Multiloop amplitudes in superstring theory

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Contents

Chapter 1

String amplitudes

In this thesis we present the construction of superstring amplitudes in genus greater than two. Our construction of the amplitudes has as starting point the conjectured equation (2.15). For that reason, the purpose of this introductory chapter is to give some intuitive justification to this equation. We sketch the construction of the string theory in the perturbative approach. We start with the bosonic case and then analyze its extensions to the supersymmetric theory. Following E. Verlinde and H. Verlinde, we try to give some motivations of the splitting in holomorphic and antiholomorphic part of the measure appearing in the integral (2.15). Moreover, we point out some unclear and not yet solved issues in the derivation of such expression. These open questions (and the considerations of Chapter 5.5.3) provide some criticisms about the general (conjectured) form of the superstrings vacuum-to-vacuum amplitudes and make necessary some deeper investigations about equation (2.15). In the next chapter we will give more details to the construction of the string perturbation theory and we will review the approach of D'Hoker and Phong, their result in genus two and their attempt to extend the result to arbitrary genus, being the starting point of our analysis.

1.1 Fermionic string path integral and chiral factorization

In this section we reassume the first attempts to correctly define the path integral in superstring theory. These studies and those of D'Hoker and Phong lead to the conjectured splitted form (2.15) for the superstring vacuum-to-vacuum amplitudes. We will follow the exposition of [VV2].

The first triumph in the perturbative theory was reached for the bosonic string. The cornerstone was the result of Belavin and Knizhnik [BK,BK2]. In these papers the authors consider the g-loop amplitudes of closed bosonic strings in the critical dimensions. They show that the integration measure is a measure on the moduli space $\bar{\mathcal{M}}_g$ of Riemann surfaces of genus g. Moreover, they proved that for $g > 1$ this measure is the squared of a holomorphic function without zeros on $\mathcal{M}_g$ and having a second order pole on degenerate surfaces. These properties determine the measure up to a constant multiple. The pole is strictly related to the presence of the tachyon in the spectrum of the bosonic theory. In addition, the authors fix their results in a rigorous way showing the connection of their computation with a deep theorem of Mumford in algebraic geometry [Mu2]. The central point is the observation that, strictly speaking, the measure is not a function, but a section of a line bundle E over M_g where the objects appearing in the explicit expression of the measure leave. Mumford showed that the line bundle E is a trivial bundle and consequently the measure is well defined. In a physical language this means that there are no gravitational anomalies and this is equivalent to the cancellation of the conformal anomaly. Since the line bundle E is trivial there is just a unique (up to a constant multiple) global section, holomorphic and everywhere non zero on $\mathcal{M}_g$ and such section will have a second order pole at the infinity of $\mathcal{M}_g$. This remarkable result can be reassumed in the following:

Theorem 1.1.1. *The integration measure in the case of the bosonic closed strings is the squared modulus of the global holomorphic section of the bundle $E = K \otimes \lambda^{13}$, divided by the natural metric on λ.*

They conclude the paper with the important conjecture "any multiloop amplitude (not only the vacuum one) in any conformal invariant string theory (such as $D = 26$ bosonic, $D = 10$ supersymmetric and heterotic strings) may be deduced from purely algebraic objects (functions or sections of some holomorphic bundles) on moduli spaces $\mathcal{M}_g$ of Riemann surfaces".

In [MYu1] Manin, using holomorphic geometry, showed that the partition function for the bosonic string can be expressed in term of classical theta functions. This is an important result because the behavior of the theta constants under the action of the modular group is well known. This observation simplify the study of the modular properties of the amplitudes. Moreover, he proposed a similar expression for the superstring theory. That expression should be considered conjectural until an analogue of the Belavin - Knizhnik theorem is proved for fermionic string. Analogous results were find in [Moo, BKMP, Mo3, GP] where the bosoinic amplitudes was written in terms of classical theta functions up to genus four.

The extension of the Belavin and Knizhnik theorem to the supersymmetric case is at the moment an open problem. There are a lot of different approaches to construct the superstring amplitudes starting from the path integral. Superstring perturbation theory starts off with the sum over fermionic surfaces, given by the integral over D massless superfield $\mathbf{X}^\mu$ coupled to two dimensional supergravity. Actually, in the covariant fermionic string one studies [Ma2] the dynamics of the string coordinates together with their world sheet spinor partners ψ^μ. One can work in local complex coordinates on the $2d$ world surface. The spinors ψ^μ in the fermionic string can be combined with the coordinates X^μ to form d functions $\mathbf{X}^\mu(z, \bar{z}, \theta, \bar{\theta})$ on $2d$ super space. Intuitively a super space is a space with anti commuting partners θ and $\bar{\theta}$ for the complex coordinates z and $\bar{z}$. In this way the action defining the theory is just the one of a collection of free super fields for the case of string in a flat background and, more generally, one can consider the propagation of string in a background geometry described by a general non linear $2d$ QFT. The condition that the background be a solution to the classical string theory is that the corresponding non linear model was super conformally invariant (conformally invariant for the bosonic case).

Like in the bosonic case, one fixes a gauge and reduces the functional integral to an integral over the moduli space $s\mathcal{M}_g$ of super Riemann surfaces [BdVH, H, Pol]. Nevertheless, in this procedure there are a lot of subtle points we will briefly reassume here. Moreover, the holomorphic factorization for the final expression of the measure appearing in the integral is not obvious. This splitting seems to depend on some properties of the super Riemann surfaces and on the choice of local coordinates. In addition, one has to take into account the global properties of the moduli space of the super Riemann surfaces. Many different approaches to menage these problems has been developed. A super Riemann surface (SRS) is an extension of the ordinary Riemann surface in which also anti commuting variables appear. A SRS is locally described by a complex coordinate z together with a complex anti commuting coordinate θ. The local coordinate neighborhoods are patched together by so called super conformal transition functions. These are defined as analytic coordinate changes $(z, \theta) \to (z', \theta')$, satisfying the super conformal condition $Dz' = \theta' D\theta'$, where $D = \partial_\theta + \theta\partial_z$ is the super derivative. The general solution to this condition is $z' = f(z) + \theta(\partial f)\epsilon(z)$, $\theta' = (\partial f)^{1/2}(\theta + \epsilon(z) + 1/2\theta\epsilon\partial\epsilon(z))$, with $f(z)$ and $\epsilon(z)$ arbitrary commuting and anti commuting analytic functions respectively. The super moduli space $s\mathcal{M}_g$ of genus g super Riemann surfaces is, for $g \geq 2$, a complex super manifold of dimension $(3g-3, 2g-2)$. We will denote with m^i the $3g-3$ commuting coordinates and with $\hat{m}^a$ the $2g-2$ anti commuting super moduli. The presence of the odd super moduli of a super Riemann surface Σ represents the main difference between Σ and an ordinary Riemann surfaces with spin structure. Actually, in the absence of odd super moduli there exists a super conformal coordinate system on Σ for which all the transition functions are of the form $z_\alpha = f_{\alpha\beta} z_\beta$ and $\theta_\alpha = (\partial f_{\alpha\beta})^{1/2}\theta_\beta$. In this way, the transition functions for the commuting variable z define an ordinary Riemann surface Σ_{ord} and those of θ are the transition functions of a spinor bundle over Σ_{ord} with spin structure Δ determined by the choice of the square roots. There are 2^{2g} non equivalent spin structures corresponding to the 2^{2g} possible $\mathbb{Z}_2$ boundary conditions, θ periodic or anti periodic, around the $2g$ primitive homology cycles of a genus g surfaces. Thus, the bosonic manifold underling the $s\mathcal{M}_g$ is the spin moduli space $\mathcal{M}_g^{spin}$ which is a 2^{2g}-fold covering of the moduli space $\mathcal{M}_g$ of ordinary Riemann surfaces. A super Riemann surface with vanishing odd super moduli is called split. Not all the super Riemann surfaces are split. Locally the odd super moduli $\hat{m}^a$ may be considered to lie in a vector bundle over the space parametrized by the even super moduli m^i. It is an open question whether this is true globally, i.e. whether $s\mathcal{M}_g$ for $g \geq 2$ can be described globally as a vector bundle over $\mathcal{M}_g^{\text{spin}}$. If so, we would be able to chose a coordinate covering $s\mathcal{M}_g$ with patching functions of the form $\tilde{m}^i = f^i(m)$ and $\tilde{\hat{m}}^a = f^a_b(m)\hat{m}^b$ and in this case the super moduli space $s\mathcal{M}_g$ itself would be a split super manifold. If this is not the case, one will need transition functions of the general form $\tilde{m}^i = f^i(m) + g^i_{ab}(m)\hat{m}^a\hat{m}^b + \cdots$ and $\tilde{\hat{m}}^a = f^a_b(m)\hat{m}^b + g^a_{bcd}(m)\hat{m}^b\hat{m}^c\hat{m}^d + \cdots$. Consequently, the nilpotent part of the m^i transition functions obstructs the existence of a (unique) projection of $s\mathcal{M}_g$ onto $\mathcal{M}_g^{\mathrm{spin}}$.

As in the bosonic case, the string measure on $s\mathcal{M}_g$ is given by the product of the matter partition function, i.e. that of the string super field $\mathbf{X}^\mu$, times the ghost partition function. The action for the super fields $\mathbf{X}^\mu$ is $S[\mathbf{X}^\mu] = c\int D\mathbf{X}^\mu \bar{D}\mathbf{X}^\mu$, where c is a suitable constant. This action leads to the equation of motion $\bar{D}D\mathbf{X}^\mu$ with general solution $\mathbf{X}^\mu(z,\bar{z},\theta,\bar{\theta}) = \mathbf{X}^\mu(z,\theta) + \bar{\mathbf{X}}^\mu(\bar{z},\bar{\theta})$ and $\mathbf{X}^\mu(z,\theta) = x^\mu(z) + \theta\psi^\mu(z)$. The commuting string coordinates $x^\mu(z)$ are chiral scalar fields and their two dimensional super partners $\psi^\mu(z)$ are Majorana-Weyl fermions. We observe that the above decomposition of the chiral scalar super field $\mathbf{X}^\mu(z,\theta)$ into component fields should be read as a local statement, i.e. within each coordinate neighborhood on the surface and only for split super Riemann surfaces it is possibly to globally decompose super conformal super fields into components. As in the bosonic case one should add to the action the part of the ghosts. The Faddeev-Popov ghosts are given by two super fields $B(z,\theta)$ and $C(z,\theta)$ and their action is $S[B,C] = c\int B\bar{D}_{-1}C$.

The construction of the vacuum amplitude for the fermionic string and the related measure on the super moduli space of super Riemann surfaces was carry on by numerous authors and with several methods: studying the properties of the differential operators appearing in the path integral or using the Selberg trace formula, see for example [BMFS,BSc,BSh,Ma2]. Following [VV2,Ma2], starting from the Polyakov path integral, one finds that the g-loop partition function of the fermionic string is:

$$\mathcal{A}_g = \int_{s\mathcal{M}_g} \prod_i d^2m^i \prod_a d^2\hat{m}^a \mathcal{Z}_{fs}(m,\bar{m},\hat{m},\bar{\hat{m}}), \tag{1.1}$$

where $\mathcal{Z}_{fs} = Z_X|Z_{BC}|^2$ with the matter part given by $Z_X = \int[d\mathbf{X}^\mu]\exp(-S[\mathbf{X}^\mu])$ and the ghost part $Z_{BC} = \int[dBdC]\exp(-S[B,C])\prod_a \delta(<\hat{\mu}_a, B>)\prod_i <\mu_i, B>$ and $\{\mu_i,\hat{\mu}_a\}$ is a basis of super Beltrami differentials representing the coordinate vector fields $\{\partial/\partial m^i, \partial/\partial\hat{m}^a\}$ on $s\mathcal{M}_g$. The above functional integrals are evaluated by expanding the fields in an orthonormal basis of eigen modes of the corresponding laplacians. The formula (1.1) is the staring point for the construction of the superstring loop amplitudes. In [VV2] the authors suggest that the expression (1.1), actually can be factorized into a holomorphic times an anti holomorphic half density on the super moduli space. This factorization would reflect the structure of the superstring Hilbert space built up by means of creation operators, which are obtained as the coefficients in the mode expansion of the super fields $\mathbf{X}^\mu$. This set of operators can be divided into a left moving sector and a right moving sector and the only interaction between the sectors is given by the zero mode p^μ and q^μ that are equal for the left and right part. This reflects the fact that the string has only one centre of mass. The consistent construction of the Hilbert space was invented by Gliozzi, Scherk and Olive [GSO], who make use of the decoupling in left and right sector, by allowing the two chiralities of the world sheet fermions $\psi^\mu(z)$ and $\psi^\mu(\bar{z})$, to be independent either in the Neveu-Schwarz or in the Ramond sector, and further to project in both the left and the right sector on even fermion parity, $(-1)^{F_L} = (-1)^{F_R} = 1$. Hence, the superstring Hilbert space has the structure:

$$\mathcal{H}_{ss} = \bigoplus_{p^\mu} \left[\mathcal{H}_p^{NS,+} \oplus \mathcal{H}_p^{R,+}\right] \otimes \overline{\left[\mathcal{H}_p^{NS,+} \oplus \mathcal{H}_p^{R,+}\right]}, \tag{1.2}$$

where $+$ indicates the projection on even fermion number. In the fermionic string theory the left and right fermions have the same boundary conditions and the even fermion parity projection is non chiral, $(-1)^{F_L+F_R} = 1$, so the fermionic Hilbert space has the form:

$$\mathcal{H}_{fs} = \bigoplus_{p^\mu} \left[\mathcal{H}_p^{NS} \otimes \overline{\mathcal{H}_p^{NS}}\right]^+ \oplus \left[\mathcal{H}_p^{R} \otimes \overline{\mathcal{H}_p^{R}}\right]^+ . \tag{1.3}$$

The different choices of NS or R fermion boundary conditions are in one to one correspondence with the spin structures on the surface Σ. Hence, the fact that for the fermionic string the left and the right fermions have the same boundary conditions implies that in its path integral formulation we have to sum over fermionic surfaces with the same spin structures for the holomorphic and anti holomorphic world sheet spinors. For the superstring, instead, the independent boundary conditions of the two fermions chiralities means that we must allow for different spin structures for the left and right fermions, and sum over each independently. This summation procedures will automatically imply the GSO projection leading to the two types of II theories, see the discussion in [SW]. However, a difficulty with the prescription for the superstrings is that a non split super Riemann surface can only be defined if it is non chiral, i.e. θ and $\bar\theta$ have the same spin structure. Thus, the chiral decoupling of the two dimensional fermions in the superstring theory implies that its amplitudes can not simply be written as integrals over the super moduli space of super Riemann surfaces. In order to obtain a definition of the sueprerstring measure, in [VV2, VV3] a procedure to isolate the contributions of the left and right movers to the fermionic string integrand was proposed. The structure of $\mathcal{H}_{fs}$ suggests that its partition function on a g-loop surface can be written as an integral over a set of loop momenta p_i^μ, and moreover, that the integrand for fixed momenta factorizes into a holomorphic and anti holomorphic function of the super moduli. They was able to show that $\mathcal{Z}_{fs}(m,\bar m,\hat m,\bar{\hat m}) = \int dp_i^\mu |\mathcal{W}_p(m,\hat m)|^2$, with $i = 1,\cdots,g$. In particular the superstring vacuum amplitude has the following form:

$$\mathcal{A}_{ss} = \int_{\mathcal{M}_g} dm d\bar m \int p_i^\mu \mathcal{Z}_p(m)\bar{\mathcal{Z}}_p(\bar m), \qquad \mathcal{Z}_p(m) = \sum_\alpha \int d\hat m^a \mathcal{W}_p^\Delta(m,\hat m), \tag{1.4}$$

where $\mathcal{W}^\Delta$ is the chirally projected fermionic string partition function, defined before and δ labels the spinor structure. Furthermore, the authors was able to explicitly perform the integral over the odd super moduli and summing over all spin structures. This is an important point. Actually, a crucial feature of superstring loop amplitudes is their space time supersymmetry. For example, it can prevent the presence of massless tadpoles. In the NSR formulation space time supersymmetry arises after performing the GSO projection on even world sheet fermion parity, which in the path integral is implemented by summing over all the spin structures. However, because the spin structure is an intrinsic part of the geometry of a super Riemann surface, this sum can only be performed after integrating over the odd super moduli. This means that whereas on $s\mathcal{M}_g$ space time supersymmetry is a symmetry relating the contribution of different super Riemann surfaces, and therefore difficult to analyze, after reducing the integrand to $\mathcal{M}_g$ it is realized as a symmetry on each individual Riemann surface.

The first step to perform the integral over the odd moduli $\hat{m}^a$ is to isolate their dependence of the string integrand. To this aim one has to know the difference between the integrand on a super Riemann surface $\Sigma_{\hat{m}}$ and the one on the split surface Σ_0 obtained from $\Sigma_{\hat{m}}$ by setting all odd moduli to zero. This difference can be computed performing a redefinition of the fields in the functional integral on Σ_0, in such a way that the new fields satisfy the boundary conditions of $\Sigma_{\hat{m}}$. Let (z, θ) and $(\tilde{z}, \tilde{\theta})$ be the local complex super coordinates on Σ_0 and $\Sigma_{\hat{m}}$ respectively. These two coordinate systems are related to each other by a so called quasi-superconformal transformation. Such coordinate transformation can be read as the operation switching on the odd supermoduli and by means of it one can relate the fields defined on Σ_0 with the ones defined on $\Sigma_{\hat{m}}$. This implies that the anti commuting super moduli are contained just in the two dimensional gravitino filed $\hat{\chi}$. Choosing suitable coordinate on $s\mathcal{M}_g$ one can expand the gravitino as $\hat{\chi}(z, \bar{z}) = \sum_{a=1}^{2g-2} \hat{m}^a \chi_a(z, \bar{z})$, where the $2g-2$ differentials $\chi_a(z, \bar{z})$ are all independent of the odd supermoduli. We will see, in the next chapter, that the necessity to make a choice of coordinates can be the origin of some troubles in the procedure and make rise to some doubts on the general approach in the computation of the string partition function. In this way one integrates out the odd moduli. An analog result on the holomorphic factorization was reached by Sonoda in [So] where the measure in the supermoduli space is given by $\mathcal{Z}(z, \bar{z}, \theta, \bar{\theta}) = \frac{1}{\det \mathrm{Im}\, \mathcal{M}_{\alpha\beta}} |F(z, \theta)|^2$. Here, $\mathcal{M}_{\alpha\beta}$ is the so called superperiod matrix and $F(z, \theta)$ depends holomorphically on z and θ. However, in that paper it is not solved the essential problem of taking the left or right moving part of $\det \mathrm{Im}\, \mathcal{M}_{\alpha\beta}$. In a slightly different way D'Hoker and Pong in [DP9,DP10] obtained the same results for the chiral factorization. They made large use of the notion of superspace and defined the functional integrals as superdeterminants. We do not enter here in the details of such a procedure because we will analyze it in the next chapter to explain the starting point of our construction of superstring partition functions. For similar computations and discussion see also [B1, B2].

In [VV2] the important issue of the dependence of the string integrand on the choice of the basis of gravitino fields is taken into account. More details on such global questions can be find in [AMS, ARS]. This is an important point since there are many possible choices for the gravitino χ_a and none of them seems to be preferred over the others. It turns out that under the variation $\chi_a \to \chi_a + \Delta\chi_a$ the integrand changes with a total derivative. Thus, the total amplitude is independent of the choice of the gravitino slice, or in other terms the integral over the super moduli is uniquely defined, if this total derivative does not contribute to the integral. This is not a trivial problem, since $s\mathcal{M}_g$ has a complicated topology and, in addition, it has a boundary. The string integrand is in general singular on the boundary of the super moduli space that describe the degenerate Riemann surfaces. Therefore, the value of the total integral of the string measure is very sensitive to variations of the choice of gravitino fields near the boundary. Moreover, an important, and not yet solved, issue is whether or not there exists some, preferred globally defined, basis of gravitino fields, which can be used to obtain a unique integrand on $\mathcal{M}_g^{spin}$. This question is closely related to the problem of whether $s\mathcal{M}g$ is a split supermanifold or not. The problem of constructing an analytically varying basis of gravitino fields, globally defined modulo linear transformations, is exactly the same as trying to find a split analytic coordinates of $s\mathcal{M}_g$. The specification "analytically" is essential because any supermanifold allows a non analytic globally split coordinate covering. If the supermoduli space $s\mathcal{M}_g$ for general genus g turns out to be not split there is no unique integrand over the moduli space $\mathcal{M}_g$: it is only defined up to total derivative. These points make the construction of the superstring amplitudes a highly non trivial problem. To overcome these problems one has to consider the complicated structure of the supermoduli space and use geometric globally defined tools to have a deeper knowledge of such space.

