INFINITE HORIZON STOCHASTIC OPTIMAL CONTROL FOR VOLTERRA EQUATIONS WITH COMPLETELY MONOTONE KERNELS

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Abstract. The aim of the paper is to study an optimal control problem on infinite horizon for an infinite dimensional integro-differential equation with completely monotone kernels, where we assume that the noise enters the system when we introduce a control. We start by reformulating the state equation into a semilinear evolution equation which can be treated by semigroup methods. The application to optimal control provides other interesting results and requires a precise description of the properties of the generated semigroup. The main tools consist in studying the differentiability of the forward-backward system with infinite horizon corresponding with the reformulated problem and the proof of existence and uniqueness of mild solutions to the corresponding Hamilton Jacobi Bellman (HJB) equation.

Key words. Abstract integro-differential equation; Analytic semigroup; Backward Stochastic differential equations; Elliptic PDEs; Hilbert spaces; Mild Solutions

AMS subject classifications. 45D05, 93E20, 60H30

1. Introduction. In recent years, many investigations have been carried out concerning HJB equations in connection with optimal control of nonlinear infinite dimensional stochastic systems. This paper attempts to use this approach to an infinite horizon control problem where the dynamics of the system is perturbed by the memory term. In this section we briefly outline the main steps of our construction; this is meant as a guideline for the reader, who will find in later sections all the details. Our starting point is a controlled stochastic Volterra equation of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} a(t-s)u(s)\mathrm{d}s = Au(t) + f(u(t)) \\ \qquad\qquad\qquad +g\left[\, r(u(t),\gamma(t)) + \dot{W}(t)\,\right], \qquad t\in[0,T] \\ u(t) = u_0(t), \qquad t\le 0 \end{cases} \tag{1.1}$$

for a process u in a Hilbert space H, where $W(t),\ t\ge 0$ is a cylindrical Wiener process defined on a suitable probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ with values into a (possibly different) Hilbert space Ξ; the kernel a is completely monotonic, locally integrable and singular at 0; A is a linear operator which generates an analytic HJB semigroup; f is a Lipschitz continuous function from H into itself; g is a bounded linear mapping from H into $L_2(\Xi,H)$ (the space of Hilbert-Schmidt operators from Ξ to H). The function r is a bounded Borel measurable mapping from $H\times\mathcal{U}$ into Ξ, with $\mathcal{U}\subseteq U$. Hence the control is modellized by the predictable process γ with values in some specified subset $\mathcal{U}$ (the set of control actions) of a third Hilbert space U. We notice that the control enters the system together with the noise. This special structure is imposed by our techniques. However the presence of the operator r allows more generality.

The optimal control problem that we wish to treat in this paper consists in minimizing the functional cost

$$\mathbb{J}(u_0,\gamma) = \mathbb{E}\int_0^\infty e^{-\lambda t}\ell(u(t),\gamma(t))\mathrm{d}t, \tag{1.2}$$

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where $\ell : H \times U \to \mathbb{R}$ is a given real continuous and bounded function and λ is any positive number.

Our first task is to provide a semigroup setting for the problem, by the state space setting first introduced by Miller in [24] and Desch and Miller [15] and recently revised, for the stochastic case, in Homan [20], Bonaccorsi and Desch [4], Bonaccorsi and Mastrogiacomo [5]. Within this approach, equation (1.1) is reformulated into an abstract evolution equation without memory on a different Hilbert space X. Namely, we rewrite equation (1.1) as

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = B\mathbf{x}(t)\mathrm{d}t + (I-B)P\,f(J\mathbf{x}(t))\mathrm{d}t \\ \qquad\qquad\qquad +(I-B)Pg(r(J\mathbf{x}(t),\gamma(t))\mathrm{d}t + \mathrm{d}W(t)) \\ \mathbf{x}(0) = x, \end{cases} \tag{1.3}$$

where the initial datum x is obtained from u_0 by a suitable transformation. Moreover, B is the infinitesimal generator of an analytic semigroup e^{tB} on X. $P : H \to X$ is a linear mapping which acts as a sort of projection into the space X. $J : D(J) \subset X \to H$ is an unbounded linear functional on X, which gives a way going from $\mathbf{x}$ to the solution to problem (1.1). In fact, it turns out that u has the representation

$$u(t) = \begin{cases} J\mathbf{x}(t), & t > 0, \\ u_0(t), & t \le 0. \end{cases}$$

For more details, we refer to the original papers [4, 20]. Further, the optimal control problem, reformulated into the state setting X, consists in minimizing the cost functional

$$\mathbb{J}(x,\gamma) = \mathbb{E}\int_0^\infty e^{-\lambda t}\ell(J\mathbf{x}(t),\gamma(t))\mathrm{d}t$$

(where the initial condition u_0 is substituted by x and the process u is substituted by $J\mathbf{x}$). It follows that γ is an optimal control for the original Volterra equation if and only if it is an optimal control for that state equation (1.3). We stress that our approach has the advantage that it naturally links the solution of a Volterra equation to a Markov process; this has important developments in view of the application to optimal control problems for Volterra equations.

To our knowledge, this paper is the first attempt to study optimal control problems for stochastic Volterra equations on an unbounded interval time horizon. In order to handle the control problem we introduce the weak reformulation; in this setting we wish to perform the standard program of synthesis of the optimal control, by proving the so-called fundamental relation for the value function and then solving in the weak sense the closed loop equation. The main step is to study the stationary HJB equation corresponding with our problem. In our case, this is given by

$$\mathcal{L}v(x) = \lambda v(x) - \psi(x, \nabla v(x)(I-B)Pg(Jx)), \qquad x \in X, \tag{HJB}$$

where $\mathcal{L}$ is a second order operator and coincides with the infinitesimal generator of the Markov semigroup corresponding to the process $\mathbf{x}$, ψ is the Hamiltonian of the problem, defined in terms of ℓ and λ a positive number. Here $\nabla h(x) \in H^\star$ denotes the Gâteaux derivative at point $x \in H$ and $\nabla^2 h$ is the second Gâteaux derivative, identified with an element of $L(H)$ (the vector space of linear and continuous functionals on H). Equations of type HJB have been studied by probabilistic approach in several finite dimensional situations [10, 14, 25, 26] and extended in infinite dimension by Fuhrman and Tessitore [19]. In particular, an equation like HJB is studied without any nondegeneracy assumption on g and a unique Gâteaux differentiable mild solution to HJB is found by an approach based on backward stochastic differential equations (BSDEs).

In this paper we study the HJB equation through the BSDE techniques initiated by Fuhrman and Tessitore in [19] and Hu and Tessitore [21]. In particular, the BSDE corresponding with our problem is given by

$$\mathrm{d}Y(\tau,x) = \lambda Y(\tau,x)\mathrm{d}t - \psi(\mathbf{x}(\tau;x), Z(\tau,x))\mathrm{d}t + Z(\tau,x)\mathrm{d}W(\tau), \tau \geq 0 \qquad \text{(BSDE)}$$

together with a suitable growth condition (which substitutes the final condition of the finite horizon case). In the above formula, $\mathbf{x}(\cdot;x)$ is the solution of the equation (1.3) starting from $x \in X$ at time $t = 0$.

The main difficulty, in our case, is due to the presence of unbounded terms such as $(I - B)Pg$, which forces us to prove extra regularity for the solution to HJB.

Our purpose is to prove that the solution of (BSDE) exists and it is unique for any positive λ and that the mild solution of the (HJB) equation is given in terms of the solution of the (BSDE). In particular, if we set $v(x) := Y(0,x)$, it follows that v is the mild solution of (HJB). As noted above, in our case it is not enough to prove that v is once (Gâteaux) differentiable to give sense to equation HJB. Indeed the occurrence of the term $\nabla v(\cdot)(I - B)P$, together with the fact that P maps H into an interpolation space of $D(B)$, forces us to prove that the map $h \mapsto \nabla v(x)(I - B)^\theta P[h]$ for suitable $\theta \in (0,1)$ extends to a continuous map on H. To do that we start proving that this extra regularity holds, in a suitable sense, for the state equation (1.3) and then that it is conserved if we differentiate in Gâteaux sense the backward equation with respect to the process $\mathbf{x}$. On the other side, we can prove that if the map $h \mapsto \nabla v(x)(I - B)^\theta[h]$ extends to a continuous function on H, then the processes $t \mapsto v(\mathbf{x}(t;x))$ and W admit joint quadratic variation in any interval $[t,T]$ and this is given by

$$\int_t^T \nabla v(\mathbf{x}(\tau;x))(I - B)Pg \, \mathrm{d}\tau.$$

This result is standard and is done by an application of the Malliavin calculus (on a finite time horizon).

We can then come back to the control problem, and using the probabilistic representation of the unique mild solution to equation (HJB) we easily show the existence of an optimal feedback law.

The paper is organized as follows: in the next section we give some useful notation and we introduce the main assumptions on the coefficients of the problem. In Section 4 we reformulate the problem into a semilinear abstract evolution equation and we study the properties of leading operator. In Section 5 we prove the first main result of the paper: we determine the existence and uniqueness of the solution of the reformulated equation (1.3). In Section 6 we turn back to the original Volterra equation (1.1) in order to establish even in this case the existence and uniqueness of the solutions. In Section 7 we give some properties of the forward equation corresponding with the reformulated uncontrolled problem. In Section 8 we study the existence, uniqueness and properties of the backward stochastic equation (BSDE). In Section 9 we proceed with the study of the HJB equation and, finally, in Section 10 we employ the results proved in the preceding sections in order to perform the standard synthesis of the optimal control.

2. Volterra equations. Integro partial differential equations are widely used in literature for modeling those economic processes where the output of an endogenous or exogenous variable depends not only on its value at the present time, but also on its values at previous time points. In particular, we recall the applications in economic growth theory (see for instance [29, 30]) and fractional diffusion-wave equations which also have economic interpretation (as suggested, for example, in [31, 32]). This last kind of equation are modeled after

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = b \frac{\partial^2}{\partial x^2} u(t,x),$$

where α is a parameter describing the order of the fractional derivative, that is taken in the sense of Caputo fractional derivative. Although theoretically α can be any number, we consider here only the case $0 < \alpha < 1$; notice that for $\alpha = 1$ the above equation represents the standard diffusion equation. The fractional derivative of u of order α is

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g_{1-\alpha}(t-s) u(s) \, \mathrm{d}s$$

where g_β is the fractional derivative kernel $g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}$. This kernel is continuous, locally integrable on $\mathbb{R}_+$ and it is completely monotone. There has been a growing interest to investigate this equation, for various reasons; we quote from [2], among others, "modeling of anomalous diffusive and subdiffusive systems" and "description of fractional random walks".

2.1. A motivating example. Next example shows an application of (1.1) to the theory of economic growth. Because growth theorists are principally interested in the long-term economic performances, the typical analysis performed is asymptotic, therefore requiring the study of infinite time horizon optimization problems (see for example, [3, Chapter 2]).

Denote by $k(t,x)$ the capital stock held by a household located at x at date t. Without capital mobility, the unique way for the household to increase $k(t,x)$ is by consuming less, thus saving more and using this saving to invest more. If we allow to capital to flow across space, $k(t,x)$ is affected by the net flows of capital to a given location or space interval. Capital spatio-temporal dynamics are described by the integro-differential parabolic equations: capital stock at (t,x) depends on the productivity, saving and investment capacity of individuals established at x and on trade as well. The following Volterra equation generalizes the budget constraint introduced in [7] and takes into account the memory effects affecting economic growth as described in [29, 30, 31, 32]. It is assumed that individuals are distributed homogeneously along the unit circle in the plane, denoted by $\mathbb{T}$. Using polar coordinates $\mathbb{T}$ can be described as the set of spatial parameters $\theta \in [0, 2\pi]$ with $\theta = 0$ and $\theta = 2\pi$ being identified. Capital is mobile along the circle $\mathbb{T}$, and its dynamics is characterized by the inclusion of the special term $\frac{\partial^2}{\partial \theta^2} k(t,\theta)$. This term takes into account the aggregate resource constraint of all the households distributed across space. In particular it is linked to the flow of capital from rich to poor regions. The total production at θ at time t is simply $y(t,\theta) = f(k(t,\theta))$, for a function f satisfying suitable assumptions (see [7] for more details) while the influence of consumption, savings, investment capacity of individuals are summarized through the presence of a nonlinear capital dependent, stochastically perturbed, source h. We thus consider the following dynamics:

$$\frac{\partial}{\partial t}\int_{-\infty}^{t} c(t-s)k(s,\theta)\mathrm{d}s - \frac{\partial^2}{\partial\theta^2}k(t,\theta) = f(k(t,\theta)) + h(t,\theta), \qquad (t,\theta)\in\mathbb{R}_+\times\mathbb{T} \tag{2.1}$$

$$k(0,\theta) = k_0(\theta) > 0, \qquad \theta\in\mathbb{T}, \tag{2.2}$$

$$k(t,0) = k(t,2\pi) = 0, \quad t\in\mathbb{R}_+. \tag{2.3}$$

We assume that h has the following form:

$$h(t,\theta) = \sigma\gamma(t) + \sigma\dot{\beta}(t).$$

Here $\sigma\in\mathbb{R}$, γ acts as the control on the system and takes into account the contribution of savings, investments and consumption to the capital motion. Moreover β is a real Wiener processes.

