UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA

Facoltà Scienze Matematiche Fisiche Naturali
Dottorato di Ricerca in Matematica del Calcolo, XXIV ciclo

Self-Adjoint Extensions for Symmetric Laplacians on Polygons

Candidate: **Luca Raimondi**

Supervisor: **Prof. Andrea Posilicano**

Introduction

By the results contained in the paper by Birman and Skvortsov "On the square summability of the highest derivatives of the solution to the Dirichlet problem in a region with piecewise smooth boundary" (see reference [1]), the Laplace operator Δ_Ω° on a plane curvilinear polygon Ω with domain the Sobolev space $H^2(\Omega)$ and homogeneous Dirichlet boundary conditions is a closed symmetric operator with deficiency indices (n, n), where n is the number of non-convex corners. Therefore on a non-convex polygon, Δ_Ω° has infinite self-adjoint extensions. Such extensions have been recently determined by means of Kreĭn's resolvent formula in [8]. The purpose of this thesis is to extend such results to the case of different, more general, boundary conditions.

In the first part of the thesis (see Chapter 2) we consider the case of mixed Dirichlet-Neumann conditions, thus allowing each side Γ_j of the polygon boundary to support either a Dirichlet or a Neumann homogeneous boundary condition. In this case, building on results by Grisvard (see [3], [4] and references therein), we have that, differently from the pure Dirichlet case, non-convexity is no more a necessary condition in order to have not zero deficiency indices. Indeed in this case the precise result is the following:
Let ω_j the interior angle at the j-th vertex S_j. If d_j denotes the contribution to the dimension of the defect space due to the vertex S_j, then

$$\begin{cases} d_j = 0\,, & 0 < \omega_j \leq \pi \\ d_j = 1\,, & \pi < \omega_j < 2\pi\,, \end{cases}$$

both in the pure Dirichlet-Dirichlet and Neumann-Neumann cases, and

$$\begin{cases} d_j = 0\,, & 0 < \omega_j \leq \frac{1}{2}\,\pi \\ d_j = 1\,, & \frac{1}{2}\,\pi < \omega_j \leq \frac{3}{2}\,\pi \\ d_j = 2\,, & \frac{3}{2}\,\pi < \omega_j < 2\pi \end{cases} \tag{1}$$

in the mixed Dirichlet-Neumann case.

Thus while in the pure Neumann case the dimensions of the defect spaces is the same as in the case of the pure Dirichlet case already studied by Birman and Skvortsov, the mixed case has a different behavior, allowing both convex cases (with vertex contribution equal to one) and non-convex cases with double vertex contribution.

After explicitly characterizing the defect subspace we determined the self-adjoint extensions by a Kreĭn's resolvent formula proceeding analogously to the pure Dirichlet case given in [8], however taking into account the double contribution due to the vertices with mixed boundary conditions.

In the second part of the thesis we further extend our analysis by allowing some sides Γ_j to support Robin boundary conditions of the kind (here n_j denotes the exterior normal at the j-th side)

$$u(x) + \alpha_j \frac{\partial u}{\partial n_j}(x) = 0\,, \quad x \in \Gamma_j\,,\ \alpha_j > 0\,.$$

While this is a deformation of the case considered in the first part, some not completely trivial calculations are necessary in order to get results similar to the ones concerning the mixed Dirichlet-Neumann case. By such calculations (which fill the entire Chapter 3) it turns out that in the Robin case the contribution d_j of the j-th vertex to the dimension of the defect space is given by the number of eigenvalues λ_j belonging to the interval $(0,1)$ of the 1-dimensional Robin boundary value problem

$$\begin{cases} -u''(\theta) = \lambda_j u(\theta)\,, & \theta \in (0,\omega_j) \\ u(0) + \alpha_j u'(0) = 0\,, & \\ u(\omega_j) - \alpha_{j+1} u'(\omega_j) = 0\,. & \end{cases}$$

Thus, by tuning the parameters α_j, one can recover results anologous to the ones in (1). However also different behaviors are possible:
1. for any $\epsilon > 0$, for any $0 < \omega_j < \epsilon$, there are parameter values which give $d_j = 1$;
2. for any $x\pi < \omega_j \leq (3/2)\pi$, $x \simeq 1.43$, there are parameter values which give $d_j = 2$.

Moreover, as expected, the d_j's converge to the ones corresponding to the mixed Dirichlet-Neumann case as the α_j's converge to either 0 or ∞ accordingly to the different possible cases.

Again as in the mixed Dirichlet-Neumann a Kreĭn's formula giving the classification of all the self-adjoint extension is provided in Chapter 4.

In the final chapter we give, to enhance the reader intuition, some simple examples regarding the case in which Ω is a wedge.

Contents

Chapter 1

Preliminary results

1.1 Sobolev spaces on Polygons

In this section we recall the basic definitions and general results[1] that will be used in the following part of this work. Let Ω be an arbitrary open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. Denoting by $L^2(\Omega)$ the space of all square integrable (complex valued) functions for the Lebesgue measure on Ω, we denote by $C_c^\infty(\Omega)$ (resp. $C_c^\infty(\bar{\Omega})$) the space of all infinitely differentiable unctions with compact support in Ω (resp. the restriction to Ω of functions in $C_c^\infty(\mathbb{R}^n)$).
Given s any real number, we shall denote by m its integral part and by σ its fractional part. According one has $s = m + \sigma$ with $0 \leq \sigma \leq 1$. Also we denote by D_i the differentiation with respect to x_i for $1 \leq i \leq n$ and for an arbitrary multi-integer $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$ with nonnegative components we set $D^\alpha = D^{\alpha_1} D^{\alpha_2} ... D^{\alpha_n}$.

Definition 1.1.1. We denote $H^s(\Omega)$ the space of all distributions u defined in Ω such that
i) $D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq m$ when $s = m$ is a nonnegative integer
ii) $u \in H^m(\Omega)$ and

$$\int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\, dxdy < \infty\,,$$

for $|\alpha| = m$ when $s = m + \sigma$ is a nonnegative and non integral integer.
We define a Hilbert norm on $H^s(\Omega)$ by

$$\|u\|_{m,\Omega} = \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^2 dx \right)^{1/2}$$

[1]Proofs can be found e.g. in [3], Chapter 1, and [4], Chapter 1.

in the case i) and by

$$\|u\|_{s,\Omega} = \left(\|u\|_{m,\Omega} + \sum_{|\alpha|=m} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} \, dxdy \right)^{1/2},$$

in case ii).

Definition 1.1.2. We denote by $H_0^s(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$.

Definition 1.1.3. We denote by $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$.

Definition 1.1.4. For every positive s we denote by $\tilde{H}^s(\Omega)$ the space of all u defined in Ω such that $\tilde{u} \in H^s(\mathbb{R}^n)$ where $\tilde{u}$ is the continuation of u by zero outside Ω.

On $\tilde{H}^s(\Omega)$ we define the norm $\|\tilde{u}\|_{s,\mathbb{R}^n}$. A simple calculation shows that this norm is equal to $\|u\|_{s,\Omega}$ when s is an integer and equivalent to

$$\|u\|_{s,\Omega} + \sum_{|\alpha|\le m} \left(\int_\Omega |D^\alpha u(x)|^2 w(x) dx \right)^{1/2}$$

when s is not an integer, where $w(x)$ is an appropriate weight. If Ω is a Lipschitz [2] bounded domain, the weight $w(x)$ is equivalent to $\rho(x)^{-2\sigma}$ where $\rho(x)$ denotes the distance from x to the boundary.

Theorem 1.1.1. *Let Ω be a Lipschitz open subset of $\mathbb{R}^n$. Then $\mathcal{C}_c^\infty(\bar{\Omega})$ is dense in $H^s(\Omega)$ for all $s \ge 0$.*

Theorem 1.1.2. *Let Ω be a Lipschitz open subset of $\mathbb{R}^n$. Then $C_c^\infty(\Omega)$ is dense in $\tilde{H}^s(\Omega)$ for all $s \ge 0$. Moreover $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ for $s \in [0, 1/2)$.*

Definition 1.1.5. We denote by $\tilde{H}^{-s}(\Omega)$ the dual space of $\tilde{H}^s(\Omega)$.

Theorem 1.1.3. *Let Ω be a bounded Lipschitz open subset or $\mathbb{R}^n$ then for every $s > 0$ there exists a continuous linear operator P_s from $H^s(\Omega)$ into $H^s(\mathbb{R}^n)$ such that*

$$P_s u\big|_\Omega = u\,,$$

for any $u \in H^s(\Omega)$.

[2]By a Lipschitz domain we mean that the boundary of Ω is locally the graph of a Lipschitz function.

Theorem 1.1.4. *Let $s' > s'' \geq 0$ and assume that Ω is a bounded Lipschitz open subset or $\mathbb{R}^n$ then the injection of $H^{s'}(\Omega)$ into $H^{s''}(\Omega)$ is compact.*

Theorem 1.1.5. *Let $s' > s'' > s''' \geq 0$ and assume that Ω s a bounded Lipschitz open subset or $\mathbb{R}^n$ then there exists a constant C (which depends on Ω, s', and s'') such that*

$$\|u\|_{s'',\Omega} \leq \epsilon \|u\|_{s',\Omega} + C\epsilon^{-(s''-s''')/(s'-s'')}\|u\|_{s''',\Omega}\,,$$

for all $u \in H^{s'}(\Omega)$.

Theorem 1.1.6. *Assume that Ω is any bounded open subset of $\mathbb{R}^n$ then there exists a constant $K(\Omega)$ which depends only on the diameter of Ω such that*

$$\|u\|_{0,\Omega} \leq K(\Omega)\big(\sum_{1\leq n}\int_\Omega |D_i u|^2 dx\big)^{1/2}$$

for any $u \in H_0^s(\Omega)$

Theorem 1.1.7. [3] *The following inclusions hold*

$$H^s(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$$

for $s < n/2$ and $q \geq 2$ such that $1/q = 1/2 - s/n$ and

$$H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$$

for any integers $k < s - n/2$.

Theorem 1.1.8. *Given Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$ and denote by $\rho(x)$ the distance from x to Γ. Then one has $u/\rho^s \in L^2(\Omega)$ for all $u \in H^s(\Omega)$ when $1 < s < 1/2$ and for all $u \in H_0^s(\Omega)$ when $1/2 < s < 1$.*

Let us now recall the well known trace theorem on an hyperplane. Given u a smooth function on $\mathbb{R}^n$ we define the function $\gamma^0 u$ by

$$\gamma^0 u(x_1, \ldots, x_{n-1}) := u(x_1, \ldots, x_{n-1}, 0)\,.$$

The density property allows one to extend γ^0 to a continuous linear operator from $H^s(\mathbb{R}^n)$ onto $H^{s-1/2}(\mathbb{R}^{n-1})$ provided $s > 1/2$. As direct consequence one has

[3] Applying Theorem 1.1.3 one obtains the same inclusions for spaces over a bounded Lipschitz domain Ω.

Theorem 1.1.9. *The mapping*

$$u \to \left(\gamma^0 u, \gamma^0 \frac{\partial u}{\partial x_n}, ..., \gamma^0 \frac{\partial^k u}{\partial x_n^k}\right)$$

defined on $C_c^\infty(\mathbb{R}^n)$ *has for* $k < s - 1/2$ *an unique continuous extension as an operator from*

$$H^s(\mathbb{R}^n) \quad \textit{onto} \quad \prod_{0\le p\le k} H^{s-p-1/2}(\mathbb{R}^{n-1})\,.$$

From now on Ω will be a polygonal domain in $\mathbb{R}^2$ where the boundary $\partial\Omega$ is given by the union the of sides Γ_j, $j = 1, \ldots N$.

Given $u \in H^s(\Omega)$ we define

$$\gamma_j^k u := \gamma^k P_s u$$

by taking a system of orthogonal coordinates $(x_1, x_2) \in \mathbb{R}^2$ with respect to which $\Gamma_j \subset \{(x_1, 0)\,, x_1 \in \mathbb{R}\}$. Given such a definition one has the following

Theorem 1.1.10. *Let* Ω *be a bounded polygonal open subset of* $\mathbb{R}^2$*, then for each* j *the mapping*

$$u \to \{\gamma_j^l\,,\ 1 \le l \le k, 1 \le j \le N\}$$

which is defined for $u \in C^\infty(\bar{\Omega})$ *has for* $k < s - 1/2$ *a unique continuous extension from*

$$H^s(\Omega) \quad \textit{into} \quad \prod_{j=1}^{N} \prod_{0\le p\le k} H^{s-p-1/2}(\Gamma_j)\,.$$

As regards the range of the application given in the previous theorem one has the following

Theorem 1.1.11. *Let* Ω *be a bounded polygonal open subset of* $\mathbb{R}^2$*. Then the mapping*

$$u \mapsto \{\gamma_j^l u\,, 0 \le l \le m-1\,, 1 \le j \le N\,\}$$

is linear continuous from $H^m(\Omega)$ *onto the subspace of*

$$\prod_{1\le j\le N} \prod_{0\le l\le m-1} H^{m-l-1/2}(\Gamma_j)$$

defined by the following conditions. Let L be any differential operator with constant coefficients and order $d \le m-1$. Denote by $P_{j,l}$ the differential operators tangential to Γ_j such that

$$L = \sum_l P_{j,l} \frac{\partial^l}{\partial n_j^l}\,,$$

where n_j denotes the exterior normal at γ_j. Then one has

i) $\sum_l (P_{j,l} f_{j,l})(S_j) = \sum_l (P_{j+1,l} f_{j+1,l})(S_j)$ *for* $d \le m-2$

ii) $\sum_l (P_{j,l} f_{j,l}) \equiv \sum_l (P_{j+1,l} f_{j+1,l})$ *at* S_j *for* $d = m-1$.

As regards "half" and 'full" Green Formula the result is the following:

Theorem 1.1.12.

$$\int_\Omega u \triangle v\, dx + \int_\Omega \nabla v \nabla u\, dx = \sum_j \int_{\Gamma_j} \gamma_j^0 u \gamma_j^1 v\, d\sigma\,, \quad u \in H^1(\Omega)\,, v \in H^2(\Omega)\,,$$

$$\int_\Omega u \triangle v\, dx - \int_\Omega v \triangle u\, dx = \sum_j \Big(\int_{\Gamma_j} \gamma_j^0 u \gamma_j^1 v\, d\sigma - \int_{\Gamma_j} \gamma_j^0 v \gamma_j^1 v\, d\sigma \Big)\,, \quad u, v \in H^2(\Omega)\,. \tag{1.1}$$

The linear maps γ_j^0 and γ_j^1 can be extended to a larger domain and this implies an extension of Green's folmulae also. Indeed, defining the *maximal domain*

$$\mathscr{D}(\triangle_\Omega^{\max}) := \{ u \in L^2(\Omega) \, : \triangle u \in L^2(\Omega) \}\,, \tag{1.2}$$

one has

Theorem 1.1.13. *Let Ω be a bounded polygonal open domain in $\mathbb{R}^2$. Then the maps γ_j^0 and $\gamma_j^1 v$ have unique continuous extensions*

$$\hat\gamma_j^0 : \mathscr{D}(\triangle_\Omega^{\max}) \to \tilde H^{-1/2}(\Gamma_j)\,,$$

$$\hat\gamma_j^1 : \mathscr{D}(\triangle_\Omega^{\max}) \to \tilde H^{-3/2}(\Gamma_j)\,.$$

Moreover

Theorem 1.1.14.

$$\int_\Omega v \triangle u dx + \int_\Omega \nabla v \nabla u dx = \sum_j \langle \gamma_j^0 u,\, \hat\gamma_j^1 v \rangle\,,$$

for every $v \in \mathscr{D}(\triangle_\Omega^{max})$ and every $u \in H^1(\Omega)$ such that

$$\gamma_j^0 u \in \tilde H^{1/2}(\Gamma_j)\,.$$

Theorem 1.1.15.

$$\int_\Omega u\triangle v dx - \int_\Omega v\triangle u dx = \sum_j \left(\langle\gamma_j^0 u,\, \hat\gamma_j^1 v\rangle - \langle\hat\gamma_j^0 v,\, \gamma_j^1 u\rangle\right),$$

for every $v \in \mathscr{D}(\triangle_\Omega^{max})$ and every $u \in H^2(\Omega)$ such that

$$\gamma_j^0 u \in \tilde H^{3/2}(\Gamma_j)\,,\ \ \gamma_j^1 u \in \tilde H^{1/2}(\Gamma_j)\,.$$

1.2 Self-adjoint extension and Kreĭn's formula

Let us consider the Hilbert space $\mathscr{H}$ equipped with the inner product $\langle\cdot,\cdot\rangle$ and the self-adjoint operator

$$A : \mathscr{D}(A) \subseteq \mathscr{H} \to \mathscr{H}\,.$$

Then we define $\mathscr{H}_A$ as the Hilbert space given by the domain $\mathscr{D}(A)$ equipped by the inner product

$$\langle\phi,\psi\rangle_A = \langle\phi,\psi\rangle + \langle A\phi, A\psi\rangle.$$

Now, given a closed subspace $\mathscr{N} \subset \mathscr{H}_A$ dense in $\mathscr{H}$, let us define S as the closed, densely defined, symmetric operator obtained by restricting A to $\mathscr{N}$. Our purpose is to describe all self-adjoint extensions of S together with their resolvent.

Since $\mathscr{N}$ is closed, $\mathscr{H}_A = \mathscr{N} \oplus \mathscr{N}^\perp$, and then $\mathscr{N}$ coincides with the kernel of the orthogonal projection onto $\mathscr{N}^\perp$. Now, since $\mathscr{N}^\perp \simeq \mathscr{H}_A/\mathscr{N}$ is an Hilbert space, without lost of generality we can always suppose that $\mathscr{N}$ coincides with the kernel of a surjective bounded linear operator:

$$\tau : \mathscr{H}_A \to \mathfrak{h},$$

with $\mathfrak{h}$ an auxiliary Hilbert space. Since this suffices for our purposes, we will suppose that $\mathfrak{h}$ is finite dimensional, thus we can pose

$$\mathfrak{h} = \mathbb{C}^n\,.$$

This meas that S has finite deficiency indices.

