^- := \lim_k s^-\left(\widetilde{\widetilde{x_{n_k}^*}}\right)$. Clearly, $s_0^+ \in [0, 1]$, $s_0^- \in [-1, 0]$ and $s_0^+ - s_0^- = 1$. We shall consider two cases.

First, suppose $s_0^+ > 0$. Then we can assume that $s^+\left(\widetilde{\widetilde{x_{n_k}^*}}\right) > 0$ for all $k \in \mathbb{N}$. Further, suppose $s_0^- < 0$. Then we can also assume that $s^-\left(\widetilde{\widetilde{x_{n_k}^*}}\right) < 0$ for all $k \in \mathbb{N}$. Define the sequence $(y_k^*)_{k \in \mathbb{N} \cup \{0\}}$ as $y_0^* = x_0^*$ and for $k \in \mathbb{N}$

$$y_k^* := \frac{s_0^+}{s^+\left(\widetilde{\widetilde{x_{n_k}^*}}\right)} \cdot \sum_{i \in \operatorname{supp}_+ \widetilde{\widetilde{x_{n_k}^*}}} \widetilde{\widetilde{x_{n_k}^*}}(i) \cdot e_i^* + \frac{s_0^-}{s^-\left(\widetilde{\widetilde{x_{n_k}^*}}\right)} \cdot \sum_{i \in \operatorname{supp}_- \widetilde{\widetilde{x_{n_k}^*}}} \widetilde{\widetilde{x_{n_k}^*}}(i) \cdot e_i^*.$$

Obviously, conditions (a), (b) and (c) are satisfied. Moreover, $s^+(y_k^*) = s_0^+$ and $s^-(y_k^*) = s_0^-$, so in order to obtain (d) it is enough to take $s^+ = s_0^+$ and $s^- = s_0^-$. We shall prove that (e) holds too. Indeed, by considering (4.1), (i) and (v), we get $\lim_k \left\|\widetilde{x_{n_k}^*}\right\| = 1$, $w^*\text{-}\lim_k \left(\sum_{i=m_{k-1}+1}^{m_k} x_{n_k}^*(i) \cdot e_i^*\right) = x_0^*$ and, consequently, $w^*\text{-}\lim_k y_k^* = x_0^* = y_0^*$, as we desired.

If $s_0^- = 0$, then $s_0^+ = 1$ and we can assume that $s^+\left(\widetilde{\widetilde{x_{n_k}^*}}\right) > 0$ for all $k \in \mathbb{N}$. We define the sequence $(y_k^*)_{k \in \mathbb{N} \cup \{0\}}$ as $y_0^* = x_0^*$ and for $k \in \mathbb{N}$ we put

$$y_k^* = \frac{s_0^+}{s^+\left(\widetilde{\widetilde{x_{n_k}^*}}\right)} \cdot \sum_{i \in \operatorname{supp}_+ \widetilde{\widetilde{x_{n_k}^*}}} \widetilde{\widetilde{x_{n_k}^*}}(i) \cdot e_i^*.$$

It is easy to see that the properties (a), (b), (c), (e), and (d) with $s^+ := s_0^+ = 1$ and $s^- := s_0^- = 0$ are satisfied.

Suppose $s_0^+ = 0$. Then $s_0^- = -1$ and we can assume that $s^-\left(\widetilde{\widetilde{x_{n_k}^*}}\right) < 0$ for all $k \in \mathbb{N}$. Now, it is enough to define $(y_k^*)_{k\in\mathbb{N}\cup\{0\}}$ as $y_0^* = -x_0^*$ and for $k \in \mathbb{N}$

$$y_k^* = -\frac{s_0^-}{s^-\left(\widetilde{\widetilde{x_{n_k}^*}}\right)} \cdot \sum_{i\in\operatorname{supp}_- \widetilde{\widetilde{x_{n_k}^*}}} \widetilde{\widetilde{x_{n_k}^*}}(i) \cdot e_i^*.$$

Then, for every $k \in \mathbb{N}$, $s^+(y_k^*) = -s_0^- = 1$. Obviously, properties (a), (b), (c), (e) and (d) are satisfied with $s^+ = 1$ and $s^- = 0$.