Problem (2.1) is a non-linear dynamical system and $k(t,\theta)$ is not, in general, a Markov process, since its future development depends on the whole history. With the choice $c(t) = g_\beta(t) = \frac{1}{\Gamma(1-\beta)}t^{-\beta}$, the above equation reads $D^\beta k(t) = \Delta k(t) + h(t)$, where D^β is the fractional derivative of order β (compare [27]).

Provided an initial distribution of physical capital $k_0(\cdot)$, a typical economic problem is to look for a control γ in order to maximize the total discounted welfare or utility of all the individuals present in the space considered over an infinite time horizon (as proposed, among the others, by [1]). A benchmark problem is to maximize the functional

$$\mathbb{J}(k_0,\gamma) = \mathbb{E}\int_0^{+\infty} e^{-\lambda t}u(\gamma(t),k(t))\mathrm{d}t\mathrm{d}\theta. \tag{2.4}$$

over all admissible bounded γ and differentiable, bounded $u:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$.

The above example will be solved in Section 10.1, where we shall see how it fits into our setting and which kind of solution this problem admits.

3. Notations and main assumptions. The norm of an element x of a Banach space E will be denoted by $|x|_E$ or simply $|x|$ if no confusion is possible. If F is another Banach space, $L(E,F)$ denotes the space of bounded linear operators from E to F, endowed with the usual operator norm.

The letters Ξ, H, U will always denote Hilbert spaces. Scalar product is denoted $\langle\cdot,\cdot\rangle$, with a subscript to specify the space, if necessary. All Hilbert spaces are assumed to be real and separable.

By a cylindrical Wiener process with values in a Hilbert space Ξ, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, we mean a family $(W(t))_{t\geq 0}$ of linear mappings from Ξ to $L^2(\Omega)$, denoted $\xi\mapsto\langle\xi,W(t)\rangle$ such that

1. for every $\xi\in\Xi$, $(\langle\xi,W(t)\rangle)_{t\geq 0}$ is a real (continuous) Wiener process;
2. for every $\xi_1,\xi_2\in\Xi$ and $t\geq 0$, $\mathbb{E}(\langle\xi_1,W(t)\rangle\langle\xi_2,W(t)\rangle) = \langle\xi_1,\xi_2\rangle$.

$(\mathcal{F}_t)_{t\geq 0}$ will denote the natural filtration of W, augmented with the family of $\mathbb{P}$-null sets. The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. All the concepts of measurability for stochastic processes refer to this filtration. By $\mathcal{B}(\Gamma)$ we mean the Borel σ-algebra of any topological space Γ.

In the sequel we will refer to the following class of stochastic processes with values in a Hilbert space K:

1. $L^p(\Omega; L^2(0,T;K))$ defines, for $T > 0$ and $p \geq 1$, the space of equivalence classes of progressively measurable processes $y : \Omega \times [0,T) \to K$, such that

$$|y|^p_{L^p(\Omega;L^2(0,T;K))} := \mathbb{E}\left[\int_0^T |y(s)|^2_K \mathrm{d}s\right]^{p/2} < \infty.$$

Elements of $L^p(\Omega; L^2(0,T;K))$ are identified up to modification.
2. $L^p(\Omega; C([0,T];K))$ defines, for $T > 0$ and $p \geq 1$, the space of equivalence classes of progressively measurable processes $y : \Omega \times [0,T) \to K$, with continuous paths in K, such that the norm

$$|y|^p_{L^p(\Omega;C([0,T];K))} := \mathbb{E}\left[\sup_{t\in[0,T]} |y(t)|^p_K\right]$$

is finite. Elements of $L^p(\Omega; C([0,T];K))$ are identified up to indistinguishability.

We also recall notation and basic facts on a class of differentiable maps acting among Banach spaces, particularly suitable for our purposes (we refer the reader to Fuhrman and Tessitore [18] or Ladas and Lakshmikantham [22, Section 1.6] for details and properties). Let now X, Y, V denote Banach spaces. We say that a mapping $F : X \to V$ belongs to the class $\mathcal{G}^1(X,V)$ if it is continuous, Gâteaux differentiable on X, and its Gâteaux derivative $\nabla F : X \to L(X,V)$ is strongly continuous.

The last requirement is equivalent to the fact that for every $h \in X$ the map $\nabla F(\cdot)h : X \to V$ is continuous. Note that $\nabla F : X \to L(X,V)$ is not continuous in general if $L(X,V)$ is endowed with the norm operator topology; clearly, if it happens, then F is Fréchet differentiable on X. It can be proved that if $F \in \mathcal{G}^1(X,V)$, then $(x,h) \mapsto \nabla F(x)h$ is continuous from $X \times X$ to V; if, in addition, G is in $\mathcal{G}^1(V,Z)$, then $G(F)$ is in $\mathcal{G}^1(X,Z)$ and the chain rule holds: $\nabla(G(F))(x) = \nabla G(F(x))\nabla F(x)$. When F depends on additional arguments, the previous definitions and properties have obvious generalizations.

Moreover, we assume the following.

Hypothesis 3.1.

1. *The kernel $a : (0,\infty) \to \mathbb{R}$ is completely monotonic, locally integrable, with $a(0+) = +\infty$. The singularity in 0 shall satisfy some technical conditions that we make precise in Section 4.*
2. *$A : D(A) \subset H \to H$ is a sectorial operator in H. Thus A generates an analytic semigroup e^{tA}.*
3. *The function $f : H \to H$ is continuous and continuously Gâteaux differentiable; moreover there exist a constant $L > 0$ such that*

$$|f(u) - f(v)| \leq L|u - v|, \qquad u, v \in H;$$

4. *The mapping g belongs to $L_2(\Xi, H)$ (the space of Hilbert-Schmidt operators from Ξ to H);*
5. *The function $r : H \times \mathcal{U} \to \Xi$ is Borel measurable and there exists a positive constant $C > 0$ such that*

$$|r(u_1,\gamma) - r(u_2,\gamma)| \leq C|u_1 - u_2|, \qquad u_1, u_2 \in H, \gamma \in \mathcal{U};$$
$$|r(u,\gamma)|_\Xi \leq C, \qquad u \in H, \gamma \in \mathcal{U}.$$

6. *The process $(W(t))_{t\geq 0}$ is a cylindrical Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with values in the Hilbert space Ξ.*

REMARK 3.2. *We notice that the assumption 3.1(3) implies that the Gateaux derivative of f is bounded, that is*

$$\|\nabla_u f(u)\|_{\mathcal{L}(H)} \leq C, \qquad u \in H.$$

The initial condition satisfies a global exponential bound as well as a linear growth bound as $t \to 0$:

HYPOTHESIS 3.3.

1. *There exist $M_1 > 0$ and $\omega > 0$ such that $|u_0(t)| \leq M_1 e^{\omega t}$ for all $t \leq 0$;*
2. *There exist $M_2 > 0$ and $\tau > 0$ such that $|u_0(t) - u_0(0)| \leq M_2|t|$ for all $t \in [-\tau, 0]$;*
3. *$u_0(0) \in H_\varepsilon$ for some $\varepsilon \in (0, 1/2)$. Here H_ε denotes the interpolation space of order ε between H and $D(A)$, i.e. $H_\varepsilon = (H, D(A))_\varepsilon$.*

Concerning the cost functional ℓ we make the following general assumptions:

HYPOTHESIS 3.4. *The function $\ell : H \times U \to \mathbb{R}$ is continuous and bounded.*

We consider the following notion of weak solution for the Volterra equation (1.1).

DEFINITION 3.5. *We say that a process $u = (u(t))_{t\geq 0}$ is a weak solution to equation (1.1) if u is an adapted, p-mean integrable (for any $p \geq 1$), continuous H-valued predictable process and the identity*

$$\int_{-\infty}^{t} a(t-s)\langle u(s), \zeta\rangle_H \mathrm{d}s = \langle \bar{u}, \zeta\rangle_H + \int_0^t \langle u(s), A^\star\zeta\rangle_H \mathrm{d}s$$
$$+ \int_0^t \langle f(u(s)), \zeta\rangle \mathrm{d}s + \int_0^t \langle g\, r(u(s), \gamma(s)), \zeta\rangle_H \mathrm{d}s + \langle gW(t), \zeta\rangle$$

holds $\mathbb{P}$-a.s. for arbitrary $t \geq 0$ and $\zeta \in D(A^\star)$, with $A^\star$ being the adjoint of the operator A and

$$\bar{u} = \int_{-\infty}^{0} a(-s)u_0(s)\mathrm{d}s.$$

4. The analytical setting. A completely monotone kernel $a : (0,\infty) \to \mathbb{R}$ is a continuous, monotone decreasing function, infinitely often derivable, such that

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} a(t) \geq 0, \quad t \in (0,\infty),\ n = 0, 1, 2, \ldots$$

By Bernstein's Theorem [27, pag. 90], a is completely monotone if and only if there exists a positive measure ν on $[0,\infty)$ such that

$$a(t) = \int_{[0,\infty)} e^{-\kappa t}\nu(\mathrm{d}\kappa), \quad t > 0.$$

From the required singularity of a at $0+$ we obtain that $\nu([0,+\infty)) = a(0+) = +\infty$ while for $s > 0$ the Laplace transform $\hat{a}$ of a verifies

$$\hat{a}(s) = \int_{[0,+\infty)} \frac{1}{s+\kappa}\,\nu(\mathrm{d}\kappa) < +\infty.$$

We introduce the quantity

$$\alpha(a) = \sup\left\{\rho \in \mathbb{R} : \int_c^\infty s^{\rho-2} \frac{1}{\hat{a}(s)} \mathrm{d}s < \infty\right\}$$

and we make the following assumption:

HYPOTHESIS 4.1. $\alpha(a) > 1/2$.

REMARK 4.2. *The function $a(t) = e^{-\omega t} t^{\alpha-1}$, where $\omega > 0$, is an example of completely monotone kernel, with Laplace transform $\hat{a}(s) = \Gamma[\alpha]\,(\omega+s)^{-\alpha}$; an easy computation shows that $\alpha(a) = 1-\alpha$, hence we satisfy assumption 4.1 whenever we take $\alpha \in (0, \frac{1}{2})$.*

REMARK 4.3. *It is known from the theory of deterministic Volterra equations that the singularity of a helps smoothing the solution. We notice that $\alpha(a)$ is independent of the choice of $c > 0$, and this quantity describes the behavior of the kernel near 0; by this way we ensure that smoothing is sufficient to keep the stochastic term tractable.*

Under the assumption of complete monotonicity of the kernel, a semigroup approach to a type of abstract integro-differential equations encountered in linear viscoelasticity was introduced in [15] and extended to the case of Hilbert space valued equations in [4]. In order to simplify the exposition we quote from [4] the main result concerning the derivation of the state equation (1.3).

We will see that this approach allows us to treat the case of semilinear, stochastic integral equations; we start for simplicity with the equation

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s &= Au(t) + \bar{f}(t), \qquad && t \in [0,T] \\ u(t) &= u_0(t), && t \le 0, \end{aligned} \tag{4.1}$$

where $\bar{f}$ belongs to $L^1(0,T;H)$ and u_0 satisfies Hypothesis 3.3. A weak solution of equation (4.1) is defined as follows:

DEFINITION 4.1. *We say that a function $u \in L^1([0,T];H)$ is a weak solution to equation (4.1) if u satisfies the identity*

$$\int_{-\infty}^t a(t-s)\langle u(s), \zeta\rangle_H \mathrm{d}s = \langle \bar{u}, \zeta\rangle_H + \int_0^t \langle u(s), A^\star\zeta\rangle_H \mathrm{d}s + \int_0^t \langle \bar{f}(s), \zeta\rangle \mathrm{d}s$$

for arbitrary $t \in [0,T]$ and $\zeta \in D(A^\star)$, with $A^\star$ being the adjoint of the operator A and

$$\bar{u} = \int_{-\infty}^0 a(-s)u_0(s)\mathrm{d}s.$$

In order to solve (4.1), we start from the following identity, which follows by Bernstein's theorem:

$$\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s = \int_{-\infty}^t \int_{[0,+\infty)} e^{-\kappa(t-s)}\,\nu(\mathrm{d}\kappa)\,u(s)\mathrm{d}s = \int_{[0,+\infty)} \mathbf{x}(t,\kappa)\,\nu(\mathrm{d}\kappa)$$

where we introduce the state variable

$$\mathbf{x}(t,\kappa) = \int_{-\infty}^t e^{-\kappa(t-s)}u(s)\,\mathrm{d}s. \tag{4.2}$$

Formal differentiation yields

$$\frac{\partial}{\partial t}\mathbf{x}(t,\kappa) = -\kappa\mathbf{x}(t,\kappa) + u(t), \tag{4.3}$$

while the integral equation (4.1) can be rewritten as

$$\int_{[0,+\infty)} (-\kappa\mathbf{x}(t,\kappa) + u(t))\,\nu(\mathrm{d}\kappa) = Au(t) + \bar{f}(t). \tag{4.4}$$

Now, the idea is to use equation (4.3) as the state equation, with $B\mathbf{x} = -\kappa\mathbf{x}(\kappa) + u$, while (4.4) enters in the definition of the domain of B.

In the following we want to give a formal description of the arguments above by introducing the suitable state space X for $\mathbf{x}(t,\cdot)$ and suitable operators.