In the following

$$\mathscr{K}(L), \quad \mathscr{R}(L), \quad \rho(L),$$

will be respectively the kernel, the range and the resolvent set of a given linear operator L. Given τ as above one has

$$S = A|\mathscr{K}(\tau), \qquad \mathscr{R}(\tau) = \mathbb{C}^n, \qquad \overline{\mathscr{K}(\tau)} = \mathscr{H}. \tag{1.3}$$

For any $z \in \rho(A)$ we define the resolvent of A, i.e. the bounded linear operator from $\mathscr{H}_A$ to $\mathscr{H}$ as:

$$R_z := (-A + z)^{-1} \tag{1.4}$$

For any element of $\rho(A)$ we also consider the continuous linear map:

$$G_z := (\tau R_{\bar{z}})^* : \mathbb{C}^n \to \mathscr{H}, \tag{1.5}$$

that is injective being τ surjective. One has that (1.3) are equivalent to:

$$\mathscr{R}(G_z) \cap \mathscr{D}(A) = \{0\}. \tag{1.6}$$

Furthermore, by the first resolvent identity one has ([6], lemma 2.1):

$$(z - w)R_w G_z = G_w - G_z, \tag{1.7}$$

$$\mathscr{R}(G_w - G_z) \in \mathscr{D}(A). \tag{1.8}$$

Let us now consider a family of linear operators $\Gamma : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$(\Gamma_z)^* = \Gamma_{\bar{z}}, \tag{1.9}$$

$$\Gamma_z - \Gamma_w = (z - w)G^*_{\bar{w}} G_z. \tag{1.10}$$

Let us observe that this class is nonempty, in fact by (1.7) and the definition of $\Gamma(z)$ one can prove (see [6], lemma 2.2) that each of these families differs by a *z-independent*, symmetric operator from the family $\hat{\Gamma}_w(z)$ defined as

$$\hat{\Gamma}_w(z) := \tau\left(\frac{G_w + G_{\bar{w}}}{2} - G_z\right), \quad w \in \rho(A).$$

Notice that $\hat{\Gamma}_w(z)$ is well defined by (1.8).
In the case $0 \in \rho(A)$, the easiest choice (the one we will take in the following chapters) is

$$\Gamma_z = \tau(G_0 - G_z) \equiv z G^*_0 G_z\,. \tag{1.11}$$

Given the orthogonal projector

$$\Pi : \mathbb{C}^n \to \mathbb{C}^n$$

we pose

$$\mathbb{C}^n_\Pi := \mathscr{R}(\Pi),$$

and for any symmetric operator:

$$\Theta : \mathbb{C}^n_\Pi \to \mathbb{C}^n_\Pi,$$

we can define the linear operator

$$\Gamma_{z,\Pi,\Theta} := (\Theta + \Pi\, \Gamma_z \Pi) : \mathbb{C}^n_\Pi \to \mathbb{C}^n_\Pi, \tag{1.12}$$

and the open set

$$Z_{\Pi,\Theta} := \{z \in \rho(A) \ : \ \det \Gamma_{\Pi,\Theta}(z) \neq 0\}.$$

Now we have the following result

Theorem 1.2.1. [4] *Let A, τ, S Π, Θ and $\Gamma_{z,\Pi,\Theta}$ as above. Then*

$$\mathbb{C}\setminus\mathbb{R} \subseteq Z_{\Pi,\Theta}$$

and the bounded linear operator:

$$R_{z,\Pi,\Theta} := R_z + G_z \Pi\, \Gamma^{-1}_{z,\Pi,\Theta} \Pi G^*_{\bar{z}}, \quad z \in Z_{\Pi,\Theta}, \tag{1.13}$$

is the resolvent of the self-adjoint extension $A_{\Pi,\Theta}$ of S defined as

$$A_{\Pi,\Theta} : \mathscr{D}(A_{\Pi,\Theta}) \subseteq \mathscr{H} \to \mathscr{H}, \qquad (-A_{\Pi,\Theta} + z)\phi := (-A + z)\phi_z,$$

$$\mathscr{D}(A_{\Pi,\Theta}) := \{\phi \in \mathscr{H} \ : \ \phi = \phi_z + G_z \Pi\, \Gamma^{-1}_{z,\Pi,\Theta} \Pi \tau \phi_z, \ \phi_z \in \mathscr{D}(A)\}.$$

The definition is z-independent and the decomposition appearing in $\mathscr{D}(A_{\Pi,\Theta})$ is univocal.

Proof. By (1.9) e (1.10) we have

$$|\zeta \cdot \Gamma_{z,\Pi,\Theta}\zeta|^2 \geq \operatorname{Im}(z)^2 \|G_z \zeta\|^4$$

for any $\zeta \in \mathbb{C}_\Pi$. So $\det \Gamma_{z,\Pi,\Theta} \neq 0$ for any $z \in \mathbb{C}\backslash\mathbb{R}$. Now by (1.10), from [6] we have that $R_{z,\Pi,\Theta}$ satisfies the resolvent identity

$$(z - w) R_{w,\Pi,\Theta} R_{z,\Pi,\Theta} = R_{w,\Pi,\Theta} - R_{z,\Pi,\Theta} \tag{1.14}$$

and by (1.9),

$$R^*_{z,\Pi,\Theta} = R_{\bar{z},\Pi,\Theta}\,. \tag{1.15}$$

Furthermore by (1.6), $R_{z,\Pi,\Theta}$ is injective. Then the extension

$$A_{\Pi,\Theta} := z - R^{-1}_{z,\Pi,\Theta}$$

is well defined on

$$\mathscr{D}(A_{\Pi,\Theta}) := \mathscr{R}(R_{z,\Pi,\Theta}),$$

and z-independent, respectively symmetric, by (1.14), respectively by (1.15). Finally it is self-adjoint since $\mathscr{R}(-A_{\Pi,\Theta} \pm i) = \mathscr{H}$ by construction. □

[4] See [6], theorem 2.1

Corollary 1.2.2. *Suppose that* $0 \in \rho(A)$. *Then*

$$A_{\Pi,\Theta} : \mathscr{D}(A_{\Pi,\Theta}) \subseteq \mathscr{H} \to \mathscr{H}, \qquad A_{\Pi,\Theta}\phi = A\phi_0,$$

$$\mathscr{D}(A_{\Pi,\Theta}) := \{\phi \in \mathscr{H} \,:\, \phi = \phi_0 + G_0\xi_\phi,\ \phi_z \in \mathscr{D}(A)\,,\ \xi_\phi \in \mathbb{C}_\Pi\,,\ \Pi\tau\phi_0 = \Theta\xi_\phi\}.$$

Notwithstanding the easy proof the self-adjoint extensions provided above exhaust the class of all self-adjoint extension of the symmetric operator S:

Theorem 1.2.3. [5] *The family of* $A_{\Pi,\Theta}$*, given by Theorem 1.2.1, coincides with the family* $\mathfrak{F}$ *of all self-adjoint extensions of the symmetric operator* S. *Thus* $\mathfrak{F}$ *can be parameterized by bundle*

$$p : E(\mathbb{C}^n) \to P(\mathbb{C}^n),$$

where $P(\mathbb{C}^n)$ *denotes the set of orthogonal projectors in* $\mathbb{C}^n$ *and* $p^{-1}(\Pi)$ *denotes the set of symmetric operators in* $\mathbb{C}^n_\Pi$. *In particular the set of symmetric operators on* $\mathbb{C}^n$, *i.e.* $p^{-1}(1)$, *parameterize the extensions such that* $\mathscr{D}(A_{1,\Theta}) \cap \mathscr{D}(A) = \mathscr{N}$, *also called relatively prime extensions.*

The next result [6] give us informations about the spectrum and eigenfunctions of $A_{\Pi,\Theta}$

Theorem 1.2.4.

$$\lambda \in \sigma_p(A_{\Pi,\Theta}) \cap \rho(A) \quad \Leftrightarrow \quad 0 \in \sigma_p(\Gamma_{\lambda,\Pi,\Theta}),$$

where $\sigma_p(\cdot)$ *denotes the point spectrum. Moreover*

$$G_\lambda : \mathscr{K}(\Gamma_{\lambda,\Pi,\Theta}) \to \mathscr{K}(-A_{\Pi,\Theta} + \lambda)$$

is a bijection for any $\lambda \in \sigma_p(A_{\Pi,\Theta}) \cap \rho(A)$.

The next theorem provides the quadratic forms corresponding to the self-adjoint extension given above. Since this suffices for our purposes we suppose here that $0 \in \rho(A)$.

Theorem 1.2.5. *Let*

$$F : \mathscr{D}(F) \times \mathscr{D}(F) \subseteq \mathscr{H} \times \mathscr{H} \to \mathbb{R}$$

be the quadratic form associated to $-A$ *and suppose that*

$$\mathscr{R}(G_0) \cap \mathscr{D}(F) = \{0\}\,.$$

[5] See [7], section 3
[6] See [7], section 2

Then

$$F_{\Pi,\Theta} : \mathscr{D}(F_{\Pi,\Theta}) \times \mathscr{D}(F_{\Pi,\Theta}) \subseteq \mathscr{H} \times \mathscr{H} \to \mathbb{R}\,,$$

$$\mathscr{D}(F_{\Pi,\Theta}) = \{\phi \in \mathscr{H} \,:\, \phi = \phi_0 + G_0\xi_\phi\,,\ \phi_0 \in \mathscr{D}(F)\,,\ \xi_\phi \in \mathbb{C}^n_\Pi\}\,,$$

$$F_{\Pi,\Theta}(\phi,\psi) = F(\phi_0,\psi_0) + \Theta\xi_\phi\cdot\xi_\psi\,,$$

is the quadratic form associated to $-A_{\Pi,\Theta}$*.*

Proof. Let $L : \mathscr{D}(L) \subseteq \mathscr{H} \to \mathscr{H}$ be the linear operator associated to $F_{\Pi,\Theta}$, i.e.

$$\mathscr{D}(L) :=$$
$$\{\phi \in \mathscr{D}(F^{\Pi,\Theta}) \,:\, \exists\tilde\phi \in \mathscr{H}\,\text{s.t.}\,\forall\psi \in \mathscr{D}(F_{\Pi,\Theta})\,,\ F_{\Pi,\Theta}(\phi,\psi) = \langle\tilde\phi,\psi\rangle_{\mathscr{H}}\}\,,$$

$$L\phi := \tilde\phi\,.$$

Since $\mathscr{D}(F) \subseteq \mathscr{D}(F_{\Pi,\Theta})$, $\mathscr{D}(F_{\Pi,\Theta})$ is dense and so L is well-defined.

By the definition of $\mathscr{D}(F_{\Pi,\Theta})$ and by taking, in the definition of $\mathscr{D}(L)$, at first $\xi_\psi = 0$ and then $\psi = G_0\xi_\psi$, one gets that $\phi = \phi_0 + G_0\xi_\phi \in \mathscr{D}(L)$ if and only if there exists $\tilde\phi$ such that

$$\forall\psi_0 \in \mathscr{D}(F)\,,\quad F(\phi_0,\psi_0) = \langle\tilde\phi,\psi_0\rangle_{\mathscr{H}}$$

and

$$\forall\xi \in \mathbb{C}^n_\Pi\,,\quad \Theta\xi_\phi\cdot\xi = \langle\tilde\phi, G_0\xi\rangle_{\mathscr{H}}\,.$$

Thus $\phi_0 \in \mathscr{D}(A)$, $L\phi = -A\phi_0$, and

$$\langle\tilde\phi, G_0\xi\rangle_{\mathscr{H}} = -\langle A\phi_0, G_0\xi\rangle_{\mathscr{H}} = -(\tau R_0 A\phi_0)\cdot\xi = (\tau\phi_0)\cdot\xi = (\Pi\tau\phi_0)\cdot\xi\,.$$

This gives $\Pi\tau\phi_0 = \Theta\xi_\phi$, and so $L = -A_{\Pi,\Theta}$. □

Chapter 2

Self-Adjoint Extensions of Symmetric Laplacians with Mixed Dirichlet-Neumann Boundary Conditions

We start this sections recalling some important results about the Laplace operator on a polygon with mixed boundary conditions at the boundary[1].
Let $\Omega \subset \mathbb{R}^2$ be a plane bounded open *curvilinear polygon*. This means that the boundary $\partial\Omega$ is a piecewise smooth closed curve with no cups points. The point where such a curve fails to be differentiable are called *vertices*. To simplify the exposition we further suppose that a such curve coincides with a broken line in a neighborhood of each vertex. If the whole boundary is made of broken lines we says that Ω is a *classical polygon*.
We also assume that Ω is connected and simply connected domain and we will denote each open smooth segment of $\partial\Omega$ (i.e. its sides) by Γ_j where the index j ranges from 1 to some integer N. These segments are numbered in such a way that Γ_{j+1} follows Γ_j according to the positive orientation. We also denote by S_j the vertex which is the endpoint of Γ_j.
Furthermore we define n_j (resp. t_j) as the unit outward normal (resp. tangent) vector on Γ_j and by ω_j the measure of the interior angle at S_j. Polar coordinates (r_j, θ_j) with origin at S_j will be used. Such coordinates are choosen in such a way that Γ_j is on the half-axis $\theta = \omega_j$ and Γ_{j+1} is on the half-axis $\theta = 0$. Then we introduce the cartesian coordinates attached to each corner with vertex S_j as

$$x_j = r_j \cos\theta\,, \quad y_j = r_j \sin\theta\,,$$

[1]All the missing proofs can be found in [3], Sections 2.1 - 2.3.

accordingly Γ_{j+1} is a subsegment of the line $y_j = 0$.

In considering mixed boundary conditions, it is useful to fix a partition of $\{1, ..., N\}$ (the set numbering the vertices) by the subsets $\mathscr{D}$ and $\mathscr{N}$ defined according to the following rule:

- $j \in \mathscr{D}$ if Γ_j supports a Dirichlet boundary condition;
- $j \in \mathscr{N}$ if Γ_j supports a Neumann boundary condition.

We also will consider the sets $\mathscr{D}^2$, $\mathscr{N}^2$, $(\mathscr{N}, \mathscr{D})$, $(\mathscr{D}, \mathscr{N})$, defined by

- $j \in \mathscr{D}^2$ if both Γ_j and Γ_{j+1} support Dirichlet boundary conditions;
- $j \in \mathscr{N}^2$ if both Γ_j and Γ_{j+1} support Neumann boundary conditions;
- $j \in (\mathscr{N}, \mathscr{D})$ if Γ_j supports a Neumann boundary condition and Γ_{j+1} supports a Dirichlet boundary condition;
- $j \in (\mathscr{D}, \mathscr{N})$ if Γ_j supports a Dirichlet boundary condition and Γ_{j+1} supports a Neumann boundary condition.

Finally we introduce the set corresponding to mixed boundary conditions:

$$\mathscr{M} := (\mathscr{D}, \mathscr{N}) \cup (\mathscr{N}, \mathscr{D})$$

and for later convenience we also define

$$\mathscr{M}_1 := \{j \in \mathscr{D}^2 \cup \mathscr{N}^2 \; : \; \omega_j > \pi\} \cup \{j \in \mathscr{M} \; : \; \omega_j > \pi/2\} \, ,$$

$$\mathscr{M}_2 := \{j \in \mathscr{M} \; : \; \omega_j > \frac{3}{2}\,\pi\} \, .$$

We will always suppose that

$$\mathscr{D} \neq \emptyset \tag{2.1}$$

Let $\triangle_\Omega$ be the distributional Laplace operator on the curvilinear polygon Ω and let us define

$$\triangle_\Omega^{\max} : \mathscr{D}(\triangle_\Omega^{\max}) \subset L^2(\Omega) \to L^2(\Omega) \, , \quad \triangle_\Omega^{\max} u := \triangle_\Omega u$$

where $\mathscr{D}(\triangle_\Omega^{\max})$ is defined in (1.2).