Nevertheless the aforementioned problems, for the genus two case D'Hoker and Phong [DP1,DP2,DP3,DP4] were able to write explicitly the vacuum to vacuum amplitude in term of theta constants. Their construction and analysis was the starting point for our construction of the superstring measure for $g \leq 5$. For these reasons we review in some details the D'Hoker and Phong strategy in the next chapter.

Chapter 2

Perturbative formulation of string theory

In this chapter we, first, review in some details the construction of the partition function in the bosonic case. Actually, this is the starting point for the generalization to the supersymmetric theory. We exploit the complex geometric construction recalling the approach and the theorem of Belavin and Knizhnik, the result of Beilinson and Manin and their connection with the Mumford theorem. Then, we will generalize the bosonic theory with the supersymmetric one. To build the supersymmetric lagrangian one introduces the gravitino, the superpartner of the metric, and the Majorana spinors, the super partner of the coordinate fields. We will introduce the partition function and we recall how the functional integration is performed. As anticipate in the previous chapter, in this procedure there are some ambiguities leading to a dependence of the result on the specific choice of the basis for the gravitino. D'Hoker and Phong proposed a strategy to solve these problems and to define correctly the superstring amplitudes. We reassume their strategy and we indicate some aspects of this approach that hardly generalize to arbitrary genus greater than two.

2.1 The bosonic history

To understand the strategy underling the construction of vacuum-to-vacuum superstring amplitudes in the NRS formalism, we think it is a good choice to start with the main rigorous results for the bosonic case. For simplicity we will work only with closed strings.

In flat Minkowski space the Polyakov action on a Riemann surface Σ_g of genus g and with metric h is

$$\begin{aligned} I_g(X,h) &= \frac{1}{4\pi\alpha'}\int_{\Sigma_g} d^2z\sqrt{h}\; h^{ab}\partial_a X\cdot\partial_b X, \\ X &: \Sigma_g \hookrightarrow \mathbb{R}^D. \end{aligned}$$

It formally selects the weight measure for the path integral formulation of the bosonic partition function

$$Z^g_{bos} = \int [Dh_{ab}][DX] \exp(-I(X,h)), \tag{2.1}$$

where the functional sums are performed over all possible metrics over Σ_g and over all maps $X : \Sigma_g \hookrightarrow \mathbb{R}^D$. The whole partition function obviously involves a sum over all genera. When the path integral is well defined, one can extend it to compute amplitudes involving vertex operators for example momenta K_i^μ of mass $-m_i^2 = K_i \cdot K_i$

$$\left\langle \prod_{i=1}^N V_i(K_i) \right\rangle = \int [Dh_{mn}][DX] \exp(-I(X,h)) \prod_{i=1}^N V_i(K_i),$$

where, for example,

$$V_{-1}(K) = \int d^2z \sqrt{h}\; e^{iK\cdot X}, \qquad \text{for the Tachyon,}$$
$$V_0(K) = \int d^2z \sqrt{h}\; \partial_a X \cdot \partial^a X e^{iK\cdot X}, \qquad \text{for the Graviton.}$$

To define the path integral the main idea is to make use of the very large symmetry group of the classical theory that is the semidirect product $G = \mathrm{Weyl}(\Sigma_g) \ltimes \mathrm{Diff}(\Sigma_g)$ of the group of Weyl transformations times the group of diffeomorphisms of the Riemann surface Σ_g. If M is the set of all possible Riemannian metrics over Σ_g, then the moduli space for conformal classes of Riemann surfaces is $\mathcal{M}_g = M/G$. It is a finite dimensional complex manifold with holomorphic dimension

$$\dim_{\mathbb{C}} \mathcal{M}_g = \begin{cases} 0 & \text{if} \quad g = 0 \\ 1 & \text{if} \quad g = 1 \\ 3g-3 & \text{if} \quad g \geq 2. \end{cases}$$

At infinitesimal level, diffeomorphisms can be thought as locally generated by vector fields v and scalar fields ω:

$$\mathrm{Diff}(\Sigma_g) \quad : \quad \delta h_{ab} = \nabla_a v_b + \nabla_b v_a;$$
$$\mathrm{Weyl}(\Sigma_g) \quad : \quad \delta h_{ab} = 2\omega h_{ab}.$$

One expects to be able to reduce the path integral to a finite dimensional integral over $\mathcal{M}_g$. Indeed, the functional summation over the X fields can be easily performed being the action a quadratic functional of X. The Gaussian integration is then expressed in terms of the determinant of the Laplacian associated to the given metric h_{ab} over Σ_g, and which can be computed, for example, by means of a zeta function regularization. However, in computing the determinant one have to be careful about the existence of zero modes which must dropped out. Such regularization breaks the conformal invariance making the procedure anomalous so that the machinery fails to work. However, the anomaly disappears when the target space dimension is $D = 26$ and the path integral is hopefully well defined. This is a well known result which can be obtained in many ways and we will not review it here. We will rather look at the approach evidencing the complex geometry underling the Riemann surfaces in place of the spectral properties of the related Laplacian. This is indeed the approach introduced by Belavin and Knizhnik [BK], Beilinson and Manin [BM] and adopted by d'Hoker and Phong [DP7] in order to finally solve the problem of computing two loop superstring amplitudes.

2.1.1 Path integral and complex geometry

Let us now collect here some technical points which permits to reexpress the path integral formula in terms of some geometric data, suitable for obtaining the subsequent main theorems, see [DP7].

We have to evaluate the path integral (2.1) where the measure is defined by the metrics

$$\|\delta h\|^2 = \int_{\Sigma_g} \sqrt{\det h} h^{ab} h^{cd} \delta h_{ac} \delta h_{bd} d^2\zeta,$$
$$\|\delta X\|^2 = \int_{\Sigma_g} \sqrt{\det h} \delta X \cdot \delta X d^2\zeta.$$

To sum over inequivalent metric configurations we need to specify some coordinates over the moduli space. This means that we need to distinguish between symmetry transformations (diffeomorphisms and conformal transformations) and genuine transformations. A particular interesting choice is given by introducing isothermal coordinates, which are such that $ds^2 = 2h_{z\bar{z}} dz d\bar{z}$. Recall that to a given metric it is associated a complex structure ${J_a}^b = h^{\frac{1}{2}} \varepsilon_{ac} h^{ac}$, so that isothermal coordinates are determined by solving the Beltrami's equations

$${J_a}^b \frac{\partial z}{\partial \zeta^b} = i \frac{\partial z}{\partial \zeta^a}.$$

Deformations of the complex structure are parameterized with Beltrami differentials ${\mu_{i\bar{z}}}^z$, $i = 1, 2, \ldots, 3g-3$ so that

$$\delta h_{\bar{z}\bar{z}} = \sum_{i=1}^{3g-3} t^i h_{z\bar{z}} {\mu_{i\bar{z}}}^z, \tag{2.2}$$

for certain complex parameters t^i. It follows that two deformations μ, $\tilde{\mu}$ are equivalent in $\mathcal{M}_g$ if ${\tilde{\mu}_{\bar{z}}}^{\ z} - {\mu_{\bar{z}}}^{\ z} = \partial_{\bar{z}} v^z$ for some vector field v^z. It then results that the tangent moduli space is

$$T\mathcal{M}_g = \{\text{Beltrami differentials } {\mu_{i\bar{z}}}^z\}/\{\text{Range of } \partial_{\bar{z}} \text{ over vector fields }\}$$

The cotangent bundle is parameterized by quadratic differentials Φ_{zz} which are non singularly paired to Beltrami differentials by

$$\langle \mu | \Phi \rangle = \int d^2 {\mu_{\bar{z}}}^{\ z} \Phi_{zz}.$$

It is not difficult to see from this that the quadratic differentials are orthogonal to the G-transformations in the following sense. Let be (ω, v) a generator of an infinitesimal symmetry transformation: if the metric is $h = 2e^{2\phi}dzd\bar{z}$, then

$$\delta_{(\omega,v)}h = 2e^{2\phi}\bar{\partial}v(d\bar{z})^2 + 2e^{2\phi}\partial\bar{v}(dz)^2 + 2[2\omega e^{2\phi} + \partial(e^{2\phi}v) + \bar{\partial}(e^{2\phi}\bar{v})]dzd\bar{z}.$$

Note that the squared norm of this variation is

$$\|\delta_{(\omega,v)}h\|^2 = \int_{\Sigma_g} 2e^{2\phi}(2\omega + e^{-2\phi}\partial(e^{2\phi}v) + e^{-2\phi}\bar{\partial}(e^{2\phi}\bar{v}))^2 d^2\zeta + \int_{\Sigma_g} 2e^{2\phi}(\partial\bar{v})(\bar{\partial}v)d^2\zeta,$$

from which one easily computes the Jacobian of the transformation. It then follows that the metric deformations orthogonal to $\delta_{(\omega,v)}h$ are the ones of the form

$$\delta_\perp h = \Phi_{zz} + \bar{\Phi}_{\bar{z}\bar{z}}$$

where Φ are the holomorphic quadratic differentials ($\bar{\partial}\Phi = 0$).
The space of holomorphic quadratic differentials is a particular case of the space V^n of holomorphic n-differentials, holomorphic covariant tensors of rank n, for n positive, end contravariant of rank $-n$ for negative n. V^n is naturally provided with the hermitian product

$$\langle \tau, \tau' \rangle_n = \int_{\Sigma_g} (e^{2\phi})^{1-n}\bar{\tau}\tau' d^2\zeta.$$

Moreover, on V^n it acts the n-Laplacian

$$\Delta_n \tau = -\frac{1}{2}e^{2(n-1)\phi}\partial(e^{-2n\phi}\partial\tau).$$

A choice of a basis μ^i, $i = 1, \ldots, 3g-3$, of Beltrami differentials, so that the metric, according to (2.2), takes the form $h = 2e^{2\phi}|dz + \sum_i t^i\mu_i d\bar{z}|^2$, determine a dual basis Φ^j of V^2 such that $\langle \Phi^j, \mu_i \rangle = \delta^i_j$. Then, the matrix $G^{ij} = \langle \Phi^i, \Phi^j \rangle_2$ is invertible with inverse G_{ij} and

$$\mu_i = \frac{1}{2}e^{-2\phi}G_{ij}\bar{\Phi}^j.$$

Then we finally get

$$\|\delta_\perp h\|^2 = G_{ij}\delta t^i \delta \bar{t}^j,$$

and we are able to compute the Jacobian transformation

$$[Dh_{ab}] = \det\Delta_{-1}[D\omega][Dv]|\det\langle \mu_i|\Phi^j\rangle|^2 \det G_{ij}(1/2)^{3g-3}dt^1\wedge\ldots\wedge dt^{3g-3}\wedge d\bar{t}^1\wedge\ldots\wedge d\bar{t}^{3g-3}.$$

Dropping the gauge volume given by integration over ω and v, and performing the Gaussian integral[1] in X, we obtain the desired expression for the bosonic partition function

$$Z^g_{BOS} = \int d^{3g-3}t\; d^{3g-3}\bar{t}\; \frac{|\det\langle \mu_i|\Phi^j\rangle|^2}{\det\langle \Phi^i|\Phi^j\rangle} \det\Delta_{-1}\left(\frac{\det\Delta_0}{\int h_{z\bar{z}}d^2\zeta}\right)^{-\frac{D}{2}}. \tag{2.3}$$

[1]Which gives $(\det'\Delta_0/\int\sqrt{h}d^2\zeta)^{-\frac{D}{2}}$, where the prime means that zero modes do not contribute to the determinant, but contribute to the volume factor.

It is also possible to give a gauge invariant formulation, adding ghosts $b = b_{zz}dz^2$, $c = c^z dz^{-1}$, $I_{gh} = \frac{1}{2\pi}\int d^2z(b_{zz}\partial_{\bar{z}}c^z + \bar{b}_{\bar{z}\bar{z}}\partial_z\bar{c}^{\bar{z}})$ which satisfy

$$\int [D(b\bar{b}c\bar{c})]e^{-I_{gh}} \left| \prod_{i=1}^{3g-3} \langle \mu_i | b \rangle \right|^2 = \det(\bar{\partial}_2^\dagger \bar{\partial}_2) \frac{|\det\langle \mu_i | \Phi_j \rangle|^2}{\det\langle \Phi_i | \Phi_j \rangle}$$

and then

$$Z^g_{BOS} = \int [D(b\bar{b}c\bar{c}X)] \prod_{i=1}^{3g-3} |\langle \mu_i | b \rangle|^2 e^{-(I_g + I_{gh})}. \tag{2.4}$$

The expression (2.3) is generically anomalous, presenting two kinds of anomalies. The first one is crucial for defining the theory and is the conformal anomaly, which breaks invariance of Z^g_{BOS} under Weyl transformations. By means of Heat-Kernel methods one can show that[2], if $\Phi^{(n)} \in V^n$,

$$\delta_\omega \log \frac{\det \Delta_n}{\det\langle \Phi_i^{(n)} | \Phi_j^{(n)} \rangle \det\langle \Phi_i^{(1-n)} | \Phi_j^{(1-n)} \rangle} = -\frac{6n^2 - 6n + 1}{6\pi} \int_{\Sigma_g} d^2z \sqrt{h} R \omega,$$

where R is the worldsheet scalar curvature. Using this in $\delta_\omega Z_{BOS}$ we see that the conformal anomaly vanish if $D = 26$. Thus, conformal invariance constraints the theory to work with a 26 dimensional space (at least in the case when the target space is the flat Minkowski space).

The second anomaly is the so called Holomorphic anomaly. The Beltrami equation says that the operator $\partial_{\bar{z}}$ depends holomorphically on the moduli: $\partial_{\bar{z}} \to \partial_{\bar{z}} - \mu_{\bar{z}}{}^z \partial_z$. Now, expression (2.4) makes evident that Z^g_{BOS} should be expected to behave as the square modulus of an holomorphic function. An obstacle for this to happen comes from the Belavin-Knizhkin theorem:

$$\delta_{\bar{\mu}}\delta_\mu \log \frac{\det(\bar{\partial}_n^\dagger \bar{\partial}_n)}{\det\langle \Phi_i^{(n)} | \Phi_j^{(n)} \rangle \det\langle \Phi_i^{(1-n)} | \Phi_j^{(1-n)} \rangle} = -\frac{6n^2 - 6n + 1}{6\pi} \int_\Sigma d^2z \nabla_{\bar{z}}\mu \nabla_z \bar{\mu}\omega.$$

Note however that, exactly as for the conformal anomaly, the holomorphic anomaly vanishes if $D = 26$. This fortunate coincidence (if this is the case) permits to eliminate both anomalies simultaneously. The importance to cancel the holomorphic anomaly for bosonic strings is not an evident fact as, however it is an advantageous fact because as it permits to reexpress the partition function in terms of global objects, as it has been shown in the remarkable paper of Belavin and Knizhkin [BK], and in a work of Beilinson and Manin [BM] and Manin [MYu1, MYu2].

A starting point is a theorem of Mumford [Mu2] which was able to proof that the linear bundle $U = K \otimes \lambda^{-13}$ is a holomorphically trivial bundle over $\mathcal{M}_g$. Here K is the canonical bundle over $\mathcal{M}_g$, that is the highest wedge power of the cotangent bundle. A local basis is $\Phi_1 \wedge \ldots \wedge \Phi_{3g-3}$. Similarly, λ is the Hodge bundle over $\mathcal{M}_g$, the highest

[2] To be more precise, a diverging additive term depending only on the worldsheet volume appears, but it is innocuous and can be adsorbed in a constant counterterm in the bosonic string action.

wedge power of the holomorphic cotangent bundle, which is generated by the Abelian differentials $\omega_1 \wedge \ldots \wedge \omega_g$. As a consequence, U admits an essentially unique global holomorphic section ψ_g, the Mumford section. Expressed in terms of a local basis the Mumford theorem says that it exists a unique holomorphic function over $\mathcal{M}_g$ such that

$$\psi_g = F \frac{\Phi^1 \wedge \ldots \wedge \Phi^{3g-3}}{(\omega_1 \wedge \ldots \wedge \omega_g)^{13}}$$

is a global section of U. Moreover, ψ_g is nonvanishing everywhere, and meromorphic at infinity with an order two pole.
The Belavin-Knizhkin theorem implies that the bosonic partition function, apart from a constant, is given by the square modulus of the Mumford section:

$$Z_{BOS} = c_g \int |F|^2 (-i)^g \Phi_1 \wedge \ldots \wedge \Phi_{3g-3} \wedge \bar{\Phi}_1 \wedge \ldots \wedge \bar{\Phi}_{3g-3} | \det \int_{\Sigma_g} \bar{\omega}_I \wedge \omega_J |^{-13}. \quad (2.5)$$

The Mumford theorem has been proved by Beilinson and Manin [BM] in a strongest form which permits to be more explicit. Moreover, Manin [MYu1, MYu2] has been able to provide explicit expressions in terms of theta functions. Recall that a canonical basis A_I, B_I, $I = 1, 2, \ldots, g$ for the homology of Σ_g, $A_I \cap A_J = B_I \cap B_J = 0$, $A_I \cap B_J = \delta_{IJ}$, one can associate a canonical basis ω_I for the Abelian differentials by

$$\int_{A_I} \omega_J = \delta_{IJ}.$$

Then

$$\Omega_{IJ} := \int_{B_I} \omega_J$$

define the period matrix which is an element of the Siegel upper half-plane $\mathbb{H}_g$, that is symmetric with imaginary part positive definite. The column of Ω define an integer lattice T_Ω and an associated Jacobian variety $J_\Omega = \mathbb{C}^g / T_\Omega$. Conversely, from Ω one can recover the starting Riemann surface. Inserting into the algebraic geometric expression (2.5) one finds:

$$Z^g_{BOS} = c_g \int | \prod_{I \leq J} d\Omega_{IJ} |^2 \det(Im\Omega)^{-13} |\Phi(\Omega)|^{-2},$$

$\Phi = 1/F$. The partition function Z^g_{BOS} is independent from the choice of the canonical basis[3]. Two different canonical basis differ by a symplectic transformation

$$\begin{pmatrix} A'_I \\ B'_J \end{pmatrix} = M \begin{pmatrix} A_I \\ B_J \end{pmatrix},$$

$$M = \begin{pmatrix} U & T \\ V & Z \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

[3]The local expression for the global section depend on such a choice, but not the global section itself.

Correspondingly the period matrix transform as

$$\Omega \longrightarrow (U\Omega + T)(V\Omega + Z)^{-1}.$$

It then follows that necessarily Φ is a modular form of weight $12 - g$

$$\Phi(\Omega) \longrightarrow \Phi(\Omega)(\det(V\Omega + Z))^{12-g}.$$

For example, at genus $g = 2$, $\Phi(\Omega)$ has weight ten. A theorem due to Igusa states that at genus two, modular forms realize a polynomial ring with four generators ψ_i, of weights $i = 4, 6, 10, 12$. Thus we must have

$$\Phi(\Omega) = \alpha\psi_4\psi_6 + \beta\psi_{10}.$$

On the other hand, the partition function must satisfy the clustering condition, that is if we pinch the surface Σ_2 separating it into the union of two tori $\Sigma_{g=2} \to \Sigma_1(\Omega_1) \cup \Sigma_1(\Omega_2)$, with complex parameters Ω_1 and Ω_2, then the partition function factorizes as $Z_2 = Z_1 Z_1$. This selects $\alpha = 0$ so that

$$\Phi(\Omega) = \psi_{10} = \prod_{\delta \text{ even}} \theta[\delta](0, \Omega)$$

and

$$Z^g_{BOS} = c_2 \int |\prod_{I \leq J} d\Omega_{IJ}|^2 \prod_{\delta \text{ even}} |\theta[\delta](0,\Omega)|^{-2} (\det Im\Omega)^{-13}.$$

Here $\theta[\delta](z, \Omega)$, $z \in \mathbb{C}^g$ (the covering space of the Jacobian variety J), $\Omega \in \mathbb{H}_g$, are the theta function with characteristic[4] $[\delta] = [{a \atop b}] \equiv a + \Omega b$, $a, b \in \mathbb{Z}^g$ defined by

$$\theta[\delta](z, \Omega) = e^{i\pi b \cdot (z + \frac{a}{2}) + i\frac{\pi}{4} b \cdot \Omega b} \theta(z + a + \Omega b, \Omega), \qquad \theta(z, \Omega) = \sum_{m \in \mathbb{Z}^g} e^{2i\pi m \cdot z + i\pi m \cdot \Omega m}.$$

2.1.2 Conclusion

What we have learned from the bosonic history is that, passing through the algebraic-geometric description, one is led to a global description of the amplitudes measure. This provides a rigorous and almost well defined expression, the only impediment being represented by the divergence due to the pole at infinity of the section. This can be indeed imputed to the presence of the tachyon in the bosonic spectrum.