Definition 4.2. *Let X denote the space of all Borel measurable functions $\mathbf{y} : [0,+\infty) \to H$ such that the seminorm*

$$\|\tilde{\mathbf{y}}\|_X^2 := \int_{[0,+\infty)} (\kappa+1)|\mathbf{y}(\kappa)|_H^2\,\nu(\mathrm{d}\kappa)$$

is finite. We shall identify the classes $\mathbf{y}$ with respect to equality almost everywhere in ν.

Definition 4.3. *We let the operator $Q : L^1((-\infty,0];H) \to L^\infty([0,+\infty);H,\mathrm{d}\nu)$ be given by*

$$\mathbf{x}(0,\kappa) = Qu_0(\kappa) = \int_{-\infty}^0 e^{-\kappa s}u_0(s)\,\mathrm{d}s. \tag{4.5}$$

The operator Q maps the initial value of the stochastic Volterra equation to the initial value of the abstract state equation. Different initial conditions of the Volterra equation generate different initial conditions of the state equation. It has been proved in [4, Lemma 3.16] that Q maps $L^1(-\infty,0;H)\cap L^\infty(-\infty,0;H)$ into X. In particular, Hypothesis 3.3 is necessary in order to have a greater regularity on the inial value of the state equation. In fact in this case [4, Lemma 3.16 (3)]) shows that Qu_0 belongs to X_η for $\eta \in (0,\frac{1}{2})$.

Remark 4.4. *We stress that under our assumptions we are able to treat, for instance, initial conditions for the Volterra equation of the following form*

$$u_0(t) = \begin{cases} 0, & t \in (-\infty,-\delta); \\ \bar{u} & t \in [-\delta,0] \end{cases}$$

provided $\bar{u}$ has a suitable regularity.

We introduce a rigorous definition for the leading operator of the reformulated state equation (4.4).

Definition 4.4. *Let $\mathbf{x} \in X$ and $u \in H$. We define the operators $\tilde{J} : D(\tilde{J}) \subset X \to X$ by*

$$D(\tilde{J}) := \left\{ \mathbf{x} \in X : \exists u \in H s.t. \int_{[0,\infty)} (\kappa+1)|\kappa\mathbf{x}(\kappa) - u|^2\nu(\mathrm{d}\kappa) < +\infty \right\}$$
$$\tilde{J}\mathbf{x} := u;$$

$B : D(B) \subset X \to X$

$$D(B) := \left\{ \mathbf{x} \in D(\tilde{J}) | \tilde{J}\mathbf{x} \in D(A), \int_{[0,\infty)} -\kappa \mathbf{x}(\kappa) + \tilde{J}\mathbf{x}(\kappa)\nu(\mathrm{d}\kappa) = A\tilde{J}\mathbf{x} \right\}$$

$$(B\mathbf{x})(\kappa) := -\kappa \mathbf{x}(\kappa) + \tilde{J}\mathbf{x}$$

and $P : H \to X$

$$(Pu)(\kappa) := \frac{1}{\kappa + 1} R(\hat{a}(1), A)u,$$

$R(\hat{a}(1), A)$ *being the resolvent* $R(\hat{a}(1), A) := (\hat{a}(1) - A)^{-1}$.

We quote from [4] the main result concerning the state space setting for stochastic Volterra equations in infinite dimensions and the properties of the linear operators introduced above.

Theorem 4.5 (State space setting). *Let A, a, W, $\alpha(a)$ be as in Hypotheses 3.1 and 4.1; choose numbers $\eta \in (0,1)$, $\theta \in (0,1)$ such that*

$$\eta > \frac{1}{2}(1 - \alpha(a)), \quad \theta < \frac{1}{2}(1 + \alpha(a)), \quad \theta - \eta > \frac{1}{2}. \tag{4.6}$$

Let X, Q, B, $\tilde{J}$ be defined as in Definitions 4.2, 4.3 and 4.4. Then

1) *the operator $B : D(B) \subset X \to X$ is a densely defined sectorial operator generating an analytic semigroup e^{tB};*
2) *the operator $\tilde{J} : D(B) \to D(A)$ is onto and admits a unique extension J as a continuous linear operator $J : X_\eta \to H$, where X_η denotes the real interpolation space $X_\eta := (X, D(B))_{\eta,2}$;*
3) *the linear operator P maps H continuously into X_θ.*

Proof. The assertion of the theorem can be proved by following [4, Section 3]. □

Our idea is to rewrite the semilinear inhomogeneous integral equation (4.1) into an abstract evolution equation on the state space X, by using the linear operators introduced above. More precisely, equation (4.1) can be reformulated as

$$\begin{aligned} &\mathbf{x}'(t) = B\mathbf{x}(t)\mathrm{d}t + (I - B)P\bar{f}(t), \qquad t \geq 0 \\ &\mathbf{x}(0) = Qu_0. \end{aligned} \tag{4.7}$$

Since B is the generator of an analytic semigroup in X, we can give a meaning to the solution of (4.7) in a mild sense as

$$\mathbf{x}(t) = e^{(t-s)B}\mathbf{x}(0) + \int_s^t e^{(t-\sigma)B}(I - B)P\bar{f}(\sigma)\mathrm{d}\sigma. \tag{4.8}$$

In the following we will explore the relation between the abstract state space and Problem (4.1).

Theorem 4.6. *We assume Hypotheses 3.1 (i-iii), 3.3, $\bar{f} \in L^1(0,T;H)$ and let X, Q, B, P be defined as in Definition 4.2, 4.3, 4.4 and J as in Theorem 4.5 (ii); choose numbers $\eta \in (0,1)$, $\theta \in (0,1)$ such that*

$$\eta > \frac{1}{2}(1 - \alpha(a)), \quad \theta < \frac{1}{2}(1 + \alpha(a)), \quad \theta - \eta > \frac{1}{2}.$$

Set $x = Qu_0$. If $\mathbf{x}$ given by (4.8) is the solution in mild sense of the abstract Cauchy problem (4.7), then the following assertions hold:

1. $\mathbf{x}(t) \in L^1_{loc}([0,\infty); X_\eta)$, *thus* $J\mathbf{x}(t)$ *is well defined almost everywhere, and* $J\mathbf{x} \in L^1_{loc}([0,\infty); H)$;
2. *The function* $u: [0,\infty) \to H$, *defined by*

$$u(t) = \begin{cases} u_0(t), & if\ t < 0, \\ J\mathbf{x}(t) & if\ t \geq 0, \end{cases} \tag{4.9}$$

is the unique weak solution of (4.1).

Proof. The result is a direct consequence of [4, Theorem 2.7]. We notice that in the aforementioned result the authors prove the existence and uniqueness of an integrated solution, which is, clearly also a weak solution (compare [4, Definition 2.6] and Definition 4.1 above). □

It is remarkable that B generates an analytic semigroup, since in this case we have a powerful theory of optimal regularity results at our disposal. In particular, besides the interpolation spaces X_θ, we may construct the extrapolation space X_{-1}, which is the completion of X with respect to the norm $\|\mathbf{x}\|_{-1} := \|(B-I)^{-1}\mathbf{x}\|_X$.

Under some additional conditions it is possible to prove the exponential stability of the semigroup e^{tB}. As we will see, this is especially needed in the study of the optimal control problem with infinite horizon. In particular we impose the following:

Hypothesis 4.5. *There exists* $\sigma > 0$ *such that the function* $e^{\sigma t} a(t)$ *is completely monotonic.*

Consequently, we obtain that B is of negative type.

Proposition 4.6. *The real parts of the spectrum of* B *are bounded by some* $\omega_0 < 0$. *Consequently the semigroup* e^{tB} *decays exponentially.*

Proof. We proceed as in Bonaccorsi and Desch [4]. All we need to show is that 0 is in the resolvent set of B. Once this is proved, we obtain that the spectral bound is negative since the spectrum is confined to a sector. Now we notice that $\hat{a}(s)$ exists in the set $\mathbb{C} \setminus (-\infty, -\sigma]$ and 0 is in the resolvent set of A, by assumption. Moreover, we can give the explicit expression of $R(s,B)$, which is

$$[R(s,B)\mathbf{x}](\kappa) = \frac{1}{s+\kappa}\left[\mathbf{x}(\kappa) + R(s\hat{a}(s))\int_{[0,\infty)} \frac{\kappa}{s+\kappa}\mathbf{x}(\kappa)\nu(\mathrm{d}\kappa)\right]$$

(the calculation is straightforward and can be found in [4, Lemma 3.5]). Then it is easily seen that $R(s,B)$ can be extended to a neighborhood of 0. □

The semigroup e^{tB} extends to X_{-1} and the generator of this extension, that we denote B_{-1}, is the unique continuous extension of B to an isometry between X and X_{-1}. See for instance [16, Definition 5.4] for further details.

Remark 4.7.

1. *In the sequel, we shall always denote the operator with the letter* B, *even in case where formally* B_{-1} *should be used instead. This should cause no confusion, due to the similarity of the operators.*
2. *We notice that the interpolation spaces* $X_\gamma, \gamma \in (0,1)$ *and the domains of fractional powers of* B *are linked by the following inclusion*

$$D((-B)^{\gamma+\varepsilon}) \subset X_\gamma, \varepsilon > 0,$$

(see [13, Proposition A.15]). Hence, if x *is any element of* $X_\gamma, \gamma \in (0,1)$, *we can find sufficiently small* $\varepsilon > 0$ *such that*

$$\|x\|_\gamma \leq \|x\|_{D((-B)^{\gamma+\varepsilon})} = \|(I-B)^{\gamma+\varepsilon}x\|_X.$$

We will frequently make use of the above inequality in the following, especially when we need to prove that an element of X is, instead in X_η.

3. *We notice that, since $B - \omega_0$ is the infinitesimal generator of the analytic semigroup of contraction $(e^{-\omega_0 t}e^{tB})_{t\geq 0}$ on X with $0 \in \rho(B-\omega_0)$, we have that $X_\theta = (D(-B)^\theta)$ for any $\alpha \in (0,1)$ and there exists $M > 0$ such that*

$$\|(B-\omega_0)e^{-\omega_0 t}e^{tB}\|_{\mathcal{L}(X_\theta, X_\eta)} \leq M t^{-1+\theta-\eta}, \qquad t \geq 0.$$

Moreover, the norm in X_θ is equivalent to the norm $x \mapsto \|(-B)^\theta x\|_X$ on $D((-B)^\theta)$. The above properties and estimate are well explained in [20, pgg. 23,25 Theorems 1.4.27, Lemma 1.4.15, Corollary 1.4.30(ii)]. For more details see also [23, pages 114, 97, 120, Theorems 4.3.5, 4.2.6, Corollary 4.3.12]. We assume, for simplicity, that $\omega_0 > -1$, so that the same estimate holds with $B - I$ instead of $B - \omega_0$, i.e.

$$\|(B-I)e^{tB}\|_{\mathcal{L}(X_\theta, X_\eta)} \leq M_T t^{-1+\theta-\eta}, \qquad t \in [0,T],$$

where M_T is a positive constant depending only on T. In the next pages we will make use of this inequality frequently and of the equivalence between interpolation spaces and the domains of the fractional power of B, especially when studying the forward equation, its differentiability and the Malliavin regularity.

5. The state equation: existence and uniqueness. In this section, motivated by the construction in Section 4, we shall establish the existence and uniqueness result for the following stochastic uncontrolled Cauchy problem on the space X defined in Section 4:

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = B\mathbf{x}(t)\mathrm{d}t + (I-B)Pf(J\mathbf{x}(t))\mathrm{d}t \\ \qquad\qquad +(I-B)Pg\, r(J\mathbf{x}(t), \gamma(t))\mathrm{d}t + (I-B)Pg\,\mathrm{d}W(t) \\ \mathbf{x}(0) = x, \end{cases} \tag{5.1}$$

for $0 \leq t \leq T$ and initial condition $x \in X_\eta$. The above expression is only formal since the coefficients do not belong to the state space. However, we can give a meaning to the mild form of the equation:

Definition 5.1. *We say that a continuous, X-valued, predictable process $\mathbf{x} = (\mathbf{x}(t))_{t\geq 0}$ is a (mild) solution of the state equation* (1.3) *if $\mathbb{P}$-a.s.,*

$$\begin{aligned} \mathbf{x}(t) = e^{(t-s)B}x &+ \int_s^t e^{(t-\sigma)B}(I-B)Pf(J\mathbf{x}(\sigma))\mathrm{d}\sigma \\ &+ \int_0^t e^{(t-\sigma)B}(I-B)P\, r(J\mathbf{x}(\sigma), \gamma(\sigma))\mathrm{d}\sigma + \int_s^t e^{(t-\sigma)B}(I-B)Pg\,\mathrm{d}W(\sigma). \end{aligned}$$

Let us state the main existence result for the solution of equation (1.3).

Theorem 5.2. *Under Hypotheses 3.1, 3.3, for an arbitrary predictable process γ with values in $\mathcal{U}$, for every $0 \leq t \leq T$ and $x \in X_\eta$, there exists a unique adapted process $\mathbf{x} \in L^p_{\mathcal{F}}(\Omega, C([0,T]; X_\eta))$ solution of* (1.3)*. Moreover, the estimate*

$$\mathbb{E} \sup_{t\in[0,T]} ||\mathbf{x}(t)||^p_\eta \leq C(1 + ||x||^p_\eta) \tag{5.2}$$

holds for some positive constant C independent of T.