We introduce the spaces $V(\Omega)$ and $V^2(\Omega)$, respectively containing the variational and the strong solutions of the boundary value problem

$$\begin{cases} \Delta_\Omega u = f \, , & f \in L^2(\Omega) \\ \gamma_j^0 u = 0 \, , & j \in \mathscr{D} \\ \gamma_j^1 u = 0 \, , & j \in \mathscr{N} \, , \end{cases}$$

defined by

$$V(\Omega) = \{u \in H^1(\Omega) \,:\, \gamma_j^0 u = 0\,,\ j \in \mathscr{D}\} \tag{2.2}$$

and

$$V^2(\Omega) = \{u \in H^2(\Omega) \,:\, \gamma_j^0 u = \gamma_i^1 u = 0\,,\ j \in \mathscr{D}\,, i \in \mathscr{N}\}\,. \tag{2.3}$$

The results given by the next theorems will be useful in the following:

Theorem 2.0.6. *The space $H^m(\Omega) \cap V(\Omega)$ is dense in $V(\Omega)$ for every $m > 1$.*

Theorem 2.0.7. *The space $H^m(\Omega) \cap V^2(\Omega)$ is dense in $V^2(\Omega)$ for every $m > 1$.*

For any function in $V^2(\Omega)$ a Caccioppoli's type inequality holds true:

Theorem 2.0.8. *Assume that Ω is a bounded polygonal open subset of $\mathbb{R}^2$ and that* (2.1) *holds. Then there exists a constant C_Ω such that*

$$\forall u \in V^2(\Omega)\,, \quad \|u\|_{H^2(\Omega)} \leq C_\Omega \|\Delta_\Omega u\|_{L^2(\Omega)}\,. \tag{2.4}$$

As a direct consequence of *Poincaré inequality* (see Theorem 1.1.6) $V(\Omega)$ is a Hilbert space for the scalar product induced by the bilinear form

$$F(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx\,, \quad u\,, v \in V(\Omega)\,.$$

This allow us to apply the *Lax-Milgram Theorem* and to conclude[2] that there exists a unique self-adjoint operator

$$\triangle_\Omega^F : \mathscr{D}(\triangle_\Omega^F) \subseteq V(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

such that

$$F(u,v) = \langle -\Delta_\Omega^F u, v\rangle_{L^2(\Omega)}\,, \quad u \in \mathscr{D}(\Delta_\Omega^F)\,, v \in V(\Omega)\,.$$

On the other hand by Theorem 2.0.8 and by Green's Formula the linear operator

$$\triangle_\Omega^\circ : V^2(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)\,, \quad \triangle_\Omega^\circ u := \triangle u\,,$$

is closed and symmetric. Thus a natural question arises: is $\triangle_\Omega^\circ$ self-adjoint? Equivalently: does $\triangle_\Omega^\circ$ coincide with $\triangle_\Omega^F$?
The answer to the previous question depends on the shape of Ω and in order

[2] see e.g. [2], Chapter IV, Section 1.

to answer to this question we need some more definition. For any vertex S_j we consider the measure ω_j of the corresponding interior angle and define[3]

$$n_1 := \#\mathscr{M}_1 \,, \quad n_2 := \#\mathscr{M}_2$$

In the following we will see that the deficiency indices of Δ°_Ω are both equal to $n_1 + n_2$.
Consequently if $n_1 + n_2 \neq 0$ we have $\mathscr{D}(\triangle^F_\Omega) \neq V^2(\Omega)$. This is an immediate consequence of the fact that for every $j \in \mathscr{D}^2$ (respectively $j \in \mathscr{N}^2$) the function

$$r_j^{\pi/\omega_j} \, \sin \frac{\pi}{\omega_j} \theta_j \,,$$

(respectively $r_j^{\pi/\omega_j} \, \cos \frac{\pi}{\omega_j} \theta_j$) belongs to $\mathscr{K}(\triangle_{\max}) \cap H^1(W)$, where W is the wedge

$$W = \{(x,y) \equiv (r_j \cos\theta_j \,, r_j \sin\theta_j) \;:\; 0 \le r_j < R \,, 0 < \theta_j < \omega_j \} \,,$$

but fails to be in $H^2(W)$ when $\pi/\omega_j < 1$.
In the case $j \in \mathscr{M}$ the conclusion is quite similar considering for example the pair of functions

$$u_m = r_j^{\frac{(m-1/2)\pi}{\omega_j}} \, \sin \frac{(m-1/2)\pi}{\omega_j} \theta \,, \quad m = 1, 2 \,.$$

From now on we will suppose that $n_1 + n_2 \neq 0$ so that

$$V^2(\Omega) \equiv \mathscr{D}(\triangle^\circ_\Omega) \subsetneq \mathscr{D}(\triangle^F_\Omega) \,.$$

Thus any self-adjoint extension of $\triangle^\circ_\Omega$ will be a restriction of its adjoint $\triangle^{\circ *}_\Omega$. Since $\triangle^{\max}_\Omega$ is the adjoint of the restriction of $\triangle_\Omega$ to $C^\infty_c(\Omega)$, one has $\triangle^{\circ *}_\Omega \subset \triangle^{\max}_\Omega$.
Inequality (2.4) shows that $\triangle^\circ_\Omega$ is injective and has a closed range and, posing

$$N := \mathscr{R}(\Delta^\circ_\Omega)^\perp = \mathscr{K}((\Delta^\circ_\Omega)^*)$$

one has the following results, which completely characterize the linear set N:

Lemma 2.0.9. *Let $v \in N$. Then v belongs to $\mathscr{D}(\triangle^{\max}_\Omega)$ and solves the (adjoint) boundary value problem*

$$\begin{cases} \triangle v = 0 & in\ \Omega \\ \hat\gamma^0_j v = 0 & j \in \mathscr{D} \\ \hat\gamma^1_i v = 0 & i \in \mathscr{N} \,. \end{cases}$$

[3]Here $\#S$ denotes the cardinality of the set S.

Lemma 2.0.10. *Every $v \in N$ is such that*

$$\int_\Omega v \triangle \eta_j dx = 0\,, \tag{2.5}$$

for any $j \in \mathscr{N}^2$,

$$\int_\Omega v \triangle (y_j \eta_j) dx = 0\,, \tag{2.6}$$

for any $j \in (\mathscr{N}, \mathscr{D})$ with either $\omega_j = \pi/2$ or $\omega_j = 3\pi/2$,

$$\int_\Omega v \triangle (x_j \eta_j) dx = 0\,, \tag{2.7}$$

for any $j \in (\mathscr{D}, \mathscr{N})$ with either $\omega_j = \pi/2$ or $\omega_j = 3\pi/2$.

Here $\eta_j \in C_c^\infty(\overline{\Omega})$ is a truncation function which depends only on the distance to S_j and such that $\eta_j \equiv 1$ near S_j and vanishes near all sides $\overline{\Gamma}_k$, $k \neq j$.

Theorem 2.0.11. *Let $v \in \mathscr{D}(\triangle_\Omega^{\max})$ be such that*

$$\begin{cases} \triangle v = 0 & in\ \Omega \\ \hat{\gamma}_j^0 v = 0 & j \in \mathscr{D} \\ \hat{\gamma}_i^1 v = 0 & i \in \mathscr{N}\,. \end{cases}$$

and that it fulfill the conditions in Lemma 2.0.10. Then $v \in N$.

Lemma 2.0.12. *Let $v \in N$ then $v \in C^\infty(\bar{\Omega} \backslash V)$ where V is any neighborhood of the corners S_j.*

Denoting by $\mathcal{I} \subset L^2(\Omega)$ the set of function satisfying the conditions appearing in Lemma 2.0.10, we can resume the results above by stating the following

Theorem 2.0.13.

$$\mathscr{K}((\triangle_\Omega^\circ)^*) = \mathscr{K}(\triangle_\Omega^{\max}) \cap \mathscr{K} \cap \mathcal{I}\,,$$

where $\mathscr{K} := \{u \in \mathscr{D}(\triangle_\Omega^{\max}) \,:\, \hat{\gamma}_j^0 u = \hat{\gamma}_i^1 u = 0\,,\ j \in \mathscr{D}\,,\ i \in \mathscr{N}\}$

Moreover, by Theorem 2.3.7 in [3], one has that

Theorem 2.0.14.

$$\dim \mathscr{K}((\triangle_\Omega^\circ)^*) = n_1 + n_2\,. \tag{2.8}$$

More precisely each corner contributes to the dimension of $\mathscr{K}(\triangle_\Omega^{\circ^*})$ as follows:

- *the contribution of a Dirichlet corner (i.e. $j \in \mathscr{D}^2$) is 0 if $\omega_j \leq \pi$ and 1 if $\omega_j > \pi$;*

- *the contribution of a Neumann corner (i.e. $j \in \mathscr{N}^2$) is 0 if $\omega_j \leq \pi$ and 1 if $\omega_j > \pi$;*

- *the contribution of a mixed-corner (i.e. $j \in \mathscr{M}$) is 0 if $\omega_j \leq \pi/2$, 1 if $\pi/2 < \omega \leq 3/2\pi$ and 2 if $\omega_j > 3/2\pi$.*

In order to better characterize the above kernel we introduce some more definitions. We consider function $C_c^2(\Omega)$ depending only on the radial variable as follows: given $R_1 < R_2 < R$,

$$\chi_j = 1\,,\ if\ r < R_1\,,\ \chi_j = 0\,,\ if\ r > R_2\ \forall k \in \mathscr{D} \cup \mathscr{N}\,,$$

For any vertex $S_j \in \partial\Omega$ such that $j \in \mathscr{D} \cup \mathscr{N}$ we consider the disc

$$D_j^R = \{x \in \mathbb{R}^2\ :\ \|x - S_j\| < R\}$$

and define the wedge

$$\begin{aligned} W_j^R &:= \Omega \cap D_j^R \\ &\equiv \{(x,y) \equiv (r\cos\theta_j,\, r\cos\theta_j)\ :\ 0 \leq r \leq R,\, 0 < \theta_j < \omega_j\}\,, \end{aligned}$$

where we choose R in such way that $W_j^R \cap W_h^R = \emptyset$ for $j \neq h$.
On any disk D_j centered at S_j we define the functions $u_{jm}^{\mp}$

$$u_{j1}^{\mp} = \frac{1}{\sqrt{\pi}} r^{\mp\pi/\omega_j} \sin\big(\frac{\pi}{\omega_j}\theta_j\big)\,,\ \ \forall j \in \mathscr{D}^2\,,$$

$$u_{j1}^{\mp} = \frac{1}{\sqrt{\pi}} r^{\mp\pi/\omega_j} \cos\big(\frac{\pi}{\omega_j}\theta_j\big)\,,\ \ \forall j \in \mathscr{N}^2\,,$$

and (here $m = 1, 2$)

$$u_{jm}^{\mp} = C_{jm} r^{\mp\frac{(m-1/2)}{\omega_j}\pi} \sin\left(\frac{(m-1/2)}{\omega_j}\pi\,\theta_j\right)\,,\ \ \forall j \in (\mathscr{D}, \mathscr{N})\,,$$

$$u_{jm}^{\mp} = C_{jm} r^{\mp\frac{(m-1/2)}{\omega_j}\pi} \sin\left(\frac{(m-1/2)}{\omega_j}\,\pi(\omega_j - \theta_j)\right)\,,\ \ \forall j \in (\mathscr{N}, \mathscr{D})\,,$$

with

$$C_{jm} = \sqrt{\frac{2(4-m)}{3\pi}}\,, \qquad m = 1, 2\,. \tag{2.9}$$

With such a choice we have that the functions $\chi_j u_{jm}^{\mp}$ are in $L^2(\Omega)$ and $L^2(\Omega)$-orthogonal.

Lemma 2.0.15. *Let us define*

$$s^0_{km} := \chi_k u^+_{km}\,, \quad s_{km} := \chi_k u^-_{km}\,, \quad \sigma_{km} := s_{km} - (\triangle^F_\Omega)^{-1}\triangle_\Omega s_{km}\,.$$

Then
1)

$$s^0_{km} \in \mathscr{D}(\triangle^F_\Omega)\,, \quad s_{km} \in \mathscr{D}(\triangle^{\max}_\Omega)\,;$$

2) σ_{km} *is the unique function in* $\mathscr{K}(\triangle^{\circ *}_\Omega)$ *such that*

$$\sigma_{km} - s_{km} \in \mathscr{D}(\triangle^F_\Omega)\,;$$

3) the σ_{km}*'s are linearly independent;*
4)

$$\langle \sigma_{hj}\,, -\triangle^F_\Omega s^0_{km}\rangle_{L^2(\Omega)} = \delta_{kh}\delta_{mj}\,.$$

5) The $\triangle^F_\Omega s^0_{km}$*'s are orthogonal and thus linearly independent.*

Proof. 1) follows by noticing that $u^\mp_{km}$ is harmonic near any vertex S_k and $C^\infty(\mathbb{R}^2 \setminus S_k)$.
2) follows by $\mathscr{K}(\triangle^F_\Omega) = \{0\}$ and by Theorem 1.1, noticing that $u^\mp_{km} \in \mathscr{K} \cap \mathcal{I}$.
3) Take the coefficients c_k, $\tilde{c}_k$ such that

$$\sum_{k\in\mathscr{M}_1} c_k\chi_k\sigma_{k1} + \sum_{k\in\mathscr{M}_2} \tilde{c}_k\chi_k\sigma_{k2} = 0\,,$$

then

$$(\triangle^F_\Omega)^{-1}\triangle_\Omega\left(\sum_{k\in\mathscr{M}_1} c_k\chi_k u^-_{k1} + \sum_{k\in\mathscr{M}_2} \tilde{c}_k\chi_k u^-_{k2}\right) =$$

$$= \sum_{k\in\mathscr{M}_1} c_k\chi_k u^-_{k1} + \sum_{k\in\mathscr{M}_2} \tilde{c}_k\chi_k u^-_{k2}\,.$$

This gives $c_k = \tilde{c}_k = 0$ for all k, since the s_{km}'s are linearly independent and do not belong to $\mathscr{D}(\triangle^F_\Omega)$.
4) First we consider the case of Neumann corner in which $k \in \mathscr{N}^2$. Posing $W^2_k := W^{R_2} \setminus W^{R_1}$ one has

$$\langle \sigma_{km}\,, \triangle^F_\Omega s^0_{km}\rangle_{L^2(\Omega)} = \langle \sigma_k\,, \triangle^F_\Omega s^0_k\rangle_{L^2(\Omega)}$$

$$= \langle \chi_k u^-_k\,, \triangle^F_\Omega \chi_k u^+_k\rangle_{L^2(W_k)} - \langle \triangle_\Omega \chi_k u^-_k\,, \chi_k u^+_k\rangle_{L^2(W_k)}$$

$$= \int_{W_k} \chi_k u^-_k\left(\chi''_k u^+_k + \left(1 - \frac{2\pi}{\omega_k}\right)\frac{1}{r}\chi'_k u^+_k\right)dx$$

$$- \int_{W_k} \chi_k u_k^+ \left(\chi_k'' u_k^- + \left(1 + \frac{2\pi}{\omega_k}\right) \frac{1}{r} \chi_k' u_k^- \right) dx$$

$$= \frac{2}{\omega_k} \int_{R_1}^{R_2} 2\chi_k' \chi_k \, dr \int_0^{\omega_k} \cos^2 \left(\frac{\pi}{\omega_k} \theta \right) d\theta = -\frac{2}{\pi} \int_0^{\pi} \cos^2 \theta \, d\theta \; = -1$$

The case $k \in \mathscr{D}^2$ is the same.
Let us consider the case $j \in \mathscr{M}$. We will first study the case $j = m$ and without loss of generality we can omit this index assuming that is equal to 1. Consequently we have

$$\langle \sigma_k \, , \triangle_\Omega^F s_k^0 \rangle_{L^2(\Omega)} = \langle \chi_k u_k^- \, , \triangle_\Omega^F \chi_k u_k^+ \rangle_{L^2(W_k)} - \langle \triangle_\Omega \chi_k u_k^- \, , \chi_k u_k^+ \rangle_{L^2(W_k)} =$$

$$= C_k^2 \int_{W_k} \chi_k u_k^- \left(\chi_k'' u_k^+ + \left(1 + \frac{\pi}{\omega_k}\right) \frac{1}{r} \chi_k' u_k^+ \right) dx +$$

$$-C_k^2 \int_{W_k} \chi_k u_k^+ + \left(\chi_k'' u_k^- + \left(1 - \frac{\pi}{\omega_k}\right) \frac{1}{r} \chi_k' u_k^- \right) dx$$

$$= C_k^2 \frac{\pi}{\omega_k} \int_{R_1}^{R_2} 2\chi_k' \chi_k \, dr \int_0^{\omega_k} \sin^2 \left(\frac{\pi}{2\omega_k} \theta \right) d\theta = -1$$

It remains to show the $L^2(\Omega)$-orthogonality between σ_{km} and $\triangle_\Omega^F s_{kj}^0$. We assume $m \neq j$ and suppose $m = 1$ and $j = 2$. So

$$\langle \sigma_{k1} \, , \triangle_\Omega^F s_{k2}^0 \rangle_{L^2(\Omega)} = \langle \sigma_{k1} \, , \triangle_\Omega^F s_{k2}^0 \rangle_{L^2(W_k^2)}$$
$$= C_{km} C_{kj} \frac{4\pi}{\omega_k} \int_{W_k} \frac{1}{r} u_{k2}^+ \chi_k \chi_k' u_{k1}^- \, dx$$

$$= C_{km} C_{kj} \frac{4\pi}{\omega_k} \int_{R_1}^{R_2} r^{\frac{\pi}{\omega_k}} \chi_k \chi_k' \, dr \int_0^{\omega_k} \sin \left(\frac{3\pi}{2\omega_k} \theta \right) \sin \left(\frac{\pi}{\omega_k} \theta \right) d\theta \, .$$

The orthogonality then follows by the value of the last integral.
5) follows noticing that

$$\langle \triangle_\Omega s_{k1}^0, \triangle_\Omega s_{k2}^0 \rangle_{L^2(\Omega)} \; =$$

$$= \langle \chi_k'' u_{k1}^+ + \chi_k' u_{k1}^+ \frac{1}{r} \, , \chi_k'' u_{k2}^+ + \chi_k' u_{k2}^+ \frac{1}{r} \rangle_{L^2(W_k)}$$

and that in any mixed corner S_k the angular part of each $u_{km}^{\mp}$ is orthogonal to the other ones. $\square$

Now (2.8) can be specified:

Theorem 2.0.16. *For any* $u \in \mathscr{K}((\triangle_\Omega^\circ)^*)$ *there exists unique* $\xi^u \in \mathbb{C}^{n_1+n_2}$, $\xi^u \equiv (\xi_1^u, \ldots, \xi_{n_1}^u, \tilde{\xi}_1^u, \ldots, \tilde{\xi}_{n_2}^u)$ *such that*

$$u = \sum_{j \in \mathscr{M}_1} \xi_j^u \sigma_{j1} + \sum_{j \in \mathscr{M}_2} \tilde{\xi}_j^u \sigma_{j2}\,.$$

By (2.8) and Theorem 2.0.16 one obtains:

Theorem 2.0.17.

$$\mathscr{D}((\triangle_\Omega^\circ)^*) = \{\, u \in \mathscr{D}(\triangle_\Omega^{\max}) \;:\; u \in \mathscr{K}\}$$

$$= \{\, u \in L^2(\Omega) : u = u_0 + \sum_{j \in \mathscr{M}_1} \xi_j^u \sigma_{j1} + \sum_{j \in \mathscr{M}_2} \tilde{\xi}_j^u \sigma_{j2}\,,\ u_0 \in \mathscr{D}(\triangle_\Omega^F)\,,\ \xi^u \in \mathbb{C}^{n_1+n_2}\,\}$$

$$= \{u \in L^2(\Omega) : u = u_0 + \sum_{j \in \mathscr{M}_1} \xi_j^u s_{j1} + \sum_{j \in \mathscr{M}_2} \tilde{\xi}_j^u s_{k2}\,,\ u_0 \in \mathscr{D}(\triangle_\Omega^F)\,,\ \xi^u \in \mathbb{C}^{n_1+n_2}\,\}\,.$$

Proof. 1) For the first equality it's sufficient to prove $\mathscr{D}(\triangle_\Omega^{\circ *}) \subseteq \mathscr{K}$. For any $v \in \mathscr{D}(\triangle_\Omega^{\circ *})$, remembering that $H^2(\Omega)$ is dense in $\mathscr{D}(\triangle_\Omega^{\circ *})$ let us take the sequence $\{v_n\} \subset \mathscr{D}(\triangle_\Omega^{\circ *}) \cap H^2(\Omega)$ such that $v_n \to v$.
By the Green Formula we have:

$$0 = \int_\Omega \triangle u\, v_n\, dx \;-\; \int_\Omega u\, \triangle v_n\, dx$$

$$= \sum_{j \in \mathscr{N}} \int_{\Gamma_j} \hat{\gamma}_j^0 u\, \gamma_j^1 v_n\, dx \;-\sum_{j \in \mathscr{D}} \int_{\Gamma_j} \gamma_j^0 v_n\, \hat{\gamma}_j^1 u\, dx \qquad \forall n\,.$$