Step 3. *A construction of the sequence* $(x_m)_{m\in\mathbb{N}}$. Let $(y_k^*)_{k\in\mathbb{N}\cup\{0\}}$, $s^- \in (-1, 0]$ and $s^+ \in (0, 1]$ be as above. By using (a) and (e), we can choose $x_1 \in B_X$ and $k_1 \in \mathbb{N}$ such that $y_0^*(x_1) > 1 - \frac{s^+}{8}$ and $y_k^*(x_1) > 1 - \frac{s^+}{8}$ for all $k \geq k_1$. Next, using (a) and (c) we can choose $x_2 \in B_X$ such that $y_0^*(x_2) > 1 - \frac{s^+}{8^2}$ and $y_{k_1}^*(x_2) > 1 - \frac{s^+}{8^2}$. Moreover, the property (e) implies that there is $k_2 > k_1$ such that for all $k \geq k_2$ we have $y_k^*(x_2) > 1 - \frac{s^+}{8^2}$. Further, using (a), (c) and (e), we can choose $x_3 \in B_X$ and $k_3 > k_2$ such that $y_0^*(x_3) > 1 - \frac{s^+}{8^3}$, $y_{k_1}^*(x_3) > 1 - \frac{s^+}{8^3}$, $y_{k_2}^*(x_3) > 1 - \frac{s^+}{8^3}$ and $y_k^*(x_3) > 1 - \frac{s^+}{8^3}$ for all $k \geq k_3$. Continuing this inductive procedure, we construct a sequence $(x_m)_{m\in\mathbb{N}} \subset B_X$ and a subsequence $(y_{k_n}^*)_{n\in\mathbb{N}}$ of $(y_k^*)_{k\in\mathbb{N}\cup\{0\}}$ such that $y_{k_n}^*(x_m) > 1 - \frac{s^+}{8^m}$ for all $m, n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ we put $z_n^* = y_{k_n}^*$. Then the sequence $(z_n^*)_{n\in\mathbb{N}}$ has the following properties:

- (a') for every $n \in \mathbb{N}$ the set $\operatorname{supp}_+ z_n^*$ is nonempty, $\operatorname{supp} z_n^*$ is finite, and $\max\operatorname{supp} z_n^* < \min\operatorname{supp} z_{n+1}^*$,
- (b') $z_n^*(x_m) > 1 - \frac{s^+}{8^m}$ for all $m, n \in \mathbb{N}$,
- (c') for every $n \in \mathbb{N}$, $s^+(z_n^*) = s^+$ and $s^-(z_n^*) = s^-$.

Step 4. *A construction of the sequences* $(e_{n_k}^*)_{k\in\mathbb{N}}$ *and* $(\varepsilon_m)_{m\in\mathbb{N}}$. Let $(z_n^*)_{n\in\mathbb{N}}$, s^- and s^+ be as above. For each $m, n \in \mathbb{N}$ we define the set

$$E_m^{(n)} = \left\{ i \in \operatorname{supp}_+ z_n^* : e_i^*(x_m) > 1 - \frac{1}{2^m} \right\}.$$

Then, using (c'), we have

$$\begin{aligned}
\sum_{i\in\operatorname{supp}_+ z_n^*} z_n^*(i)\cdot e_i^*(x_m) &= \sum_{i\in E_m^{(n)}} z_n^*(i)\cdot e_i^*(x_m) + \sum_{i\in(\operatorname{supp}_+ z_n^*)\setminus E_m^{(n)}} z_n^*(i)\cdot e_i^*(x_m) \\
&\leq \sum_{i\in E_m^{(n)}} z_n^*(i) + \left(1 - \frac{1}{2^m}\right)\cdot \sum_{i\in(\operatorname{supp}_+ z_n^*)\setminus E_m^{(n)}} z_n^*(i) \\
&= \left(1 - \frac{1}{2^m}\right)\cdot \sum_{i\in\operatorname{supp}_+ z_n^*} z_n^*(i) + \frac{1}{2^m}\cdot \sum_{i\in E_m^{(n)}} z_n^*(i) \\
&= \left(1 - \frac{1}{2^m}\right)\cdot s^+ + \frac{1}{2^m}\cdot \sum_{i\in E_m^{(n)}} z_n^*(i).
\end{aligned}$$

On the other hand, using (b') and (c'), we get

$$\begin{aligned}\sum_{i\in\operatorname{supp}_+ z_n^*} z_n^*(i)\cdot e_i^*(x_m) &= \sum_{i\in\operatorname{supp} z_n^*} z_n^*(i)\cdot e_i^*(x_m) - \sum_{i\in\operatorname{supp}_- z_n^*} z_n^*(i)\cdot e_i^*(x_m)\\ &> 1-\frac{s^+}{8^m}+s^- = 1-\frac{s^+}{8^m}-1+s^+ = \left(1-\frac{1}{8^m}\right)\cdot s^+.\end{aligned}$$