Proof. The proof of the above theorem proceeds, basically, on the same lines as the proof of Theorem 3.2 in Bonaccorsi and Mastrogiacomo [5] (2009) and is therefore omitted. ◻

REMARK 5.3. *In the following it will be also useful to consider the uncontrolled version of equation* (1.3)*, namely:*

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = B\mathbf{x}(t)\mathrm{d}t + (I-B)Pf(J\mathbf{x}(t))\mathrm{d}t + (I-B)Pg\,\mathrm{d}W(t) \\ \mathbf{x}(0) = x. \end{cases} \tag{5.3}$$

We will refer to (5.3) *as the forward equation. We then notice that existence and uniqueness for the above equation can be treated in an identical way as in the proof of Theorem 5.2.*

6. The controlled stochastic Volterra equation. As a preliminary step for the sequel, we state two results of existence and uniqueness for (a special case of) the original Volterra equation. The proofs can be found in [6, Section 2] and in [12, Section 4].

PROPOSITION 6.1. *For any $T > 0$, the linear equation*

$$\begin{gathered} \frac{d}{dt}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s = Au(t), \qquad t \in [0,T] \\ u(t) = 0, \qquad t \le 0. \end{gathered} \tag{6.1}$$

has a unique solution $u \equiv 0$.

Now we deal with the existence and uniqueness of the Stochastic Volterra equation with non-homogeneous terms. The result extends Theorem 2.14 in [4], where the case $f(t) \equiv 0$ is treated, by considering the case where f is a deterministic function in $L^1(0,T;H)$ and g is as in Hypothesis 3.1.

PROPOSITION 6.2. *In our assumptions, let $x_0 \in X_\eta$ for some $\frac{1-\alpha(a)}{2} < \eta < \frac{1}{2}\alpha(a)$. Given the process*

$$\mathbf{x}(t) = e^{tB}x_0 + \int_0^t e^{(t-s)B}(I-B)Pf(s)\,\mathrm{d}s + \int_0^t e^{(t-s)B}(I-B)Pg\,\mathrm{d}W(s) \tag{6.2}$$

we define the process

$$u(t) = \begin{cases} J\mathbf{x}(t), & t \ge 0, \\ u_0(t), & t \le 0. \end{cases} \tag{6.3}$$

Then $u(t)$ is a weak solution to problem

$$\begin{gathered} \frac{d}{dt}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s = Au(t) + f(t) + g\dot{W}(t), \qquad t \in [0,T] \\ u(t) = u_0(t), \qquad t \le 0. \end{gathered} \tag{6.4}$$

After the preparatory results stated above, here we prove that main result of the existence and uniqueness of solutions of the original controlled Volterra equation (1.1).

Theorem 6.3. *Assume Hypothesis 3.1 and 3.3. Let γ be an admissible control and $\mathbf{x}$ be the solution to problem (1.3)) (associated with γ) in X_η with η satisfying the assumptions of Theorem 5.2. Then the process*

$$u(t) = \begin{cases} u_0(t), & t \le 0 \\ J\mathbf{x}(t), & t \in [0,T] \end{cases} \tag{6.5}$$

is the unique solution of the stochastic Volterra equation

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s &= Au(t) + f(t,u(t)) + g\,\left[r(u(t),\gamma(t)) + \dot W(t)\right], \quad t\in[0,T] \\ u(t) &= u_0(t), \quad t \le 0. \end{aligned} \tag{6.6}$$

Proof. We propose to fulfill the following steps: first, we prove that the affine equation

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s &= Au(t) + f(t,\tilde u(t)) + g\,\left[r(\gamma(t),\tilde u(t)) + \dot W(t)\right], \quad t\in[0,T] \\ u(t) &= u_0(t), \quad t \le 0, \end{aligned} \tag{6.7}$$

defines a contraction mapping $\mathbf{Q} : \tilde u \mapsto u$ on the space $L^p(\Omega; C([0,T];H))$. Therefore, equation (6.6) admits a unique solution.

Then we show that the process u defined in (6.5) satisfies equation (6.6). Accordingly, by the uniqueness of the solution, the thesis of the theorem follows.

First step. We proceed by defining the mapping

$$\mathbf{Q} : L^p(\Omega; C([0,T];H)) \to L^p(\Omega; C([0,T];H))$$

where $\mathbf{Q}(\tilde u) = u$ is the solution of the problem

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s &= Au(t) + g(t,\tilde u(t))\,\left[r(\tilde u(t),\gamma(t)) + \dot W(t)\right], \qquad t\in[0,T] \\ u(t) &= u_0(t), \qquad t \le 0. \end{aligned} \tag{6.8}$$

Let $\tilde u_1, \tilde u_2$ be two processes belonging to $L^p(\Omega; C([0,T];H))$ and take $u_1 = \mathbf{Q}(\tilde u_1)$ and $u_2 = \mathbf{Q}(\tilde u_2)$. It follows from Proposition 6.2, that, if $\mathbf{x}_i, i = 1,2$ are the processes defined as

$$\mathbf{x}_i(t) = e^{tB}x + \int_0^t e^{(t-s)B}(I-B)Pgr(s,\tilde u_i(s),\gamma(s))\,\mathrm{d}s \\ + \int_0^t e^{(t-s)B}(I-B)Pg\,\mathrm{d}W(s),$$

then the processes $w_i(t)$ $(i=1,2)$ defined through the formula

$$w_i(t) = \begin{cases} J\mathbf{x}_i(t), & t\in[0,T] \\ u_0(t), & t \le 0 \end{cases}$$

satisfy the stochastic Volterra equations for any $t\in[0,T]$,

$$\frac{d}{dt}\int_{-\infty}^{t} a(t-s)w_i(s)\,\mathrm{d}s = Aw_i(t) + f(t,\tilde u_i(t)) + g\,\left[r(\tilde u_i(t),\gamma(t)) + \dot W(t)\right],$$

with initial condition

$$u(t) = u_0(t), \qquad t \le 0.$$

By the uniqueness of the stochastic homogeneous Volterra equation stated in Proposition 6.1, we have $w_i = u_i, i = 1, 2$. Now define $U(t) := u_1(t) - u_2(t)$. We have

$$U(t) = \begin{cases} J(\mathbf{x}_1(t) - \mathbf{x}_2(t)), & t \in [0, T] \\ 0, & t \le 0; \end{cases}$$

and

$$\mathbb{E} \sup_{t\in[0,T]} e^{-\beta p t} |U(t)|^p \le \|J\|^p_{L(X_\eta, H)} \mathbb{E} \sup_{t\in[0,T]} e^{-\beta p t} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^p_\eta.$$

The quantity on the right hand side can be treated as in Theorem 5.2 and the claim follows.

Second step. It follows from the previous step that there exists at most a unique solution u of problem (6.7); hence it only remains to prove the representation formula (6.5) for u.

Let $\tilde f(t) = f(t, J\mathbf{x}(t)) + gr(J\mathbf{x}(t), \gamma(t))$; it is a consequence of Proposition 6.2 that u, defined in (6.5), is a weak solution of the problem

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^t a(t-s) u(s)\, \mathrm{d}s = Au(t) + \tilde f(t) + g\dot W(t), \qquad t \in [0, T] \\ u(t) = u_0(t), \qquad t \le 0, \end{aligned} \tag{6.9}$$

and the definition of $\tilde f$ implies that u is a weak solution on $[0, T]$ of

$$\frac{d}{dt} \int_{-\infty}^t a(t-s) u(s)\, \mathrm{d}s = Au(t) + f(t, J\mathbf{x}(t)) + g\left[r(J\mathbf{x}(t), \gamma(t)) + \dot W(t) \right], \tag{6.10}$$

with initial condition

$$u(t) = u_0(t), \qquad t \le 0,$$

which is problem (6.6). □

7. The regularity of the forward equation. In this section we state some properties of the forward equation (1.3) corresponding with our problem.

In the following, $\mathbf{x}$ will denote the solution of the uncontrolled equation (1.3). Under the assumption introduced in the previous sections we can give the regular dependence of $\mathbf{x}$ on the initial condition x. This result will be used later in order to characterize the solution of the stationary HJB in terms of the solution of a suitable stochastic backward differential equation and, consequently, to characterize the optimal control for our problem.

Proposition 7.1. *Under Hypotheses 3.1 and 4.5, for any $p \ge 1$ the following holds:*

1. *The map $x \mapsto \mathbf{x}(t; x)$ defined on X_η and with values in $L^p(\Omega, C([0, T]; X_\eta))$ is continuous.*

2. *The map $x \mapsto \mathbf{x}(t;x)$ has, at every point $x \in X_\eta$, a Gâteaux derivative $\nabla_x \mathbf{x}(\cdot;x)$. The map $(x,h) \mapsto \nabla_x \mathbf{x}(\cdot;x)[h]$ is a continuous map from $X_\eta \times X_\eta \to L^p(\Omega, C([0,T];X_\eta))$ and, for every $h \in X_\eta$, the following equation holds $\mathbb{P}$-a.s.:*

$$\nabla_x \mathbf{x}(t;x)[h] = e^{tB}h + \int_0^t e^{\tau B}(I-B)P\,\nabla_u f(J\mathbf{x}(\tau;x))J\nabla_x \mathbf{x}(\tau;x)[h]\mathrm{d}\tau. \tag{7.1}$$

Finally, $\mathbb{P}$-a.s., we have

$$|\nabla_x \mathbf{x}(t;x)[h]| \leq C\|h\|_\eta, \tag{7.2}$$

for all $t > 0$ and some $C > 0$.

Proof. Points 1 and 2 are proved, for instance, in [12, Proposition 6.2]. To prove the (7.2) we simply notice that

$$\left\| \int_0^t e^{\tau B}(I-B)P\,\nabla_u f(J\mathbf{x}(\tau;x))J\nabla_x \mathbf{x}(\tau;x)[h]\mathrm{d}\tau \right\|_\eta \\ \leq |P|_{L(H,X_\theta)}\|\nabla_u f\|_{L(H,H)}|J|_{L(X_\eta,H)} \int_0^t e^{-\omega s}s^{-1-\eta+\theta}|\nabla_x \mathbf{x}(s;x)[h]|\mathrm{d}s.$$

By application of Gronwall's lemma we thus obtain

$$|\nabla_x \mathbf{x}(t;x)[h]| \leq C\|h\|_\eta, \quad t \geq 0, \quad \mathbb{P}-a.s.$$

where C is a positive constant independent of t and x. □

As we will see later, an important point in order to study the HJB equation corresponding with our problem is that of extending the map $h \mapsto \nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]$ - a priori defined on $X_{1-\theta}$ - to the whole space X. This result is stated below.

Proposition 7.2. *Under assumptions 3.1 and 4.5 the map $h \mapsto \nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]$ - a priori defined on $X_{1-\theta}$ - can be extended to the whole space X and it is continuous from $[0,T] \times X_\eta \times X$ to $L^\infty(\Omega; C([0,T];X_\eta))$ for any $T > 0$. Finally, there exists a constant*

$$|\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]| \leq C\|h\|_\eta, \tag{7.3}$$

for all $t \geq 0, x \in X_\eta, h \in X$, with C independent of t and x.

Proof. We proceed by proving that the norm of $\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg$ in the space of linear operators on X is finite. In fact, taking Proposition 7.1 into account we see that, for any $h \in X_{1-\theta}$, the process $\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]$ satisfies the following equation:

$$\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}[h] = e^{tB}(I-B)^{1-\theta}h \\ + \int_0^t e^{(t-\tau)B}(I-B)P\,\nabla_u f(J\mathbf{x}(\tau;x))J\nabla_x \mathbf{x}(\tau;x)(I-B)^{1-\theta}[h]\mathrm{d}\tau.$$

Hence, recalling that $\nabla_x f$ is bounded and Proposition 4.6, we can estimate $|\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]|$ as follows:

$$|\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]| \leq e^{-\omega t}t^{\theta-1} \\ + C_f\|J\| \int_0^t |\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pg[h]|e^{-(t-\tau)\omega}(\tau-t)^{\theta-1}\mathrm{d}\tau.$$

Now the bound (7.3) follows by an easy application of Gronwall's lemma, while for continuity, we can refer to [12, Proposition 6.2]. □

In the rest of the section we introduce an auxiliary space that will turn to be useful when dealing with the formulation of the fundamental relation for the value function of our control problem. More precisely, for any $x \in X_\theta$ and $t \in [0,T]$ we introduce the space

$$\mathcal{D} := \left\{ h \in X_{1-\theta} \subset X : \ (I-B)^{1-\theta} h \in X_\eta \right\}, \tag{7.4}$$

In the following we shall prove that the space $\mathcal{D}$ is dense in X.

Proposition 7.3. *The space $\mathcal{D}$ defined in* (7.4) *is dense in X.*

Proof. We recall that the linear operator B is densely defined (see Theorem (4.5)) and that $D(B)$ is contained in all real interpolation spaces X_ρ, $\rho \in (0,1)$. We notice that if $h \in D(B)$ we have

$$\|(I-B)^{1-\theta} h\|_\eta = \|(I-B)^{1-\theta+\eta} h\|_X < \infty.$$

This implies that $D(B) \subset \mathcal{D}$ so that the claim follows. □

8. The backward equation on an infinite horizon. In this section we consider the backward stochastic differential equation in the unknown (Y,Z):

$$Y(\tau) = Y(T) + \int_\tau^T (\lambda Y(\sigma)) - \psi(\mathbf{x}(\sigma;x), Z(\sigma))\mathrm{d}\sigma + \int_\tau^T Z(\sigma)\mathrm{d}W(\sigma), \tag{8.1}$$

where $0 \le \tau \le T < \infty$, $\lambda > 0$, $\mathbf{x}(\cdot;x)$ is the solution of the uncontrolled equation (5.1) and ψ is the Hamiltonian function relative to the control problem described in Section 1. More precisely, for $x \in X_\eta, z \in \Xi^\star$ we have

$$\psi(x,z) := \inf\left\{\ell(Jx,\gamma) + z\cdot r(Jx,\gamma) : \ \gamma \in \mathcal{U}\right\}. \tag{8.2}$$

We require the following assumption on ψ:

Hypothesis 8.1.