Since the last equality holds for every $u \in \mathscr{D}(\triangle_\Omega^\circ)$, this means that $v_n \in \mathscr{K}$, for every n. Then by continuity of trace operators one obtains

$$\hat{\gamma}_i^1 v = \lim_{n\to\infty} \gamma_i^1 v_n = 0 = \lim_{n\to\infty} \gamma_j^0 v_n = \hat{\gamma}_j^0 v \quad \forall j \in \mathscr{D}\,,\ i \in \mathscr{N}$$

and so $v \in \mathscr{K}$. For any $v \in \mathscr{D}(\triangle_\Omega^{\max}) \cap \mathscr{K}$ let us define $v_0 := (\triangle_\Omega^F)^{-1} \triangle_\Omega v$. Noticing that $v - v_0 \in \mathscr{K}(\triangle_\Omega^{\circ *})$, the converse inclusion is consequence of Theorem 2.0.16.
2) For the second identity let us again define $v_0 := (\triangle_\Omega^F)^{-1} \triangle_\Omega v$, for any $v \in \mathscr{D}(\triangle_\Omega^{\circ *})$. Since $v - v_0 \in \mathscr{K}(\triangle_\Omega^{\circ *})$, by Theorem 2.0.16 one has $\mathscr{D}(\triangle_\Omega^{\circ *}) \subseteq \mathscr{D}(\triangle_\Omega^F) + \mathscr{K}(\triangle_\Omega^{\circ *})$. The reverse inclusion follows from $\triangle_\Omega^\circ \subset \triangle_\Omega^F$ which gives $\mathscr{D}(\triangle_\Omega^F) \subset \mathscr{D}(\triangle_\Omega^{\circ *})$ proving the the statement.
3)Finally the last identity follows from point 2 in Lemma 2.0.15. $\square$

The next theorem gives a decomposition for $\mathscr{D}(\triangle_\Omega^F)$:

Theorem 2.0.18.

$$\mathscr{D}(\triangle_\Omega^F) = \{u \in L^2(\Omega)\,,\,:\ u = u_\circ + \sum_{k\in\mathscr{M}_1} \zeta_k^u s_{k1}^0 + \sum_{k\in\mathscr{M}_2} \tilde{\zeta}_k^u s_{k2}^0$$

$$u_\circ \in \mathscr{D}(\triangle_\Omega^\circ)\,,\ \zeta^u \equiv (\zeta_1^u, \dots, \zeta_{n_1}^u, \tilde{\zeta}_1^u, \dots, \zeta_{n_2}^u) \in \mathbb{C}^{n_1+n_2}\,\}\,.$$

Proof. Given $\mathscr{D}(\triangle_\Omega^F)$, since the $\triangle_\Omega^F s_{km}^0$'s are linearly independent by 5) in Lemma 2.0.15 and not orthogonal to $\mathscr{K}(\triangle_\Omega^{\circ *})$ by 4) in the same lemma, the decomposition $L^2(\Omega) = \mathscr{R}(\triangle_\Omega^\circ) \oplus \mathscr{K}(\triangle_\Omega^{\circ *})$ implies that there exists unique $u_\circ \in \mathscr{D}(\triangle_\Omega^\circ)$ such that

$$\triangle_\Omega^F u = \sum_{k\in\mathscr{M}_1} \zeta_k \triangle_\Omega^F s_{k1}^0 + \sum_{k\in\mathscr{M}_2} \tilde{\zeta}_k s_{k2}^0\,,$$

and the proof is done. □

Corollary 2.0.19.

$$\mathscr{D}(\triangle_\Omega^{\circ *}) = \{u \in L^2(\Omega)\,,\,:\ u = u_\circ + \sum_{k\in\mathscr{M}_1}^{n} (\zeta_k^u s_{k1}^0 + \xi_k^u s_{k1}) + \sum_{k\in\mathscr{M}_2} (\tilde{\zeta}_k^u s_{k2}^0 + \tilde{\xi}_k^u s_{k2})\,,$$

$$u_\circ \in \mathscr{D}(\triangle_\Omega^\circ)\,,\ \zeta^u\,, \xi^u \in \mathbb{C}^{n_1+n_2}\,\}\,.$$

Lemma 2.0.20. *The linear map*

$$\tau_\Omega : \mathscr{D}(\triangle_\Omega^F) \to \mathbb{C}^{n_1+n_2} \qquad \tau_\Omega u := \zeta^u$$

is well defined, surjective and continuous.

Proof. By Theorem 2.0.19 $\mathscr{D}(\triangle_\Omega^F) = \mathscr{D}(\triangle_\Omega^\circ) + \mathscr{V}_+$ where $\mathscr{V}_+$ is the (n_1+n_2)-dimensional vector space generated by s_{km}^0. Since $\mathscr{V}_+$ is closed and $\mathscr{D}(\triangle_\Omega^\circ) \cap \mathscr{V}_+ = 0$, τ_Ω is the composition of the continuous projection $P : \mathscr{D}(\triangle_\Omega^F) \to \mathscr{V}_+$ with the continuous map which identifies $\mathscr{V}_+$ with $\mathbb{C}^{n_1+n_2}$. □

Next results provide an alternative definition of τ_Ω:

Lemma 2.0.21. *For all $k \in \mathscr{M}_1 \backslash \mathscr{N}^2$ there exists constants*[4] *C_{kl} such that*

$$\xi_k^u = \lim_{R\downarrow 0} \frac{C_{kl}}{R^{2-\pi/\omega_k\beta}} \langle \chi_k^R,\, u\rangle_{L^2(\Omega)}\,, \quad u \in \mathscr{D}((\Delta_\Omega^\circ)^*)\,,\ k \in \mathscr{M}_l\,,$$

where χ_k^R is the characteristic function of the wedge W_k^R and $\beta = 1$ if $k \in \mathscr{D}^2$, $\beta = 1/2$ if $k \in \mathscr{M}$ and $\omega_k < \pi$ and $\beta = 3/2$ otherwise.
In the case $k \in \mathscr{M}_1 \cap \mathscr{N}^2$ similar results hold with u and χ_k^R replaced respectively by $\partial^2 u/\partial r^2$ and the characteristic function of the half wedge (i.e. $0 < \theta < \omega/2$).

[4] Such constants can be made explicit: see the proof below.

Proof. Let us define the constants C_{kl} by

$$\int_{W_k^R} u_{kl}^{\pm}(r,\theta) r dr d\theta = \frac{R^{2\pm\pi/\omega_k}}{C_{kl}}\,, \quad k \in \mathscr{D}^2 \cup \mathscr{M}\,,$$

and

$$\int_{W_k^R} \partial_r^2 u_{kl}^{\pm}(r,\theta) r dr d\theta = \frac{R^{\pm\pi/\omega_k}}{C_{kl}}\,, \quad k \in \mathscr{N}^2\,.$$

The proof of is then concluded noticing that the last integral is equivalent to the first, in the case $k \in \mathscr{D}^2$, and by

$$\int_0^R \int_0^{\omega_k} |u_0(r,\theta)| r dr d\theta \le \frac{\sqrt{\omega_k}}{2} R^2 \left(\int_0^R \int_0^{\omega_k} |u_0(r,\theta)|^2 \frac{dr d\theta}{r} \right)^{1/2},$$

for all $u_0 \in \mathscr{D}(\triangle_\Omega^F)$, being the last integral finite since $u_0 \in H_0^1(\Omega)$ (see e.g. [3], Theorem 1.2.15).

□

By the previous theorem and the above lemma there follows that $\mathscr{D}(\triangle_\Omega^\circ) = \mathscr{K}(\tau_\Omega)$. Thus we can write all self-adjoint extension of $\triangle_\Omega^\circ$, together their resolvent, by using general theory of Krein's resolvent formula as provided in Chapter 1, Section 2. To this end we give the following

Lemma 2.0.22. *If*

$$G_0 : \mathbb{C}^{n_1+n_2} \to L^2(\Omega)\,, \quad G_0 := -(\tau_\Omega(\triangle_\Omega^F)^{-1})^*$$

then, posing $\xi \equiv (\xi_1, \dots \xi_{n_1}, \tilde{\xi}_1, \dots, \tilde{\xi}_{n_2})$,

$$G_0 \xi = \sum_{k \in \mathscr{M}_1} \xi_k \sigma_{k1} + \sum_{k \in \mathscr{M}_2} \tilde{\xi}_k \sigma_{k2}\,.$$

Proof. By Lemma 2.0.15 one has

$$\langle \sigma_{km}\,, \triangle_\Omega^F u \rangle_{L^2(\Omega)} = \langle \triangle_\Omega^{\circ *} \sigma_{km}\,, u_\circ \rangle_{L^2(\Omega)} + \delta_{1m} \sum_{j \in \mathscr{M}_1} \zeta_j^u \langle \sigma_{km}\,, \triangle_\Omega^F s_{j1}^0 \rangle_{L^2(\Omega)} +$$

$$+ \delta_{2m} \sum_{j \in \mathscr{M}_2} \tilde{\zeta}_j^u \langle \sigma_{km}\,, \triangle_\Omega^F s_{j2}^0 \rangle_{L^2(\Omega)} = -(\delta_{1m} \zeta_k^u + \delta_{2m} \tilde{\zeta}_k^u)\,.$$

Thus, by the definition of τ_Ω,

$$\langle G_0 \xi\,, u \rangle_{L^2(\Omega)} = -\langle \xi\,, \tau_\Omega (\triangle_\Omega^F)^{-1} u \rangle_{L^2(\Omega)}$$

$$= \sum_{k \in \mathscr{M}_1} \xi_k \langle \sigma_{k1}\,, u \rangle_{L^2(\Omega)} + \sum_{k \in \mathscr{M}_2} \tilde{\xi}_k \langle \sigma_{k2}\,, u \rangle_{L^2(\Omega)}\,.$$

□

By (1.7) and (1.11) one has then

$$\begin{aligned}G_z\xi =&(1 - zR_z)G_0\xi\\ =&\sum_{k\in\mathscr{M}_1} \xi_k(\sigma_{k1} - z(-\Delta_\Omega^F + z)^{-1}\sigma_{k1}) + \sum_{k\in\mathscr{M}_2} \tilde\xi_k(\sigma_{k2} - z(-\Delta_\Omega^F + z)^{-1}\sigma_{k2})\\ =&\sum_{k\in\mathscr{M}_1} \xi_k(s_{k1} - (-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)s_{k1})\\ &+\sum_{k\in\mathscr{M}_2} \tilde\xi_k(s_{k2} - (-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)s_{k2})\end{aligned}$$

and

$$\Gamma_z : \mathbb{C}^{n_1+n_2} \to \mathbb{C}^{n_1+n_2}\,, \quad (\Gamma_z)_{ik} = z(G_0^* G_z)_{ik}\,,$$

has the block decomposition

$$\Gamma_z^{lm} : \mathbb{C}^{n_l} \to \mathbb{C}^{n_m}\,, \quad l = 1, 2\,,\ m = 1, 2\,,$$

where Γ_z^{lm} is represented by the matrix

$$(\Gamma_z^{lm})_{ik} = z\langle\sigma_{il}, \sigma_{km} - z(-\Delta_\Omega^F + z)^{-1}\sigma_{km}\rangle_{L^2(\Omega)}\,, \quad i \in \mathscr{M}_l\,,\ k \in \mathscr{M}_m\,.$$

In conclusion, by the results provided in Section 1.2 one then obtains the following

Theorem 2.0.23. *Any self-adjoint extension of* Δ_Ω° *is of the kind*

$$\Delta_\Omega^{\Pi,\Theta} : \mathscr{D}(\Delta_\Omega^{\Pi,\Theta}) \subset L^2(\Omega) \to L^2(\Omega)\,, \quad \Delta_\Omega^{\Pi,\Theta}u := \Delta_\Omega u\,,$$

$$\mathscr{D}(\Delta_\Omega^{\Pi,\Theta}) := \{u \in \mathscr{D}((\Delta_\Omega^\circ)^*)\ :\ \xi^u \in \mathbb{C}_\Pi^{n_1+n_2}\,,\ \Pi\zeta^u = \Theta\xi^u\}\,,$$

where $(\Pi, \Theta) \in \mathsf{E}(\mathbb{C}^{n_1+n_2})$*. Moreover*

$$(-\Delta_\Omega^{\Pi,\Theta} + z)^{-1} = (-\Delta_\Omega^F + z)^{-1} + G_z\Pi\,(\Theta + \Pi\,\Gamma_z\Pi)^{-1}\Pi\,G_{\bar z}^*\,.$$

The quadratic form corresponding to $-\Delta_\Omega^{\Pi,\Theta}$ *is given by*

$$F_\Omega^{\Pi,\Theta} : \mathscr{D}(F_\Omega^{\Pi,\Theta}) \times \mathscr{D}(F_\Omega^{\Pi,\Theta}) \subset L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}\,,$$

$$F_\Omega^{\Pi,\Theta}(u, v) = \langle\nabla u_0, \nabla v_0\rangle_{L^2(\Omega)} + f_\Theta(\xi^u, \xi^v)\,,$$

$$\mathscr{D}(F_\Omega^{\Pi,\Theta}) = \{u \in L^2(\Omega)\ :\ u = u_0 + \sum_{k\in\mathscr{M}_1} \xi_k^u\,\sigma_{k1} + \sum_{k\in\mathscr{M}_2} \tilde\xi_k^u\,\sigma_{k2}\,,$$

$$u_0 \in V(\Omega)\,,\ \xi^u \in \mathbb{C}_\Pi^{n_1+n_2}\}\,,$$

where f_Θ *is the quadratic form corresponding to* Θ*.*

Chapter 3

Symmetric Laplacians with Mixed Robin Boundary Conditions in a Polygon

In this chapter we study the qualitative properties of the solutions of the Laplace equations under mixed Robin boundary conditions on a polygon Ω. Let we start defining the set of indices j characterized by Robin boundary conditions:

- $j \in \mathscr{R}$ if Γ_j supports a Robin boundary condition;
- $\mathscr{R}^2 := \{j \,:\, j \in \mathscr{R}\,, j+1 \in \mathscr{R}\}$;
- $(\mathscr{D}, \mathscr{R}) := \{j \,:\, j \in \mathscr{D}\,, j+1 \in \mathscr{R}\}$;
- $(\mathscr{R}, \mathscr{D}) := \{j \,:\, j \in \mathscr{R}\,, j \in \mathscr{D}\}$;
- $(\mathscr{N}, \mathscr{R}) := \{j \,:\, j \in \mathscr{N}\,, j+1 \in \mathscr{R}\}$;
- $(\mathscr{R}, \mathscr{N}) := \{j \,:\, j \in \mathscr{R}\,, j \in \mathscr{N}\}$.

Now introduce the variational solution u of the Laplace equation

$$\triangle u = f \quad on\, \Omega\,, \tag{3.1}$$

with boundary conditions

$$\hat{\gamma}_j^0 v = 0\,, \quad j \in \mathscr{D}\,, \qquad \hat{\gamma}_j^1 v = 0\,, \quad j \in \mathscr{N}\,,$$

$$\hat{\gamma}_j^0 u + \alpha_j \hat{\gamma}_j^1 u = 0\,, \quad j \in \mathscr{R}\,, \tag{3.2}$$

where f is a given function and $\alpha_j > 0$.

According to (2.1), the Poincaré inequality holds and the *space of variational solutions* $V(\Omega)$, defined in 2.2, is an Hilbert space for the scalar product

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx + \sum_{j\in\mathscr{R}} \frac{1}{\alpha_j} \int_{\Gamma_j} \gamma_j^0 u \gamma_j^0 v \, d\sigma \,. \tag{3.3}$$

Therefore, assuming $f \in L^2(\Omega)$, the mapping

$$v \mapsto \int_\Omega f v dx \,,$$

is a continuous linear form on $V(\Omega)$ and by Riesz Theorem there exists a unique $u \in V(\Omega)$ such that

$$a(u,v) = -\int_\Omega f v dx \,, \tag{3.4}$$

for every $v \in V(\Omega)$. We want now to see in what sense u solves the mixed Robin problem introduced above.
According to (3.4) if $v \in \mathcal{C}_c^\infty(\bar{\Omega})$ one has that (3.1) holds in the sense of distributions. Moreover, being $u \in V(\Omega)$ one as automatically that $\gamma_j^0 u = 0$ for any $j \in \mathscr{D}$.
Then by Theorem 1.1.14 since $u \in \mathscr{D}(\triangle_\Omega^{max})$, for very $v \in V(\Omega)$ such that $\hat{\gamma}_j^0 v \in \tilde{H}^{1/2}(\Gamma_j)$ one has

$$\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega f v dx + \sum_j \langle \hat{\gamma}_j^1 u ,\, \hat{\gamma}_j^0 v \rangle \,,$$

and consequently

$$\sum_{j\in\mathscr{N}} \langle \hat{\gamma}_j^1 u ,\, \hat{\gamma}_j^0 v \rangle + \sum_{j\in\mathscr{R}} \langle \hat{\gamma}_j^1 u ,\, \hat{\gamma}_j^0 v \rangle + \sum_{j\in\mathscr{R}} \frac{1}{\alpha_j} \langle \hat{\gamma}_j^0 u ,\, \hat{\gamma}_j^0 v \rangle = 0 \,.$$

So that if $j \in \mathscr{N}$ one has that, being $\hat{\gamma}_j^0 v$ an arbitrary function, it must be

$$\hat{\gamma}_j^1 u = 0 \,, \quad j \in \mathscr{N}$$

and

$$\hat{\gamma}_j^0 u + \alpha_j \hat{\gamma}_j^1 u = 0 \,, \quad j \in \mathscr{R} \,.$$

A converse statement is also true[1]:

[1] The proof follows by contradiction and by [3], Theorem 2.1.1.