The above implies that

$$\left(1-\frac{1}{8^m}\right)\cdot s^+ < \left(1-\frac{1}{2^m}\right)\cdot s^+ + \frac{1}{2^m}\cdot \sum_{i\in E_m^{(n)}} z_n^*(i),$$

so

$$\sum_{i\in E_m^{(n)}} z_n^*(i) \geq \left(1-\frac{1}{4^m}\right)\cdot s^+.$$

The above calculations also show that for any $m, n \in \mathbb{N}$ the set $E_m^{(n)}$ is nonempty and

$$\sum_{i\in(\operatorname{supp}_+ z_n^*)\setminus E_m^{(n)}} z_n^*(i) \leq \frac{1}{4^m}\cdot s^+. \tag{4.2}$$

For each $m, n \in \mathbb{N}$ we define the set $F_m^{(n)} = \bigcap_{j=1}^{m} E_j^{(n)}$. Obviously, for every $n \in \mathbb{N}$, $F_1^{(n)} \supseteq F_2^{(n)} \supseteq F_3^{(n)} \supseteq \dots$. We claim that for every $n \in \mathbb{N}$ and $m \in \mathbb{N}$ the set $F_m^{(n)}$ is nonempty. Indeed, suppose that there is $n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $F_m^{(n)} = \emptyset$. Then, $\operatorname{supp}_+ z_n^* = \bigcup_{j=1}^{m}(\operatorname{supp}_+ z_n^* \setminus E_j^{(n)})$, so taking into account (4.2) and (c'), we obtain a contradiction,

$$\frac{1}{2}s^+ > \sum_{j=1}^{m}\frac{1}{4^j}s^+ \geq \sum_{j=1}^{m}\ \sum_{i\in(\operatorname{supp}_+ z_n^*)\setminus E_j^{(n)}} z_n^*(i) \geq \sum_{i\in\operatorname{supp}_+ z_n^*} z_n^*(i) = s^+.$$

Since for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ the set $F_m^{(n)}$ is nonempty and, in view of (a'), $F_1^{(n)}$ is finite, we conclude that the set $G^{(n)} := \bigcap_{m=1}^{\infty} F_m^{(n)}$ is nonempty, for every $n \in \mathbb{N}$. Clearly,

$$G^{(n)} = \bigcap_{m=1}^{\infty} E_m^{(n)} = \left\{ i \in \operatorname{supp}_+ z_n^* : e_i^*(x_m) > 1-\frac{1}{2^m} \text{ for all } m \in \mathbb{N}\right\}.$$

Moreover, using (a') we see that $G^{(i)} \cap G^{(j)} = \emptyset$ provided $i \neq j$. Hence, the set $\bigcup_{j=1}^{\infty} G^{(j)}$ is infinite. Take any $\sigma(\ell_1, X)$-convergent subsequence $(e_{n_k}^*)_{k\in\mathbb{N}}$ of $(e_n^*)_{n\in\bigcup_{j=1}^{\infty} G^{(j)}}$. Then, for every $m \in \mathbb{N}$ and $k \in \mathbb{N}$, we have $e_{n_k}^*(x_m) > 1 - \frac{1}{2^m}$. Apply now **The Final Step**, with $\varepsilon_m = \frac{1}{2^m}$. The proof of (1) $\Longrightarrow$ (4) is finished.

□

Remark 4.2. The spaces "bad" W_f and W_g in the statements (5) and (6) of Theorem 4.1 cannot be replaced by the space c (see Example 2.4).

We conclude the paper by pointing out an issue related to our results that still remain as open problem. Let X be a predual of ℓ_1. Theorem 3.7 implies that the existence of an isometric copy of a "bad" W_f in X ensures the failure of the $\sigma(\ell_1, X)$-FPP. On the other hand, Theorem 4.1 provides a necessary and sufficient condition for the failure of the $\sigma(\ell_1, X)$-FPP based on the existence of a quotient of X isometric to a "bad" W_f. Taking into account these two facts, a natural question still unanswered is whether the lack of the $\sigma(\ell_1, X)$-FPP implies that X contains an isometric copy of a "bad" W_f.

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