1. *ψ is uniformly Lipschitz continuous in z, with Lipschitz constant K, that is:*
$$|\psi(x,z_1) - \psi(x,z_2)| \le K\|z_1 - z_2\|_{\Xi^*}.$$
2. $\sup_{x\in X} |\psi(x,0)| := M < \infty$.
3. *The map ψ is Gâteaux differentiable on $X_\eta \times \Xi^\star$ and the maps $(x,h,z) \mapsto \nabla_x\psi(x,z)[h]$ and $(x,z,\zeta) \mapsto \nabla_z\psi(x,z)[\zeta]$ are continuous on $X_\eta \times X \times \Xi^\star$ and $X_\eta \times \Xi^\star \times \Xi^\star$ respectively. Moreover, we have $|\nabla_x\psi(x,z)[h]| \le C|h|$, for all $h \in X$, $x \in X_\eta$, $z \in \Xi^\star$.*

The existence and uniqueness of solution to (8.1) under Hypothesis 8.1 was first studied (even though for more general coefficients ψ) by Briand and Hu in [9] and successively by Royer in [28]. Their result is valid when W is a finite dimensional Wiener process but the extension to the case in which W is a Hilbert-valued Wiener process is immediate.

In our context the result reads as follows:

Proposition 8.2. *Assume Hypothesis 3.4 and 8.1. Then we have:*

1. *For any $x \in X_\eta$, there exists a solution (Y,Z) to BSDE* (8.1) *such that Y is a continuous process bounded by $\frac{M}{\lambda}$, and $Z \in L^2(\Omega; L^2(0,\infty;\Xi))$ with $\mathbb{E}\int_0^\infty e^{-2\lambda s}|Z(s)|^2\mathrm{d}s < \infty$. Moreover, the solution is unique in the class of processes (Y,Z) such that Y is continuous and uniformly bounded, and Z belongs to $L^2_{loc}(\Omega, L^2(0,\infty;\Xi^\star))$. In the following we will denote such a solution by $Y(\cdot;x)$ and $Z(\cdot;x)$.*

2. *Denoting by $(Y(\cdot;x,n), Z(\cdot;x,n))$ the unique solution of the following BSDE (with finite horizon)*

$$Y(\tau;x,n) = \int_\tau^n (\psi(\mathbf{x}(\sigma;x), Z(\sigma)) - \lambda Y(\sigma))\mathrm{d}\sigma + \int_\tau^n Z(\sigma;x,n)\mathrm{d}W(\sigma), \tag{8.3}$$

then $|Y(\tau;x,n)| \leq \frac{M}{\lambda}$ and the following convergence rate holds:

$$|Y(\tau;x,n) - Y(\tau;x)| \leq \frac{M}{\lambda} \exp\{-\lambda(n-\tau)\}.$$

Moreover,

$$\mathbb{E}\int_0^\infty e^{-2\lambda\sigma}\|Z(\sigma;x,n) - Z(\sigma;x)\|^2\mathrm{d}\sigma \;\to\; 0.$$

3. *For all $T>0$ and $p>1$, the map $x \mapsto (Y(\cdot;x)|_{[0,T]}, Z(\cdot;x)|_{[0,T]})$ is continuous from X_η to the space $L^p(\Omega, C([0,T];\mathbb{R})) \times L^p(\Omega, L^2([0,T];\Xi^\star))$.*

We need to study the regularity of $Y(\cdot;x)$. More precisely we would like to show that $Y(0;x)$ is Gâteaux differentiable with respect to the initial condition x and that both $Y(0;x)$ and $\nabla_x Y(0;x)$ turn out to be bounded.

The following result is one of the crucial points of the paper.

THEOREM 8.3. *Under Hypotheses 3.1, 4.5, 3.4 and 8.1, the map $x \mapsto Y(0;x)$ is Gâteaux differentiable. Moreover, $|Y(0;x)| + |\nabla_x Y(0;x)| \leq c$.*

Proof. The argument follows essentially the proof of [21, Theorem 3.1]. An important point is the boundedness of $\nabla_x \mathbf{x}(t,x)[h]$ ($\mathbb{P}$-a.s. and for any $t \geq 0$), which was proved in Proposition 7.1. We first recall that, under our assumptions (see [18, Proposition 5.2]), the map $x \mapsto (Y(t;x,n), Z(t;x,n))$ considered in Proposition 8.2 is Gâteaux differentiable from X_η to $L^p(\Omega; C(0,T;\mathbb{R}) \times L^p(\Omega; L^2(0,T;\Xi^\star))$ for all $p \geq 2$. Denoting by $\nabla_x Y(t;x,n)[h], \nabla_x Z(t;x,n)[h]$ the partial Gâteaux derivative with respect to x in the direction $h \in X$, the processes $(\nabla_x Y(t;x,n)[h], \nabla_x Z(t;x,n)[h])_{t\in[0,n]}$ solve the equation, $\mathbb{P}$-a.s.

$$\begin{aligned}\nabla_x Y(\tau;x,n)[h] = &\int_\tau^n (\psi(\mathbf{x}(\sigma;x), Z(\sigma;x,n))\nabla_x \mathbf{x}(\sigma;x)[h]\mathrm{d}\sigma \\ &+ \int_\tau^n (\psi(\mathbf{x}(\sigma;x), Z(\sigma;x,n))\nabla_z Z(\sigma;x)[h]\mathrm{d}\sigma \\ &- \int_\tau^n \lambda\nabla_x Y(\sigma;x,n)[h]\mathrm{d}\sigma + \int_\tau^n Z(\sigma;x,n)\mathrm{d}W(\sigma),\end{aligned}$$

We notice that by the assumptions made on $\nabla_x\psi, \nabla_z\psi$ and Proposition 7.2, we have

$$|\nabla_x \mathbf{x}(t;x)[h]| \leq C|h| \qquad \textit{and} \qquad |\nabla_z \psi(\mathbf{x}(t;x), Z(t;x))| \leq C.$$

Therefore by the same arguments based on Girsanov transform as in [8, Lemma 3.1], we obtain

$$\sup_{t\in[0,n]} |\nabla_x Y(t;x,n)| \leq C|h|, \quad \mathbb{P}-a.s.$$

and, again as in the proof of [8, Lemma 3.1], applying Itô formula to $e^{-2\lambda t}|\nabla_x Z(t;x,n)|^2$, we get:

$$\mathbb{E}\int_0^\infty e^{-2\lambda t}(|\nabla_x Y(t;x,n)h|^2 + |\nabla_x Z(t;x,n)h|^2)\mathrm{d}t < C|h|^2.$$

Let now $\mathcal{M}^{2,-2\lambda}$ be the Hilbert space of all couples of $(\mathcal{F}_t)_{t\geq 0}$-adapted processes (y,z), where y has values in $\mathbb{R}$ and $z \in \Xi^\star$, such that

$$\mathbb{E}\int_0^\infty e^{-2\lambda t}(|y(t)|^2 + |z(t)|^2)\mathrm{d}t < C|h|^2.$$

Fix now $x \in X_\eta$ and $h \in X$. Then there exists a subsequence of

$$\{(\nabla_x Y(t;x;n)h, \nabla_x Z(t;x,n)h, \nabla_x Y(0;x;n)h)\}$$

such that $\{(\nabla_x Y(t;x;n)h, \nabla_x Z(t;x,n)h)\}$ converges weakly to $(U^1(x,h), V^1(x,h))$ in $\mathcal{M}^{2,-2\lambda}$ and $\nabla_x Y(0;x,n)h$ converges to $\xi(x,h) \in \mathbb{R}$. Proceeding as in [21, Theorem 3.1], we see that the convergence of $\nabla_x Y(t;x;n)h$ is, in reality, in $L^2(0;T;\mathbb{R})$ for all $T > 0$ and, moreover, that $\lim_{n\to\infty} \nabla_x Y(0;x;n)h$ exists and coincides with the value at 0 of the process $> U(0;x)h$ defined by the following equation:

$$\begin{aligned} U(t;x)h = U(0;x)h &- \int_0^t \nabla_x \psi(\mathbf{x}(s;x), Z(s;x))\nabla_x \mathbf{x}(s;x)[h]\mathrm{d}s \\ &+ \int_0^t \nabla_z \psi(\mathbf{x}(s;x), Z(s;x))V^1(s;x)[h]\mathrm{d}t \\ &- \lambda\int_0^t U(s;x)h\mathrm{d}s + V^1(s;x)[h]\mathrm{d}W(s). \end{aligned}$$

Moreover, it holds that $U^1(t;x)h = U(t;x)h$ for any fixed $h \in X$. Summarizing, we have that $U(0;x)h = \lim_{n\to\infty} \nabla_x Y(0;x,n)h$ exists, it is linear and verifies $U(0;x)h \leq C|h|$ for every h fixed. Finally, it is continuous in x for every h fixed. Finally, for $t > 0$ we have

$$\begin{aligned} &\lim_{t\to 0}\frac{Y(0;x+th) - Y(0;x)}{t} = \lim_{t\to 0}\lim_{n\to\infty}\frac{Y(0;x+th,n) - Y(0;x,n)}{t} \\ &= \lim_{t\to 0}\lim_{n\to\infty}\int_0^1 \nabla_x Y(0;x+th\theta)h\mathrm{d}\theta \\ &= \lim_{t\to 0}\int_0^1 Y(0;x+th\theta)\mathrm{d}\theta = U(0;x)h, \end{aligned}$$

and the claim is proved. □

Starting from the Gâteaux derivatives of Y and Z, we introduce suitable auxiliary processes which allow us to express Z in terms of ∇Y and $(I-B)^{1-\theta}$ and then get the fundamental relation for the optimal control problem introduced in Section 1. The main point is to prove that the mappings $(h,x) \mapsto \nabla_x Y(t;x)(I-B)^{1-\theta}Pg[h]$ and $(h,x) \mapsto \nabla_x Z(t;x)(I-B)^{1-\theta}Pg[h]$ are well defined as operators from $X \times X$ respectively in $L^\infty(\Omega; C([0,T];\mathbb{R}))$ and $L^\infty(\Omega; C([0,T];\Xi^\star))$.

Proposition 8.4. *For every $p \geq 2$, $\beta < 0$, $x \in X_\eta$, $h \in X$ there exist two processes*

$$\{\Pi(t;x)[h] :\ t \geq 0\} \qquad \textit{and} \quad \{Q(t;x)[h] :\ t \geq 0\}$$

with $\Pi(t;x)[h] \in L^p(\Omega; C([0,T]; X_\eta))$ *and* $Q(\cdot;x)[h] \in L^p(\Omega; C([0,T];\Xi^\star))$ *for any* $T>0$ *and such that if* $x \in X_\eta$*, then* $\mathbb{P}$*-a.s. the following identifications hold:*

$$\Pi(t;x)[h] = \nabla_x Y(t;x)(I-B)^{1-\theta}[h], \qquad t \geq 0; \tag{8.4}$$

$$Q(t;x)[h] = \nabla_x Z(t;x)(I-B)^{1-\theta}[h], \qquad t \geq 0. \tag{8.5}$$

Moreover, the map $(x,h) \mapsto \Pi(\cdot;x)[h]$ *is continuous from* $X_\eta \times X$ *to* $L^p(\Omega; C_\beta([0,T];\mathbb{R}))$ *and the map* $(x,h) \mapsto Q(\cdot;x)[h]$ *is continuous from* $X_\eta \times X$ *into* $L^p(\Omega; C_\beta([0,T];\Xi^\star))$ *and both maps are linear with respect to* h*. Finally, there exists a positive constant* C *such that*

$$\mathbb{E}\|\Pi(0;x)\| \leq C. \tag{8.6}$$

Proof. For any $n \in \mathbb{N}$, we introduce a suitable stochastic differential equation on $[0,n]$ which should give a sequence $(\Pi^{(n)}(\cdot;x)[h], Q^{(n)}(\cdot;x)[h])$ of approximating processes for $(\Pi(\cdot;x), Q(\cdot;x))$; more precisely, for fixed $p \geq 2$, $x \in X_\eta$ and $h \in X$ we consider the equation

$$\begin{aligned}\Pi^{(n)}(t;x)[h] = \int_t^n \nu^{(n)}(t;x)[h]\mathrm{d}t + \int_t^n \nabla_z \psi(\mathbf{x}(t;x), Z(t;x)) Q^{(n)}(t;x)[h]\mathrm{d}t \\ -\lambda \int_t^n \Pi^{(n)}(t;x)[h]\mathrm{d}t + \ Q^{(n)}(t;x)[h]\mathrm{d}W(t) \qquad t \in [0,n]\end{aligned} \tag{8.7}$$

where

$$\nu^{(n)}(t;x)[h] = \mathbf{1}_{[0,n]}(t)\nabla_x \psi(\mathbf{x}(t;x), Z(t;x))\nabla_x \mathbf{x}(t;x)(I-B)^{1-\theta}Pgh.$$