Theorem 3.0.24. *Assume that Ω is a bounded polygonal open subset of $\mathbb{R}^2$ and that $u \in H^1(\Omega)$ solves (in distributional sense) the equation $\Delta_\Omega u = f$ together with the boundary conditions $\gamma_j^0 u = 0$ if $j \in \mathscr{D}$, $\hat{\gamma}_j^1 u = 0$ if $j \in \mathscr{N}$ and $\hat{\gamma}_j^0 + \alpha_j \hat{\gamma}_j^1 u = 0$ if $j \in \mathscr{R}$. Then $u \in V(\Omega)$ and is the unique solution of*

$$\int_\Omega \nabla u \nabla v dx + \sum_{j \in \mathscr{R}} \frac{1}{\alpha_j} \int_{\Gamma_j} \hat{\gamma}_j^0 u \hat{\gamma}_j^0 v d\sigma = - \int_\Omega f v dx \,,$$

for every $v \in V(\Omega)$.

Now let us consider the space of strong solutions

$$W^2(\Omega) :=$$

$$\{u \in H^2(\Omega) \,:\, \gamma_j^0 u = 0 \,,\ j \in \mathscr{D} \,,\ \gamma_j^1 u = 0 \,,\ j \in \mathscr{N} \,,\ \gamma_j^0 + \alpha_j \gamma_j^1 u = 0 \,,\ j \in \mathscr{R}\} \,.$$

Then

Theorem 3.0.25. *The space $W^m(\Omega) := W^2(\Omega) \cap H^m(\Omega)$ is dense in $W^2(\Omega)$, $\forall m \geq 2$.*

Proof. Let us suppose that there exist $u_0 \in W^2(\Omega)$ such that

$$\forall v \in W^m(\Omega) \,, \quad \|u_0 - v\|_{H^2(\Omega)} > \delta > 0 \,, \tag{3.5}$$

and for simplicity let us assume, without loss of generality, that exists an unique index $k \in \mathscr{R}$. Now let us consider the decomposition

$$u_0 = u_0^0 + u_0^1 \in V^{0,2}(\Omega) \oplus V^{1,2}(\Omega) \,,$$

where we defined, for $i = 0, 1$,

$$V^{i,2}(\Omega) = \{u \in H^2(\Omega) : \ \hat{\gamma}_j^0 u = \hat{\gamma}_i^1 u = 0 \,, \forall j \in \mathscr{D} \,, i \in \mathscr{N}\} \,.$$

By Theorem 2.0.7 there exist two sequences $\{v_{0,n}\} \subset V^{0,m}(\Omega)$ and $\{v_{1,n}\} \subset V^{1,m}(\Omega)$ such that

$$\lim_{n \to +\infty} \|u_0 - v_n\|_{H^2(\Omega)} = 0 \,, \quad v_n := v_{0,n} + v_{1,n} \,.$$

Obviously $v_n \notin W^m(\Omega)$, but, by the continuity of the trace maps $\hat{\gamma}_k^0$ and $\hat{\gamma}_k^1$, for any $\epsilon > 0$ the sequence $\{v_n\}$ definitively belongs to

$$W^{m,\epsilon}(\Omega) :=$$

$$\{u \in H^m(\Omega) \,: \hat{\gamma}_j^0 u = \hat{\gamma}_i^1 u = 0 \,, |\hat{\gamma}_k^0 u + \alpha_k \hat{\gamma}_k^1 u| \leq \epsilon \,,\ \forall j \in \mathscr{D} \,, i \in \mathscr{N} \,, k \in \mathscr{R}\} \,.$$

Since

$$\bigcap_{\epsilon>0} W^{m,\epsilon}(\Omega) = W^m(\Omega)\,,$$

there exists $v \in W^m(\Omega)$ such that

$$\|v - v_n\|_{H^2(\Omega)} \leq \delta/2\,,$$

for any n sufficiently large. Hence in conclusion there exists n sufficiently large such that

$$\|u_0 - v\|_{H^2(\Omega)} \leq \|u_0 - v_n\|_{H^2(\Omega)} + \|v_n - v\|_{H^2(\Omega)} \leq \delta\,,$$

inconsistent with (3.5) □

Now we define

$$\triangle_\Omega^\circ : W^2(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)\,, \quad \triangle_\Omega^\circ u := \Delta_\Omega u$$

and we look for a generalization of Theorem 2.0.8.

Let us first consider the following inequality:[2]

$\forall \epsilon > 0\ \exists K_\epsilon > 0$ such that $\forall u \in H^1(\Omega)$

$$\sum_j \int_{\Gamma_j} |\gamma_j^0 u|^2 d\sigma \leq \epsilon \int_\Omega \|\nabla u\|^2 dx + K_\epsilon \int_\Omega |u|^2 dx\,. \tag{3.6}$$

Then we have the following

Lemma 3.0.26. *If $u \in W^2(\Omega)$ then then there exists $C_\Omega > 0$ such that*

$$\int_\Omega \Big(|\partial^2_{xx} u|^2 + 2|\partial^2_{xy} u|^2 + |\partial^2_{yy} u|^2 \Big) dx \leq C_\Omega \Big(\int_\Omega |\triangle u|^2 dx + \int_\Omega |\nabla u|^2 dx \Big)$$

Proof. It is sufficient to estimate $\int_\Omega 2\partial^2_{xy} u \partial^2_{xy} u dx$. By integrating by parts, one has

$$\int_\Omega 2\partial^2_{xy} u \partial^2_{xy} u dx = 2\int_\Omega \partial^2_{xx} u \partial^2_{yy} u dx + \sum_j 2 \int_{\Gamma_j} \big(\partial_x u \partial^2_{xy} u n_y - \partial_x u \partial^2_{yy} u n_x dx \big) \tag{3.7}$$

where n_x and n_y denotes the components of the normal vector n_j to Γ_j. If $j \in \mathscr{N}$ or $j \in \mathscr{D}$ the last term is zero by Lemma 2.2.2 of [3].

Then let us suppose that $j \in \mathscr{R}$. Without loss of generality we can chose a

[2]See [5], equation 2.25 page 49.

system of coordinates x, y such that Γ_j is represented by equation $x = 0$. Then

$$\frac{\partial u}{\partial x}(0, y) = \alpha_j u(0, y), \tag{3.8}$$

and on Γ_j one has

$$\frac{\partial^2 u}{\partial x \partial y}(0, y) = \alpha_j \frac{\partial u}{\partial y}(0, y).$$

Substituting this identity in the last integral of (3.7) (noticing that according to the chosen coordinates system $n_x = 0$ and $n_y = 1$) one obtains

$$2\int_{\Gamma_j} \left(\partial_x u \partial^2_{xy} u n_y - \partial_x u \partial^2_{yy} u n_x dx\right) = 2\alpha_j \int_{\Gamma_j} \partial_x u \partial_y u d\sigma .$$

Then (3.6) implies that

$$2\int_{\Gamma_j} \partial_x u \partial_y u d\sigma \leq \int_{\Gamma_j} (\partial_x u)^2 (\partial_y u)^2 d\sigma$$

$$= \int_{\Gamma_j} \|\nabla u\|^2_{L^2(\Omega)} d\sigma \ \leq \epsilon \int_\Omega (\partial^2_{xx} u)^2 + 2(\partial^2_{xy} u)^2 + (\partial^2_{yy} u)^2 dx$$

$$+K_\epsilon \int_\Omega \|\nabla u\|^2_{L^2(\Omega)} dx .$$

This implies that

$$\int_\Omega 2(\partial^2_{xy} u)^2 dx \leq 2\int_\Omega \partial^2_{xx} u \partial^2_{yy} u dx$$

$$+\left(\sum_{j\in\mathscr{R}} \alpha_j\right)\left(\epsilon\int_\Omega (\partial^2_{xx} u)^2 + 2(\partial^2_{xy} u)^2 + (\partial^2_{yy} u)^2 dx + K_\epsilon \int_\Omega \|\nabla u\|^2_{L^2(\Omega)} dx\right).$$

Now, since that

$$2\int_\Omega \partial^2_{xx} u \partial^2_{yy} u dx \leq \int_\Omega (\partial^2_{xx} u)^2 + (\partial^2_{yy} u)^2 dx ,$$

choosing ϵ sufficiently small in order to have

$$\epsilon \sum_{j\in\mathscr{R}} \alpha_j < 1$$

one has

$$\int_\Omega 2(\partial^2_{xy} u)^2 dx \leq$$

$$\frac{\epsilon \sum_{j\in\mathscr{R}} \alpha_j + 1}{1 - \epsilon \sum_{j\in\mathscr{R}} \alpha_j} \int_\Omega (\triangle u)^2 dx + \frac{K_\epsilon \epsilon \sum_{j\in\mathscr{R}} \alpha_j}{1 - \epsilon \sum_{j\in\mathscr{R}} \alpha_j} \int_\Omega \|\nabla u\|^2_{L^2(\Omega)} dx .$$

□

Now by (1.1.5) $\forall\epsilon > 0$ there exists $c > 0$ such that

$$\|\nabla u\|_{L^2(\Omega)} < \epsilon \|u\|_{H^2(\Omega)} + \frac{c}{\epsilon} \|u\|_{L^2(\Omega)} .$$

Since, by our hypothesis $\alpha_j > 0$ for all $j \in \mathscr{R}$, there follows that $\langle -\Delta_\Omega u, u\rangle_{L^2(\Omega)} \geq \lambda \|u\|^2_{L^2(\Omega)}$ with $\lambda > 0$, applying Lemma 3.0.26 one gets the following

Theorem 3.0.27. *If $u \in W^2(\Omega)$ then $\exists C_\Omega > 0$ such that*

$$\|u\|_{H^2(\Omega)} \leq C_\Omega \|\triangle_\Omega u\|_{L^2(\Omega)} .$$

Now, as in Chapter 2, we look for a characterization of $N := \mathscr{K}((\Delta^\circ_\Omega)^*)$. The first step is the following.

Lemma 3.0.28. *Let $v \in N$, then v belongs to $\mathscr{D}(\triangle^{max}_\Omega)$ and is solution of the following (adjoint) boundary value problem*

$$\begin{cases} \triangle v = 0 & in\, \Omega \\ \hat{\gamma}^0_j v = 0\,, & j \in \mathscr{D}\,, \\ \hat{\gamma}^1_j v = 0\,, & j \in \mathscr{N}\,, \\ \hat{\gamma}^0_j v + \alpha_j \hat{\gamma}^1_j v = 0 & j \in \mathscr{R}\,. \end{cases}$$

Proof. According to the definition of N any $v \in N$ is a square integrable function in Ω such that

$$\int_\Omega v \triangle u dx = 0\,, \quad \forall u \in W^2(\Omega)\,.$$

In particular this is true for every $u \in \mathcal{C}^\infty_c(\Omega)$ and consequently v is harmonic. This implies that $v \in \mathscr{D}(\triangle^{max}_\Omega)$ and it remains to check the boundary conditions.
By Theorem 1.1.11 we know that given any $\varphi \in C^\infty_c(\Gamma_j)$ for $j \in \mathscr{D}$, and $\psi_j \in C^\infty_c(\Gamma_j)$ for $j \in N$ there exists $u \in H^2(\Omega)$ such that

$$\hat{\gamma}^0_j u = \varphi\,, \quad \hat{\gamma}^1_j u = 0\,, \quad j \in \mathscr{N}\,,$$

$$\hat{\gamma}^0_j u = 0\,, \quad \hat{\gamma}^1_j u = \psi_j \quad j \in \mathscr{D}\,.$$

It remains to control the case $j \in \mathscr{R}$. Then fixed φ_j, ψ_j and $\phi^{1,2}_k$ in $\mathcal{C}^\infty_c(\Omega)$ we know that $\exists u_1 \in H^2(\Omega)$ such that

$$i)\, \hat{\gamma}^0_j u_1 = 0\,, \quad \hat{\gamma}^1_j u_1 = \varphi/2 \quad j \in \mathscr{D}\,,$$

$$ii)\, \hat{\gamma}^0_j u_1 = \psi/2\,, \quad \hat{\gamma}^1_j u_1 = 0\,, \quad j \in \mathscr{N}\,,$$

and for the mixed conditions

$$iii)\,\hat{\gamma}^0_k u_1 = \phi^1_k\,,\ \ \hat{\gamma}^1_k u_1 = 0\,.$$

In the same way one can find $u_1 \in H^2(\Omega)$ such that $i)\,,ii)$ hold and

$$iv)\,\hat{\gamma}^0_k u_2 = 0\,,\ \ \hat{\gamma}^1_k u_2 = \alpha_k\phi^2_k\,.$$

Then one conclude the case $k \in \mathscr{R}$ putting $u = u_1 + u_2$. This allows one to apply the Green Formula

$$\int_\Omega u\triangle v dx - \int_\Omega v\triangle u = \sum_j (\langle\hat{\gamma}^0_j u\,,\hat{\gamma}^1_j v\rangle - \langle\hat{\gamma}^0_j v\,,\hat{\gamma}^1_j u\rangle)\,.$$

Since v is harmonic and $u \in W^2(\Omega)$ the integrals on Ω vanishes and taking into account the boundary conditions on u we have

$$0 = \sum_{j\in\mathscr{N}}\langle\varphi_j\,,\hat{\gamma}^1_j v\rangle - \sum_{j\in\mathscr{D}}\langle\hat{\gamma}^0_j v\,,\psi_j\rangle + \sum_{k\in\mathscr{R}}(\langle\phi^1_k\,,\hat{\gamma}^1_k v\rangle + \langle\hat{\gamma}^0_k v\,,\alpha_k\phi^2_k\rangle)$$

and posing $\phi^2_k(x) \equiv -\phi^1_k(x)$ on Γ_k, one has

$$\sum_{j\in\mathscr{N}}\langle\varphi_j\,,\hat{\gamma}^1_j v\rangle - \sum_{j\in\mathscr{D}}\langle\hat{\gamma}^0_j v\,,\psi_j\rangle + \sum_{k\in\mathscr{R}}\langle\phi^1_k\,,\hat{\gamma}^0_j v + \alpha_j\hat{\gamma}^1_j v\rangle\ = 0\,,$$

for any $\psi_j \in C^\infty_c(\Gamma_j)$, $j \in \mathscr{N}$, $\varphi_j \in C^\infty_c(\Gamma_j)$, $j \in \mathscr{D}$ and $\phi^1_k \in C^\infty_c(\Gamma_k)$, $k \in \mathscr{R}$. Then

$$\hat{\gamma}^1_j v = 0\,,\quad for\, j \in \mathscr{N}\ \ \hat{\gamma}^0_j v = 0\,,\quad j \in \mathscr{D}\,.$$

and

$$\hat{\gamma}^0_j v + \alpha_j\hat{\gamma}^1_j v\,,\quad k \in \mathscr{R}\,.$$

□

Lemma 3.0.28 shows that $v \in N$ is a weak solution of the homogeneous adjoint problem, however it does not characterize completely N.
To this purpose we need to the following results:

Lemma 3.0.29. *Every $v \in N$ is such that*

$$\int_\Omega v\triangle\,\eta_j dx = 0\,, \tag{3.9}$$

for any $j \in \mathscr{N}^2$,

$$\int_\Omega v\triangle(y_j\eta_j)dx = 0\,, \tag{3.10}$$

for any $j \in (\mathscr{N}, \mathscr{D})$ *with* $\omega_j = 3/2\pi$, *or* $\pi/2$,

$$\int_\Omega v \triangle (x_j \eta_j) dx = 0\,, \tag{3.11}$$

for any $j \in (\mathscr{D}, \mathscr{N})$ *with* $\omega_j = 3\pi/2$, *or* $\pi/2$, *and* $\eta_j \in C_c^\infty(\overline{\Omega})$ *is the truncation function which depends only on the distance to* S_j *and such that* $\eta_j \equiv 1$ *near* S_j *and vanishes near all* $\overline{\Gamma}_k$.
Now we have to consider the case $j \in \mathscr{R}$. *Taking* $0 < R_1 < R_2$ *such that for any* j

$$i)\, \eta_j(r) \equiv 1\,, \ on\, B_{R_1}(S_j) \cap \Omega\,,$$

$$ii)\, supp\{\eta_j\} \subset B_{R_2}(S_j) \cap \Omega\,,$$

and $\chi_j^{R_2}(r)$ *the characteristic function of the set* $B_{R_2}(S_j) \cap \Omega$, *then for any couple* $(\omega_1\,, \omega_2)$ *such that* $0 < \omega_1 < \omega_2 < \omega_j$, $\forall j$, *we have that any* $v \in N$ *is such that for any* $u \in W^2(\Omega)$ *one has*

$$\int_\Omega v \triangle \Phi^{\mathscr{R}}_{j,\,u} dx = 0\,, \tag{3.12}$$

for any $j \in \mathscr{R}$, *where*

$$\begin{aligned}\Phi^{\mathscr{R}}_{j,\,u}(r,\,\theta) = \big[&\beta_1^j\, \theta^2(\theta-\omega_j)^2(\theta-\omega_2)(r^2 - \partial_\theta u(r,\omega_1)) \\ &+\, \beta_2^j\, \theta^2(\theta-\omega_j)^2(\theta-\omega_1)^2\, r\partial_r u(S_j)\,\big]\eta_j \cdot \chi_j^{R_2}(r)\,,\end{aligned} \tag{3.13}$$

with

$$\beta_1^j = \big[(\omega_1-\omega_j)^2(\omega_1-\omega_2)\omega_1^2\big]^{-1}\, and\ \beta_2^j = \big[(\omega_2-\omega_j)^2(\omega_2-\omega_1)\omega_2^2\big]^{-1}.$$

Proof. The proof of (3.9), (3.10) and (3.11) is given by Lemma 2.0.10, (3.12) follows noticing that $\Phi^{\mathscr{R}}_{j,\,u}(r,\theta) \in W^2(\Omega)$ by its definition. □

Theorem 3.0.30. *Let* $v \in \triangle_\Omega^{max} \cap W^2(\Omega)$ *such that* v *is harmonic in* Ω *and assume that* v *satisfyies the conditions of Lemma 3.0.29, then* $v \in N$.

Proof. The case $\mathscr{R} = \emptyset$ is given by Theorem 2.0.11 and therefore we can suppose $j \in \mathscr{R}$. Our thesis consist to show that

$$\int_\Omega v \triangle u dx = 0 \tag{3.14}$$

for every $u \in W^2(\Omega)$.
By Theorem 3.0.25 it suffices to consider the case $u \in W^2(\Omega) \cap H^4(\Omega) \subset C^2(\bar{\Omega})$.