By means of the GSO projection the tachyon disappears from the spectrum of supersymmetric strings so that we could expect that a similar treatment extended to the supersymmetric case should lead to a completely well defined expression for the partition function. Unfortunately supersymmetry makes all things much more difficult and actually an analogue global description is not yet available. We will see now why this happens and a possible strategy proposed by D'Hoker and Phong and worked out at genus two [DP1], [DP2], [DP3], [DP4].

[4] In the second member a, b ar thought as row vectors and in the last as column vectors.

2.2 Supersymmetric strings

We can try to generalize directly the bosonic realization to the supersymmetric case. On the Riemann surface Σ_g there are 2^{2g} possible spin structures. For any choice among them, one can define spinor fields over the surface. To the metric h_{ab} one can then define its superpartner χ_a^α, the gravitino. The coordinate fields x^μ have as superpartners Majorana spinors ψ^μ. For any fixed spin structure one can then define the superstring action

$$I_{g,\delta} = \frac{1}{4\pi\alpha'}\int_{\Sigma_g} d^2z\sqrt{h}[\frac{1}{2}h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu - \frac{i}{2}\psi^\mu\gamma^\alpha D_\alpha\psi_\mu - \frac{1}{2}\psi^\mu\gamma^a\gamma^\alpha\chi_a\partial_\alpha x_\mu$$
$$+\frac{1}{8}\psi^\mu\gamma^a\gamma^b\chi_a(\chi_b\psi_\mu)] + \lambda\mathcal{X}(\Sigma_g),$$

where $\mathcal{X}$ is the Euler characteristic. Here we reserved Greek indices for tangent directions and Latin indices for flat direction. In particular the metric is related to a zweibein e_α^a via the usual relation $g_{\alpha\beta} = e_\alpha^a e_\beta^b\delta_{ab}$. The gamma matrices satisfy $\{\gamma^a,\gamma^b\} = -2\delta^{ab}$. The covariant derivative is $D_\alpha\psi_\mu = \partial_\alpha - \frac{i}{2}\omega_\alpha\gamma^1\gamma^2$, ω being the spin connection. The spinor fields behave as half-integer powers of differentials so that we can write

$$\psi = \psi_+ dz^{\frac{1}{2}}, \qquad \bar{\psi} = \psi_- d\bar{z}^{\frac{1}{2}}, \qquad \chi = \chi_{\bar{z}}^+ d\bar{z}dz^{-\frac{1}{2}}, \qquad \bar{\chi} = \chi_z^- dz d\bar{z}^{-\frac{1}{2}}.$$

The symmetries of the action are thus extended to supersymmetries by adding to diffeomorphisms Diff(Σ_g)

$$\begin{aligned}
\delta_v e_\alpha^a &= v^\beta\partial_\beta e_\alpha^a + e_\beta^a\partial_\alpha v^\beta,\\
\delta_v\chi_a &= v^\beta\partial_\beta\chi_a + \chi_b e_a^\alpha\partial_\alpha v^b,\\
\delta_v x^\mu &= v^\alpha\partial_\alpha x^\mu,\\
\delta_v\psi^\mu &= v^\alpha\partial_\alpha\psi^\mu,
\end{aligned}$$

and Weyl transformations Weyl(Σ_g)

$$\begin{aligned}
\delta_\omega e_\alpha^a &= \omega e_\alpha^a,\\
\delta_\omega\chi_a &= \frac{\omega}{2}\chi_a,\\
\delta_\omega x^\mu &= 0,\\
\delta_\omega\psi^\mu &= -\frac{\omega}{2}\psi^\mu,
\end{aligned}$$

the supersymmetry transformations Susy(Σ_g)

$$\begin{aligned}
\delta_\epsilon e_\alpha^a &= i\epsilon\gamma^a\chi_\alpha,\\
\delta_\epsilon\chi_a &= 2D_a\epsilon,\\
\delta_\epsilon x^\mu &= \epsilon\psi^\mu,\\
\delta_\epsilon\psi^\mu &= -\frac{i}{2}(\gamma^\beta\epsilon)(\partial_\beta x^\mu - \frac{1}{2}\chi_\beta\psi^\mu),
\end{aligned}$$

and super-Weyl transformations $\mathrm{SW}(\Sigma_g)$

$$\begin{aligned}\delta_\lambda e^a_\alpha &= 0,\\ \delta_\lambda \chi_a &= \gamma_a \lambda,\\ \delta_\lambda x^\mu &= 0,\\ \delta_\lambda \psi^\mu &= 0.\end{aligned}$$

Here ϵ and λ are spinors and all spinor index, which we have omitted everywhere, are contracted in an obvious way along the $NW - SE$ convention.
One has thus to compute the partition function (for fixed spin structure δ)

$$Z^g_\delta = \int [Dh_{\alpha\beta}][D\chi_a][Dx^\mu][D\psi^\mu] \exp(-I_{g,\delta}). \tag{2.6}$$

The measure is the one inherited by the bosonic metric plus the metric for spinor deformations

$$\|\delta\psi\|^2 = \int_{\Sigma_g} \delta\bar{\psi}^\mu \delta\psi_\mu \sqrt{h} d^2\zeta, \qquad \|\delta\chi_a\|^2 = \int_{\Sigma_g} \delta\bar{\chi}_a \delta\chi_b h^{ab} \sqrt{h} d^2\zeta.$$

Moreover, it is convenient to introduce a norm in the space of supersymmetry deformations

$$\|\epsilon\|^2 = \int_{\Sigma_g} \bar{\epsilon}\epsilon \sqrt{h} d^2\zeta.$$

Again, we can use the huge symmetry group to reduce the path integral to a finite dimensional integration. The moduli space to be considered is now

$$\mathcal{SM}_g = (\{h_{\alpha\beta}\} \times \{\chi_a\})/(\mathrm{Diff}(\Sigma_g) \times \mathrm{Weyl}(\Sigma_g) \times \mathrm{Susy}(\Sigma_g) \times \mathrm{SW}(\Sigma_g)). \tag{2.7}$$

To better understand it one can study its tangent bundle (the super Teichmüller space). Locally it splits into the usual bosonic Teichmüller space plus a novel part described by the gravitinos deformations. Under symmetry transformations gravitinos deform as

$$\delta_{(v,\omega,\epsilon,\lambda)} \chi_a = \frac{\omega}{2} \chi_a + \gamma_a \lambda + v^\beta \partial_\beta \chi_a + \chi_\beta e^\alpha_a \partial_\alpha v^\beta + 2D_a \epsilon.$$

Genuine deformations $\delta_\perp \chi_a$ are then defined by the spinor deformations orthogonal to symmetry deformations. Let us define the operator $P_{\frac{1}{2}}$ sending $\frac{1}{2}$ spinors to $\frac{3}{2}$ spinors, defined by

$$[P_{\frac{1}{2}}\epsilon]_a := 2D_a\epsilon + \gamma_a \gamma^\beta D_\beta \epsilon.$$

Its adjoint $P^\dagger_{\frac{1}{2}}$, acting on $\frac{3}{2}$ spinors, is defined by

$$\langle P^\dagger_{\frac{1}{2}} \delta\chi, \epsilon\rangle = \langle \delta\chi, P_{\frac{1}{2}}\epsilon\rangle,$$

where the scalar products are the ones associated to the metrics defined above. It results that $\{\delta_\perp \chi\} = \mathrm{Ker} P^\dagger_{\frac{1}{2}}$. Moreover $\mathrm{Ker} P^\dagger_{\frac{1}{2}} = V^{\frac{3}{2}}$, the holomorphic $\frac{3}{2}$-differentials.

Indeed, it happens that $V^2 \oplus V^{\frac{3}{2}}$ can be seen as a superspace, so that the true deformations of gravitinos are measured by the superpartners of the quadratic differentials. An application of the Riemann-Roch theorem shows that the super moduli space is a complex super manifold of dimension[5] $(3g-3|2g-2)$. We will not investigate this further, all details can be found in the lecture notes of D'Hoker and Phong [DP7], see also Nelson [GN1, GN2] for a deeper analysis of super moduli spaces. We only mention that, correspondingly, the Beltrami differential will be extended to super-Beltrami differentials whose odd components $\Theta_{a,u}$, similarly to the even ones, will parameterize the gravitino deformations so that, in local coordinates $(t^i, \theta^u) \in \mathbb{C}^{(3g-3|2g-2)}$,

$$\Theta_{a,u} = \frac{\partial \chi_a}{\partial \theta^u} + \frac{1}{2}\gamma_a \gamma^b \frac{\partial \chi_b}{\partial \theta^u},$$

and the orthogonal deformations are

$$\delta_\perp \chi_a = \Gamma_{vw} \langle \Pi^w, \Theta_u \rangle \Pi^v_a \theta^u.$$

Here $\{\Pi^u_a\}_{u=1}^{2g-2}$ is a basis of $V^{\frac{3}{2}}$, and Γ_{vw} is the inverse matrix of $\Gamma^{vw} = \langle \Pi^v, \Pi^w \rangle$. Proceeding as for the bosonic case one finally arrive to the expression

$$Z^g_\delta = \int [Dx^\mu][D\psi^\mu] d\mu_t d\mu_\theta \det \Delta_{-1} \frac{|\det\langle \Phi^i, \mu^j \rangle|^2}{\det\langle \Phi^i, \Phi^j \rangle} \det \Delta_{-\frac{1}{2}} \frac{|\det\langle \Pi^u, \Theta^v \rangle|^2}{\det\langle \Pi^u, \Pi^v \rangle} e^{-I_{g,\delta}}, \quad (2.8)$$

where $d\mu_t = dt^{3g-3} d\bar{t}^{3g-3}$ and $d\mu_\theta = d\theta^{2g-2} d\bar{\theta}^{2g-2}$. Note that we have yet integrated out the symmetry group degrees of freedom. Again, this can be verified computing the Weyl variation. One obtains that in this case anomalies disappear if $D = 10$. Using the symmetry transformations, we can project the gravitinos on the $V^{\frac{3}{2}}$ part so that

$$\chi_a = \sum_{u=1}^{2g-2} \zeta_u \Pi^u_a,$$

where ζ_u are fermionic coordinates, which we will use in place of the θ^u (this give not any nontrivial contribution to the Jacobian). In particular this imply $\gamma^a \chi_a = 0$ so that the action takes the simpler form[6]

$$I_{g,\delta} = \int_{\Sigma_g} d^2 z \sqrt{h} [\frac{1}{2} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu - \frac{i}{2} \psi^\mu \gamma^\alpha D_\alpha \psi_\mu + \psi^\mu \chi^\alpha \partial_\alpha x_\mu - \frac{1}{4} \psi^\mu \chi^a \psi_\mu \chi_a].$$

We can now first integrate over the x^μ configurations. This gives

$$\int [Dx^\mu] e^{-I_\delta} = \left(\frac{\det \Delta_0}{\int h_{z\bar{z}} d^2\zeta} \right)^{-\frac{D}{2}} e^{\frac{i}{2} \int \sqrt{h} \psi^\mu \gamma^\alpha D_\alpha \psi_\mu d^2 z} e^{-\frac{1}{2} \sum_{u,v} \zeta_u \zeta_v K^{uv}},$$

$$K^{uv} = \int d^2 z d^2 z' \sqrt{h(z,\bar{z}) h(z',\bar{z}')} \psi^\mu \Pi^{u,a}(z) \psi_\mu \Pi^{v,b}(z') \Sigma_{a,b}(z,z'),$$

$$\Sigma_{a,b}(z,z') = \frac{\partial^2}{\partial z^a \partial z'^b} \Delta_0^{-1} - \frac{1}{2} h_{ab} \delta(z,z').$$

[5]We mention here only the case of genus $g \geq 2$.

[6]We also set $4\pi\alpha' = 1$ for simplicity, and omitted the topological term.

Integration over ζ_u then gives the Pfaffian of K, and, finally, integration over ψ^μ gives

$$Z^g_\delta = \int d\mu_t \det\Delta_{-1} \frac{|\det\langle\Phi^i,\mu^j\rangle|^2}{\det\langle\Phi^i,\Phi^j\rangle} \det\Delta_{-\frac{1}{2}} \frac{|\det\langle\Pi^u,\Theta^v\rangle|^2}{\det\langle\Pi^u,\Pi^v\rangle} \left(\frac{\det\Delta_0}{\int h_{z\bar{z}} d^2\zeta}\right)^{-5} \cdot(\det{}'(\gamma^a D_a))^5 \int [D\psi^\mu_0] < \mathrm{Pfaff} K >_{\psi'}, \qquad (2.9)$$

where a prime indicates that zero modes are dropped, ψ^μ_0 are zero modes of $\gamma^a D_a$ and $< \mathrm{Pfaff} K >_{\psi'}$ indicates expectation value.

2.2.1 An unfortunate history

Expression (2.9) is conformally invariant and apparently gives the desired result. However, here is where the problems start. A first problem is to realize a chiral splitting. This is a first point where the absence of holomorphic anomaly should be very helpful. Indeed, the obtained expression is only for fixed spin structure δ. To define the full super string theory model one need to sum up over all spin structures taking account of the GSO projection in order to eliminate the tachyon. But GSO projection acts separately on the chiral modes so that the splitting becomes essential.

But even before to solve this problem, one must recognize that (2.9) is ambiguous.

One expects for the partition function to be independent from the choice of the parametrization of the moduli, that is from the choice of the super Beltrami differentials. Any change should eventually add to the integral some boundary terms which should vanish. But this is not what happen: the boundary terms do not generically vanish.

The ambiguity was first noted by Verlinde and Verlinde [VV1]. In [MM], Moore and Morozov analyzed the problem on the light of some consistency conditions superstring theories should satisfy: modular invariance, vanishing of the cosmological constant, and nonrenormalization theorems. In particular, they have computed the difference between type II and heterotic partition functions for genus two surfaces, showing that they differ by a positive term, so that they seem to be not simultaneously consistent. The physical origin of the ambiguity has been further investigated in [ARS], se also Morozov and Perelomov [MP]. Here they computed the $g = 2$ heterotic partition function by choosing an explicit basis for the super Beltrami differentials, represented by δ-functions supported on fixed points z_u ($u = 1, 2$). It then results an explicit dependence on the points in the sense that changing the points gives rise to a shift of the integrand by a differential term which does not vanish on the boundary. The boundary of the moduli space contains two disjoint tori with two marked points p_u. There, it is shown that the ambiguity disappears if one choose z_u in such the way that $z_u = p_u$ on the boundary.

A decisive analysis of the ambiguities can be found in [AMS], where global issues are considered. They found that the superstring measure is a total derivative so that all problems are related to the boundary conditions. Many peculiarities of the ambiguities are put in light but the analysis do not provide a prescription able to eliminate them. We will demand to the cited literature all details and will not discuss it here further.

2.2.2 The D'Hoker and Phong strategy

A proposal for solving the ambiguity problem comes from D'Hoker and Phong [DP7]. The main idea is that the problems have origin in a wrong choice for the slice parametrization, that is the choice of the metric to select the bosonic component of the slice is not a good one. Suppose that a slice is selected by a choice $(h_{\alpha\beta}, \chi_\alpha)$. After a supersymmetric transformation one obtains a new coordinatization $(\tilde{h}_{\alpha\beta}, \tilde{\chi}_\alpha)$. If the metric were a good selection for the bosonic components, then the projector $\phi : (h_{\alpha\beta}, \chi_\alpha) \mapsto h_{\alpha\beta}$ would be supersymmetry preserving in the sense that the supersymmetry transformation would induce a diffeomorphism (eventually composed with a conformal transformation) $h_{\alpha\beta} \mapsto \tilde{h}_{\alpha\beta}$. But this results not to be the case and in general $h_{\alpha\beta}$ and $\tilde{h}_{\alpha\beta}$ are not related by a bosonic symmetry.
Their main idea is to substitute the metric with the period matrix associated to the Riemann surface to be considered. As we said, the choice of a canonical basis for holomorphic differentials, associated to a given symplectic basis $\{A_I, B_J\}$ of $H_1(\Sigma_g, \mathbb{Z})$ defines a $g \times g$ period matrix $\Omega = \{\Omega_{IJ}\} \in M_g(\mathbb{C})$. It can be shown that the period matrix lies in the Siegel upperhalf space

$$\Omega \in \mathbb{H}_g := \{\tau \in M_g(\mathbb{C}) : \tau = {}^t\tau,\ \mathrm{Im}\tau > 0\}. \tag{2.10}$$

The important fact is that Torelli's theorem stating that Σ_g is completely characterized by its period matrix.
After introducing a suitable superdifferential description of super-Riemann surfaces, D'Hoker and Phong introduce the concept of super holomorphic differentials $\hat{\omega}_I = \omega_{I0} + \theta\omega_{I1}$, θ being an odd variable, which are associated to a symplectic basis by

$$\oint_{A_I} dz d\theta \hat{\omega}_J = \delta_{IJ},$$

and define the super period matrix

$$\hat{\Omega}_{IJ} = \oint_{B_I} dz d\theta \hat{\omega}_J.$$

Indeed, they showed that

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int d^z d^2 z' \omega_{I0}(z) \chi_{\bar{z}}^+ S_\delta(z, z') \chi_{\bar{z}'}^+ \omega_{J0}(z'),$$

where S_δ is the Szegö kernel, the unique solution of the equation

$$\partial_{\bar{z}} S_\delta(z, z') + \frac{1}{8\pi} \chi_{\bar{z}}^+ \int d^2 w \chi_{\bar{w}}^+ \partial_z \partial_w \ln E(z, w) S_\delta(w, z') = 2\pi\delta(z, z'),$$

and $E(z, w)$ is the prime form. $\hat{\Omega}$ is indeed supersymmetric and does not suffer the defects of the metric. In a long series of remarkable papers, they have been able to prove that the super period matrix prescription provides a well defined result for the amplitudes. Most of calculation are explicitly developed for the genus two case, but they argued that in principle they should work at any g. For $g = 2$, in particular, slice independence has been verified as well as the nonrenormalization theorems. After integrating over the odd moduli, a nice expression for the $g = 2$ whole vacuum amplitude was found to be

$$Z^2 = \int_{\mathcal{M}_2} (\det \mathrm{Im}\, \Omega)^{-5} \sum_{\delta\delta'} c_{\delta\delta'} d\mu[\delta](\Omega) \wedge \overline{d\mu[\delta'](\Omega)}, \tag{2.11}$$

where $c_{\delta\delta'}$ are phases realizing the right GSO projection and

$$d\mu[\delta](\Omega) = \frac{\theta^4[\delta](0,\Omega)\Xi_6[\delta](\Omega)}{16\pi^6\psi_{10}(\Omega)} \prod_{I\leq J} d\Omega_{IJ}, \tag{2.12}$$

$$\Xi_6[\delta](\Omega) := \sum_{1\leq i<j\leq 3} \langle\nu_i|\nu_j\rangle \prod_{k=4,5,6} \theta^4[\nu_i + \nu_j + \nu_k](0,\Omega)\ , \tag{2.13}$$

where each even spin structure[7] δ is written as a sum of three distinct odd spin structures $\delta = \nu_1 + \nu_2 + \nu_3$ and ν_4, ν_5, ν_6 denote the remaining three distinct odd spin structures, and

$$\langle\kappa|\lambda\rangle := e^{\pi i(a_\kappa\cdot b_\lambda - b_\kappa\cdot a_\lambda)}, \qquad \kappa = [{}^{a_\kappa}_{b_\kappa}], \ \ \lambda = [{}^{a_\lambda}_{b_\lambda}].$$

Finally, ψ_{10} is the Igusa form

$$\psi_{10} = \prod_{\delta \text{ even}} \theta[\delta](0,\Omega). \tag{2.14}$$

D'Hoker and Phong claimed that after integrating over odd moduli:

$$Z^g = \int_{\mathcal{M}_g} (\det Im\Omega)^{-5} \sum_{\Delta\Delta'} c_{\Delta\Delta'} d\mu[\Delta](\Omega) \wedge \overline{d\mu[\Delta'](\Omega)} \tag{2.15}$$

with

$$d\mu[\Delta](\Omega) = d\mu_{BOS}(\Omega)\Xi_8(\Delta),$$
$$\Delta = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad a, b \in \mathbb{Z}_2^g,$$

and $\Xi_8[\Delta](\Omega)$ are equivariant modular forms[8] (the symplectic group acts on the characteristics Δ also). In [DP5] a detailed analysis of the $g = 2$ results led D'Hoker and Phong to make an ansatz for the measure at genus 3. The genus three bosonic measure is

$$\mathrm{d}\mu_B^{(3)} = \frac{c_3}{\Psi_9(\Omega)} \prod_{I\leq J} \mathrm{d}\Omega_{IJ}$$

[7]In what follow we will indicate spin structures of any genus with Δ, just for the case $g = 2$ we will use, as D'Hoker and Phong, δ.