The solution to (8.7) exists (see [12, Proposition 7.5]) and the maps $x \mapsto \Pi^{(n)}(\cdot;x)[h]$ are continuous from $X_\eta \times X$ to $L^p(\Omega; C([0,n]; X_\eta))$ and $Q^{(n)}(\cdot;x)[h] \in L^p(\Omega; C([0,n];\Xi^\star))$. Further, if $(Y(\cdot;n,x), Z(\cdot;n,x))$ are the processes introduced in Proposition 8.2, then the following identifications hold

$$\Pi(t;x)^{(n)}[h] = \nabla_x Y(t;x,n)(I-B)^{1-\theta}[h], \qquad t \in [0,n]; \tag{8.8}$$

$$Q(t;x)^{(n)}[h] = \nabla_x Z(t;x,n)(I-B)^{1-\theta}[h], \qquad t \in [0,n]; \tag{8.9}$$

for any $x \in X_\eta$ and h in X. Hence, for any $n \in \mathbb{N}$ the functions $(\Pi^{(n)}(t;x), Q^{(n)}(t;x))_{t\in[0,n]}$ extend the folloqing mappings:

$$h \mapsto \nabla_x Y(t;x,n)(I-B)^{1-\theta}Pg[h] \qquad h \mapsto \nabla_x Z(t;x,n)(I-B)^{1-\theta}Pg[h], \qquad t \in [0,n].$$

Moreover, we have the estimates

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,n]} \|\Pi^{(n)}(t;x)\|_\eta^p e^{-p\beta t} + \mathbb{E}\left(\int_0^n e^{-2\beta r}|\Pi^{(n)}(r;x)[h]|^2\mathrm{d}r\right)^{p/2} \\ + \mathbb{E}\left(\int_0^n e^{-2\beta r}\|Q^{(n)}(r;x)[h]\|_{\Xi^\star}^2\mathrm{d}r\right) < \infty,\end{aligned} \tag{8.10}$$

for suitable $\beta > 0$. It remains to prove that the processes $(\Pi^{(n)}(\cdot;x), Q^{(n)}(\cdot;x))$ converge to some pair $(\Pi(\cdot;x), Q(\cdot;x))$ and that the identifications (8.4) and (8.5) hold.

For the convergence of $(\Pi^{(n)}(\cdot;x), Q^{(n)}(\cdot;x))$, we notice that $(\Pi^{(n)}(\cdot;x), Q^{(n)}(\cdot;x))$ solves a BSDE with bounded coefficients. In fact, by the assumption made on $\nabla_x\psi, \nabla_z\psi$ and Proposition 7.2, we have

$$|\nu^{(n)}(t;x)h| \leq C|h| \qquad \textit{and} \qquad |\nabla_z\psi(\mathbf{x}(t;x), Z(t;x))| \leq C.$$

Hence, following [21, Theorem 3.1] or Theorem 8.3 above, we conclude that $\Pi(0;x)h = \lim_{n\to\infty} \Pi^{(n)}(0;x)h$ exists, it is linear, verifies $|\Pi(0;x)h| \leq C|h|$ for every h fixed and it is continuous in x for every h fixed. Finally, since on $X_{\theta-1}$ the processes $(\Pi^{(n)}(\cdot;x), Q^{(n)}(\cdot;x))$ extend the processes

$$(\nabla_x Y(t;x,n)(I-B)^{1-\theta}[h], \nabla_x Z(t;x,n)(I-B)^{1-\theta}[h]),$$

on the same space we have

$$\Pi(0;x)h = \lim_{n\to\infty} \nabla_x Y(0;x,n)(I-B)^{1-\theta}Pg[h] = \nabla_x Y(0;x)(I-B)^{1-\theta}[h].$$

□

COROLLARY 8.5. *Setting $v(x) = Y(0;x)$, we have that v is Gâteaux differentiable with respect to x on X_η and the map $(x,h) \mapsto \nabla v(x)[h]$ is continuous.*

Moreover, for $x \in X_\eta$ the linear operator $h \mapsto \nabla v(x)(I-B)^{1-\theta}[h]$ - a priori defined for $h \in \mathcal{D}$ - has an extension to a bounded linear operator from X into $\mathbb{R}$, that we denote by $[\nabla v(I-B)^{1-\theta}](x)$.

Finally, the map $(x,h) \mapsto [\nabla v(I-B)^{1-\theta}Pg](x)$ is continuous as a mapping from $X_\eta \times X$ into $\mathbb{R}$ and there exists $C > 0$ such that

$$|[\nabla v(I-B)^{1-\theta}](x)[h]| \leq C|h|_X, \tag{8.11}$$

for $x \in X_\eta, h \in X$.

Proof. First of all, we notice that $Y(0;x)$ is deterministic, since $(\mathcal{F}_t)_{t\geq 0}$ is generated by the Wiener process W and $Y(\tau)$ is $\mathcal{F}_\tau$-adapted.

Moreover, in Theorem 8.3 we proved that the map $x \mapsto Y(\cdot;x)$ is continuous and Gâteaux differentiable with values in $L^p(\Omega; C([0,T];\mathbb{R}))$; consequently, it follows that $x \mapsto v(x) = Y(0;x)$ is Gâteaux differentiable with values in $\mathbb{R}$.

Next, we notice that $\Pi(0;x) = \nabla_x Y(0;x)(I-B)^{1-\theta}[h]$. The existence of the required extension and its continuity are direct consequences of Proposition 8.4 and estimate (8.11) follows from (8.6). □

We are now in the position to give a meaning to the expression $\nabla_x Y(t;x)(I-B)^{1-\theta}$ and, successively, to identify it with the process $Z(t;x)$. To this end we quote from [12] a preliminary result where we investigate the existence of the joint quadratic variation of W with a process of the form $\{w(t,\mathbf{x}(t)) : t \in [0,T]\}$ for a given function $w : [0,T] \times X \to \mathbb{R}$, on an interval $[0,s] \subset [0,T)$. In order to simplify the exposition we omit the proof. We only notice that the proof requires the study of the Malliavin derivative of $\mathbf{x}$ and this can be done in the same way as in [12, Section 6.1].

PROPOSITION 8.6. *Suppose that $w \in C([0,T) \times X_\eta; \mathbb{R})$ is Gâteaux differentiable with respect to $\mathbf{x}$, and that for every $s < T$ there exist a constant K (possibly depending on s) such that*

$$|w(t,x)| \leq K, \quad |\nabla w(t,x)| \leq K, \quad t \in [0,s], x \in X. \tag{8.12}$$

Let η and θ satisfy the conditions (4.6) in Theorem 4.5. Assume that for every $t \in [0,T)$, $x \in X_\eta$ the linear operator $k \mapsto \nabla w(t,x)(I-B)^{1-\theta}k$ (a priori defined for $k \in$

$\mathcal{D}$) has an extension to a bounded linear operator $X \to \mathbb{R}$, that we denote by $[\nabla w(I-B)^{1-\theta}](t,x)$. Moreover, assume that the map $(t,x,k) \mapsto [\nabla w(I-B)^{1-\theta}](t,x)k$ is continuous from $[0,T)\times X_\eta \times X$ into $\mathbb{R}$. For $t \in [0,T)$, $x \in X_\eta$, let $\{\mathbf{x}(t;s,x), t \in [s,T]\}$ be the solution of equation (5.3). Then the process $\{w(t,\mathbf{x}(t;s,x)), t \in [s,T]\}$ admits a joint quadratic variation process with W^j, for every $j \in \mathbb{N}$, on every interval $[s,t] \subset [s,T)$, given by

$$\int_s^t [\nabla w(I-B)^{1-\theta}](r,\mathbf{x}(r;s,x))(I-B)^\theta Pge_j \mathrm{d}r.$$

The above result allows to identify the process $Z(\cdot;x)$ with $[\nabla v(I-B)^{1-\theta}](\mathbf{x}(\cdot;x))(I-B)^\theta Pg$, as we can see in the result below.

Corollary 8.7. *For every $x \in X_\eta$ we have, $\mathbb{P}$-a.s.*

$$Y(t;x) = v(\mathbf{x}(t;x)), \qquad \textit{for all } t \geq 0; \tag{8.13}$$

$$Z(\cdot,x) = [\nabla v(I-B)^{1-\theta}](\mathbf{x}(\cdot;x))(I-B)^\theta Pg(\mathbf{x}(\cdot;x)), \ \textit{for almost all } t \geq 0. \tag{8.14}$$

Proof. We need to consider the equation (which is slightly more general than (1.3))

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = B\mathbf{x}(t)\mathrm{d}t + (I-B)P\,f(J\mathbf{x}(t))\mathrm{d}t \\ \qquad\qquad\qquad +(I-B)Pg(u(t))(r(J\mathbf{x}(t),\gamma(t))\mathrm{d}t + \mathrm{d}W(t)) \\ \mathbf{x}(s) = x, \end{cases} \tag{8.15}$$

for t varying $[s,\infty)$. We set $\mathbf{x}(t) = x$ for $t \in [0,s)$ and we denote by $(\mathbf{x}(t;s,x))_{t\geq 0}$ the solution, to indicate dependence on x and s. By an obvious extension of the results concerning the BSDE (8.1) stated above, we can solve the backward equation (8.1) with $\mathbf{x}$ given by (8.15); we denote the corresponding solution by $(Y(t;s,x), Z(t;s,x))$ for $t \geq 0$. Thus $(Y(t;x), Z(t;x))$ coincides with $(Y(t;0,x), Z(t;0,x))$ that occurs in the statement of Proposition 8.2.

Now let us prove (8.13). We start from the well-known equality: for $0 \leq r \leq T$, $\mathbb{P}$-a.s.

$$\mathbf{x}(t;s,x) = \mathbf{x}(t;r,\mathbf{x}(r;s,x)), \qquad \text{for all } t \in [s,T].$$

It follows easily from the uniqueness of the backward equation (8.1) that $\mathbb{P}$-a.s.

$$Y(t;s,x) = Y(t;r,\mathbf{x}(r;s,x)), \qquad \text{for all } \ t \in [s,T].$$

In particular, for every $0 \leq r \leq s \leq t$,

$$\begin{aligned} Y(t;s,\mathbf{x}(s;r,x)) &= Y(t;r,x), \qquad \text{for } t \in [s,\infty), \\ Z(t;s,\mathbf{x}(s;r,x)) &= Z(t;r,x), \qquad \text{for } t \in [s,\infty). \end{aligned}$$

Since the coefficients of (8.15) do not depend on time, we have

$$\mathbf{x}(\cdot;0,x) \overset{(d)}{=} \mathbf{x}(\cdot+t;t,x), \qquad t \geq 0,$$

where $\overset{(d)}{=}$ denotes equality in distribution. As a consequence we obtain

$$(Y(\cdot;0,x), Z(\cdot;0,x)) \overset{(d)}{=} (Y(\cdot+t;t,x), Z(\cdot+t;t,x)), \qquad t \geq 0,$$

where both sides of the equality are viewed as random elements with values in the space $C(\mathbb{R}_+;\mathbb{R}) \times L^2_{loc}(\mathbb{R}_+;\Xi^\star)$. In particular

$$Y(0;0,x) \stackrel{(d)}{=} Y(t;t,x),$$

and since they are both deterministic, we have

$$Y(0;0,x) = Y(t;t,x), \qquad x \in X_\eta, t \geq 0,$$

so that we arrive at (8.13).

To prove (8.14) we consider the joint quadratic variation of $(Y(t;x))_{t\in[0,T]}$ and W on an arbitrary interval $[0,t] \subset [0,T)$; from the backward equation (8.1) we deduce that it is equal to $\int_0^t Z(r;x)\mathrm{d}r$. On the other side, the same result can be obtained by considering the joint quadratic variation of $(v(\mathbf{x}(t;x)))_{t\in[0,T]}$ and W. Now by an application of Proposition 8.6 (whose assumptions hold true by Corollary 8.5) we are led to the identity

$$\int_0^t Z(r;x)\mathrm{d}r = \int_0^t [\nabla v(I-B)^{1-\theta}](r;\mathbf{x}(r;x))(I-B)^\theta Pg(\mathbf{x}(r;x))\mathrm{d}r,$$

and (8.14) is proved. □

9. The stationary HJB equation. Now we proceed as in [19]. Let us consider again the solution $\mathbf{x}(t;x)$ of equation (1.3) and denote by $(P_t)_{t\geq 0}$ its transition semigroup:

$$P_t[h](x) = \mathbb{E}h(\mathbf{x}(t;x)), \qquad x \in X_\eta, 0 \leq t,$$

for any bounded measurable $h : X_\eta \to \mathbb{R}$. We notice that by the bound (5.2) this formula is true for every h with polynomial growth. In the following P_t will be considered as an operator acting on this class of functions.

Let us denote by $\mathcal{L}$ the generator of P_t:

$$\mathcal{L}[h](x) = \frac{1}{2}\mathrm{Tr}[(I-B)Pg\nabla^2 h(x)g^* p^*(I-B)^*] + \langle Bx + (I-B)Pf(Jx), \nabla h(x)\rangle,$$

where ∇h and $\nabla^2 h$ are first and second Gâteaux derivatives of h at the point $x \in X$ (here we are identified with elements of X and $L(X)$ respectively). This definition is formal, since it involves the terms $(I-B)Pg$ and $(I-B)Pf$ which - *a* priori - are not defined as elements of $L(X)$ and the domain of $\mathcal{L}$ is not specified.