Obviously for any $j \in \mathscr{R} \setminus \mathscr{R}^2$ one has $u(S_j) = 0$ and the same holds for every $j \in \mathscr{R}^2$ such that $\alpha_j \neq \alpha_{j+1}$, in fact

$$\begin{gathered} \lim_{r \to 0} |u(r, 0)| = |u(S_j)| = \lim_{r \to 0} |u(r, \omega_j)| \\ \Downarrow \\ \alpha_{j+1} \lim_{r \to 0} |\partial_\theta u(r, 0)| = |u(S_j)| = \alpha_j \lim_{r \to 0} |\partial_\theta u(r, \omega_j)| \,. \end{gathered} \tag{3.15}$$

Therefore let us consider the case $u(S_j) = 0$, for any $j \in \mathscr{R}$ defining

$$w(r, \theta) = u - \sum_{j \in \mathscr{R}} \Phi_j^{\mathscr{R}}(r, \theta) \,,$$

so that, by Lemma 3.0.29,

$$\int_\Omega v \triangle u dx = \int_\Omega v \triangle w dx \,.$$

By construction in any vertex characterized by Robin conditions, one has

$$\lim_{r \downarrow 0} \frac{1}{r} \, w(r \,, \omega_1) = \lim_{r \downarrow 0} \frac{1}{r} \left[\partial_\theta u(r \,, \omega_1) - \partial_\theta \Phi_j(r \,, \omega_1) \right] = 0 \,. \tag{3.16}$$

By the same considerations one has

$$\partial_r w(r \,, \omega_2) \equiv 0 \,, \; \forall r > 0, \tag{3.17}$$

i.e. the gradient ∇w must be zero at the vertex S_j, being

$$\nabla w(r \,, \omega_1) \perp \nabla w(r \,, \omega_2) \,, \; \forall r > 0 \,.$$

Now let us suppose $\alpha_{j+1} = \alpha_j = \alpha$ and in particular $u(S_j) \neq 0$, then

$$u_j^0(r, \theta) := \frac{u(S_j)}{\omega_j^2} (\theta - \omega_j)^2 \, e^{A(\omega_j, \alpha)} \,,$$

$$A(\omega_j, \, \alpha) = -\frac{\omega + 2\alpha}{\omega \alpha}$$

in order to have $u_j^0 \in W^2(\Omega)$ and $u_j^0(r, 0) \to u(S_j)$, for $r \downarrow 0$.
Then it is sufficient to reproduce the above argumentation substituting $u(r, \theta)$ with

$$u(r, \theta) - u_j^0(r, \theta) \,.$$

Thus one can always consider the function $w(r, \theta)$ such that

$$\gamma_j w \in \tilde{H}^{3/2}(\Gamma_j) \,, \; \textit{and} \; \gamma_j(\partial w / \partial n_j) \in \tilde{H}^{1/2}(\Gamma_j)$$

and the Green formula of Theorem 1.1.15 may be applied to w and v. Finally we conclude

$$\int_\Omega v\triangle w dx = \sum_{j\in\mathscr{R}} \left(\langle\gamma_j w,\, \gamma_j(\partial v/\partial n_j)v\rangle_{H^{1/2}(\Gamma_j)} - \langle\gamma_j v,\, \gamma_j(\partial w/\partial n_j)\rangle_{H^{1/2}(\Gamma_j)}\right)$$

that coincides, being $\gamma_j w = 0$ for $j \in \mathscr{D}$, and $\partial w/\partial n_j = 0$, for any $j \in \mathscr{N}$, with

$$\sum_{j\in\mathscr{R}^2} \left(\langle\gamma_j w,\, \gamma_j(\partial v/\partial n_j)\rangle_{H^{1/2}(\Gamma_j)} - \langle\gamma_j v,\, \gamma_j(\partial w/\partial n_j)\rangle_{H^{1/2}(\Gamma_j)}\right) = 0\,,$$

being respectively $\gamma_j w = -\alpha_j\gamma_j(\partial w/\partial n_j)$ and $\gamma_j v = -\alpha_j\gamma_j(\partial v/\partial n_j)$. □

Lemma 3.0.31. *Let $v \in N$ then $v \in C^\infty(\bar\Omega\backslash V)$ where V is any neighborhood of the corner S_j.*

Proof. For $j \in \mathscr{D}$ and$j \in \mathscr{N}$ the proof is given by Lemma 2.0.12 so it suffices to consider the case $j \in \mathscr{R}$.
Noticing that v is harmonic and therefore smooth inside Ω, we must prove the smoothness of v near any of the Γ_j. For our purpose, is sufficient consider only the case with boundary conditions given by

$$\hat\gamma_j^0 u + \alpha_j\hat\gamma_j^1 u = 0\,. \tag{3.18}$$

Now we perform a change of coordinates axes in order to consider the segment Γ_j on the axis $\{x_2 = 0\}$ and such that Ω is above Γ_j.
Then we can introduce the cut-off function $\phi \in \mathcal{C}_c^\infty(\bar\Omega)$ whose support does not intersect any sides $\bar\Gamma_k$ with $k \neq j$ and such that it is x_2-independent for small values of x_2. Noticing that with this choice ϕ does not intersect any of the corners, we shall now investigate on ϕv.
The function $\omega = \phi v$ belongs to $L^2(\mathbb{R}^2_+)$ where $\mathbb{R}^2_+ = \{x_2 < 0\}$. In addition ω is solution of

$$\begin{cases} -\triangle\omega + \omega = f & in\ \mathbb{R}^2_+ \\ \hat\gamma_j^0\omega + \alpha_j\hat\gamma_j^1\omega = 0\,, & on\ \{x_2 = 0\}\,, \end{cases}$$

where, according to the choice of ϕ

$$f = \left(\phi v - 2\frac{\partial\phi}{\partial x_1}\frac{\partial v}{\partial x_1} - (\triangle\phi)v\right)\,,$$

and there follows that $f \in L^2(\mathbb{R}_+;\, H^{-1}(\mathbb{R}))$ if we agree to see f as a vector-valued function of x_2. This will allow us to show that $\omega \in H^1(\mathbb{R}^2_+)$ as a first

step.
We replace ω with $R\omega$ where R is the inverse operator of $(1 - D_1^2)^{1/2}$ so that

$$Rw = F_1^{-1}(1 + \xi_1^2)F_1\omega\,,$$

where F_1 denotes the Fourier transform in x_1. By Lemma 2.3.2.5. in [4] one has $R\omega \in L^2(\mathbb{R}^2_+)$ and

$$\begin{cases} -\triangle R\omega + R\omega = Rf & in\ \mathbb{R}^2_+ \\ \hat{\gamma}_j^0 R\omega + \alpha_j \hat{\gamma}_j^1 R\omega = 0\,, & on\ \{x_2 = 0\} \end{cases}$$

where $Rf \in L^2(\mathbb{R}^2_+)$. Then noticing that for any domain $\Omega_1 \subset \Omega$ with a smooth boundary containing the support of ϕ and such that $\Gamma_j \subset \partial\Omega_1$, the solution of the above problem is such that

$$R\omega|_{\Omega_1} \in H^2(\Omega_1)\,,$$

one deduces that $R\omega \in H^2(\mathbb{R}^2_+)$ and $\omega \in H^1(\mathbb{R}^2_+)$. Varying ϕ an j one has

$$v \in H^1(\Omega \setminus V)\,,$$

where V is a neighborhood of the vertices of Ω.
Now we reiterate the previous steps of the proof. Since we know that v belongs to $H^1(\Omega \setminus V)$, we also know that $f \in L^2(\mathbb{R}^2_+)$. Thus applying one more time Lemma 2.3.2.5. in [4] to ω, instead to $R\omega$ one has

$$\omega|_{\Omega_1} \in H^2(\Omega_1)\,,$$

and consequently

$$v \in H^2(\Omega \setminus V)\,,$$

where V is a neighborhood of the vertices of Ω.
Finally, repeated application of Theorem 2.5.1.1. in [4] in which Ω is replaced by Ω_1 as above, shows that

$$v \in H^{k+2}(\Omega \setminus V)\,,$$

for every positive integer k and by Sobolev Imbedding Theorem one concludes. □

Now we shall study the behavior of $v \in N$ near the corners. For technical purpose we shall need the eigenfunctions of the operator

$$\varphi \longmapsto -\varphi''\,,$$

under various boundary conditions on $(0,\omega_j)$. More precisely let us define the unbounded operator Λ_j in $\mathscr{H}_j := L^2(0,\omega_j)$ as follows

$$\Lambda_j\varphi = -\varphi''$$

where $\mathscr{D}(\Lambda_j)$ is given by:

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi(0) = \varphi(\omega_j) = 0\}\,,\quad j \in \mathscr{D}^2$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi'(0) = \varphi'(\omega_j) = 0\}\,,\quad j \in \mathscr{N}^2$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi(0)+\alpha_j\varphi'(0) = \varphi(\omega_j)-\alpha_{j+1}\varphi'(\omega_j) = 0\}\,,\quad j \in \mathscr{R}^2\,,$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi(0) = \varphi'(\omega_j) = 0\}\,,\quad j \in (\mathscr{N},\mathscr{D})\,,$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi'(0) = \varphi(\omega_j) = 0\}\,,\quad j \in (\mathscr{D},\mathscr{N})\,,$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi(0) + \alpha_j\varphi'(0) = \varphi'(\omega_j) = 0\}\,,\quad j \in (\mathscr{R},\mathscr{N})\,,$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi'(0) = \varphi(\omega_j) - \alpha_{j+1}\varphi'(\omega_j) = 0\}\,,\quad j \in (\mathscr{N},\mathscr{R})\,,$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi(0) = \varphi(\omega_j) - \alpha_{j+1}\varphi'(\omega_j) = 0\}\,,\quad j \in (\mathscr{D},\mathscr{R})\,,$$

$$\mathscr{D}(\Lambda_j) = \{\varphi \in H^2(0,\omega_j)\ :\ \varphi(0) + \alpha_j\varphi'(0) = \varphi(\omega_j) = 0\}\,,\quad j \in (\mathscr{R},\mathscr{D})\,,$$

The operator λ_j is self-adjoint, has a discrete spectrum and is strictly positive under the following hypotheses that we assume from now on:

$$\alpha_j > 0\,, \alpha_{j+1} > 0\,,\quad \alpha_j + \alpha_{j+1} \neq \omega_j\,,$$

We shall denote by $\varphi_{j,m}$, $m \geq 1$, the normalized eigenfunction and by $\lambda^2_{j,m}$, $m \geq 1$ the corresponding eigenvalues in increasing order. We thus have

$$-\varphi''_{j,m} = \lambda^2_{j,m}\varphi_{j,m}$$

where $\varphi_{j,m} \in \mathscr{D}(\Lambda_j)$ for every m.
With Dirichlet and Neumann boundary conditions these eigenfunction and eigenvalues are well known. We have

$$\varphi_{j,m}(\theta) = \sqrt{2/\omega_j}\sin(\theta\lambda_{j,m})\,,\quad \lambda_{j,m} = m\pi/\omega_j\,,\quad j \in \mathscr{D}^2\,.$$

$$\varphi_{j,m}(\theta) = \sqrt{2/\omega_j}\sin(\theta\lambda_{j,m})\,,\quad \lambda_{j,m} = [m-1/2]\pi/\omega_j\,,\quad j \in (\mathscr{N},\mathscr{D})\,,$$

$$\varphi_{j,m}(\theta) = \sqrt{2/\omega_j}\sin([\omega_j-\theta]\lambda_{j,m})\,,\quad \lambda_{j,m} = [m-1/2]\pi/\omega_j\,,\quad j \in (\mathscr{D},\mathscr{N})\,,$$

$$\varphi_{j,m}(\theta) = \sqrt{2/\omega_j}\cos(\theta\lambda_{j,m})\,,\quad \lambda_{j,m} = [m-1]\pi/\omega_j\,,\quad j \in \mathscr{N}^2\,, m \geq 2\,.$$

In the case of Robin boundary conditions one has

$$\varphi_{j,m}(\theta) = \sin(\lambda_{j,m}\theta) - \alpha_j \lambda_{j,m} \cos(\lambda_{j,m}\theta)\,, \quad j \in \mathscr{R}^2\,,$$

with $\lambda_{j,m} \neq k\pi/2\omega_j$ solutions of

$$\tan(\lambda_{j,m}\omega_j) = \frac{\lambda_{j,m}(\alpha_j + \alpha_{j+1})}{1 - \alpha_j \alpha_{j+1} \lambda_{j,m}^2} \tag{3.19}$$

For mixed Robin-Dirichlet conditions we have

$$\varphi_{j,m}(\theta) = \sin(\lambda_{j,m}\theta)\,, \quad j \in (\mathscr{R}, \mathscr{D})\,,$$
$$\varphi_{j,m}(\theta) = \sin[\lambda_{j,m}(\theta - \omega_j)]\,, \quad j \in (\mathscr{D}, \mathscr{R})\,,$$

with $\lambda_{j,m} \neq k\pi/2\omega_j$ solution of

$$\tan(\lambda_{j,m}\omega_j) = \lambda_{j,m}\alpha\,, \tag{3.20}$$

where α coincides with the unique not zero coefficient according to the Robin conditions.
Finally with mixed conditions Robin-Neumann one has

$$\varphi_{j,m}(\theta) = \cos(\lambda_{j,m}\theta)\,, \quad j \in (\mathscr{R}, \mathscr{N})$$
$$\varphi_{j,m}(\theta) = \cos[\lambda_{j,m}(\theta - \omega_j)]\,, \quad j \in (\mathscr{N}, \mathscr{R})\,,$$

where $\lambda_{j,m} \neq k\pi/2\omega_j$ is solution of

$$\tan(\lambda_{j,m}\omega_j) = \frac{1}{\lambda_{j,m}\alpha} \tag{3.21}$$

where α is defined as above.
Notice that, since Λ_j is symmetric, the eigenfunctions $\varphi_{j,m}$ are orthogonal. Using the polar coordinates (r,θ) with origin at S_j, we see that any $v \in N$ is solution of

$$\partial^2 v/\partial r^2 + r^{-1}\partial v/\partial r + r^{-2}\partial^2 v/\partial\theta^2 = 0\,, \quad 0 < \theta < \omega_j\,, 0 < r < \rho$$

for opportune ρ such that D_ρ does not cut any side of Ω but Γ_j and Γ_{j+1}.

The boundary conditions at the sides $\theta = 0$ and $\theta = \omega_j$ depends on which set the index j belongs to.

These boundary conditions are meaningful since v is regular for $r > 0$ by Lemma 3.0.31. In addition since

$$v(re^{i\theta}) \in H^2(0, \omega_j)\,, \quad \forall 0 < r < \rho\,,$$

it follows that

$$v(re^{i\theta}) \in \mathscr{D}(\Lambda_j)\,, \quad \forall 0 < r < \rho\,.$$

This allows us to rewrite the equation for v as

$$\partial^2/\partial r^2 + r^{-1}\partial/\partial r - r^{-2}\Lambda_j = 0\,,\ \ 0 < r < \rho \tag{3.22}$$

if we see v as an infinitely differentiable vector-valued function of r with values in$\mathscr{D}(\Lambda_j)$. This implies that v can be expanded on the eigenfunctions $\varphi_{j,m}$ in the following fashion.

Theorem 3.0.32. *Let $v \in C^\infty(0,\rho;\mathscr{D}(\Lambda_j))$ be a solution of equation 3.22 and assume that $v \in L^2(\mathscr{D}_\rho)$ then*

$$v(re^{i\theta}) = \sum_{m\geq 2} 2^{\alpha_m} r^{\lambda_{j,m}}\varphi_{j,m}(\theta) + \sum_{0<\lambda_{j,m}<1} 2^{\beta_m} r^{-\lambda_{j,m}}\varphi_{j,m}(\theta)\,,$$

where α_m and β_m are real numbers such that

$$|\alpha_m| \leq L\, m^{1/2\rho - \lambda j,m}$$

and L is a constant depending only on v.

Proof. [3] Since the sequence $\varphi_{j,m}$ for $m > 1$ is an orthonormal basis for $\mathscr{H}_j$ we have

$$v(re^{i\theta}) = \sum_{m\geq 1} v_m(r)\varphi_{j,m}(\theta)\,,$$

where

$$v_m(r) = \int_0^{\omega_j} v(re^{i\theta})\varphi_{j,m}(\theta)d\theta\,. \tag{3.23}$$

However being v differentiable in r with values in $\mathscr{D}(\Lambda_j)$ the differential equation (3.22) implies that

$$v''_m(r) + r^{-1}v'_m(r) - \lambda^2_{j,m}r^{-2}v_m(r) = 0\,,\ 0 < r < \rho\,.$$

Solving this last differential equation we see that (notice that $\lambda_{j,m} > 0$)

$$v_m(r) = \alpha_m r^{\lambda_{j,m}} + \beta_m r^{-\lambda_{j,m}}\,.$$

On the other hand since v belongs to $L^2(D_\rho)$ it follows from identity (3.23) that

$$v_m(r)^2 \leq \int_0^{\omega_j} |v(re^{i\theta})|^2 r dr \leq \|v\|^2\,.$$

[3] In this proof we use the same argumentation given in the proof of Proposition 2.3.5. in [3].

This implies that $\beta_m = 0$ when $\lambda_{j,m} \geq 1$ and that

$$|\alpha_m|^2 \int_0^\rho r^{2\lambda_{j,m}+1} dr = |\alpha_m|^2 \rho^{2\lambda_{j,m}+2}/[2\lambda_{j,m}+2] \leq \|v\|^2 \,,$$

for $\lambda_{j,m} \geq 1$. This complete the proof. $\square$

We shall now try bound the dimension of N. The previous proposition gives precise enough information on the behavior of $v \in N$ near of the corners S_j. We have thus to match these expansions together in order to obtain global information on v in Ω.