[8]Will be much more detailed in the next sections.

where $\Psi_9^2(\Omega)$ is a Siegel modular form of weight 18 for $Sp(6,\mathbb{Z})$. By analogy with the $g=2$ case, D'Hoker and Phong [DP5] proposed that the genus three chiral superstring measure is of the form

$$\mathrm{d}\mu[\Delta] = \frac{\theta[\Delta]^4(0,\Omega)\Xi_6[\Delta](\Omega)}{8\pi^4\Psi_9(\Omega)}\prod_{I\leq J} d\Omega_{IJ},$$

and they gave three constraints on the functions $\Xi_6[\Delta^{(3)}](\tau^{(3)})$. Note that they assume $\Xi_8[\Delta](\Omega) \equiv \theta[\Delta]^4(0,\Omega)\Xi_6[\Delta](\Omega)$. However, they did not succeed in finding functions which satisfy all their constraints [DP6].
The failure in finding such a solution is not simply due to the formidable problem of searching it in a very big space of modular forms without making use of a systematic procedure, but mainly on the fact that it does not exist, as we will prove in Section 4.4.5 and in Section 4.4.7 where the representation theory of group is employed (see also [CDG1] Section 4.4 or [DvG] Section 8.3). In particular, as was also remarked by Morozov, the main obstacle in finding a solution was the too strong and prejudicious imposition for the measure to be proportional to the fourth power of $\theta[\Delta](0,\Omega)$. After eliminating this condition and adapting the D'Hoker and Phong constraints to the most general ansatz, led us to determine the existence of a unique solution at genus three and at genus four and to find a solution, even though not unique, for the genus five case. This will be the main argument of the rest of this thesis.

2.3 The general ansatz

Our starting point consists in assuming the validity of (2.11) to be true. This is a crucial point so that some criticisms have been to be considered. Before discussing such points, let us finish to expose our approach. As discussed by Morozov [Mo1] there are two strategies to deal with superstring measures. The first one is the more direct one, that is by direct integration of odd moduli after holomorphic factorization. The second one is to use general considerations to deduce a reasonable guess for the measure and then to use its properties to determine its final form. We will follow this second approach.
We will assume the problem of computing the bosonic measure as resolved and well known. We have seen that the bosonic partition function density is

$$|d\mu_{BOS}^{(g)}|^2(\det\mathrm{Im}\Omega)^{-13} = |\prod_{I\leq J} d\Omega_{IJ}|^2\det(\mathrm{Im}\Omega)^{-13}|\Phi(\Omega)|^{-2}.$$

Invariance under modular transformations requires precise modular properties for $d\mu_{BOS}$. This structure is very strongly supported by global issues in algebraic and complex geometry. For super symmetric strings one instead obtains the much more less supported expression (2.11). If the Belavin-Knizhnik and Manin-Mumford arguments really play a crucial role in superstring theory too, then, looking again at modular properties, it is natural to expect for the superstring measure at fixed chirality to be related to the bosonic measure (Mumford form) by the relation

$$d\mu^{(g)}[\Delta^{(g)}] = \Xi_8^{(g)}[\Delta^{(g)}](\Omega) d\mu_{BOS}^{(g)}, \tag{2.16}$$

so that all dependence on the spin structure, that is the characteristic Δ, is codified in the factor $\Xi_8[\Delta](\Omega)$. Let as look at the known examples. We specify the genus by an apex (g), for example $\Delta^{(1)}$ means a $g = 1$ characteristic. At genus one the chiral measure is

$$d\mu[\Delta^{(1)}] = c^{(1)} \theta[\Delta^{(1)}]^4(\Omega^{(1)}) \eta^{12}(\Omega^{(1)})\, d\mu_B^{(1)}, \tag{2.17}$$

$$d\mu_{BOS}^{(1)} = \frac{1}{(2\pi)^{12}\eta^{24}(\Omega^{(1)})} d\Omega^{(1)}. \tag{2.18}$$

Then $\Xi_8[\Delta^{(1)}](\Omega^{(1)}) = \theta[\Delta^{(1)}]^4(\Omega^{(1)})\eta^{12}(\Omega^{(1)})$ is a modular form of weight 8 on a certain subgroup of $SL(2,\mathbb{Z})$ and $\eta(\Omega)$ is the Dedekind function, see Section 5.10.1.
For genus two D'Hoker and Phong obtained

$$d\mu[\Delta^{(2)}] = c^{(2)} \theta[\Delta^{(2)}]^4(\Omega^{(2)}) \Xi_6[\Omega^{(2)}](\Omega^{(2)})\, d\mu_{BOS}^{(2)}, \tag{2.19}$$

$$d\mu_{BOS}^{(2)} = \frac{c_2}{\Psi_{10}(\Omega^{(2)})} \prod_{i \leq j} d\Omega_{ij}, \tag{2.20}$$

where Ξ_6 has been defined in the previous section and

$$\Xi_8[\Delta^{(1)}](\Omega^{(1)}) = \theta[\Delta^{(2)}]^4(\Omega^{(2)}) \Xi_6[\Delta^{(2)}](\Omega^{(2)}) \tag{2.21}$$

is indeed a modular form of weight 8 on a suitable subgroup of $Sp(4,\mathbb{Z})$. It also has the right behavior at the boundary of the moduli space. For example, in the limit where the genus two Riemann surface Σ_2 splits as the union of two elliptic curves $\Sigma_1[\Omega_1^{(1)}]$ and $\Sigma_1[\Omega_2^{(1)}]$ with moduli given by $\Omega_1^{(1)} = \Omega_{11}^{(2)}$ and $\Omega_2^{(1)} = \Omega_{22}^{(2)}$, then the measure separates as the product of the genus one measures. Such limiting behavior, which we will call the clustering property, is a fundamental property to be satisfied by the right superstring measures.
Following the declared strategy we are now ready to state a general guess for the supersymmetric invariant measure at any genera, t.i. for the functions $\Xi_8[\Delta^{(g)}](\Omega^{(g)})$. This consists in three points:

i. The functions $\Xi_8[\Delta^{(g)}]$ are holomorphic on the Siegel upper halfplane $\mathbb{H}_g$.

ii. Under the action of the symplectic group $Sp(2g,\mathbb{Z})$ on $\mathbb{H}_g$, they should transform as follows:

$$\Xi_8[M \cdot \Delta^{(g)}](M \cdot \Omega) = \det(C\Omega + D)^8 \Xi_8[\Delta^{(g)}](\Omega), \tag{2.22}$$

for all $M \in Sp(2g,\mathbb{Z})$. Here the affine action of M on the characteristic $\Delta^{(g)}$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \left[{a \atop b} \right] := \left[{c \atop d} \right], \qquad \begin{pmatrix} {}^t c \\ {}^t d \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} {}^t a \\ {}^t b \end{pmatrix} + \begin{pmatrix} (C\,{}^tD)_0 \\ (A\,{}^tB)_0 \end{pmatrix} \mod 2 \tag{2.23}$$

where $N_0 = (N_{11}, \ldots, N_{gg})$ is the diagonal of the matrix N.

iii. The restriction of these functions to 'reducible' period matrices is a product of the corresponding functions in lower genus. More precisely, let

$$D_{k,g-k} := \left\{ \Omega_{k,g-k} := \begin{pmatrix} \Omega_k & 0 \\ 0 & \Omega_{g-k} \end{pmatrix} \in \mathbb{H}_g \, : \, \Omega_k \in \mathbb{H}_k, \; \Omega_{g-k} \in \mathbb{H}_{g-k} \right\} \cong \mathbb{H}_k \times \mathbb{H}_{g-k}.$$

Then we require that for all k, $0 < k < g$,

$$\Xi_8\left[\begin{smallmatrix} a_1 \dots a_k \, a_{k+1} \dots a_g \\ b_1 \dots b_k \, b_{k+1} \dots b_g \end{smallmatrix}\right](\Omega_{k,g-k}) = \Xi_8\left[\begin{smallmatrix} a_1 \dots a_k \\ b_1 \dots b_k \end{smallmatrix}\right](\Omega_k)\Xi_8\left[\begin{smallmatrix} a_{k+1} \dots a_g \\ b_{k+1} \dots b_g \end{smallmatrix}\right](\Omega_{g-k})$$

for all even characteristics $\Delta^{(g)} = \left[\begin{smallmatrix} a_1 \dots a_g \\ b_1 \dots b_g \end{smallmatrix}\right]$ and all $\Omega_{k,g-k} \in D_{k,g-k}$.

Note that our guess coincide with the one of D'Hoker and Phong apart from the fact that we do not require the functions $\Xi_8[M \cdot \Delta^{(g)}]$ to factorize as the products of $\theta[\Delta^{(3)}]^4(\Omega^{(3)})$ times some equivariant modular form of weight 6. This is exactly what will permit us to make success in finding a solution at genus 3, 4 and 5. These solutions are unique for genus three and four.
A direct tackling of this constraints is quite formidable and can explain the failure of D'Hoker and Phong in finding a solution (or recognizing that their ansatz was to strong). For this reasons we will take advantage of the theory of induced representations: we will build up representations of the modular group on the space of forms starting from the representations given by a suitable subspace left invariant by a certain subgroup of the entire modular group. This way to proceed resemble the method used by Wigner to classify the irreducible representations of the Poincaré group induced from the representations of the little group.
It is clear that this program requires an adequate exposition of the mathematical tools necessary to reach the target. We will provide all the necessary mathematical background in the next section. Before to do it, let us stop for a moment to expose the promised criticisms to formula (2.15).

2.3.1 Criticism to the main formula

To obtain the result (2.15) one has to integrate over the odd coordinates of the moduli space. This is the super moduli space of super Riemann surfaces. To this aim one needs a "splitting" of the super-Riemann surface $\hat{\Sigma}$. For example, at $g = 2$ and for even spin structures this should be done as follows:

- find a basis for super Abelian differentials ω_1, ω_2, with only even part;
- take periods $\hat{\Omega}$ and the Jacobian variety $J = \mathbb{C}^2/\hat{\Omega}$;
- take for Σ the Riemann surface having J as Jacobian;

then $\hat{\Sigma} \longrightarrow \Sigma$ is a fibration and the point in $\mathcal{M}_2$ define the splitting of the super moduli space. For $g > 2$ the situation is quite complicated, and it is hard to argue that a similar

splitting should work in this case.[9] Generically, it happens that odd differentials do not exist. However, some odd differential may exist for special complex structures. In this case the Jacobian variety is no more well defined and this procedure breaks down. But let us assume that there were not odd differentials, so that $J = \mathbb{C}^g/\hat{\Omega}$ is well defined. But there is not any particular reason to believe for it to be the Jacobian variety of ordinary Riemann surface Σ_g: its periods can differ from those of an ordinary Riemann surface by terms that are bilinear in fermionic moduli.
To make this more clear, let us recall that the ordinary period matrix Ω is a point in $\mathbb{H}_g$ (see Section 2.4 for some definitions). On this space it acts the modular group $\Gamma_g := Sp(2g, \mathbb{Z})$. Following [vG1], let us define $A_g := \Gamma_g \backslash \mathbb{H}_g$. Torelli's theorem than ensures that the natural holomorphic map $j : \mathcal{M}_g \to A_g$ is injective. If $\mathbb{J}_g^0 \subset \mathbb{H}_g$ is the set of all period matrices of genus g Riemann surfaces, the Jacobian locus is its closure $\mathbb{J}_g$ in $\mathbb{H}_g$. It can be shown that $\mathbb{J}_g - \mathbb{J}_g^0 \subset \mathbb{H}_g$ consists of block diagonal matrix whose diagonal blocks are period matrices of lower dimensional Riemann surfaces. As we have seen, in string theory these select the degenerate limits which play an important role in computing the amplitudes (clustering property). Therefore, we are really interested in considering the Jacobi locus $\mathbb{J}_g$ or better its image j_g in A_g. It results that $\mathbb{J}_g$ is a $(3g-3)$-dimensional complex subvariety of $\mathbb{H}_g$. The Schottky problem is related to the question of what points of $\mathbb{H}_g$ are in $\mathbb{J}_g$. In Table 2.1 we report the dimensions of these manifolds for inreasing g. From this it follows that the Schottky problem is trivial up to

g	$\dim\mathbb{H}_g$	$\dim\mathcal{M}_g$
0	0	0
1	1	1
2	3	3
3	6	6
4	10	9
g	$g(g+1)/2$	$3g-3$

Table 2.1: Dimensions of $\mathbb{H}_g$ and $\mathcal{M}_g$.

genus three. To ensure the validity of (2.11) at genus 3 we then should worry only about the existence of odd differentials. To higher genera the situation complicates because the codimension of $\mathbb{J}_g$ in $\mathbb{H}_g$ increases quadratically with g, so that we cannot expect, without some strong motivation, for the super period matrix $\hat{\Omega}$ to lie in $\mathbb{J}_g$. Thus, even though we will see that the ansatz provide a solution for the $g = 3, 4$ cases, a much deeper investigation must be devoted to understand (2.11) or improve.

[9]This observations was pointed out to us by Ed. Witten to which we are very grateful for his explanations.

2.4 Moduli space and Schottky problem

We conclude this chapter with a brief summary on some topics about the moduli space of Riemann surfaces, for some more details see [vG1].

Let C be a Riemann surface of genus g and consider the homology group $H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. A symplectic basis of $H_1(C,\mathbb{Z})$ is a basis $\{\alpha_1,\cdots,\alpha_g,\beta_1,\cdots,\beta_g\}$ satisfying $(\alpha_i,\alpha_j) = (\beta_i,\beta_j) = 0$ and $(\alpha_i,\beta_j) = \delta_{ij}$. Let $H^0(C,\Omega_C)$ be the g-dimensional complex vector space of holomorphic one forms on C. Given a path $\gamma \in C$ and an $\omega \in H^0(C,\Omega_C)$ one can compute the integral $\int_\gamma \omega$ and, if γ is a closed path, the integral depends only on the homology class of γ. It can be shown that given a symplectic basis for $H_1(C,\mathbb{Z})$ then there is a unique basis $\{\omega_1,\cdots,\omega_g\}$ of $H^0(C,\Omega_C)$ such that $\int_{\alpha_i}\omega_j = \delta_{ij}$. We now use the β_j to define a complex $g\times g$ matrix, the period matrix of C, $\tau = (\tau_{ij}) \in M_g(\mathbb{C})$ with $\tau_{ij} := \int_{\beta_i}\omega_j$, where ω_i is an element of the basis of $H^0(C,\Omega_C)$ determined from the symplectic basis. Torelli's theorem asserts that one can recover the Riemann surface from its period matrix. The Schottky problem basically asks for equations which determine the period matrices of Riemann surfaces among all $g \times g$ matrices. Period matrices have two properties: they are symmetric and $\mathrm{Im}(\tau)$, the imaginary part of τ, which is a symmetric, real, $g\times g$ matrix, defines a positive definite quadratic form on $\mathbb{R}^g$: ${}^t x(\mathrm{Im}\,\tau)x > 0$, for all $x \in \mathbb{R}^g$; briefly one writes $\mathrm{Im}(\tau) > 0$. This leads to the definition of the Siegel upper half plane $\mathbb{H}_g := \{\tau \in M_g(\mathbb{C}) :\ {}^t\tau = \tau, \mathrm{Im}(\tau) > 0\}$. Thus if τ is the period matrix of a Riemann surface, then $\tau \in \mathbb{H}_g$. One can show that $\mathbb{H}_g$ is a complex manifold of dimension $\frac{1}{2}g(g+1)$.

To define the period matrix of a Riemann surface we had to choose a symplectic basis and two such basis are related by an element of the symplectic group Γ_g. The symplectic group acts on $\mathbb{H}_g$ and the period matrix of Riemann surfaces are a Γ_g-orbit in $\mathbb{H}_g$. Thus one can study the images of period matrices under the quotient map $\pi : \mathbb{H}_g \to A_g := \Gamma_g\backslash\mathbb{H}_g$. The moduli space M_g of Riemann surfaces is a variety whose points correspond to isomorphism classes of Riemann surfaces. Then we have a well defined holomorphic map: $j : M_g \to A_g$, $[X] \mapsto \Gamma_g\tau$, where τ is a period matrix of X. This map is injective from Torelli's theorem. The Schottky problem can now be reformulated as the problem of finding equations for the image of j.

Let $J_g^0 \subset \mathbb{H}_g$ be the set of period matrices of Riemann surfaces. Its image in A_g is $j(M_g) = \mathrm{Image}(J_g^0 \to A_g = \Gamma_g\backslash\mathbb{H}_g)$. We have the diagram:

$$\begin{array}{ccc} J_g^0 & \xhookrightarrow{\;i\;} & \mathbb{H}_g \\ & & \big\downarrow{\scriptstyle \pi} \\ M_g & \xrightarrow{\;j\;} & A_g := \Gamma_g\backslash\mathbb{H}_g \end{array}$$

where i is the immersion map of J_g^0 in $\mathbb{H}_g$ and $J(M_g) = \pi(i(J_g^0))$. The subvariety J_g^0 and $j(M_g)$ are not closed and one defines the Jacobi locus J_g as the closure of J_g^0 in $\mathbb{H}_g$. A $\tau \in \mathbb{H}_g$ will be called decomposable if τ lies in the Γ_g-orbit of matrices in diagonal block form. The set $J_g - J_g^0$ in $\mathbb{H}_g$ consists of decomposable matrices, the diagonal blocks being period matrices of Riemann surfaces of lower genus. From Teichmüller theory one knows that the subset J_g is actually an irreducible subvariety of $\mathbb{H}_g$ of dimension $3g-3$, for $g>1$ and for $g=1$ one has $\mathbb{H}_1 = J_1 = J_1^0$. The Table 5.1 shows that the Schottky problem is trivial for $g \leq 3$. This shows why for $g \leq 3$, as expected, the forms $\Xi_8^{(g)}[0^{(g)}]$ are defined on the whole $\mathbb{H}_g$.

Chapter 3

Mathematical background

In order to describe the construction of the string amplitudes from an axiomatic point of view we need to develop some mathematical tools. In this section we will introduce the symplectic group and modular forms, the theta functions and their transformation properties under the action of the symplectic group.