In this section we address the solvability of the nonlinear stationary Kolmogorov equation:

$$\mathcal{L}[v(\cdot)](x) = \lambda v(x) - \psi(x, \nabla v(x)(I-B)Pg), \quad x \in X. \qquad (HJB)$$

This is a nonlinear elliptic equation for the unknown function $v : X_\eta \to \mathbb{R}$. We define the notion of the solution of the (HJB) by means of the variation of constant formula:

Definition 9.1. *We say that a function $v : X_\eta \to \mathbb{R}$ is a mild solution of the HJB equation* (HJB) *if the following conditions hold:*

1. *$v \in C(X_\eta;\mathbb{R})$ and there exist $C \geq 0$ such that $|v(x)| \leq C,\ x \in X_\eta$;*
2. *v is Gâteaux differentiable with respect to x on X_η and the map $(x,h) \mapsto \nabla v(x)[h]$ is continuous $X_\eta \times X_\eta \to \mathbb{R}$;*

3. *For all $x \in X_\eta$ the linear operator $k \mapsto \nabla v(x)(I-B)^{1-\theta}k$ (a priori defined for $k \in \mathcal{D}$) has an extension to a bounded linear operator on $X \to \mathbb{R}$, that we denote by $[\nabla v(I-B)^{1-\theta}](x)$.*
   Moreover the map $(x,k) \mapsto [\nabla v(I-B)^{1-\theta}](x)k$ is continuous $X_\eta \times X \to \mathbb{R}$ and there exist constants $C \geq 0$ such that

$$\|[\nabla v(I-B)^{1-\theta}](x)\|_{L(X)} \leq C, \qquad x \in X_\eta.$$

4. *The following equality holds for every $x \in X_\eta$:*

$$v(x) = e^{-\lambda T} P_T[v](x) - \int_0^T e^{-\lambda s} P_s \left[\psi([\nabla v(I-B)^{1-\theta}](\cdot)(I-B)^\theta Pg]\right)(x)\mathrm{d}s. \tag{9.1}$$

REMARK 9.2. *We notice that, by assumption, $|\psi(x,z)| \leq C$; moreover, we saw in the preceding sections how to give a meaning to the term*

$$\nabla v(x)(I-B)Pg.$$

Hence if v is a function satisfying the bound required in Definition 9.1,3, we have

$$\left|\psi(x, [\nabla v(I-B)^{1-\theta}](x)(I-B)^\theta Pg)\right| \leq C$$

and formula (9.1) *is meaningful.*

Now we are ready to prove that the solution of the equation (HJB) can be defined by means of the solution of the BSDE associated with the control problem (1.3).

THEOREM 9.3. *Assume Hypothesis 3.1, 3.3, 3.4 and 8.1; then there exists a unique mild solution of the HJB equation. The solution is given by the formula*

$$v(x) = Y(0;0,x) = Y(t;t,x), \qquad t \geq 0, \tag{9.2}$$

where $(\mathbf{x}, Y, Z)$ is the solution of the forward-backward system (1.3) *and* (8.1)*.*

Proof. We start by proving the existence. In particular, we prove that v, given by (9.2), is a solution of HJB. Hence, we set:

$$v(x) := Y(0;0,x).$$

By Corollary 8.7 the function v has the regularity properties stated in Definition 9.1. In order to verify that equality (9.1) holds, we first fix $x \in X_\eta$. We notice that

$$\psi(\cdot, [\nabla v(I-B)^{1-\theta}](\cdot)(I-B)^\theta Pg)(x) = \psi(\cdot, [\nabla Y(I-B)^{1-\theta}](\cdot)(I-B)^\theta Pg)(x)$$

and we recall that

$$[\nabla v(I-B)^{1-\theta}](\mathbf{x}(t;0,x))(I-B)^\theta Pg = Z(t;0,x), \qquad 0 \leq t.$$

Hence

$$\begin{aligned} P_t \left[\psi(\cdot, [\nabla v(I-B)^{1-\theta}](\cdot)(I-B)^\theta Pg)\right] (\mathbf{x}(t;0,x)) \\ = \mathbb{E}\left[\psi(\mathbf{x}(t;0,x), Z(t;0,x))\right]. \end{aligned} \tag{9.3}$$

On the other hand, applying the Itô formula to the backward equation it gives

$$
\begin{aligned}
e^{-\lambda t}Y(t;t,x) - e^{-\lambda T}Y(T;t,x) + \int_t^T e^{-\lambda r} Z(r;t,x)\mathrm{d}W(r) \\
= -\int_t^T e^{-\lambda r}\psi(\mathbf{x}(r;t,x), Z(r;t,x))\mathrm{d}r,
\end{aligned}
$$

for any $0 \le t \le T < \infty$. Taking the expectation and recalling Corollary 8.7 again we obtain

$$
e^{-\lambda t}v(x) = e^{-\lambda T}\mathbb{E}v(\mathbf{x}(T;t,x)) - \mathbb{E}\int_t^T e^{-\lambda r}\psi(\mathbf{x}(r;t,x), Z(r;t,x))\mathrm{d}r
$$

and substituting in the integral the expression obtained in (9.3) we get the required equality (9.1). This completes the proof of the existence part.

Now we consider the uniqueness of the solution. Let v denote a mild solution. We look for a convenient expression for the process $v(\mathbf{x}(r;t,x))$, $0 \le t \le r \le T < \infty$. By (9.1) and the definition of P_t, for any $y \in X$ we have

$$
\begin{aligned}
&v(y) = e^{-\lambda(T-t)}\mathbb{E}\left[v(\mathbf{x}(T-t;0,y))\right] \\
&- \int_0^{T-t} e^{-\lambda r}\mathbb{E}\left[\psi(\mathbf{x}(r;0,y), [\nabla v(I-B)^{1-\theta}](r,\mathbf{x}(r;0,y))(I-B)^{\theta}Pg)\right]\mathrm{d}r.
\end{aligned} \tag{9.4}
$$

Set $y = \mathbf{x}(t;0,x)$. Then equality (9.4) rewrites as

$$
\begin{aligned}
&v(\mathbf{x}(t;0,x)) = e^{-\lambda(T-t)}\mathbb{E}\left[v(\mathbf{x}(T-t;0,\mathbf{x}(t;0,x)))\right] \\
&-\int_0^{T-t} e^{-\lambda r}\mathbb{E}\left[\psi(\mathbf{x}(r;0,\mathbf{x}(t;0,x)), [\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,\mathbf{x}(t;0,x)))(I-B)^{\theta}Pg)\right]\mathrm{d}r.
\end{aligned}
$$

Moreover, recalling that for any $r \in [t,T]$ the following equality

$$
\mathbf{x}(r;t,\mathbf{x}(t;0,x)) = \mathbf{x}(r;0,x)
$$

holds $\mathbb{P}$-a.s., we obtain the identity

$$
\begin{aligned}
&v(\mathbf{x}(t;0,x)) = e^{-\lambda(T-t)}\mathbb{E}\left[v(\mathbf{x}(T;t,x))\right] \\
&- \int_0^{T-t} e^{-\lambda r}\mathbb{E}\left[\psi(\mathbf{x}(r+t;0,x), [\nabla v(I-B)^{1-\theta}](\mathbf{x}(r+t;0,x))(I-B)^{\theta}Pg)\right]\mathrm{d}r.
\end{aligned}
$$

Since $\mathbf{x}(t;0,x)$ is $\mathcal{F}_t$-measurable, we can replace the expectation by the conditional expectation given $\mathcal{F}_t$:

$$
\begin{aligned}
&e^{-\lambda t}v(\mathbf{x}(t;0,x)) = e^{-\lambda T}\mathbb{E}^{\mathcal{F}_t}\left[v(\mathbf{x}(T;t,x))\right] \\
&- \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{-\lambda r}\psi(\mathbf{x}(r+t;0,x), [\nabla v(I-B)^{1-\theta}](\mathbf{x}(r+t;0,x))(I-B)^{\theta}Pg\mathrm{d}r\right],
\end{aligned}
$$

and, by change of variable, we get:

$$\begin{aligned}
e^{-\lambda t}v(\mathbf{x}(t;0,x)) &= e^{-\lambda T}\mathbb{E}^{\mathcal{F}_t}\left[v(\mathbf{x}(T;t,x))\right] \\
&\quad - \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \psi(\mathbf{x}(r;0,x),[\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,x))(I-B)^{\theta}Pg\mathrm{d}r\right] \\
&= \mathbb{E}^{\mathcal{F}_t}[\xi] \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[\int_0^t \psi(\mathbf{x}(r;0,x),[\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,x))(I-B)^{\theta}Pg\mathrm{d}r\right],
\end{aligned}$$

where we have defined

$$\begin{aligned}
\xi &:= e^{-\lambda T}v(\mathbf{x}(T;t,x)) \\
&\qquad\qquad - \int_0^T \psi(\mathbf{x}(r;0,x),[\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,x))(I-B)^{\theta}Pg)\mathrm{d}r.
\end{aligned}$$

Since we assume polynomial growth for v and ∇v, therefore ξ is square integrable. Since $(\mathcal{F}_t)_{t\geq 0}$ is generated by the Wiener process W, it follows that there exists $\tilde{Z}\in L^2_{\mathcal{F}}(\Omega\times[0,T];L_2(\Xi;\mathbb{R}))$ such that

$$\mathbb{E}^{\mathcal{F}_t}[\xi] = \mathbb{E}[\xi] + \int_0^t \tilde{Z}(r)\mathrm{d}W(r), \qquad t\in[0,T].$$

An application of the Itô formula gives

$$\begin{aligned}
v(\mathbf{x}(t;0,x)) &= \mathbb{E}[\xi] + \int_0^t e^{\lambda r}\tilde{Z}(r)\mathrm{d}W(r) + \lambda\int_0^t v(\mathbf{x}(r;0,x)\mathrm{d}r \\
&\quad + \int_0^t \psi(r,\mathbf{x}(r;t,x),[\nabla v(I-B)^{1-\theta}](r,\mathbf{x}(r;t,x))(I-B)^{\theta}Pg\mathrm{d}r. \qquad (9.5)
\end{aligned}$$

We conclude that the process $v(\mathbf{x}(t;s,x))$, $t\in[s,T]$ is a real continuous semimartingale.

For $\xi\in\Xi$, let us define W^{ξ} by $W^{\xi}(t)=\langle\xi,W(t)\rangle$ and let us consider the joint quadratic variation process of W^{ξ} with both sides of (9.5). Applying Proposition 8.6, we obtain, $\mathbb{P}$-a.s.,

$$\int_0^t [\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,x))(I-B)^{\theta}Pg\mathrm{d}r = \int_0^t \tilde{Z}(r)\mathrm{d}r.$$

Therefore, for a.a. $t\in[0,T]$, we have $\mathbb{P}$-a.s. $[\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,x))(I-B)^{\theta}Pg = \tilde{Z}(r)$, so substituting into (9.5) we obtain, for $t\in[0,T]$,

$$\begin{aligned}
v(\mathbf{x}(t;0,x)) &= v(x) + \int_0^t [\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;t,x))(I-B)^{\theta}Pg\mathrm{d}W(r) \\
&+ \int_0^t \lambda v(\mathbf{x}(r;0,x)\mathrm{d}r + \int_0^t \psi(\mathbf{x}(r;t,x),[\nabla v(I-B)^{1-\theta}](\mathbf{x}(t;0,x))(I-B)^{\theta}Pg\mathrm{d}r.
\end{aligned}$$

Comparing with the backward equation (8.1) we notice that the pairs

$$(Y(t;0,x),Z(t;0,x))$$

and

$$(v(\mathbf{x}(t;0,x)), [\nabla v(I-B)^{1-\theta}](\mathbf{x}(r;0,x))(I-B)^{\theta} Pg = \tilde{Z}(r))$$

solve the same equation. By uniqueness, we have $Y(t;0,x) = v(\mathbf{x}(t;0,x)), t \in [0,T]$, and setting $t=0$ we obtain $Y(t;t,x) = v(x)$. □

10. Synthesis of the optimal control. In this section we proceed with the study of the optimal control problem associated with the stochastic Volterra equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} a(t-s)u(s)\mathrm{d}s = Au(t) + f(u(t)) \\ \qquad\qquad +g(u(t))\,[\,r(u(t),\gamma(t)) + \dot{W}(t)\,], \qquad t \in [0,T] \\ u(t) = u_0(t), \qquad t \leq 0 \end{cases} \tag{10.1}$$

for a process u with values in the Hilbert space H. Here f and r are the nonlinear functions introduced in Hypothesis 3.1 and $\gamma = \gamma(\omega, t)$ is the control variable, which is assumed to be an a.s. predictable real-valued processes γ $\mathcal{F}_t$-adapted that satisfy the constraint $\gamma(t) \in \mathcal{U}$, $\mathbb{P}$-a.s. for a.e. $t \geq 0$, where $\mathcal{U}$ is a fixed subset of U. The optimal control that we wish to treat consists in minimizing a cost functional of the form

$$\mathbb{J}(u_0, \gamma) = \mathbb{E} \int_0^{\infty} e^{-\lambda t} \ell(u(t), \gamma(t))\mathrm{d}t, \tag{10.2}$$

over all admissible controls where $\ell :\ H \times U \to \mathbb{R}$ is a given continuous and bounded real-valued function.