Lemma 3.0.33. *For each j and each $\lambda_{j,m} \in (0,1)$ there exists $\sigma_{j,m} \in N$ such that*

$$\sigma_{j,m} - \eta_j r^{-\lambda_{j,m}} \varphi_{j,m}(\theta) \in H^1(\Omega) \,.$$

Proof. For $j \notin \mathscr{R}$ the proof is given by Lemma 2.3.6 in[3] so that we shall consider only the case $j \in \mathscr{R}$. Then let $j \in \mathscr{R}$. Denoting now the function $\eta_j r^{-\lambda_{j,m}} \varphi_{j,m}(\theta)$ by $u_{j,m}$, we have

$$\triangle u_{j,m} = f_{j,m} \in \mathcal{C}_c^\infty(\bar{\Omega})$$

and

$$\hat{\gamma}_j^0 u_{j,m} = \hat{\gamma}_i^1 u_{j,m} = 0 \,, \;\; j \in \mathscr{D} \,, i \in \mathscr{N} \,,$$
$$\hat{\gamma}_j^0 u_{j,m} + \alpha_j \hat{\gamma}_j^1 u_{j,m} = 0 \,, \; j \in \mathscr{R} \,.$$

By Theorem 3.0.24 and the above argumentation there exists a unique $v_{j,m} \in H^1(\Omega)$ variational solution of problem

$$\begin{cases} \triangle v_{j,m} = f_{j,m} \\ \hat{\gamma}_j^0 v_{j,m} = 0 \,, & j \in \mathscr{D} \,, \\ \hat{\gamma}_j^1 v_{j,m} = 0 \,, & j \in \mathscr{N} \,, \\ \hat{\gamma}_j^0 u_{j,m} + \alpha_j \hat{\gamma}_j^1 u_{j,m} = 0 \,, & j \in \mathscr{R} \end{cases}$$

meaning $v_{j,m} \in V(\Omega)$ and according to (3.3)

$$a(v_{j,m},\, h) = -\int_\Omega h \triangle u_{j,m} \, dx$$

for any $h \in V(\Omega)$. Then the conclusion follows by setting

$$\sigma_{j,m} = u_{k,m} - v_{j,m} \,. \tag{3.24}$$

and proving that this is an element of N.
Obviously $\sigma_{j,m} \in \mathscr{D}(\triangle_\Omega^{max}) \cap W^2(\Omega)$ and is harmonic in the interior of Ω, so

that it remains to verify the orthogonality conditions of Lemma 3.12, *i.e.* for any $u \in W^2(\Omega) \cap C^2(\Omega)$

$$0 = \int_\Omega \sigma_{j,m} \triangle \Phi^{\mathscr{R}}_{u,j} dx = \int_\Omega u_{j,m} \triangle \Phi^{\mathscr{R}}_{u,j} dx - \int_\Omega v_{j,m} \triangle \Phi^{\mathscr{R}}_{u,j} dx \,.$$

Integrating by parts the last integral

$$= \int_\Omega u_{j,m} \triangle \Phi^{\mathscr{R}}_{u,j} dx \,+\, \int_\Omega \nabla \Phi^{\mathscr{R}}_{u,j} \nabla v_{j,m} dx \,.$$

So that, remembering that $\Phi^{\mathscr{R}}_{u,j}$ satisfies the homogeneous Dirichlet and Neumann conditions by construction, and integrating by parts again it suffices to show that

$$0 = \int_\Omega u_{j,m} \triangle \Phi^{\mathscr{R}}_{u,j} dx \,. \tag{3.25}$$

Now notice that the orthogonality conditions of Lemma 3.12 do not change substituting $\Phi^{\mathscr{R}}_{u,j}$ with

$$\hat{\Phi}^{\mathscr{R}}_{u,j} := \Phi^{\mathscr{R}}_{u,j}(r,\theta) \cdot (\theta - \alpha) \,,$$

where $\alpha \in \mathbb{R}$. Then we have to prove that

$$\alpha \langle u_{j,m},\, \triangle \Phi^{\mathscr{R}}_{u,j} \rangle_{L^2(\Omega)} = \langle u_{j,m},\, \theta \,\triangle \Phi^{\mathscr{R}}_{u,j} \rangle_{L^2(\Omega)} + 2 \,\langle u_{j,m},\, \frac{1}{r^2}\, \partial_\theta \Phi^{\mathscr{R}}_{u,j} \rangle_{L^2(\Omega)}$$

Noticing that $\partial_\theta u(r,\, \omega_1)$ goes to zero for $r \downarrow 0$ at least linearly one has that the two last integrals are finite and choosing

$$\alpha_{j,u} = \frac{\langle u_{j,m},\, \theta \,\triangle \Phi^{\mathscr{R}}_{u,j} \rangle_{L^2(\Omega)} - 2 \langle u_{j,m},\, 1/r^2 \partial_r \Phi^{\mathscr{R}}_{u,j} \rangle_{L^2(\Omega)}}{\langle u_{j,m},\, \triangle \Phi^{\mathscr{R}}_{u,j} \rangle_{L^2(\Omega)}} \,,$$

one then concludes the proof. □

Thus we have the following

Theorem 3.0.34.

$$\dim(N) = \sum_j \#\{\lambda_{j,m}\,; 0 < \lambda_{j,m} < 1\} \,.$$

Proof. We will consider only the case $\mathscr{R} \neq \emptyset$. By Theorem 3.0.32 one has an expansion for any $v \in N$ near each corner. Then, substituting $\sigma_{j,m}$ to $r^{\lambda_{j,m}} \varphi_{j,m}(\theta)$ in this expansion gives

$$\sum_{m \geq 2} \alpha_m r^{\lambda_{j,m}} \varphi_{j,m}(\theta) + \sum_{0 < \lambda_{j,m} < 1} \beta_m \sigma_{j,m} \in H^1(D_\rho) \,.$$

by Lemma 3.0.33. Then we check that

$$\sum_{m\geq 2} \alpha_m r^{\lambda_{j,m}} \varphi_{j,m}(\theta) \in H^1(D_{\rho'}) \tag{3.26}$$

for every $\rho' < \rho$. Indeed denoting by w the series in 3.26 one has

$$\partial_r w = \sum_{m\geq 2} \alpha_m \lambda_{j,m} r^{\lambda_{j,m}-1} \varphi_{j,m}(\theta)$$

$$r^{-1}\partial_\theta w = \sum_{m\geq 2} \alpha_m \lambda_{j,m} r^{\lambda_{j,m}-1} \varphi_{j,m}(\theta)$$

and consequently, according to the bounds for α_m in Theorem 3.0.32

$$\|\nabla w\| \leq C_{\omega_j} \sum_{m\geq 2} |\alpha_m| \lambda_{j,m} r^{\lambda_{j,m}-1} / \rho^{\lambda_{j,m}}$$

for a constant C_{ω_j} depending only by ω_j. Then it is square integrable in $D_{\rho'}$ for $\rho' < \rho$. Summing up, this shows

$$v(r\, e^{i\theta}) - \sum_{0<\lambda_{j,m}<1} \beta_m \sigma_{j,m} \in H^1(D_\rho)\,,$$

at each corner. To make the notation consistent, at this stage, we must reintroduce the subscript j everywhere. Thus there exists numbers $\beta_{j,m}$ such that

$$v - \sum_{0<\lambda_{j,m}<1} \beta_m \sigma_{j,m}$$

is of class H^1 near S_j.
By Lemma 3.0.31 it follows that globally holds

$$w = v - \sum_j \sum_{0<\lambda_{j,m}<1} \beta_m \sigma_{j,m} \in H^1(\Omega),.$$

We shall conclude the proof by showing that w vanishes. Indeed we already know that $w \in N \cap H^1(\Omega)$.
Theorem 3.0.24 shows that

$$a(w,\, v) = 0\,,$$

for ever $v \in V(\Omega)$ hence $w \equiv 0$. This shows that

$$v = \sum_j \sum_{0<\lambda_{j,m}<1} \beta_m \sigma_{j,m} \in H^1(\Omega),.$$

In other words v is a linear combination of the $\sigma_{j,m}$ with $0 < \lambda_{j,m} < 1$ and these functions are linearly independent. □

Denoting as before by d_j the contribute of vertex S_j to the dimension of N, by Theorem 3.0.34, and looking for the solutions of equations (3.19), (3.20), (3.21), one has that with Robin boundary conditions it is possible to have results of the same kind of the ones obtained in Chapter 2 regarding $j \in \mathscr{M}$. However, by a simple graphical analysis the of equation (3.19), it is possible to find also different behaviors. For example taking $\alpha_j = \omega_j$, $\alpha_{j+1} = \alpha_\epsilon$, $\alpha_\epsilon \downarrow 0$ as $\epsilon \downarrow 0$, one has the following

Lemma 3.0.35. *For any $\epsilon > 0$, for any $\omega_j < \epsilon$ it is possible to choose α_j and α_{j+1} in such a way that $d_j = 1$.*

Moreover, taking $\alpha_j = \theta_0$ where $\theta_0 = 4.494309...$ is the first positive solution of the equation $\tan\theta = \theta$, and $\alpha_{j+1} = \alpha_\epsilon$ as before, one gets the following result

Lemma 3.0.36. *For any $x\pi < \omega_j < (3/2)\pi$, $x \simeq 1.43$, it is possible to choose α_j and α_{j+1} in such a way that $d_j = 2$.*

As expected when $\alpha_j\,, \alpha_{j+1}$ go to infinity (respectively to zero) the case $j \in \mathscr{R}^2$ converges to the case $j \in \mathscr{N}^2$, (respectively to $j \in \mathscr{D}^2$) with the first solution disappearing to zero and the second going to π/ω. Similarly in the case $j \in \mathscr{R}^2$ with coefficients $\alpha_j\,, \alpha_{j+1}$ such that $\alpha_j\alpha_{j+1} = 1$, if one coefficient goes to zero and the other one to infinity, then the case $j \in \mathscr{R}^2$ converges to the case $j \in \mathscr{M}$, with the two solutions λ_k that converge to $(1/2 + k)\pi/\omega$ for $k = 0, 1$.

Chapter 4

Self-adjoint Extensions for Symmetric Laplacians with Mixed Robin Boundary Conditions

It this chapter we extend the results given in chapter 2 to the case of mixed Robin boundary conditions.

First of all, denoting by $\mathcal{I}' \subset \mathcal{I}$ the linear set of $u \in L^2(\Omega)$ satisfying the orthogonally conditions of Lemma 3.0.29, we can immediately generalize Theorem 2.0.13:

Theorem 4.0.37.

$$\mathscr{K}(\triangle_\Omega^{\circ *}) = \mathscr{K}(\triangle_\Omega^{\max}) \cap \mathscr{K} \cap \mathcal{I}',$$

where

$$\mathscr{K} := \{u \in \mathscr{D}(\triangle_\Omega^{\max}) \,:\, \hat{\gamma}_j^0 u + \alpha_j \hat{\gamma}_j^1 u = \hat{\gamma}_k^0 u = \hat{\gamma}_h^1 u\,,\ k \in \mathscr{D}\,, h \in \mathscr{R}\,, j \in \mathscr{N}\}\,.$$

Proof. The proof of this theorem follows by Theorem 2.0.13 in the case $j \notin \mathscr{R}$ and by Lemma 3.0.29 and Theorem 3.0.30 in the remanent cases. □

As a direct consequence of *Poincaré inequality* (see Theorem 1.1.6) and the continuity results on trace operators given in chapter 1, $V(\Omega)$ is a Hilbert space for the scalar product inducted by the bilinear form

$$F(u,v) := \int_\Omega \nabla u \cdot \nabla v dx + \sum_{j \in \mathscr{R}} \frac{1}{\alpha_j} \int_{\Gamma_j} \gamma_j^0 u \gamma_j^0 v \, d\sigma \,.\,, \quad u\, v \in V(\Omega)\,.$$

This allow us to apply the *Lax-Milgram Theorem* and conclude[1] that exists an unique self-adjoint operator defined as

$$\triangle_\Omega^F : \mathscr{D}(\triangle_\Omega^F) \subseteq V(\Omega) \subset L^2(\Omega) \to L^2(\Omega)\,,$$

such that

$$F(u,v) = \langle -\Delta_\Omega^F u, v\rangle_{L^2(\Omega)}\,, \quad \forall u \in \mathscr{D}(\Delta_\Omega^F)\,, v \in V(\Omega)\,.$$

On the other hand if we consider $W^2(\Omega)$ defined in (2.3), by the estimate

$$\forall u \in W^2(\Omega)\,, \quad \|u\|_{H^2(\Omega)} \le c_\Omega \|\triangle u\|_{L^2(\Omega)}\,,$$

provided in Theorem 2.0.8, there follows by the Green's Formula, that

$$\triangle_\Omega^\circ : W^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)\,, \quad \triangle_\Omega^\circ u := \triangle u\,,$$

is a closed symmetric operator. Similarly to the discussion made in Chapter 2 we investigate about the difference between $\triangle_\Omega^\circ$ and the Friedrichs extension Δ_Ω^F. In particular we will show as the Robin vertices give a contribute to the kernel of $(\triangle_\Omega^\circ)^*$.
We denote by $\mathscr{M}_1'$ the set of indices $j \in \mathscr{R}$ such that the contribution of S_j to the dimension of N is either one or two and by $\mathscr{M}_2'$ the ones which give contribution two. Then we pose

$$\tilde{\mathscr{M}}_1 := \mathscr{M}_1 \cup \mathscr{M}_1'\,, \quad \tilde{\mathscr{M}}_2 := \mathscr{M}_2 \cup \mathscr{M}_2'\,,$$

$$\tilde{n}_1 := \#\tilde{\mathscr{M}}_1\,, \quad \tilde{n}_2 := \#\tilde{\mathscr{M}}_2\,.$$

Then one has the generalization of Theorem 2.0.14 given by

Theorem 4.0.38.

$$\dim \mathscr{K}((\triangle_\Omega^\circ)^*) = \tilde{n}_1 + \tilde{n}_2\,.$$

Now on any disk D_k centered at S_k we introduce the functions $u_{jm}^{\mp}$ defined as

$$u_{j,m}^{\pm}(r,\theta) := C_{jm} r^{\pm\lambda_{jm}}\, \varphi_{j,m}(\theta)\,, \quad j \in \mathscr{R}\,,$$

with C_{jm} a normalization constant such that

$$C_{j,m}^2 \int_0^{\omega_j \lambda_{j,m}} |\sin(\theta) \, - \alpha^j \lambda_{j,m} \cos(\theta)|^2 d\theta = 1/2\,, \tag{4.1}$$

where $\varphi_{j,m}$ is defined for $j \in \mathscr{R}$ accordingly to the results found in Chapter 3. In particular one has $m = 1$ if $0 < \lambda_{j,1} < 1 \le \lambda_{j,2}$, whereas $m = 1,2$ if $0 < \lambda_{j,1} < \lambda_{j,2} < 1$. Then one has the exact analogue of Lemma 2.0.15:

[1]See e.g. [2], Chapter IV, Section 1.

Lemma 4.0.39. *Let us define*

$$s^0_{km} := \chi_k u^+_{km}\,, \quad s_{km} := \chi_k u^-_{km}\,, \quad \sigma_{km} := s_{km} - (\triangle^F_\Omega)^{-1}\triangle_\Omega s_{km}\,.$$

Then 1)

$$s^0_{km} \in \mathscr{D}(\triangle^F_\Omega)\,, \quad s_{km} \in \mathscr{D}(\triangle^{\max}_\Omega)\,;$$

2) σ_{km} is the unique (up to multiplication by constant) function in $\mathscr{K}(\triangle^{\circ *}_\Omega)$ such that

$$\sigma_{km} - s_{km} \in \mathscr{D}(\triangle^F_\Omega)\,;$$

3) the σ_{km}'s are linearly independent;
4)

$$\langle \sigma_{hj}\,, -\triangle^F_\Omega s^0_{km}\rangle_{L^2(\Omega)} = \delta_{kh}\delta_{mj}\,.$$

5) The $\triangle^F_\Omega s^0_{km}$'s are orthogonal and thus linearly independent.

Proof. The proof are similar to the ones given for Lemma 2.0.15. The only prove point 4 in the case $k \in \mathscr{R}$: Let us start with the case of pure Robin corner in which $k \in \mathscr{R}^2$. First of all we can assume the case in which the corner contribute is equal to 1. Then posing $W^2_k := W^{R_2} \setminus W^{R_1}$ one has

$$\langle \sigma_{km}\,, \triangle^F_\Omega s^0_{km}\rangle_{L^2(\Omega)} = \langle \sigma_k\,, \triangle^F_\Omega s^0_k\rangle_{L^2(\Omega)}$$

$$= \langle \chi_k u^-_k\,, \triangle^F_\Omega \chi_k u^+_k\rangle_{L^2(W_k)} - \langle \triangle_\Omega \chi_k u^-_k\,, \chi_k u^+_k\rangle_{L^2(W_k)}$$

$$= \int_{W_k} \chi_k u^-_k \left[\chi''_k u^+_k + \left(1 + 2\lambda_k\right)\frac{1}{r}\chi'_k u^+_k\right] dx$$

$$- \int_{W_k} \chi_k u^+_k \left(\chi''_k u^-_k + \left(1 - 2\lambda_k\right)\frac{1}{r}\chi'_k u^-_k\right) dx$$

$$= 2\lambda_k C^2_k \int_{R_1}^{R_2} 2\chi'_k \chi_k\, dr \int_0^{\omega_k} \varphi^2_k \theta d\theta$$

$$= -2\lambda_k C^2_k \int_0^{\omega_k} |\sin(\lambda_k\theta)\, - \alpha^k \lambda_k \cos(\lambda_k\theta)|^2 d\theta = -1\,.$$

It remains to show the $L^2(\Omega)$-orthogonality between σ_{km} and $\triangle^F_\Omega s^0_{kj}$ in the same Robin corner. So assuming $m \neq j$ we can suppose without loos of generality $m = 1$ and $j = 2$. Then

$$\langle \sigma_{km}\,, \triangle^F_\Omega s^0_{km}\rangle_{L^2(\Omega)} = \langle \sigma_k\,, \triangle^F_\Omega s^0_k\rangle_{L^2(\Omega)} = 0$$

by orthogonality of the φ_{km}'s.
Let us consider the mixed-corner $j \in (\mathscr{R}, \mathscr{D})$. We will first study the case $j = m$ and without loss of generality we can omit this index assuming that is equal to 1. Consequently we have

$$\begin{aligned} &\langle \sigma_k \,, \triangle_\Omega^F s_k^0 \rangle_{L^2(\Omega)} \\ &= \langle \chi_k u_k^- \,, \triangle_\Omega^F \chi_k u_k^+ \rangle_{L^2(W_k)} - \langle \triangle_\Omega \chi_k u_k^- \,, \chi_k u_k^+ \rangle_{L^2(W_k)} \\ &= 2C_k^2 \lambda_k \int_{R_1}^{R_2} 2\chi_k' \chi_k \, dr \int_0^{\omega_k} \sin^2(\lambda_k \theta) \, d\theta = -1 \end{aligned}$$

Now, assuming $m \neq j$, again we can suppose $m = 1$ and $j = 2$, then

$$\langle \sigma_{k1} \,, \triangle_\Omega^F s_{k2}^0 \rangle_{L^2(\Omega)} = 0$$

by orthogonality of the φ_{km}'s. Finally we can conclude noticing that the same argumentation holds for cases of Robin-Neumann mixed conditions and for $j \in (\mathscr{D}, \mathscr{R})$. □

All the results provided in Chapter 2 following Lemma 2.0.15 can be extended to the Robin case. The statements and the relative proofs remain the same: it suffices to replace n_1, n_2 with $\tilde{n}_1$, $\tilde{n}_2$ and $\mathscr{M}_1$, $\mathscr{M}_2$ with $\tilde{\mathscr{M}}_1$, $\tilde{\mathscr{M}}_2$.