3.1 The symplectic group

The symplectic group $\mathrm{Sp}(2g, F)$ of degree $2g$ over a field F is the group of $2g \times 2g$ matrices with entries in F satisfying:

$$ME^tM = E, \tag{3.1}$$

with $E = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ the canonical symplectic form and I_g the g-dimensional identity matrix. Note that the symplectic form has determinant $+1$, its inverse is $E^{-1} = {}^tE = -E$ and that a symplectic matrix M is always invertible and its inverse is $M^{-1} = E^{-1\,t}ME$. Also the product of two symplectic matrices is symplectic: suppose $M = M_1M_2$, with M_1 and M_2 symplectic thus $ME^tM = M_1M_2E^tM_2{}^tM_1 = M_1E^tM_1 = E$. This shows that $\mathrm{Sp}(2g, F)$ with matrix multiplication is a group. Directly from the definition, it follows that the determinant of a symplectic matrix is ± 1, but it turns out that this determinant is always positive. To see this one uses the identity[1] $\mathrm{Pf}({}^tMEM) = \det(M)\,\mathrm{Pf}(E)$, since ${}^tMEM = E$ and $\mathrm{Pf}(E) \neq 0$ it follows that $\det(M) = 1$. Thus, the symplectic group is a subgroup of the special linear group $\mathrm{SL}(2g, F)$. For a block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ the condition to be symplectic is equivalent to the conditions

$$\begin{aligned} A^tB &= B^tA \\ C^tD &= D^tC \\ A^tD - B^tC &= I. \end{aligned}$$

[1]Let $A = (a_{ij})$ be a $2n \times 2n$ skew-symmetric matrix, the Pfaffian of A, $\mathrm{Pf}(A)$, is defined as: $\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(21-1),\sigma(2i)}$, where S_{2n} is the symmetric group and $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ.

More abstractly, the symplectic group can be defined as the group of linear transformations of a $2n-$dimensional vector space over a field F which preserve a nondegenerate, skew-symmetric, bilinear form.

When the symplectic matrices take value in $\mathbb{Z}$ we will write $\Gamma_g := \mathrm{Sp}(2g,\mathbb{Z})$. Even though $\mathbb{Z}$ is not a field Γ_g is a group and we define its congruence subgroup of level n as

$$\Gamma_g(n) := \{M \in \Gamma_g : M \equiv I \bmod n\}. \tag{3.2}$$

These are normal subgroups of Γ_g. A subgroup H of a group G is normal if it is invariant under conjugation, i.e. $ghg^{-1} \in H$ for any $g \in G$ and $h \in H$. If $M \in \Gamma_g$ and $N \in \Gamma_g(n)$ we have that MNM^{-1} is again in $\Gamma_g(n)$ because the matrix elements outside the diagonal are $\frac{k\times 0 \bmod n}{\det M}$ that is again $0 \bmod n$ for all $k \in \mathbb{Z}$ and the matrix elements on the diagonal are $\frac{\det(M)\times 1 \bmod b + k\times 0 \bmod n}{\det(M)}$, for a certain $k \in \mathbb{Z}$, and this is again $1 \bmod n$, thus $\Gamma_g(n)$ is a normal subgroup of Γ_g.

The case $n = 2$ is of particular interest for the applications to string theory:

$$\Gamma_g(2) = \ker(\Gamma_g := \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow \mathrm{Sp}(2g) := \mathrm{Sp}(2g,\mathbb{F}_2)), \tag{3.3}$$

where $\mathbb{F}_2$ is the field with two elements. The reduction mod two map, as proved by Igusa, is surjective, so that we have $\mathrm{Sp}(2g) \cong \Gamma_g/\Gamma_g(2)$. This group is clearly finite and its order is [J]: $|\mathrm{Sp}(2g)| = 2^{2g-1}(2^{2g}-1)|\mathrm{Sp}(2g-2)| = 2^{2g-1}(2^{2g}-1)2^{2g-3}(2^{2g-2}-1)\cdots 2(2^2-1)$.

3.2 The action of $\mathrm{Sp}(2g)$ on the theta characteristics

In order to construct string amplitudes we will use certain special functions called *theta functions with characteristic*. There is a natural action of the symplectic group on such theta functions. This can be used to construct a special class of functions called modular forms. Before introducing theta functions and modular forms we define here theta characteristics and study the action of the symplectic group on them.

The finite field $\mathbb{F}_2^{2g}$ has 2^{2g} elements called *period characteristics* and the group Γ_g naturally acts linearly on them through its quotient $\mathrm{Sp}(2g) = \Gamma_g/\Gamma_g(2)$, the action being simply given by the matrix product on the column vectors of $\mathbb{F}_2^{2g}$. A *theta g-characteristic*, or simply a *theta characteristic*, is a $2\times g$ array $\begin{bmatrix} a_1 & \cdots & a_g \\ b_1 & \cdots & b_g \end{bmatrix}$, with b_i, $a_i \in \mathbb{F}_2$. We can define a (non linear) action of the symplectic group on the theta characteristics. Following the abstract definition of Section 3.1, the group $\mathrm{Sp}(2g)$ fixes a nondegenerate skew-symmetric form E on the $\mathbb{F}_2$-vector space $V = \mathbb{F}_2^{2g}$. Choosing a symplectic basis for V, which is a basis $e_1,\cdots,e_{2g}$ of V such that $E(e_1,e_j) = 0$ unless $|i-j| = g$ and then $E(e_i,e_j) = 1$, we obtain:

$$E : V\times V \longrightarrow \mathbb{F}_2, \qquad E(v,w) := v_1w_{g+1} + \cdots + v_gw_{2g} + v_{g+1}w_1 + \cdots + 2_{2g}w_g, \tag{3.4}$$

where $v,w \in \mathbb{F}_2^{2g}$. More compactly, we can write $E((v',v''),(w',w'')) = {}^tv'w'' + {}^tv''w'$ and occasionally we will write $v = \binom{v'}{v''}$, where v', v'' are then considered as row vectors.

Let us consider the quadratic form q on the vector space V whose associated bilinear form is E. This is the map:

$$q : V \longrightarrow \mathbb{F}_2, \qquad q(v+w) = q(v) + q(w) + E(v,w). \tag{3.5}$$

It is not hard to verify that for each choice of a_i and $b_i \in \mathbb{F}_2$ the function

$$q(v) = v_1 v_{g+1} + v_2 v_{g+2} + \cdots + v_g v_{2g} + a_1 v_1 + \cdots + a_g v_g + b_1 v_{g+1} + \cdots + b_g v_{2g} \tag{3.6}$$

satisfies $q(v+w) = q(v) + q(w) + E(v,w)$ and that any quadratic form associated to E is of this form. With the compact notations, we can write $q(v) = {}^t v' v'' + av' + bv''$, with row vectors $a = (a_1, \cdots, a_g)$ and $b = (b_1, \cdots, b_g)$. We are now able to give the precise definition of theta characteristics. The *theta characteristic* Δ_q *associated to the quadratic form* q is defined as:

$$\Delta_q := \begin{bmatrix} a_1\, a_2 \ldots a_g \\ b_1\, b_2 \ldots b_g \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{3.7}$$

To introduce the notion of parity of a theta characteristic we define $e(\Delta_q) := (-1)^{\sum_{i=1}^{g} a_i b_i} \in \{1, -1\}$ and we say that Δ_q is even if $e(\Delta_q) = +1$ and odd elsewhere.
One can verify that:

$$e(\Delta_q) 2^g = \sum_{v \in V} (-1)^{q(v)}. \tag{3.8}$$

It follows that $q(v)$ has $2^{g-1}(2^g+1)$ zeroes in V if Δ_q is even and has $2^{g-1}(2^g-1)$ zeroes if Δ_q is odd. To show this consider an even characteristic Δ_q, from (3.8) we obtain $2^g = z - p$, where z is the number of v for which $q(v)$ is $0 \pmod 2$ and p is the number of v for which $q(v)$ is $1 \pmod 2$. Hence, we obtain the two equations:

$$\begin{aligned} 2^g &= z - p \\ 2^{2g} &= z + p, \end{aligned}$$

solving this system one obtains the number of zeroes for $q(v)$ for an even Δ_q. The case Δ_q odd is similar but the first equation of the system becomes $-2^g = z - p$.

For any genus g there are 2^{2g} theta characteristics of which $2^{g-1}(2^g+1)$ are even and $2^{g-1}(2^g-1)$ odd. This can be shown as follows ([RF], Chapter 1, Theorem 1). In genus one there are three even characteristics: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and one odd $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If one borders an even (odd) $(g-1)$-characteristic on the right by an even 1-characteristic, one gets a even (odd) g-characteristic. Instead, bordering an even (odd) $(g-1)$-characteristic by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives an odd (even) g-characteristic. Thus, if e_i and o_i are the cardinals of the even and odd i-characteristics respectively, then:

$$\begin{aligned} e_g &= 3e_{g-1} + o_{g-1} \\ o_g &= 3o_{g-1} + e_{g-1}. \end{aligned} \tag{3.9}$$

Adding the two equations, considering that $e_1 + o_1 = 2^2$, and by induction one obtains:

$$e_g + o_g = 2^2(e_{g-1} + o_{g-1}) = 2^{2g}, \tag{3.10}$$

this prove that in genus g there are exactly 2^{2g} characteristics. Subtracting the two equations (3.9) and again by induction we obtain:

$$e_g - o_g = 2(e_{g-1} - o_{g-1}) = 2^g. \tag{3.11}$$

Solving (3.10) and (3.11) for e_g and o_g we obtain:

$$\begin{aligned} e_g &= 2^{g-1}(2^g + 1) \\ o_g &= 2^{g-1}(2^g - 1). \end{aligned}$$

The group $\mathrm{Sp}(2g)$ acts naturally on the characteristics by:

$$(g \cdot q)(v) := q(g^{-1}v), \qquad (g \in \mathrm{Sp}(2g),\ v \in V). \tag{3.12}$$

This action is transitive on both the set of even and odd characteristics which are the two orbits of the action.

3.3 Modular forms

We recall here that the Siegel upper half space $\mathbb{H}_g$ is the space of complex $g \times g$ symmetric matrices with positive imaginary part. We can see $\mathbb{H}_g$ as a higher dimensional generalization of the half upper complex plane (i.e. the set of complex numbers with positive imaginary part):

$$\mathbb{H}_g := \{\tau \in M_g(\mathbb{C}) :\ {}^t\tau = \tau,\ \mathrm{Im}(\tau) > 0\}. \tag{3.13}$$

A Siegel modular form of genus g, weight k and level n is a holomorphic function on the Siegel upper half space such that:

$$f : \mathbb{H}_g \longrightarrow \mathbb{C}, \qquad f(M \cdot \tau) = \det(C\tau + D)^k f(\tau) \quad \forall M \in \Gamma_g(n), \tag{3.14}$$

plus, for $g = 1$, the requirement that f is holomorphic at ∞. The action of Γ_g on the Siegel upper half space is given by

$$M \cdot \tau := (A\tau + B)(C\tau + D)^{-1}, \qquad M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}), \quad \tau \in \mathbb{H}_g. \tag{3.15}$$

The set of all Siegel modular forms of genus g, weight k and level n form a finite dimensional complex vector space denoted $M_k(\Gamma_g(n))$. For the applications to string theory we are mainly interested to the case $n = 2$. The finite group $Sp(2g)$ has a representation

$$\rho \equiv \rho_k : Sp(2g) \longrightarrow GL(M_k(\Gamma_g(2)))$$

on this vector space defined by

$$(\rho(g^{-1})f)(\tau) := \det(C\tau + D)^{-k} f(M \cdot \tau), \tag{3.16}$$

where $M \in \Gamma_g$ is any representative of the equivalence class of $g \in Sp(2g)$ and $f \in M_k(\Gamma_g(2))$ (note that $\det(C\tau + D)^{-k} f(M \cdot \tau) = f(\tau)$ for $M \in \Gamma_g(2)$, thus the action of $M \in \Gamma_g$ factors over $\Gamma_g/\Gamma_g(2) = Sp(2g)$). The equality $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in Sp(2g)$ follows from $(MN) \cdot \tau = M \cdot (N \cdot \tau)$ and $\gamma(MN, \tau) = \gamma(M, N \cdot \tau)\gamma(N, \tau)$ where $\gamma(M, \tau) := \det(C\tau + D)$. This shows that ρ effectively defines a group representation.

3.3.1 Theta constants with characteristic

A powerful tool to determine modular forms on $\Gamma_g(2)$ is provided by theta constants with characteristic. Let $\Delta = [^a_b]$ be an even characteristic, then one defines a function, called *theta constant*, on the Siegel space $\mathbb{H}_g$ by

$$\theta[^a_b](\tau) := \sum_{m\in\mathbb{Z}^g} e^{\pi i((m+a/2)\tau^t(m+a/2)+(m+a/2)^tb)} \tag{3.17}$$

so m is a row vector and $\sum a_i b_i \equiv 0 \bmod 2$. Then, for all $M \in Sp(2g,\mathbb{Z})$, one has ([I1], V.1, Corollary):

$$\theta[M\cdot\Delta](M\cdot\tau) = \kappa(M)e^{2\pi i\phi_\Delta(M)}\det(C\tau+D)^{1/2}\theta[\Delta](\tau), \tag{3.18}$$

with:

$$\phi_\Delta(M) = \sum_{k,l=1}^{g} \tfrac{-1}{8}\Big(({}^tDB)_{kl}a_ka_l - 2({}^tBC)_{kl}a_kb_l + ({}^tCA)_{kl}b_kb_l\Big)+$$
$$\tfrac{1}{4}(({}^tD)_{kl}a_k - ({}^tC)_{kl}b_k)(A^tB)_{ll},$$

and $\kappa(M)$ a constant independent on the characteristic. Here the action of $M \in \mathrm{Sp}(2g,\mathbb{Z})$ on the characteristic Δ is given by (2.23). The formula (3.18) is called *transformation formula.* This formula is explicit except for the constant $\kappa(M)$. See [RF] for the expression of $\kappa(M)$ or, in case of squared theta constants, see [I1]. Note that

$$\theta[\Delta](M^{-1}\cdot\tau) = \theta[M^{-1}M\cdot\Delta](M^{-1}\cdot\tau) = c_{M^{-1},\Delta,\tau}\theta[M\cdot\Delta](\tau) \tag{3.19}$$

where $c_{M^{-1},\Delta,\tau}$ collects the non-relevant part. Thus the action of M basically maps $\theta[\Delta]$ to $\theta[M\cdot\Delta]$.

The action of Γ_g on the theta characteristics defined in (2.23) corresponds to its action on the quadratic forms on V defined in Section 3.2. To show this we have to proof explicitly that

$$(M\cdot q_\Delta)(v) = q_{M\cdot\Delta}(v). \tag{3.20}$$

From the definition of the action of $\mathrm{Sp}(2g)$ on the quadratic forms $(M\cdot q_\Delta)(v) = q_\Delta(M^{-1}v)$ then we must verify the relation $q_\Delta(M^{-1}v) = q_{M\cdot\Delta}(v)$. From the definition of Γ_g we have $ME^tM = E$ and taking the inverse of both sides we obtain $M^{-1} = -E^tME$, so:

$$M^{-1}v = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}\begin{pmatrix} v' \\ v'' \end{pmatrix} = \begin{pmatrix} {}^tDv' - {}^tBv'' \\ -{}^tCv' + {}^tAv'' \end{pmatrix}. \tag{3.21}$$

Using $q_\Delta(v) = q_{[^a_b]}((^{v'}_{v''})) = {}^tv'v'' + av' + bv''$, we get

$$q_\Delta(M^{-1}v) = {}^t({}^tDv' - {}^tBv'')(-{}^tCv' + {}^tAv'') + a({}^tDv' - {}^tBv'') + b(-{}^tCv' + {}^tAv''). \tag{3.22}$$

The non-linear part is

$${}^tv'(-D\,{}^tC)v' + {}^tv'(D\,{}^tA)v'' + {}^tv''(B\,{}^tC)v' - {}^tv''(B\,{}^tA)v''. \tag{3.23}$$

As M is symplectic, $D\,{}^tC$ is symmetric, hence only the terms $(D\,{}^tC)_{ii}(v'_i)^2$ survives mod 2. But $(v'_i)^2 \equiv v'_i$ mod 2 and thus ${}^tv'(-D\,{}^tC)v' \equiv (D\,{}^tC)_0 v'$ mod 2. Similarly, $v''(B\,{}^tA)v'' \equiv (B\,{}^tA)_0 v''$ mod 2. Next, $B\,{}^tC \equiv I + A\,{}^tD$ mod 2 so that ${}^tv'(D\,{}^tA)v'' + {}^tv''(B\,{}^tC)v' \equiv {}^tv'v''$ mod 2. Thus, we find that

$$q_\Delta(M^{-1}v) \;=\; {}^tv'v'' + (a\,{}^tD - b\,{}^tC + (D\,{}^tC)_0)v' + (-a\,{}^tB + b\,{}^tA + (B\,{}^tA)_0)v'' \qquad (3.24)$$

so $q_\Delta(M^{-1}v) = q_{M\cdot\Delta}(v)$, as desired. This clarifies the definition of the action given in (2.23).

3.4 The subgroup $O^+(2g)$ of $\mathrm{Sp}(2g)$

The stabilizer subgroup of the (even) characteristic $[0] := \left[\begin{smallmatrix}0\cdots0\\0\ldots0\end{smallmatrix}\right]$ is the subgroup $\Gamma_g(1,2)$, a special case of a series of subgroups of Γ_g:

$$\Gamma_g(n,2n) \;:=\; \{M \in \Gamma_g(n) :\, \mathrm{diag}A\,{}^tB \equiv \mathrm{diag}C\,{}^tD \equiv 0 \bmod 2n\,\}. \qquad (3.25)$$

In case n is even $\Gamma_g(n,2n)$ is a normal subgroup of $\Gamma_g \equiv \Gamma_g(1)$. We call $O^+(2g)$ the image of $\Gamma_g(1,2)$ in $\mathrm{Sp}(2g)$

$$O^+(2g) \;:=\; \Gamma_g(1,2)/\Gamma_g(2) \qquad (\subset Sp(2g)). \qquad (3.26)$$

As $Sp(2g)$ acts transitively on the even theta characteristics, there is a natural bijection

$$Sp(2g)/O^+(2g) \;\longrightarrow\; \{\Delta :\, \Delta \text{ even}\,\}, \qquad hO^+(2g) \;\longmapsto\; h\cdot[0], \qquad (3.27)$$

for $h \in \mathrm{Sp}(2g)$. In particular, $[Sp(2g) : O^+(2g)] = 2^{g-1}(2^g+1)$. One has $O^+(2) \cong \mathbb{Z}/2\mathbb{Z}$ and $O^+(4)$ is isomorphic to the subgroup of the symmetric group $S_6 \cong Sp(4)$ consisting of all permutations σ such that $\sigma(\{1,2,3\}) \subset \{1,2,3\}$ or $\sigma(\{1,2,3\}) = \{4,5,6\}$. Thus $S_3 \times S_3$ is a subgroup of index two in $O^+(4)$ and $|O^+(4)| = (3!)\cdot(3!)\cdot 2 = 72$. One has $O^+(6) \cong S_8$, the symmetric group of order 8! and $O^+(8)$ is the quotient of the subgroup of elements of[2] $W(E_8)$ with determinant $+1$ in the standard 8-dimensional representation, by its center, generated by $-I$.

3.5 Theta constants and the Heisenberg group

In this section we will study the relation between modular forms and theta constants. We will see that the modular group acts on the theta constant projectively instead of linearly. Recognizing the action of a finite Heisenberg group will help us to obtain modular forms from suitable polynomials in theta constants.

The theta constants are *almost modular* forms of weight 1/2 on $\Gamma_g(4,8)$; due to the presence of the constant $\kappa(M)$ we used the expression "almost modular". This can be shown by using the "transformation formula"

$$\theta[\Delta](M\cdot\tau) = \theta[MM^{-1}\Delta](M\cdot\tau) = \kappa(M)e^{2\pi i\Phi_\Delta(M)}\det(C\tau+D)^{1/2}\theta[M^{-1}\Delta](\tau).$$

[2] The Weyl group of E_8.