We will work under assumptions 3.1 and 8.1.

To handle the control problem, we first restate equation (10.1) in an evolution setting and we provide the synthesis of the optimal control by using the forward backward system approach.

As it has been proved in Section 4, given a control process γ and any $u_0 \in H$, we can rewrite the problem (10.1) in the following abstract form

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = B\mathbf{x}(t)\mathrm{d}t + (I-B)f(J\mathbf{x}(t))\mathrm{d}t \\ \qquad\qquad +(I-B)Pg(r(J\mathbf{x}(t),\gamma(t))\mathrm{d}t + \mathrm{d}W(t)) \\ \mathbf{x}(0) = x. \end{cases} \tag{10.3}$$

Here X is a suitable separable Hilbert space, B is a densely defined sectorial operator on a domain $D(B) \subset X$, P is a linear operator from H with values into a real interpolation space X_θ ($\theta \in (0,1)$) between $D(B)$ and X, and J is a linear operator from X_η ($\eta \in (0,1)$) into H; finally $x \in X_\eta$ (see Theorem 4.5 for more details).

In this setting the cost functional will depend on x and γ and is given by

$$\mathbb{J}(x, \gamma) = \mathbb{E} \int_0^{\infty} e^{-\lambda s} \ell(J\mathbf{x}(s), \gamma(s))\mathrm{d}s \tag{10.4}$$

(with an abuse of notation we still denote the rewritten cost functional as $\mathbb{J}$). We notice that for all $\lambda > 0$ the cost functional is well defined and $\mathbb{J}(x,\gamma) < \infty$ for all $x \in X_\eta$ and all a.c.s. There are different ways to give precise meaning to the above problem; one of them is the so called *weak formulation* and will be specified below.

In the weak formulation the class of *admissible control system* (a.c.s.) is given by the set $\mathbb{U} := (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0}, \hat{\mathbb{P}}, \hat{W}, \hat{\gamma})$, where $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ is a complete probability space;

the filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$ verifies the usual conditions, the process $\hat{W}$ is a Wiener process with respect to the filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$, and the control $\hat{\gamma}$ is an $\mathcal{F}_t$-predictable process taking values in some subset $\mathcal{U}$ of X with respect to the filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$. With an abuse of notation, for given $x \in X_\eta$, we associate to every a.c.s. a cost functional $\mathbb{J}(x, \mathbb{U})$ given by the right side of (10.4). Although formally the same, it is important to note that now the cost is a functional of the a.c.s. and not a functional of $\hat{\gamma}$ alone. Any a.c.s. which minimizes $\mathbb{J}(x, \cdot)$, if it exists, is called optimal for the control problem starting from x at time t in the weak formulation. The minimal value of the cost is then called the optimal cost. Finally we introduce the value function $V: X_\eta \to \mathbb{R}$ of the problem as:

$$V(x) = \inf_{\gamma\in\mathcal{U}} \mathbb{J}(x, \gamma), \quad x \in X_\eta,$$

where the infimum is taken over all a.c.s. $\mathbb{U}$.

Abstract optimal control problems on infinite horizon and in infinite dimension have been exhaustively studied by Fuhrman and Tessitore in [18, 19] and Hu and Tessitore in [21], compare Theorem 5.1. Within their approach, the existence of an optimal control is related to the existence of the solution of a suitable forward backward system (FBSDE), that is a system in which the coefficients of the backward equation depend on the solution of the forward equation. Moreover, the optimal control can be selected using a feedback law given in terms of the solution to the corresponding FBSDE. In the following we adapt the theory developed by Fuhrman and Tessitore to our problem and prove that this theory applies to stochastic Volterra equation, provided the problem is reformulated as in Section 4.

We recall that the Hamiltonian corresponding to our control problem is given by

$$\psi(x, z) = \inf_{\gamma\in\mathcal{U}} \left\{\ell(Jx, \gamma) + z \cdot r(Jx, \gamma)\right\}, \qquad x \in X_\eta \ z \in \Xi^\star,$$

and we define the set of minimizers

$$\Gamma(x, z) = \left\{\gamma \in \mathcal{U} : \ell(Jx, \gamma) + z \cdot r(Jx, \gamma) = \psi(x, z)\right\}, \\ t \in [0, T],\ x \in X_\eta,\ z \in \Xi^\star. \tag{10.5}$$

For further use we require an additional property of the function ψ:

Hypothesis 10.1. *The set $\Gamma(x, z)$ is nonempty for all $x \in X_\eta$, $z \in \Xi^\star$.*

Example 10.2. If $U = \Xi$, $\mathcal{U}$ is the ball $\{u \in U : |u|_\Xi \leq C\}$ for some fixed $C > 0$, $r = \mathrm{Id}$ and $\ell(x, \gamma) = \ell_0(|\gamma|^\alpha) + \ell_1(Jx)$ with $\ell_0 \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ convex, $\ell_0'(0) > 0, \alpha > 1$, $\ell_1 \in \mathcal{G}^1(H, \mathbb{R})$ with $|\nabla\ell_1(Jx)h| \leq L|h|$ for suitable constant $L > 0$, and all $x, h \in H$ then by easy computations ψ is in $\mathcal{G}^1(H, \Xi^*)$ and is uniformly Lipschitz both in x and z (thus verifies Hypothesis 8.1). Moreover, $\Gamma(x, z) = \{-\nabla_z\psi(x, z)\}$ turns out to be always a singleton and a continuous function of z only.

We notice that for all $\lambda > 0$ the cost functional is well defined and $\mathbb{J}(x, \gamma) < \infty$ for all $x \in X_\eta$ and all a.c.s. $\mathbb{U}$.

By Theorem 9.3, for all $\lambda > 0$ the stationary HJB equation relative to the above stated problem, namely

$$\mathcal{L}v(x) = \lambda v(x) - \psi(x, \nabla v(x)(I - B)Pg), \quad x \in X_\eta, \tag{HJB}$$

admits a unique mild solution, in the sense of Definition 9.1. Here $\mathcal{L}$ is the infinitesimal generator of the Markov semigroup corresponding to the process $\mathbf{x}$:

$$\mathcal{L}\phi(x) = \frac{1}{2}\mathrm{Tr}((I - B)Pgg^*P^*(I - B)^*\nabla^2\phi(x)) + \langle Bx + (I - B)Pf(x), \nabla\phi(x)\rangle.$$

The relevance of the HJB equation to our control problem is explained in the following main result of this section:

THEOREM 10.3. *Assume Hypotheses 8.1 and 10.1 and suppose that $\lambda > 0$. Then the following holds:*

1. *For all a.c.s. $\mathbb{U} := (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}})_{t\geq 0}, \hat{\mathbb{P}}, (\hat{W}_t)_{t\geq 0}, \hat{\gamma})$ we have $\mathbb{J}(x,\gamma) \geq v(x)$.*
2. *The equality holds if and only if the following feedback law is verified by $\hat{\gamma}$ and $\hat{\mathbf{x}}$:*

$$\hat{\gamma}(t) = \Gamma(\hat{\mathbf{x}}(t), [\nabla v(\hat{\mathbf{x}}(t))(I-B)^{1-\theta}(\mathbf{x}(t))(I-B)^{\theta} Pg), \qquad \hat{\mathbb{P}} - a.s.\ for\ a.e.\ t \geq 0. \tag{10.6}$$

Finally, there exists at least an a.c.s. $\mathbb{U}$ verifying (10.6). *In such a system, the closed loop equation admits a solution*

$$\begin{cases} \mathrm{d}\hat{\mathbf{x}}(t) = B\hat{\mathbf{x}}(t)\mathrm{d}t + (I-B)Pf(\hat{\mathbf{x}}(t))\mathrm{d}t + \\ \qquad (I-B)Pg\left(r(\hat{\mathbf{x}}(t), \Gamma(\hat{\mathbf{x}}(t), [\nabla v(\hat{\mathbf{x}}(t))(I-B)^{1-\theta}](I-B)^{\theta} Pg))\mathrm{d}t + \mathrm{d}\hat{W}(t)\right), \quad t \geq 0 \\ \hat{\mathbf{x}}(0) = x \in X_\eta, \end{cases} \tag{10.7}$$

and if $\hat{\gamma}(t) = \Gamma(\hat{\mathbf{x}}(t), [\nabla v(\hat{\mathbf{x}}(t))(I-B)^{1-\theta}](I-B)Pg)$ then the couple $(\hat{\gamma}, \hat{\mathbf{x}})$ is optimal for the control problem.

Proof. The proof follows from the same arguments used in the proof of [21, Theorem 5.1]. Notice that in this case by Theorem 9.3 we have $Z(t;s,x) = [\nabla v(I-B)^{1-\theta}](t, \mathbf{x}(t;s,x))(I-B)^{\theta} Pg$ and the role of G in [21, Theorem 5.1] is here played by $(I-B)Pg$. $\square$

10.1. Application to the motivating example. We wish to apply the previous results to perform the synthesis of the optimal control described in Section 2.1: minimize the cost functional

$$-\mathbb{J}(k_0, \gamma) = -\mathbb{E}\int_0^{+\infty} e^{-\lambda t}\int_0^{2\pi} u(\gamma(t), k(t,\theta))\mathrm{d}t\mathrm{d}\theta, \tag{10.8}$$

subject to

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{-\infty}^{t} c(t-s)k(s,\theta)\mathrm{d}s - \frac{\partial^2}{\partial\theta^2}k(t,\theta) = \\ &\qquad\qquad f(k(t,\theta)) + \sigma[r(k(t,\theta),\gamma(t)) + \dot{\beta}(t)] \qquad\qquad (t,\theta) \in \mathbb{R}_+ \times \mathbb{T} \\ &k(0,\theta) = k_0(\theta) > 0, \qquad \theta \in \mathbb{T}, \\ &k(t,0) = k(t,2\pi) = 0, \quad t \in \mathbb{R}_+. \end{aligned} \tag{10.9}$$

The admissible controls are all a.s. predictable real-valued processes γ $\mathcal{F}_t$-adapted that satisfy the constraint $\gamma(t) \in \mathcal{U}$, $\mathbb{P}$-a.s. for a.e. $t \geq 0$, where $\mathcal{U}$ is a fixed subset of U. We take u in the expression (10.8) depending only on k and such that it is a bounded and differentiable utility function. We take f to be the identity operator, i.e. $f(k(\cdot)) = k(\cdot)$. We also take r depending only on the control variable γ and being given by the identity operator, i.e. $r(k,\gamma) = \gamma$, $\gamma \in \mathcal{U}$.

The above problem falls under the scope of the general results proved in this section letting $H = L^2(\mathbb{T})$, $U = \Xi = \mathbb{R}$, $\mathcal{U} := [-C, C] \subset U$ for some fixed $C > 0$ and $W(t) = \beta(t)$, $t \geq 0$. $L^2(\mathbb{T})$ is the set of functions $\phi : \mathbb{T} \to \mathbb{R}$ s.t. $\int_0^{2\pi} |\phi(\theta)|^2 \mathrm{d}\theta < +\infty$. We define the coefficient $g \in L_2(\Xi, H)$ as

$$[g(\xi)](\theta) = \sigma\xi \quad \xi \in \Xi = \mathbb{R}.$$

We also consider the differential operator associated with the dynamics (10.9), namely

$$Ak(\theta) = \frac{\partial^2}{\partial\theta^2} k(\theta).$$

In order to define the domain of A we recall the following spaces of real functions defined on $\mathbb{T}$:

$$H^1(\mathbb{T}) := \left\{k \in L^2(\mathbb{T}) : \exists k' \text{ in weak sense and belongs to } L^2(\mathbb{T})\right\}$$
$$H^2(\mathbb{T}) := \left\{k \in L^2(\mathbb{T}) : \exists k' \text{ in weak sense and belongs to } H^1(\mathbb{T})\right\}$$

Now we the take as domain of A the space $D(A) := H^2(\mathbb{T})$. The operator A is well-defined on $H^2(\mathbb{T})$. In particular, A is closed on $D(A)$ and is sectorial on the space H since it corresponds to the differential operator A acting on functions $\phi : [0, 2\pi] \to \mathbb{R}$ satisfying the boundary conditions $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$ (see Chapters 7 and 8, Section 3, in [11]).

With the above definitions, our problem can be written in the following form:

$$\text{minimize} \left[-\mathbb{E}\int_0^{+\infty} e^{-\lambda t}\ell(k(t))\mathrm{d}t\right] \qquad \text{over all a.c.s}$$

subject to

$$\frac{\partial}{\partial t}\int_{-\infty}^{t} c(t-s)k(s)\,\mathrm{d}s - Ak(t) = f(k(t)) + g\,\left[\gamma(t) + \dot{W}(t)\right].$$

It is easy to verify that Hypothesis 3.1, 8.1 and 10.1 hold (see also Example 10.2). Then Theorem 10.3 can be applied to obtain the synthesis of the optimal control in terms of a feedback law.

Aknowledgment. The author would like to express her gratitude to Marco Fuhrman, Federica Masiero and Gianmario Tessitore for stimulating discussions and useful comments. The author would also express her gratitude to the anonymous reviewers for their careful reading of the manuscript and their comments and suggestions.

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