In conclusion, by the results contained in Section 1.2, one gets the following

Theorem 4.0.40. *Any self-adjoint extension of Δ_Ω° is of the kind*

$$\Delta_\Omega^{\Pi,\Theta} : \mathscr{D}(\Delta_\Omega^{\Pi,\Theta}) \subset L^2(\Omega) \to L^2(\Omega) \,, \quad \Delta_\Omega^{\Pi,\Theta} u := \Delta_\Omega u \,,$$

$$\mathscr{D}(\Delta_\Omega^{\Pi,\Theta}) := \{ u \in \mathscr{D}((\Delta_\Omega^\circ)^*) \, : \, \xi^u \in \mathbb{C}_\Pi^{\tilde{n}_1 + \tilde{n}_2} \,, \; \Pi \zeta^u = \Theta \xi^u \} \,,$$

where $(\Pi, \Theta) \in \mathsf{E}(\mathbb{C}^{\tilde{n}_1 + \tilde{n}_2})$. Moreover

$$(-\Delta_\Omega^{\Pi,\Theta} + z)^{-1} = (-\Delta_\Omega^F + z)^{-1} + G_z \Pi \, (\Theta + \Pi \, \Gamma_z \Pi)^{-1} \Pi \, G_{\bar{z}}^* \,.$$

The quadratic form corresponding to $-\Delta_\Omega^{\Pi,\Theta}$ is given by

$$F_\Omega^{\Pi,\Theta} : \mathscr{D}(F_\Omega^{\Pi,\Theta}) \times \mathscr{D}(F_\Omega^{\Pi,\Theta}) \subset L^2(\Omega) \times L^2(\Omega) \to \mathbb{R} \,,$$

$$F_\Omega^{\Pi,\Theta}(u, v) = \langle \nabla u_0, \nabla v_0 \rangle_{L^2(\Omega)} + \sum_{k \in \mathscr{R}} \frac{1}{\alpha_k} \langle \gamma_k^0 u_0, \, \gamma_k^0 v_0 \rangle_{L^2(\Gamma_k)} + \, f_\Theta(\xi_u, \xi_v) \,,$$

$$\mathscr{D}(F_\Omega^{\Pi,\Theta}) = \{ u \in L^2(\Omega) \, : \, u = u_0 + \sum_{k \in \tilde{\mathscr{M}}_1} \xi_k^u \, \sigma_{k1} + \sum_{k \in \tilde{\mathscr{M}}_2} \tilde{\xi}_k^u \, \sigma_{k2} \,,$$

$$u_0 \in V(\Omega) \,, \; \xi^u \in \mathbb{C}_\Pi^{\tilde{n}_1 + \tilde{n}_2} \} \,,$$

where f_Θ is the quadratic form corresponding to Θ.

Chapter 5

Examples

In this last chapter we give some simple examples in the case where Ω is the wedge

$$\mathsf{W} = \{(r,\theta) \,:\, 0<r<1\,,\ 0<\theta<\pi/\beta\}\,, \quad \beta>1/2\,.$$

5.1 Case 1: Dirichlet boundary conditions

Let us consider the case of a non-convex wedge, i.e. we take $\beta < 1$, and let $\Delta^\circ_{\mathsf{W}}$ be the restriction of Δ to $H^2(\mathsf{W}) \cap H^1_0(\mathsf{W})$. By the results given in Chapter 2 we know that the kernel of $(\Delta^\circ_{\mathsf{W}})^*$ is one dimensional and is generated by σ, the unique (up to the multiplication by a constant) square integrable solution of the boundary value problem

$$\begin{cases} \Delta\sigma(r,\theta)=0\,, & (r,\theta)\in\mathsf{W}\,,\\ \sigma(r,0)=0\,, & 0<r<1\,,\\ \sigma(r,\pi/\beta)=0\,, & 0<r<1\,,\\ \sigma(1,\theta)=0\,, & 0<\theta<\pi/\beta\,. \end{cases}$$

Thus

$$\sigma(r,\theta)=\frac{1}{\sqrt{\pi}}\left(\frac{1}{r^\beta}-r^\beta\right)\sin\beta\,\theta\,.$$

Analogously we define σ_z by solving the boundary value problem

$$\begin{cases} \Delta\sigma(r,\theta)=z\sigma(r,\theta)\,, & (r,\theta)\in\mathsf{W}\,,\\ \sigma(r,0)=0\,, & 0<r<1\,,\\ \sigma(r,\pi/\beta)=0\,, & 0<r<1\,,\\ \sigma(1,\theta)=0\,, & 0<\theta<\pi/\beta\,. \end{cases}$$

Thus

$$\sigma_z(r,\theta)$$
$$=\frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{z}}{2}\right)^{\beta}\Gamma(1-\beta)\left(J_{-\beta}(\sqrt{z}\,r)-\frac{J_{-\beta}(\sqrt{z}\,)}{J_{\beta}(\sqrt{z})}\,J_{\beta}(\sqrt{z}\,r)\right)\sin\beta\,\theta\,,$$

where J_ν denotes the Bessel function of order ν and Γ denotes Euler's gamma function. Here the constants are chosen in order to have $\sigma_z \to \sigma$ as $z \to 0$. Notice that σ_z is single-valued. Indeed, by

$$J_\nu(z)=\left(\frac{z}{2}\right)^{\nu}\frac{\tilde{J}_\nu(z^2)}{\nu\Gamma(\nu)}\,,\qquad \tilde{J}_\nu(z):=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!(k+\nu)!}\left(\frac{z}{4}\right)^{k}$$

(here $(k+\nu)! := (1+\nu)\cdots(k+\nu)$), one has

$$\sigma_z(x)=\frac{1}{\sqrt{\pi}}\left(\tilde{J}_{-\beta}(zr^2)\,\frac{1}{r^{\beta}}-\frac{\tilde{J}_{-\beta}(z)}{\tilde{J}_{\beta}(z)}\,\tilde{J}_{\beta}(zr^2)\,r^{\beta}\right)\sin\beta\,\theta\,.$$

Therefore

$$z\langle\sigma,\sigma_z\rangle_{L^2(\mathsf{W})}=1+\left(\frac{\sqrt{z}}{2}\right)^{2\beta}\frac{\Gamma(-\beta)}{\Gamma(\beta)}\,\frac{J_{-\beta}(\sqrt{z})}{J_{\beta}(\sqrt{z})}=1-Q_\beta(z)\,,$$

where we posed

$$Q_\beta(z):=\frac{\tilde{J}_{-\beta}(z)}{\tilde{J}_{\beta}(z)}\,.$$

Thus for any real α one can define a self-adjoint extension $\Delta^\alpha_{\mathsf{W}}$ of $\Delta^\circ_{\mathsf{W}}$ with resolvent kernel

$$R^\alpha_z(r,\theta;r',\theta')=R^D_z(r,\theta;r',\theta')+(\alpha-Q_\beta(z))^{-1}\,\sigma_z(r,\theta)\sigma_z(r',\theta')\,,$$

where

$$R^D_z(r,\theta;r',\theta')=\sum_{m,n\geq 1}\frac{\psi_{m,n}(r,\theta)\psi_{m,n}(r',\theta')}{\lambda^2_{m,n\beta}-z}$$

is the resolvent of the self-adjoint Dirichlet Laplacian Δ^D_Ω,

$$\psi_{m,n}(r,\theta)=2\sqrt{\frac{\beta}{\pi}}\,\frac{J_{n\beta}(\lambda_{m,n\beta}r)}{J_{n\beta+1}(\lambda_{m,n\beta})}\,\sin n\beta\theta$$

are the normalized eigenfunctions of Δ^D_Ω and $\lambda_{m,n\beta}$ denotes the m-th positive zero of $J_{n\beta}$.

5.2 Case 2: mixed Dirichlet-Neumann boundary conditions

Here we consider a non-convex wedge with $\beta < 2/3$. Let $\Delta^\circ_{\mathsf{W}}$ be the restriction on Δ to $\{u \in H^2(\mathsf{W}) : \partial_\theta u(r,0) = u(r,\pi/\beta) = u(1,\theta) = 0\}$. By the results in Chapter 2 we know that in this case the kernel of $(\Delta^\circ_{\mathsf{W}})^*$ is two dimensional and is generated by σ_k, $k = 1,2$, the two linearly independent square integrable solutions of the boundary value problem

$$\begin{cases} \Delta\sigma_k(r,\theta) = 0\,, & (r,\theta) \in \mathsf{W}\,, \\ \partial_\theta\sigma_k(r,0) = 0\,, & 0 < r < 1\,, \\ \sigma_k(r,\pi/\beta) = 0\,, & 0 < r < 1\,, \\ \sigma_k(1,\theta) = 0\,, & 0 < \theta < \pi/\beta\,. \end{cases}$$

Thus, posing $\beta_k = (k - 1/2)\beta$,

$$\sigma_k(r,\theta) = C_{km}{}^1 \left(\frac{1}{r^{\beta_k}} - r^{\beta_k}\right) \sin((k-1/2)(\pi - \beta\theta))\,,$$

Analogously we define $\sigma_{k,z}$, $k = 1,2$, as the two linearly independent solutions of the boundary value problem

$$\begin{cases} \Delta\sigma_{k,z}(r,\theta) = z\,\Delta\sigma_{k,z}(r,\theta)\,, & (r,\theta) \in \mathsf{W}\,, \\ \partial_\theta\sigma_{k,z}(r,0) = 0\,, & 0 < r < 1\,, \\ \sigma_{k,z}(r,\pi/\beta) = 0\,, & 0 < r < 1\,, \\ \sigma_{k,z}(1,\theta) = 0\,, & 0 < \theta < \pi/\beta\,. \end{cases}$$

Thus

$$\begin{aligned} &\sigma^{k,z}(r,\theta) \\ =&C_{km}\left(\frac{\sqrt{z}}{2}\right)^{\beta_k} \Gamma\left(1-\beta_k\right) \left(J_{-\beta_k}(\sqrt{z}\,r) - \frac{J_{-\beta_k}(\sqrt{z}\,)}{J_{\beta_k}(\sqrt{z})}\,J_{\beta_k}(\sqrt{z}\,r)\right)\varphi_k(\theta) \\ \equiv&C_{km}\left(\tilde{J}_{-\beta_k}(zr^2)\,\frac{1}{r^{\beta_k}} - \frac{\tilde{J}_{-\beta_k}(z)}{\tilde{J}_{\beta_k}(z)}\,\tilde{J}_{\beta_k}(zr^2)\,r^{\beta_k}\right)\varphi_k(\theta)\,.\,, \end{aligned}$$

with

$$\varphi_k(\theta) = \sin((k-1/2)(\pi - \beta\theta))\,.$$

[1]The constants C_{km} are defined in equation (2.9)

Then for $h, k = 1, 2$ one has, since the φ_k's are orthogonal,

$$z\langle \sigma_h, \sigma_{k,z}\rangle_{L^2(\mathsf{W})} = \delta_{hk}(1 - Q_{\beta_k}(z)) .$$

Thus for any 2×2 Hermitean matrix $\Theta \equiv \{\Theta_{kh}\}_{h,k=1}^2$ posing

$$M^\Theta(z) \equiv \{M^\Theta_{h,k}(z)\}_{h,k=1}^2 , \quad M^\Theta_{h,k}(z) := \Theta_{kh} + \delta_{hk} Q_{\beta_k}(z)$$

one can define a self-adjoint extension $\Delta^\Theta_{\mathsf{W}}$ of $\Delta^\circ_{\mathsf{W}}$ with resolvent kernel

$$R^\Theta_z(r, \theta; r', \theta') = R^{N,D}_z(r, \theta; r', \theta') + \sum_{h,k} [M^\Theta(z)]^{-1}_{hk} \sigma_{h,z}(r, \theta)\sigma_{k,z}(r', \theta') ,$$

where

$$R^{N,D}_z(r, \theta; r', \theta') = \sum_{m,n\geq 1} \frac{\psi_{m,n}(r, \theta)\psi_{m,n}(r', \theta')}{\lambda^2_{m,n\beta/2} - z}$$

is the resolvent of the self-adjoint Laplacian with mixed Dirichlet-Neumann boundary conditions $\Delta^{N,D}_{\mathsf{W}}$ and

$$\psi_{m,n}(r, \theta) = 2\sqrt{\frac{\beta}{\pi}} \frac{J_{n\beta/2}(\lambda_{m,n\beta/2} r)}{J_{n\beta/2+1}(\lambda_{m,n\beta/2})} \varphi_k(n\theta) ,$$

are the normalized eigenfunctions of $\Delta^{N,D}_{\mathsf{W}}$.

5.3 Case 3: Robin boundary conditions

We consider here the case of pure Robin boundary conditions at the vertex, so we let $\Delta^\circ_{\mathsf{W}}$ be the restriction on Δ to $\{u \in H^2(\mathsf{W}) : u(r, 0) + \alpha_1 \partial_\theta u(r, 0) = u(r, \pi/2) - \alpha_2 \partial_\theta(r, \pi/2) = u(1, \theta) = 0\}$. To simplify the expositions we suppose that $0 < \alpha_k \leq \pi/2$, $k = 1, 2$ [2]. By the results in Chapter 3 we know that in this case the dimension of the kernel of $(\Delta^\circ_{\mathsf{W}})^*$ is given by the number of solutions $0 < \lambda < 1$ of the equation

$$\tan(\pi\lambda/\beta) = \frac{(\alpha_1 + \alpha_2)\lambda}{1 - \alpha_1\alpha_2\lambda^2} \tag{5.1}$$

In order to present an example describing a situation not covered by the mixed Dirichlet-Neumann case, we take the convex right-angled wedge with $\beta = 2$. Thus in the Dirichlet-Neumann case $\mathscr{K}((\Delta^\circ_{\mathsf{W}})^*)$ should be zero-dimensional and $\Delta^\circ_{\mathsf{W}}$ should be self-adjoint.

[2] This hypothesis avoids eventual contributions given by the two (right-angled) corners not at the origin

By a trivial graphical analysis one can check that equation (5.1) has a solution $0<\beta'<1$ for any $\alpha_1>0$, $\alpha_2>0$ such that

$$\alpha_1\alpha_2<1\,,\quad \alpha_1+\alpha_2>\frac{\pi}{2}\,.$$

Then the kernel of $(\Delta^\circ_{\mathsf{W}})^*$ is generated by the unique (up to the multiplication by a constant) square integrable solution of the boundary value problem

$$\begin{cases}\Delta\sigma(r,\theta)=0\,, & (r,\theta)\in\mathsf{W}\,,\\ \sigma(r,0)+\alpha_1\partial_\theta\sigma(r,0)=0\,, & 0<r<1\,,\\ \sigma(r,\pi/2)-\alpha_2\partial_\theta\sigma(r,\pi/2)=0\,, & 0<r<1\,,\\ \sigma(1,\theta)=0\,, & 0<\theta<\pi/2\,.\end{cases}$$

given by

$$\sigma(r,\theta)=C_{k1}{}^3\left(\frac{1}{r^{\beta'}}-r^{\beta'}\right)\varphi'(\theta)\,,$$

where

$$\varphi'(\theta)=\sin(\beta'\theta)\;-\alpha_1\beta'\cos(\beta'\theta)\,.$$

Denoting by σ_z the square integrable solution of the boundary value problem

$$\begin{cases}\Delta\sigma(r,\theta)=z\,\sigma(r,\theta)\,, & (r,\theta)\in\mathsf{W}\,,\\ \sigma(r,0)+\alpha_1\partial_\theta\sigma(r,0)=0\,, & 0<r<1\,,\\ \sigma(r,\pi/2)-\alpha_2\partial_\theta\sigma(r,\pi/2)=0\,, & 0<r<1\,,\\ \sigma(1,\theta)=0\,, & 0<\theta<\pi/2\,.\end{cases}$$

one has

$$\begin{aligned}&\sigma_z(r,\theta)\\ =&C_{k1}\left(\frac{\sqrt{z}}{2}\right)^{\beta'}\Gamma\left(1-\beta'\right)\left(J_{-\beta'}(\sqrt{z}\,r)-\frac{J_{-\beta'}(\sqrt{z}\,)}{J_{\beta'}(\sqrt{z})}\,J_{\beta'}(\sqrt{z}\,r)\right)\varphi'(\theta)\\ =&C_{k1}\left(\tilde J_{-\beta'}(zr^2)\,\frac{1}{r^{\beta'}}-\frac{\tilde J_{-\beta'}(z)}{\tilde J_{\beta'}(z)}\,\tilde J_{\beta'}(zr^2)\,r^{\beta'}\right)\varphi'(\theta)\,.\end{aligned}$$

Then

$$z\langle\sigma,\sigma_z\rangle_{L^2(\mathsf{W})}=1-Q_{\beta'}(z)\,.$$

[3]The constants C_{k1} are defined in equation (4.1)

and for any real α one can define a self-adjoint extension $\Delta_{\mathsf{W}}^{\alpha}$ of $\Delta_{\mathsf{W}}^{\circ}$ with resolvent kernel

$$R_z^{\alpha}(r,\theta;r',\theta') = R_z^{R}(r,\theta;r',\theta') + (\alpha - Q_{\beta'})^{-1}\sigma_z(r,\theta)\sigma_z(r',\theta')\,,$$

where R_z^R is the resolvent of the self-adjoint Laplacian Δ_{Ω}^{R} with Robin boundary conditions at the vertex.

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