The exponential phase takes the value 1 if $\Phi_\Delta(M)$ is an integer. As $M \in Sp(2g,\mathbb{Z})$, also $M^{-1} \in Sp(2g,\mathbb{Z})$, thus M satisfies also ${}^tBD - {}^tDB = 0$ and ${}^tAC - {}^tCA = 0$, that mean that ${}^tDB$ and ${}^tCA$ are symmetric matrices. Hence, the integers $a_k a_l, b_k b_l$ in $a{}^tDB{}^ta + bCA{}^tb$ are multiplied by an even integer if $k \neq l$, so that they do not contribute to the exponential if ${}^tDB$ and ${}^tCA$ are $0 \bmod 4$. For $k = l$ we have $\sum_k (a_k^2({}^tDB)_{kk} + b_k^2({}^tCA)_{kk})$, but note that $a_k^2 \equiv a_k \bmod 2$. For a $g \times g$ matrix M, let $\mathrm{diag}(M)$ be the column vector $(M_{11}, M_{22}, \ldots, M_{gg})$ of diagonal entries. Then, the last term is $a\,\mathrm{diag}({}^tDB) + b\,\mathrm{diag}({}^tCA)$ which does not contribute to the phase if its value is $0 \bmod 8$. These two requests are precisely the conditions defining the subgroup $\Gamma_g(4,8)$. Note that if $M \in \Gamma_g(4,8)$ the term $2({}^tBC)_{kl}a_k b_l$ is a multiple of eight so that it does not contribute to the phase and the same hold true for the second term in $\Phi_\Delta(M)$ because $A{}^tB$ is $0 \bmod 8$. Moreover $M^{-1}\Delta = \Delta$ for all $M \in \Gamma_g(2)$ (or in some its subgroup). Just the constant $\kappa(M)$ survives.

To determine the modular forms of even weight on $\Gamma_g(2)$ it is convenient to define the 2^g (second order) theta constants:

$$\Theta[\sigma](\tau) := \theta[{}^\sigma_0](2\tau), \qquad [\sigma] = [\sigma_1\, \sigma_2\, \ldots\, \sigma_g],\ \sigma_i \in \{0,1\},\ \tau \in \mathbb{H}_g.$$

These theta constants, being evaluated in 2τ, are almost modular forms of weight 1/2 on $\Gamma_g(2,4)$:

$$\Theta[\sigma](M\cdot\tau) = \theta[{}^\sigma_0]\left(2\frac{A\tau+B}{C\tau+D}\right) = \theta[{}^\sigma_0](\tilde{M}\cdot 2\tau),$$

with $\tilde{M} = \begin{pmatrix} A & 2B \\ \frac{C}{2} & D \end{pmatrix}$ and similar considerations as the previous ones lead to the right conclusion. As we shall see, the invariants of degree $4k$ of the quotient group $\Gamma_g(2)/\Gamma_g(2,4) \cong \mathbb{F}_2^{2g}$ in the ring of polynomials in the $\Theta[\sigma]$'s are modular forms of weight $2k$ on $\Gamma_g(2)$. However, the quotient group $\Gamma_g(2)/\Gamma_g(4,2)$ doesn't act linearly on the $\Theta[\sigma](\tau)$. Using the action ρ defined in (3.16) we obtain that $\rho(g_N^{-1})\rho(g_M^{-1}) \neq \rho(g_{(MN)^{-1}})$ for the presence in the exponential of the phases $\Phi_\Delta(M) + \Phi_\Delta(N)$ which are not equal to $\Phi_\Delta(NM)$. Here g_M, g_N stand for the equivalence classes in $\Gamma_g(2)/\Gamma_g(2,4)$ of the matrices M, $N \in \Gamma_g(2)$ respectively. Thus, a term depending on the characteristic Δ remains. Moreover, $\kappa(M)\kappa(N) \neq \kappa(MN)$, but we will see how to get rid of this term. The previous considerations lead to take into account a central extension of the quotient group $\Gamma_g(2)/\Gamma_g(2,4)$, the Heisenberg group.

The finite Heisenberg group (cf. also [CvG])is defined as $H_g = \mu_4 \times \mathbb{F}_2^g \times \mathbb{F}_2^g$, where $\mu_4 = \{z \in \mathbb{C} : z^4 = 1\}$ is the multiplicative group of fourth roots of unity. The group composition law is $(s,x,u)(t,y,v) = (st(-1)^{uy}, x+y, u+v)$, with $uy = u_1y_1 + \cdots + u_gy_g$ for $u = (u_1, \cdots u_g)$, $y = (y_1, \cdots y_g) \in \mathbb{F}_2^g$. The center of H_g is the multiplicative group μ_4 and the quotients H_g/μ_4 are isomorphic to $\mathbb{Z}_2^{2g}$. Let us consider the ring of polynomials in 2^g variables X_σ, where $[\sigma] = [\sigma_1\, \sigma_2 \cdots \sigma_g]$, $\sigma_i \in \{0,1\}$. On this space we can define the (Schrödinger) representation of the Hesienberg group

$$(s,x,u)X_\sigma := s(-1)^{(x+\sigma)u}X_{\sigma+x},$$

and this action is extended to polynomials in X_σ's in an obvious way. Directly from this definition, it follows that a polynomial in the X_σ is invariant under the center μ_4 of the Heisenberg group if and only if its degree is a multiple of four, so the H_g invariant polynomials result to be modular forms of even weight. We will denote the subring of these invariants in the ring of polynomials in X_σ as $\mathbb{C}[\cdots, X_\sigma, \cdots]^{H_g}$. The action of H_g on the ring of polynomials $\mathbb{C}[\cdots, X_\sigma, \cdots]$ induces an action on the ring of the (second order) theta constants $\mathbb{C}[\cdots, \Theta[\sigma], \cdots]$ under the map $X_\sigma \mapsto \Theta[\sigma]$. The subspace of $M_{2k}(\Gamma_g(2))$ of the Heisenberg invariants is denoted by $M^\theta_{2k}(\Gamma_g(2)) \subset M_{2k}(\Gamma_g(2))$, where $M^\theta_{2k}(\Gamma_g(2)) := \mathbb{C}[\cdots, \Theta, \cdots]^{H_g}_{2k}$.

Using the two generators $(1, x, 0)$ and $(1, 0, u)$ of the Heisenberg group it is not hard to construct a basis for the space of the invariants. We fixed 1 as element of the center because the latter acts trivially on polynomials of degree four. First, each monomial in an Heisenberg invariant polynomial P is a product $\prod_{i=1}^{4n} X_{\sigma_i}$ which must be invariant under the action of $(1, 0, u)$. This means $(1, 0, u) \prod_{i=1}^{4n} X_{\sigma_i} = (-)^{(\sum_i \sigma_i)u} \prod_{i=1}^{4n} X_{\sigma_i}$, which is invariant for all u if $\sum \sigma_i = 0$. Next, P must be invariant also for the elements of the form $(1, x, 0)$, which means that all monomials in P of the type $\prod_{i=1}^{4n} X_{\sigma_i + x}$, for any $x \in \mathbb{F}_2^g$, must have the same coefficient in P. Thus a basis for the subring of the Heisenberg invariants is provided by polynomials of the form $\sum_x \prod_{i=1}^{4n} X_{\sigma_i + x}$, where $\sum \sigma = 0$.

3.5.1 Transvections

The group $Sp(2g)$ is generated by transvections t_v, for $v \in V$, which are analogous to reflections in orthogonal groups ([J], § 6.9). They are defined as:

$$t_v : V \longrightarrow V, \qquad t_v(w) := w + E(w, v)v.$$

It is straightforward to verify that $t_v \in Sp(2g)$. In fact the same formula works also for $\mathbb{Z}$ in place of $\mathbb{Z}_2$ and then defines elements in $Sp(2g, \mathbb{Z})$. As $g t_v g^{-1} = t_{g(v)}$ for $g \in Sp(2g, \mathbb{F}_2)$ and $v \in V$, the non-trivial transvections form a conjugacy class. It is not hard to prove that t_v is an involution, i.e. $t_v^2 = 1$.

Let us now determine how tranvections act on the characteristics. Let $v \in V$ and let q be a quadratic form with associated bilinear form E and characteristic Δ_q. As t_v is an involution, $q(v + w) = q(v) + q(w) + E(v, w)$ for all $v, w \in V$ and $q(av) = aq(v)$ for $a \in \mathbb{F}_2$, we have

$$(t_v \cdot q)(w) = q(t_v(w)) = q(w + E(v, w)v) = q(w) + E(v, w)q(v) + E(v, w)^2.$$

Hence we get the simple rule:

$$(t_v \cdot q)(-) = \begin{cases} q(-) & \text{if} \quad q(v) = 1, \\ q(-) + E(v, -) & \text{if} \quad q(v) = 0, \end{cases} \qquad \text{so} \quad t_{\binom{v'}{v''}} \cdot q_{\left[\substack{a \\ b}\right]} = q_{\left[\substack{a+v'' \\ b+v'}\right]}$$

in case $q_{\left[\substack{a \\ b}\right]}(\binom{v'}{v''}) = 0$.

Using the transvections we obtain the action of $Sp(2g)$ on the Heisenberg invariants in a simple manner and a computer can be used to perform all computations (see the Appendix B of [DvG] for an exhaustive discussion about transvections and also [CvG]).

3.5.2 Dimension of the space of Heisenberg invariants

We will show that the functions we need to construct superstrings amplitudes belong to the subspace of the Heisenberg invariants given by the vector space $\mathbb{C}[\cdots, X_\sigma, \cdots]_n$ of homogeneous polynomials of degree $n = 16$ in the X_σ's. Here we show that the dimensions of these spaces are given by the formula:

$$\dim(\mathbb{C}[\ldots, X_\sigma, \ldots]_{4n})^{H_g} = 2^{-2g}\left(\binom{2^g+4n-1}{4n} + (2^{2g}-1)\binom{2^{g-1}+2n-1}{2n}\right). \tag{3.28}$$

We list in Table 3.1 such dimensions for the lowest values of genus g and degree $4n$.

g / degree	4	8	12	16
1	2	3	4	5
2	5	15	35	69
3	15	135	870	3993
4	51	2244	69615	1180396

Table 3.1: Dimension of some space of Heisenberg invariants.

To prove the formula we will employ the theory of finite group representations. For fixed g let ρ_n be the representation of the Heienberg group on the vector space of homogeneous polynomials in X_σ's, as introduced before:

$$\rho_n : H_g \to GL(\mathbb{C}[\cdots, X_\sigma, \cdots]_n]). \tag{3.29}$$

Clearly, the Heisenberg invariants are the space of the trivial subrepresentation of ρ_n. Thus its dimension is

$$\dim\, \mathbb{C}[\cdots, X_\sigma \cdots]_n^{H_g} = \langle \rho_n, 1_{H_g}\rangle_{H_g}, \tag{3.30}$$

i.e. the multiplicity of the trivial representation 1_{H_g} of H_g in ρ_n. The scalar product of the characters is given by $\langle \rho_n, 1_{H_g}\rangle_{H_g} = \frac{1}{|H_g|}\sum_{h\in H_g} \mathrm{Tr}(\rho_n(h))$, where tr is the trace and $|H_g|$ is the number of elements of the group H_g. Consider the element $(1, x, u) \in \mathbb{H}_g$. If $xu = 0$, but $(x, u) \neq (0, 0)$ then $(1, x, u)$ has order two in $\mathbb{H}_g$ and the eigenvalues of (t, x, u) on $\mathbb{C}[\ldots, X_\sigma, \ldots]_1$ are t and $-t$, each with multiplicity 2^{g-1} for all $t \in \mu_4$.

If $\alpha_1, \ldots, \alpha_N$, $N = 2^g$ are the eigenvalues of (t, x, u) on $\mathbb{C}[\ldots, X_\sigma, \ldots]_1$, the eigenvalues on $\mathbb{C}[\ldots, X_\sigma, \ldots]_n$ are the $\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_N^{m_N}$ with $\sum m_i = n$. As the trace is the sum of the eigenvalues, we get with a variable X:

$$\sum_n \mathrm{Tr}(\rho_n(t,x,u))X^n = \prod_{i=1}^{N}(1-\alpha_i X)^{-1}.$$

So if $(x,u) \neq (0,0)$ we have

$$\sum_n \mathrm{Tr}(\rho_n(t,x,u))X^n = (1-i^{2a}X^2)^{-2^{g-1}} = \sum_m (-1)^{am}\binom{2^{g-1}+m-1}{m}X^{2m},$$

and in case $(x,u)=(0,0)$ the trace is just

$$\sum_n \mathrm{Tr}(\rho_n(t,0,0))X^n = \sum_n t^n(\dim \mathbb{C}[\ldots,X_\sigma,\ldots]_n)X^n = \sum_n t^n\binom{2^g+n-1}{n}X^n.$$

Thus, we obtain the anticipated formula:

$$\dim(\mathbb{C}[\ldots,X_\sigma,\ldots]_{4n})^{H_g} = 2^{-2g}\left(\binom{2^g+4n-1}{4n}+(2^{2g}-1)\binom{2^{g-1}+2n-1}{2n}\right),$$

note that non-trivial invariants have degree multiple of 4.

3.5.3 The ring of modular forms

Let us call $M^\theta_{2k}(\Gamma_g(2))$ the spaces of the Heisenberg invariants of degree $4k$ (and weight $2k$) in the second order theta functions. These are the images under the surjective maps

$$\mathbb{C}[\ldots,X_\sigma,\ldots]^{H_g}_{4k} \longrightarrow M^\theta_{2k}(\Gamma_g(2)) := \mathbb{C}[\ldots,\Theta[\sigma],\ldots]^{H_g}_{4k}, \qquad X_\sigma \longmapsto \Theta[\sigma] \tag{3.31}$$

of the Heisenberg invariant polynomials of degree $4k$. These maps define a surjective $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[\ldots,X_\sigma,\ldots]^{H_g} \longrightarrow M^\theta(\Gamma_g(2)) := \oplus_k M^\theta_{2k}(\Gamma_g(2)), \tag{3.32}$$

whose kernel is the ideal of algebraic relations between the $\Theta[\sigma]$'s. This means that a polynomial $F(\cdots,X_\sigma,\cdots)$ maps to zero if and only if $F(\cdots,\Theta[\sigma](\tau),\cdots)=0$ for all $\tau \in \mathbb{H}_g$. For $g=1,2$ there are no polynomials vanishing on the image. In case $g=3$ there is a homogeneous polynomial F_{16}, of degree sixteen in eight variables, vanishing on the image [vGvdG], so that $M^\theta(\Gamma_g(2)) = \mathbb{C}[\cdots,X_\sigma,\cdots]^{\mathbb{H}_g}/(F_{16})$. For $g \geq 4$ there are many algebraic relations between the $\Theta[\sigma]$'s, but a complete description of these relations is not known. The graded ring of modular forms of even weight on $\Gamma_g(2)$ is the normalization of the ring[3] of the $\Theta[\sigma]$'s (cf. [SM1] Thm 2, [R1], [R2]):

$$\oplus_{k=0}^{\infty} M_{2k}(\Gamma_g(2)) = (\mathbb{C}[\ldots,\Theta[\sigma],\ldots]^{H_g})^{Nor}.$$

[3]An Abelian ring A is normal if it does not contains nontrivial nilpotents and is integrally closed w.r.t. its quotient ring $Q[A]$. In other words, any polynomial equation whose coefficients are fractions in $Q[A]$ has solutions in A. A non normal ring can be normalized by takeing its closure in $Q[A]$, see [L].

In case $g = 1, 2$ there are no relations and the rings of invariants are already normal. In case $g = 3$, there is one relation given by $F_{16}(\dots, \Theta[\sigma], \dots) = 0$, and it has been shown by Runge ([R1], [R2]) that the quotient of the ring of invariants by the ideal generated by this relation is again normal. This implies that any modular form of weight $2k$ can be written as a homogeneous polynomial of degree $4k$ in the $\Theta[\sigma]$'s if $g \leq 3$, and

$$M^{\theta}_{2k}(\Gamma_g(2)) \;=\; M_{2k}(\Gamma_g(2)) \qquad \text{for} \quad g = 1, 2, 3.$$

This polynomial is unique for $g \leq 2$. For $g = 3$ it is unique if its degree is less than 15, otherwise it is unique up to the addition of $F_{16}G_{4k-16}$, where G_{4k-16} is any homogeneous polynomial of degree $4k - 16$ in the $\Theta[\sigma]$'s.

For $g > 3$ there will always be non-trivial relations and if $g > 4$ the ring $\mathbb{C}[\dots, \Theta[\sigma], \dots]^{H_g}$'s is not normal, (cf. [OSM], Theorem 6, but note that our H_g is slightly different from their one). In case the ring is not normal, there are also quotients G_{4k+d}/H_d of homogeneous polynomials in the $\Theta[\sigma]$'s, of degree $4k + d$ and d respectively, which are modular forms of weight $4k$ (but which cannot be written as a polynomial in the $\Theta[\sigma]$'s). These observations will play a crucial role in proving the uniqueness of superstring amplitudes. Actually, in genus two and three the proof of uniqueness is based on the result that every modular forms of weight 8 are polynomial in the theta constants, see section 4.3.2 and 4.4.7. For the case $g = 4$ we will be able to prove the uniqueness in a weakened form, that is by assuming the polynomiality for the superstring measures 4.5.3, in [GS] the general case is considerated.

3.6 Turning back to the classical theta constants with characteristic

To describe the spaces of modular forms $M^{\theta}_{2k}(\Gamma_g(2))$ it is convenient to use also the classical theta functions with arbitrary characteristics $\theta[\Delta]$. Recall that by this we mean that the argument is τ and not 2τ as in the relation defining the $\Theta[\sigma]$. In particular, we are interested in decomposing these spaces into irreducible representations for the group $Sp(2g)$ and we want to describe their subspaces of O^+-invariants as well as O^+-anti-invariants. These will play a crucial role in the construction of superstring measures.

3.6.1 The quadratic relations between the $\theta[\Delta]$'s and the $\Theta[\sigma]$'s

A classical formula for theta functions shows that any product of two $\Theta[\sigma]$'s is a linear combination of the $\theta[\Delta]^2$. Note that there are 2^g functions $\Theta[\sigma]$ and thus there are $(2^g + 1)2^g/2 = 2^{g-1}(2^g + 1)$ products $\Theta[\sigma]\Theta[\sigma']$. This is also the number of even characteristics, and the products $\Theta[\sigma]\Theta[\sigma']$ span the same space (of modular forms of weight 1) as the $\theta[\Delta]^2$'s, which has dimension $2^{g-1}(2^g+1)$ (see [vG2] Lemma (2.7), [RF]).

As the degree of an H_g-invariant homogeneous polynomial in the $\Theta[\sigma]$ is a multiple of four, say $4k$, it can be written as a homogeneous polynomial of degree $2k$ in the

$\theta[\Delta]^2$'s. Thus for $g \leq 3$, any element in $M_{2k}(\Gamma_g(2))$ is a homogeneous polynomial of degree $2k$ in the $\theta[\Delta]^2$'s.

The $\theta[\Delta]^2$ are the better known functions and their transformation under $\Gamma_g(1)$ is easy to understand, but the $\Theta[\sigma]$ have the advantage that they are algebraically independent for $g \leq 2$ and there is a unique relation of degree 16 for $g = 3$. In contrast, there are many quadratic relations between the $\theta[\Delta]^2$'s, for example Jacobi's relation in $g = 1$. The classical formula used here is (cf. [I1] IV.1):

$$\theta[{}^a_b]^2 \;=\; \sum_\sigma (-1)^{\sigma b}\Theta[\sigma]\Theta[\sigma + a] \tag{3.33}$$

where we sum over the 2^g vectors $\sigma \in \mathbb{F}_2^g$ and $[{}^a_b]$ is an even characteristic, so $a\,{}^t\!b = 0$ ($\in \mathbb{F}_2$). These formulae are easily inverted to give:

$$\Theta[\sigma]\Theta[\sigma + a] \;=\; \tfrac{1}{2^g}\sum_b (-1)^{\sigma b}\theta[{}^a_b]^2.$$

It is easy to see that the $\theta[\Delta]^2$ span one-dimensional subrepresentations of H_g. Indeed, using the classical formula we find

$$(s, x, u)\theta[{}^a_b]^2 = s^2(-1)^{ua+xb}\theta[{}^a_b]^2.$$

This implies that the $\theta[{}^a_b]^4$ are Heisenberg invariants and thus are in $M_2(\Gamma_g(2))$. More generally, we have:

$$\prod_i^{2k} \theta[{}^{a_i}_{b_i}]^2 \;\in\; M_{2k}(\Gamma_g(2)) \quad \text{iff} \quad \sum a_i = \sum b_i = 0 \ (\in \mathbb{F}_2).$$

For example in case $g = 1$ one has

$$\theta[{}^0_0]^2 = \Theta[0]^2 + \Theta[1]^2, \qquad \theta[{}^0_1]^2 = \Theta[0]^2 - \Theta[1]^2, \qquad \theta[{}^1_0]^2 = 2\Theta[0]\Theta[1],$$

or, equivalently,

$$\Theta[0]^2 = (\theta[{}^0_0]^2 + \theta[{}^0_1]^2)/2, \qquad \Theta[1]^2 = (\theta[{}^0_0]^2 - \theta[{}^0_1]^2)/2, \qquad \Theta[0]\Theta[1] = \theta[{}^1_0]^2/2.$$

Note that upon substituting the first three relations in Jacobi's relation $\theta[{}^0_0]^4 = \theta[{}^0_1]^4 + \theta[{}^1_0]^4$ we obtain a trivial identity in the $\Theta[\sigma]$'s.

3.7 Modular group and representations

Using the classical formula we find that $\theta[0]^4 = (\sum_\sigma \Theta[\sigma]^2)^2$, and it is clear that this functions is Heisenberg invariant and thus defines a modular form of weight 2 on $\Gamma_g(2)$. For $g \in O^+(2g)$ we have $g \cdot [0] = [0]$ and the explicit transformation formula for theta constants shows that $\theta[0]^4$ transforms by a non-trivial character which we denote by ϵ:

$$\rho(g)\theta[0]^4 \;=\; \epsilon(g)\theta[0]^4, \qquad \epsilon : O^+(2g) \longrightarrow \{\pm 1\}. \tag{3.34}$$

For $g \geq 3$, this homomorphism is the only non-trivial one dimensional representation of $O^+(2g)$ and its kernel is a simple group.

3.7.1 Thomae formula and the case $g = 2$

The case $g = 2$ is quite simple and it is interesting to expand some details. For $g = 2$ there are 16 characteristics, six odd and ten even. The odd characteristics are:

$$\nu_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \nu_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \nu_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\nu_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \nu_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \nu_6 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The even ones are:

$$\delta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \delta_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \delta_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$\delta_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad \delta_5 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \delta_6 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\delta_7 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \delta_8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \delta_9 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\delta_{10} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Note that each even characteristic can be written in two different ways as sum $\pmod 2$ of three odd characteristics and in the sums none odd characteristic is repeated [RF]. For example $\delta_1 = \nu_1 + \nu_4 + \nu_6 = \nu_2 + \nu_3 + \nu_5$. Each set of three odd characteristics that summed gives an even characteristic is called a *triad*. We report all the triads in Table 3.2. The Thomae formula [Mu, F] allows to express the fourth power of the theta

Triads	$[\delta]$
146 235	$\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$
126 345	$\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$
125 346	$\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$
145 236	$\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$
124 356	$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$
156 234	$\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$
123 456	$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$
134 256	$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$
136 245	$\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$
135 246	$\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$

Table 3.2: The two triads of odd characteristics giving the same even characteristic.

constants in term of the six branch points of the genus two Riemann surface on which they are defined

$$\theta^4[\delta] = c\,\epsilon_{S,T} \prod_{i,j\in S\; i<j} (u_i - u_j) \prod_{k,l\in T\; k<l} (u_k - u_l) =: \epsilon_{S,T} P_{S,T}, \tag{3.35}$$

where the u_i's are the six branch points, S and T contain the indices of the two triad of the odd characteristics giving the even characteristic of the theta constant. For example, for δ_4, from Table 3.2, we have $S = \{1, 4, 5\}$ and $T = \{2, 3, 6\}$, $\epsilon_{S,T}$ is a sign depending on the triads and c is a constant independent from the characteristic. Using the many relations between the $\theta[\delta]^4$'s one can determine the relative sign appearing in the Thomae formula: if $\sum_\delta a_\delta \theta[\delta]^4 = 0$ it also holds $\sum_\delta a_\delta \epsilon_{S,T} P_{S,T} = 0$, with the same a_δ. For instance, considering the relation $\theta[\delta_1]^4 - \theta[\delta_4]^4 - \theta[\delta_6]^4 - \theta[\delta_7]^4 = 0$, which can be obtained using the classical formula, we obtain the relative signs between this four $\theta[\delta]^4$'s. Using some other relations we fixe all the relative signs. We report these signs in Table 3.3. For example, we have:

146	126	125	145	124	156	123	134	136	135
235	345	346	236	356	234	456	256	245	246
δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}
-1	1	1	-1	1	-1	1	-1	-1	-1

Table 3.3: Relative signs between the $\theta[\delta]^4$'s for the thomae formula.

$$\theta[\delta_4]^4 = -c(u_1 - u_4)(u_1 - u_5)(u_4 - u_5)(u_2 - u_3)(u_2 - u_6)(u_3 - u_6), \tag{3.36}$$

and

$$\theta[0]^4 = \theta[\delta_1]^4 = -c(u_1 - u_4)(u_1 - u_6)(u_4 - u_6)(u_2 - u_3)(u_2 - u_5)(u_3 - u_5). \tag{3.37}$$

From the expression of $\theta[0]^4$ obtained with the Thomae formula we conclude that ϵ is the product of the sign character on the subgroup $S_3 \times S_3$ of $O^+(4)$ and $\epsilon(g) = 1$ if $g = (15)(24)(36)$, where we identify $O^+(4)$ with a subgroup of S_6 as in Section 3.4.

3.7.2 The space of O^+-(anti)-invariants

For applications to superstring measures, we will be particularly interested in the subspace of $O^+(2g)$-anti-invariants of weight 6

$$M_6(\Gamma_g(2))^\epsilon := \{ f \in M_6(\Gamma_g(2)) \,:\, \rho(g)f = \epsilon(g)f \quad \forall g \in O^+(2g) \}$$

and the space of O^+-invariants of weight 8

$$M_8(\Gamma_g(2))^{O^+} := \{ f \in M_8(\Gamma_g(2)) \,:\, \rho(g)f = f \quad \forall g \in O^+(2g) \}.$$

It should be noted that $Sp(2g)$ permutes the $\theta[\Delta]^{4k} \in M_{2k}(\Gamma_g(2))$, up to sign if k is odd. This fact is particular evident in genus two using the Thomae formula and the identification $\mathrm{Sp}(4) \cong S_6$. Thus it is not hard to write down some invariants or anti-invariants, but the problem is to find all of them.

3.7.3 The dimensions of the O^+-(anti)-invariants

To find all O^+-(anti)-invariants we have to know the dimensions of this space. In this section we apply the representation theory of finite groups to $\mathrm{Sp}(2g)$ in order to decompose its representations into irreducibles ones. Once the decomposition of an $\mathrm{Sp}(2g)$-representation into irreducibles is known, it is easy to find the dimension of the O^+-(anti)-invariants. The dimension of the O^+-invariants in V is the multiplicity of the trivial representation $\mathbf{1}$ of O^+ in the O^+-representation $\mathrm{Res}^{Sp}_{O^+}(V)$ (the restriction of the representation from $Sp(2g)$ to $O^+(2g)$):

$$\dim V^{O^+} \;=\; \langle\, \mathrm{Res}^{Sp}_{O^+}(V), 1\,\rangle_{O^+} \;=\; \langle\, V, Ind^{Sp}_{O^+}(1)\,\rangle_{Sp}$$

where the second equality is Frobenius reciprocity, see Section 3.8. According to Frame [Fr2] one has:

$$\mathrm{Ind}^{Sp}_{O^+}(1) \;=\; \mathbf{1} \,+\, \sigma_\theta, \qquad \dim \sigma_\theta \;=\; 2^{g-1}(2^g+1)-1 = (2^g-1)(2^g+2)/2,$$

where $\mathbf{1}$ is the trivial representation and σ_θ is an irreducible representation of $Sp(2g)$. Note that $\dim \mathrm{Ind}^{Sp}_{O^+}(\mathbf{1}) = [Sp(2g) : O^+(2g)] = 2^{g-1}(2^g+1)$. Thus if the multiplicity of $\mathbf{1}$ and σ_θ in V is n_1 and n_θ respectively, then $\dim V^{O^+} = n_1 + n_\theta$.

Similarly, the dimension of the O^+-anti-invariants in V is the multiplicity of the representation ϵ of O^+ in the O^+-representation $\mathrm{Res}^{Sp}_{O^+}(V)$:

$$\dim V^\epsilon \;=\; \langle \mathrm{Res}^{Sp}_{O^+}(V), \epsilon\,\rangle_{O^+} \;=\; \langle V, \mathrm{Ind}^{Sp}_{O^+}(\epsilon)\,\rangle_{Sp}.$$

According to Frame [Fr2], the induced representation has two irreducible components:

$$\mathrm{Ind}^{Sp}_{O^+}(\epsilon) \;=\; \rho_\theta \,\oplus\, \rho_r, \qquad \begin{cases} \dim \rho_\theta &=& (2^g+1)(2^{g-1}+1)/3, \\ \dim \rho_r &=& (2^g+1)(2^g-1)/3. \end{cases}$$

Thus if the multiplicity of ρ_θ and ρ_r in V are n_θ and n_r respectively, then $\dim V^\epsilon = n_\theta + n_r$.

3.7.4 The $\mathrm{Sp}(2g)$-representation on $M_2^\theta(\Gamma_g(2))$ and on $\mathrm{Sym}^2(M_2^\theta(\Gamma_g(2)))$

The representation ρ_2 of $Sp(2g)$ on the subspace $M_2^\theta(\Gamma_g(2)) \subset M_2(\Gamma_g(2))$ was shown to be irreducible and isomorphic to ρ_θ by van Geemen in [vG2].

Frame has shown that the $Sp(2g)$-representation $\mathrm{Sym}^2(M_2^\theta(\Gamma_g(2)))$ decomposes into irreducible representations as follows:

$$\mathrm{Sym}^2(\rho_\theta) \;=\; \mathbf{1} \,+\, \sigma_\theta \,+\, \sigma_c, \qquad \dim \sigma_c \;=\; 2^{g-2}(2^g+1)(2^g-1)(2^g+2), \tag{3.38}$$

and σ_θ as in 3.7.3. The functions $\theta[\Delta]^8$ are in $M_4^\theta(\Gamma_g(2))$. They are permuted (without signs) by $Sp(2g)$ and span the subrepresentation $\mathbf{1}+\sigma_\theta$. The trivial subrepresentation in $\mathrm{Sym}^2(M_2^\theta(\Gamma_g(2)))$ is then spanned by the invariant $\sum_\Delta \theta[\Delta]^8$. It is easy to verify that the dimension of the image of $\mathrm{Sym}^2(M_2^\theta(\Gamma_g(2)))$ is larger than $2^{g-1}(2^g+1) = \dim(\mathbf{1}+\sigma_\theta)$ for $g \geq 2$. As the multiplication map is $Sp(2g)$-equivariant it follows that $\mathrm{Sym}^2(M_2^\theta(\Gamma_g(2))) \subset M_4^\theta(\Gamma_g(2))$ (so if $f_1, \ldots, f_N$ is a basis of $M_2^\theta(\Gamma_g(2))$ then the $f_i f_j$ are linearly independent).

3.7.5 Decomposing representations of $Sp(2g)$

Given a representation of a finite group on a complex vector space, one could determine the value of the character of the representation on each conjugacy class and then use the table of irreducible characters of the group to find the decomposition of the representation. However, it is very time consuming to compute these character values in our examples. Thus we take another approach, which has the additional advantage of identifying explicitly certain subrepresentations.

There is one conjugacy class of $Sp(2g)$ which has only $2^{2g}-1$ elements, the class of the transvections t_v with $v \in \mathbb{F}_2^{2g} - \{0\}$, see 3.5.1. If $\rho : Sp(2g) \to GL(V)$ is a complex representation of $Sp(2g)$, the operator

$$C \,=\, C_\rho \,:=\, \sum_{v\neq 0} \rho(t_v) \qquad (\in GL(V))$$

obviously satisfies $\rho(g)C\rho(g)^{-1} = C$ for all $g \in Sp(2g)$. If $V = \oplus V_i^{n_i}$ is the decomposition of V into irreducible representations V_i, $V_i \not\cong V_j$ if $i \neq j$, then, by Schur's lemma, C must be scalar multiplication by a $\lambda_i \in \mathbb{C}$ on V_i. In particular, the eigenvalues of C are the λ_i with multiplicity $n_i = \dim V_i$ (but it can happen that $\lambda_i = \lambda_j$ for $i \neq j$).

To find λ_i we consider the trace of C on V_i: as the t_v, $v \neq 0$, are the elements of one conjugacy class,

$$Tr(C_{|V_i}) \,=\, (2^{2g}-1)Tr(\rho_i(t_v)) \,=\, (2^{2g}-1)\chi_i(t_v)$$

where t_v is now one specific (but arbitrary) transvection and χ_i is the character of the irreducible representation ρ_i. On the other hand,

$$Tr(C_{|V_i}) \,=\, (\dim V_i)\lambda_i, \qquad \text{hence} \quad \lambda_i \,=\, \frac{(2^{2g}-1)\chi_i(t_v)}{\dim V_i}. \tag{3.39}$$

Note that $\ker(C-\lambda I)$ will be the direct sum of the $V_i^{n_i}$ with $\lambda_i = \lambda$, so we do not only get information on the multiplicities of the irreducible constituents of ρ but also on the corresponding subspaces of V.

3.8 Restricted and induced representations

In this section we introduce restricted and induced representations and we will see as they characters are connected by the Frobenius reciprocity [Sa]. Let G be a group and H a subgroup. Suppose ρ is a matrix representation of G, then one can obtain a representation of H by the operation of restriction. The restriction of ρ to H, $\mathrm{Res}_H^G(\rho)$, is given by:

$$\mathrm{Res}_H^G(\rho(h)) := \rho(h), \tag{3.40}$$

for all $h \in H$. The restriction, actually, is a representation of H. Although ρ may be an irreducible representation of G, $\mathrm{Res}_H^G(\rho(h))$ can be reducible. We can also consider the inverse operation. The process of moving from a representation of the subgroup H to a representation of the whole G is called induction. Fix a transversal $\{t_1, \cdots, t_k\}$ for the left cosets of H, i.e. $\mathcal{H} = \{t_1H, \cdots, t_kH\}$ is a complete set of disjoint left cosets for H in G, so $G = t_1H \uplus \cdots \uplus t_kH$ and $\uplus$ denotes disjoint union, $t_i \in G$. Let σ be a representation of H, then the induced representation $\mathrm{Ind}_H^G(\sigma)$ from H to G assigns to each $g \in G$ the block matrix:

$$\mathrm{Ind}_H^G(\sigma(g)) := \sigma(t_i^{-1}gt_j) = \begin{pmatrix} \sigma(t_1^{-1}gt_1) & \cdots & \sigma(t_1^{-1}gt_k) \\ \vdots & \ddots & \vdots \\ \sigma(t_k^{-1}gt_1) & \cdots & \sigma(t_k^{-1}gt_k), \end{pmatrix} \tag{3.41}$$

and $\sigma(g)$ is the zero matrix if $g \notin H$. It can be proved that $\mathrm{Ind}_H^G(\sigma)$ is a representation of G.

Of particular interest is the induced representation of the identity $\mathbf{1}$ and it is strictly related to the coset representation. Suppose, as before, that $\mathcal{H} = \{t_1H, \cdots, t_kH\}$ is a complete set of disjoint left cosets for H in G. The group G acts on the set $\mathcal{H}$ by $g(g_iH) := (gg_i)H$, for all $g \in G$. The set $\mathcal{H}$ can be turned, as every set on which a group G acts, in a G-module as follows. Let $\mathbb{C}\mathcal{H}$ denote the vector space generated by $\mathcal{H}$ over $\mathbb{C}$, that is $\mathcal{H}$ consists of all the formal linear combinations $a_1g_1H + \cdots + a_kg_kH$, $a_i \in \mathbb{C}$. Vector addition and scalar multiplication are defined as follows:

$$(a_1g_1H + \cdots + a_kg_kH) + (b_1g_1H + \cdots + b_kg_kH) = (a_1 + b_1)g_1H + \cdots + (a_k + b_k)g_kH$$
$$c(a_1g_1H + \cdots +) = (ca_1)g_1H + \cdots + (ca_k)g_kH,$$

for $a_1, \cdots, a_k, b_1, \cdots, b_k, c \in \mathbb{C}$ and $g_1, \cdots, g_k \in \mathbb{C}$. The action of G on $\mathcal{H}$ can be extend to an action on $\mathbb{C}\mathcal{H}$ by linearity:

$$g(a_1g_1H + \cdots + a_kg_kH) = a_1(gg_1H) + \cdots + a_k(gg_kH), \tag{3.42}$$

for all $g \in G$. In this way $\mathbb{C}\mathcal{H}$ becomes a G-module of dimension $|\mathcal{H}| = k$. More generally, given a set S on which a group G acts, then the associated module $\mathbb{C}S$ is called the permutation representation associated with S and the elements of S form a basis for $\mathbb{C}S$ called the standard basis. Note that if $H = G$ then the coset representation reduces to the trivial representation. We have the following

Proposition 3.8.1. *Let H be a subgroup of G which has transversal $\{t_1 \cdots t_k\}$ with cosets $\mathcal{H} = t_1H, \cdots t_kH$. Then the matrices of $\mathrm{Ind}_H^G(\mathbf{1})$ are identical with those of G acting on the basis $\mathcal{H}$ for the coset module $\mathbb{C}\mathcal{H}$.*

Proof. Let the matrices for the representations ρ and σ of $\mathrm{Ind}_H^G(\mathbf{1})$ and $\mathbb{C}\mathcal{H}$ be $X = (x_{ij})$ and $Y = (y_{ij})$ respectively. The matrix elements of both matrices are only zeros and ones. Moreover, for any $g \in G$ $x_{ij}(g) = 1$ if and only if $t_i^{-1} g t_j \in H$. But $t_i^{-1} g t_j \in H$ if and only if $g t_j H = t_i H$ and this happens if and only if $z_{ij}(g) = 1$. □

We will now prove the reciprocity law of Frobenius, which relates inner products of restricted and induced characters. Before we need a formula for the character of an induced representation. Let G, H and t_i as in the preceding proposition. Consider a representation ρ of H with character ψ. The transversal t_i give rise to the representation $\mathrm{Ind}_H^G(\rho)$ with character χ. We recall that the character of a representation is $\chi(g) = \mathrm{Tr}\rho(g)$ and given another representation σ with character ψ one can define an inner product of χ and ψ as $\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$, where the bar stands for complex conjugation. Note that $\overline{\psi(g)} = \psi(g^{-1})$. We have $\psi(t_i^{-1} g t_i) = \psi(h^{-1} t_i^{-1} g t_i h)$ for any $h \in H$, so:

$$\chi(\mathrm{Ind}_H^G(\rho(g))) = \sum_i \psi(t_i^{-1} g t_i) = \frac{1}{|H|} \sum_i \sum_{h \in H} \psi(h^{-1} t_i^{-1} g t_i h), \tag{3.43}$$

but as h runs over H and t_i run over the transversal, the product $t_i h$ runs over all the elements of G exactly once. Thus:

$$\chi(\mathrm{Ind}_H^G(\rho(g))) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1} g x). \tag{3.44}$$

We can now prove the Frobenius reciprocity. To simplify the notation we use $\psi(\mathrm{Ind}_H^G(\rho))$ to indicate the character of the induced representation of ρ if its character is ψ and analogously for the character of the restrict representation we use $\chi(\mathrm{Res}_H^G(\sigma))$ if σ has character χ.

Theorem 3.8.1 (Frobenius Reciprocity)**.** *Let G a group and H a subgroup, ρ a representation of H with character ψ and σ a representation of G with character χ. Then*

$$\langle \psi(\mathrm{Ind}_H^G(\rho)), \chi(\sigma) \rangle = \langle \psi(\rho), \chi(\mathrm{Res}_H^G(\sigma)) \rangle, \tag{3.45}$$

where the left inner product is calculated in G and the right one in H.

Proof. We have the following identities:

$$\begin{aligned}
\langle \psi(\mathrm{Ind}_H^G(\rho)), \chi(\sigma) \rangle &= \frac{1}{|G|} \sum_{g \in G} \psi(\mathrm{Ind}_H^G(\rho)) \chi(g^{-1}) = \frac{1}{|G||H|} \sum_{x \in G} \sum_{g \in G} \psi(x^{-1} g x) \chi(g^{-1}) \\
&= \frac{1}{|G||H|} \sum_{x \in G} \sum_{y \in G} \psi(y) \chi(x y^{-1} x^{-1}) = \frac{1}{|G||H|} \sum_{x \in G} \sum_{y \in G} \psi(y) \chi(y^{-1}) \\
&= \frac{1}{|H|} \sum_{y \in G} \psi(y) \chi(y^{-1}) = \frac{1}{|H|} \sum_{y \in H} \psi(y) \chi(y^{-1}) \\
&= \langle \psi(\rho), \chi(\mathrm{Res}_H^G(\sigma)) \rangle.
\end{aligned}$$

The second identity follows from equation (3.44), in the third we posed $y = x^{-1}gx$, the fourth follows from the constancy of χ on the equivalent classes of G, the fifth because x is constant in the sum and the sixth because ψ is zero outside H. $\square$

These are some of the ingredients we will use to construct string amplitudes for $g \leq 5$, to prove their uniqueness for $g \leq 3$ and in a weaker form for $g = 4$. The explicit construction will clarify the various steps and for $g \leq 2$ we will obtain the known results.

Chapter 4

Existence and uniqueness of the forms $\Xi_8[\Delta]$

We are now ready to work out a solution for the constraints and show its uniqueness, at least for genus $g \leq 3$. For genus four the uniqueness will be proved in a weakened form that is by restricting to polynomial expressions. We will analyze the equations genus by genus. In Section 4.1 we make some remarks on the three constraints and we compare them with the ones given by D'Hoker and Phong. The last remark allows us to simplify the problem to construct the functions $\Xi_8[\Delta]$. Actually, employing the action of the modular group on the theta constants, we can construct just one, instead of $2^{g-1}(2^g+1)$, of such functions and then obtain the others by the action of the group. The genus five case will be analyzed in the next chapter.

4.1 Some remarks on the constraints

In this section we revise the three constraints of Section 2.3 for the functions $\Xi_8[\Delta]$ using the mathematical tools introduced in Chapter 3 and we compare them with those of D'Hoker and Phong (DHP) in [DP6] for the functions $\Xi_6[\Delta]$.

- *Remark on condition (ii).* The only essential difference is in the constraint (ii), the transformation request. Note that the products $\theta[\Delta]^4(\tau)\Xi_6[\Delta](\tau)$, with $\Xi_6[\Delta]$ as in the DHP constraint (ii) and $\tau \in \mathbb{H}_3$, transform in the same way as our $\Xi_8[\Delta]$ apart from a factor $\epsilon(M,\Delta)^{4+4}$. However, $\epsilon(M,\Delta)^8 = 1$, so that this difference is only apparent. Conversely, if we require for each $\Xi_8[\Delta]$ to factorize in the product of $\theta[\Delta]^4$ and another function, the latter would satisfy constraint (ii) of [DP6]. But there is not a priory any reason to assume such a factorization: our form for the transformation constraint is weaker than the one imposed in [DP6] because is just a request on the weight of the "modular forms" and it does not impose a factorized form as a product of a function of weight six times a theta constant at the fourth power. Moreover, it is equivalent to assert that the $\Xi_8[\Delta]$ are modular forms of genus g and weight 8 on $\Gamma_g(2)$.

- *Remark on condition (iii).* DHP impose the factorization condition for an arbitrary separating degeneration. The point is that any such degeneration can be obtained from the one in condition (iii) by a symplectic transformation, so that we have to consider the functions $\Xi_8[\Delta](N \cdot \tau_{k,g-k})$ for all $N \in Sp(2g, \mathbb{Z})$. By constraint (ii), this amounts to considering the function $\Xi_8[N^{-1} \cdot \Delta](\tau_{k,g-k})$ (up to an easy factor) which is indeed determined by constraint (iii).

- *Oversimplification of the problem.* In order to search for a solution of the constraints, we will now show that equation (2.22) can be used to restrict the problem to a single value of the characteristic, for which we choose $\Delta = [{}^0_0]$ with $[{}^0_0] = [{}^{0\ldots0}_{0\ldots0}]$. In particular, we will see that the problem reduces itself to restrict the constraints to simpler ones for the function $\Xi_8[{}^0_0]$. Solving the problem for $\Xi_8[{}^0_0]$ will permit us to define functions $\Xi_8[\Delta^{(g)}]$, for all even characteristics $[\Delta^{(g)}]$, which satisfy the constraints from section 2.3.
If we take M to be in the stabilizer of the null characteristic, then condition (ii) is equivalent to require for $\Xi_8[{}^0_0]$ to be a modular form on $\Gamma_g(1,2)$ of weight 8. We know that the group $Sp(2g, \mathbb{Z})$ acts transitively on the even characteristics. This means that for any even characteristic $[\Delta^{(g)}]$ there exists at least an $M \in Sp(2g, \mathbb{Z})$ such that $M \cdot [{}^0_0] = [\Delta^{(g)}]$ mod 2. Then we define
$$\Xi_8[\Delta^{(g)}](\tau) := \gamma(M, M^{-1} \cdot \tau)^8 \Xi_8[{}^0_0](M^{-1} \cdot \tau), \tag{4.1}$$
with $\gamma(M, \tau) := \det(C\tau + D)$. The definition of $\Xi_8[\Delta^{(g)}]$ does not depend on the choice of M and also satisfies the transformation constraint, we postpone the verification of this fact to the next subsection, see aslo [CDG1]. Thus, the functions $\Xi_8[\Delta^{(g)}]$, defined by equation (4.1),' verify all the constraints if $\Xi_8[{}^0_0]$ satisfies the following reduced constraints.

(ii$_0$) The function $\Xi_8[{}^0_0]$ is a modular form Ξ_8 of weight 8 on $\Gamma_g(1,2)$.

(iii$_0$)(1) For all k, $0 < k < g$, and all $\tau_{k,g-k} \in \Delta_{k,g-k}$ we have
$$\Xi_8[{}^0_0](\tau_{k,g-k}) \;=\; \Xi_8[{}^0_0](\tau_k)\Xi_8[{}^0_0](\tau_{g-k})$$
(iii$_0$)(2) If $\Delta^{(g)} = [{}^{ab\ldots}_{cd\ldots}]$ with $ac = 1$ then $\Xi_8[\Delta^{(g)}](\tau_{1,g-1}) = 0$.

(iii$_0$)(1,2) is obviously a consequence of (iii). For (iii$_0$)(1), let us consider the characteristic
$$\Delta^{(g)} = [{}^{a_1\ldots a_g}_{b_1\ldots b_g}], \qquad \Delta^{(k)} := [{}^{a_1\ldots a_k}_{b_1\ldots b_k}], \quad \Delta^{(g-k)} := [{}^{a_{k+1}\ldots a_g}_{b_{k+1}\ldots b_g}].$$
and assume that $\Delta^{(k)}$ is even, so that also $\Delta^{(g-k)}$ is even. By transitivity, there are two symplectic matrices $M_1 \in Sp(2k, \mathbb{Z})$ and $M_2 \in Sp(2(g-k), \mathbb{Z})$ such

that $M_1 \cdot \left[{0 \atop 0}\right] = [\Delta^{(k)}]$ and $M_2 \cdot \left[{0 \atop 0}\right] = [\Delta^{(g-k)}]$. We compose such matrices in a block diagonal form so defining a matrix $M \in Sp(2g, \mathbb{Z})$ which has the properties: $M \cdot (\Delta_{k,g-k}) = \Delta_{k,g-k}$ and $\Delta^{(g)} = M \cdot \left[{0 \atop 0}\right]$. As M and $\tau_{k,g-k}$ are made up of $k \times k$ and $(g-k) \times (g-k)$ blocks we get

$$\gamma(M, M^{-1} \cdot \tau_{k,g-k}) = \gamma(M_1, M_1^{-1} \cdot \tau_k)\gamma(M_2, M_2^{-1} \cdot \tau_{g-k}), \qquad \Delta^{(g)} = M \cdot \left[{0 \atop 0}\right].$$

It follows that $M^{-1} \cdot \tau_{k,g-k} \in \Delta_{k,g-k}$ is a matrix with blocks $M_1^{-1} \cdot \tau_k$ and $M_2^{-1} \cdot \tau_{g-k}$. Thus, from the constraint (iii$_0$)(1) get

$$\Xi_8\left[{0 \atop 0}\right](M^{-1} \cdot \tau_{k,g-k}) = \Xi_8\left[{0 \atop 0}\right](M_1^{-1} \cdot \tau_k)\Xi_8\left[{0 \atop 0}\right](M_2^{-1} \cdot \tau_{g-k}),$$

and then we finally have

$$\begin{aligned} \Xi_8[\Delta^{(g)}](\tau_{k,g-k}) &= \gamma(M, M^{-1} \cdot \tau)^8 \Xi_8\left[{0 \atop 0}\right](M^{-1} \cdot \tau_{k,g-k}) \\ &= \gamma(M_1, M_1^{-1} \cdot \tau_k)^8 \gamma(M_2, M_2^{-1} \cdot \tau_{g-k})^8 \Xi_8\left[{0 \atop 0}\right](M_1^{-1} \cdot \tau_k)\Xi_8\left[{0 \atop 0}\right](M_2^{-1} \cdot \tau_{g-k}) \\ &= \Xi_8[\Delta^{(k)}](\tau_k)\Xi_8[\Delta^{(g-k)}](\tau_{g-k}), \end{aligned}$$

so for such $\Delta^{(g)}$ the functions $\Xi_8[\Delta^{(g)}]$ satisfy (iii).

Independence of $\Xi_8[\Delta^{(g)}]$ on the choice of M

We verify here that the definition of $\Xi_8[\Delta^{(g)}]$ does not depend on the choice of M: if also $N \cdot \left[{0 \atop 0}\right] = [\Delta^{(g)}]$ mod 2, then $N^{-1}M$ fixes $\left[{0 \atop 0}\right]$ so $N^{-1}M \in \Gamma_g(1,2)$. To verify that

$$\gamma(M, M^{-1} \cdot \tau)^8 \Xi_8\left[{0 \atop 0}\right](M^{-1} \cdot \tau) \stackrel{?}{=} \gamma(N, N^{-1} \cdot \tau)^8 \Xi_8\left[{0 \atop 0}\right](N^{-1} \cdot \tau)$$

we let $\tau = M\tau'$, so we must verify that

$$\gamma(M, \tau')^8 \Xi_8\left[{0 \atop 0}\right](\tau') \stackrel{?}{=} \gamma(N, N^{-1}M \cdot \tau')^8 \Xi_8\left[{0 \atop 0}\right](N^{-1}M \cdot \tau').$$

As $N^{-1}M \in \Gamma_g(1,2)$ and γ satisfies the cocycle condition, we get

$$\begin{aligned} \gamma(N, N^{-1}M \cdot \tau')^8 \Xi_8\left[{0 \atop 0}\right](N^{-1}M \cdot \tau') &= \gamma(N, N^{-1}M \cdot \tau')^8 \gamma(N^{-1}M, \tau')^8 \Xi_8\left[{0 \atop 0}\right](\tau') \\ &= \gamma(M, \tau')^8 \Xi_8\left[{0 \atop 0}\right](\tau'), \end{aligned}$$

which verifies the desired identity. Finally we show that the functions $\Xi_8[\Delta^{(g)}]$ satisfy constraint (ii) of section 2.3. So with $M, \Delta^{(g)}$ as above, we must verify that for all $N \in Sp(2g, \mathbb{Z})$ we have

$$\Xi_8[N \cdot \Delta](N \cdot \tau) \stackrel{?}{=} \gamma(N, \tau)^8 \Xi_8[\Delta](\tau).$$

As $N\cdot\Delta = NM\cdot\left[{0 \atop 0}\right]$, we have:

$$\begin{aligned}\Xi_8[N\cdot\Delta](N\cdot\tau) &= \gamma(NM,(NM)^{-1}N\cdot\tau)^8\Xi_8\left[{0 \atop 0}\right]((NM)^{-1}N\cdot\tau)\\ &= \gamma(NM,M^{-1}\cdot\tau)^8\Xi_8\left[{0 \atop 0}\right](M^{-1}\tau)\\ &= \gamma(N,\tau)^8\gamma(M,M^{-1}\cdot\tau)^8\Xi_8\left[{0 \atop 0}\right](M^{-1}\cdot\tau)\\ &= \gamma(N,\tau)^8\Xi_8[\Delta](\tau),\end{aligned}$$

where we used the cocycle relation. Thus the second constraint is verified if $\Xi_8\left[{0 \atop 0}\right]$ satisfies the constraint (ii$_0$) and if the $\Xi_8[\Delta^{(g)}]$ are defined as in equation 4.1.

4.2 The case $g = 1$

In genus one there are three even characteristics and one odd. Clearly, the even ones are $\left[{0 \atop 0}\right]$, $\left[{0 \atop 1}\right]$ and $\left[{1 \atop 0}\right]$ and the odd one is $\left[{1 \atop 1}\right]$, as follows from the definitions given in Section 3.2. Using the classical theta formula (3.33), we can find the relations between the classical and the second order theta constants. Moreover, there are no algebraic relations between the $\Theta[\sigma]$'s. The dimension formula (3.28) and the fact that $M^{\theta}_{2k}(\Gamma_1(2)) = M_{2k}(\Gamma_1(2))$ show that $\dim M_{2k}(\Gamma_1(2)) = k+1$. A basis of $M_2(\Gamma_1(2))$ is given by the Heisenberg invariants $\Theta[0]^4 + \Theta[1]^4$ and $(\Theta[0]\Theta[1])^2$ and a basis of $M_{2k}(\Gamma_1(2)$ is given by homogeneous polynomials of degree k in these invariants:

$$(\Theta[0]^4+\Theta[1]^4)^k, \quad (\Theta[0]\Theta[1])^2(\Theta[0]^4+\Theta[1]^4)^{k-1}, \quad \ldots, \quad (\Theta[0]\Theta[1])^{2k}.$$

In genus one the group $\mathrm{Sp}(2,\mathbb{Z})$ is isomorphic to the special linear group $SL(2,\mathbb{Z})$ and its standard generators are $S = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. The classical transformation theory of theta functions gives:

$$\rho_2(S): \begin{cases}\theta\left[{0 \atop 0}\right]^4 \longmapsto -\theta\left[{0 \atop 0}\right]^4\\ \theta\left[{0 \atop 1}\right]^4 \longmapsto -\theta\left[{1 \atop 0}\right]^4\\ \theta\left[{1 \atop 0}\right]^4 \longmapsto -\theta\left[{0 \atop 1}\right]^4\end{cases}, \qquad \rho_2(T): \begin{cases}\theta\left[{0 \atop 0}\right]^4 \longmapsto \theta\left[{0 \atop 1}\right]^4\\ \theta\left[{0 \atop 1}\right]^4 \longmapsto \theta\left[{0 \atop 0}\right]^4\\ \theta\left[{1 \atop 0}\right]^4 \longmapsto -\theta\left[{1 \atop 0}\right]^4\end{cases}.$$

In the computation of the matrices of the $\rho_2(g)$'s w.r.t. the basis of $M_2(\Gamma_1(2))$ ($\theta\left[{0 \atop 0}\right]$ and $\theta\left[{0 \atop 1}\right]$, for example) one has to apply the Jacobi relation $\theta\left[{0 \atop 0}\right]^4 = \theta\left[{1 \atop 0}\right]^4 + \theta\left[{0 \atop 1}\right]^4$. As follows from section 3.7.4 and from the table of characters of S_3 reported in Table 4.1, $M_2(\Gamma_1(2))$ is the unique irreducible two dimensional representation of the symmetric group S_3 that can be identified with the representation $\rho_\theta = \rho[21]$, hence $M_{2k}(\Gamma_1(2)) \simeq \mathrm{Sym}^k(\rho[21])$. The group $O^+(2)$ is the group of order two generated by the image of $S \in SL(2,\mathbb{Z})$ in $Sp(2)$.

We now study to the decomposition of $M_6(\Gamma_1(2))$ because some functions we will extensively use in the construction of superstring measures belong to this space. The representation on $M_6(\Gamma_1(2)) = \mathrm{Sym}^3(M_2(\Gamma_1(2)))$ can be decomposed in irreducible representations of S_3 as:

$$M_6(\Gamma_1(2)) \cong \rho_{[3]} \oplus \rho_{[2,1]} \oplus \rho_{[1^3]},$$

S_3	$C_{1,1,1}$	$C_{2,1}$	C_3
$\rho_{[1^3]}$	1	1	1
$\rho_{[3]}$	1	-1	1
$\rho_{[2,1]}$	2	0	-1

Table 4.1: Table of the characters of the group S_3.

where $\rho_{[1^3]}$ is the sign representation of S_3 and $\rho_{[3]}$ is the trivial representation. One obtains this decomposition using the techniques we will expose for the more enlightening case $g = 2$, see section 4.3.1.

It is easy to verify that the three irreducible representations are generated by:

$$\begin{aligned}
\rho_{[3]} &= \langle\, \Theta[0]^{12} - 33\Theta[0]^8\Theta[1]^4 - 33\Theta[0]^4\Theta[1]^8 + \Theta[1]^{12}\rangle, \\
\rho_{[1^3]} &= \langle\, \eta^{12}\,\rangle, \\
\rho_{[2,1]} &= \{(a+b)\theta[{}^0_0]^{12} + a\theta[{}^0_1]^{12} + b\theta[{}^1_0]^{12} \,:\, a, b \in \mathbb{C}\},
\end{aligned}$$

where we used a classical formula for the Dedekind η function: $\eta^3 = \theta[{}^0_0]\theta[{}^0_1]\theta[{}^1_0]$, so

$$\begin{aligned}
\eta^{12} &= \theta[{}^0_0]^4\theta[{}^0_1]^4\theta[{}^1_0]^4 \\
&= (\Theta[0]^2 + \Theta[1]^2)^2(\Theta[0]^2 - \Theta[1]^2)^2(2\Theta[0]\Theta[1])^2.
\end{aligned}$$

The two dimensional subspace of $O^+(2)$-anti-invariants, see section 3.7.3, is:

$$M_6(\Gamma_g(2))^\epsilon = \langle \eta^{12},\, f_{21} := 2\theta[{}^0_0]^{12} + \theta[{}^0_1]^{12} + \theta[{}^1_0]^{12}\,\rangle,$$

the function f_{21} lies in the two-dimensional $\rho_{[21]}$ irreducible subrepresentation[1]. We list here some other modular forms that can be expressed in terms of f_{21} and η^{12} that we will use later.

$$\begin{aligned}
\theta^{12}[{}^0_0] &= \tfrac{1}{3}f_{21} + \eta^{12}, \\
\theta[{}^0_0]^4(\theta[{}^0_0]^8 + \theta[{}^0_1]^8 + \theta[{}^1_0]^8) &= \tfrac{2}{3}f_{21}, \\
\theta[{}^0_0]^{12} + \theta[{}^0_1]^{12} + \theta[{}^1_0]^{12} &= \tfrac{2}{3}f_{21} - \eta^{12}, \\
\theta[{}^0_0]^4\theta[{}^0_1]^8 + \theta[{}^0_0]^4\theta[{}^1_0]^8 &= \tfrac{1}{3}f_{21} - \eta^{12}.
\end{aligned}$$

In genus one it is well known that the modular forms $\Xi_8[\Delta]$ are given by $\Xi_8[\Delta] = \theta[\Delta]^4\eta^{12}$. The function $\Xi_8[{}^0_0] = \theta[{}^0_0]^4\eta^{12}$ is a modular form of weight eight on $\Gamma_1(1,2)$.

[1] From this we choose the "strange" name f_{21}.

4.3 The case $g = 2$

At genus two there are ten even characteristics which correspond to ten theta constants. From the Table 3.1 we know that the ring of the Heisenberg invariants of degree four is a five dimensional space. It is generated by the fourth power of the theta constants, as can be shown using the classical formula (and a computer to make the computations faster). A useful basis for this space, obtained as explained in section 3.5, is provided by the following five homogeneous polynomials $p_0, \cdots, p_4$ of degree four in the $\Theta[\sigma]$'s:

$$\mathbb{C}[\dots,\Theta[\sigma],\dots]^{H_2} = \mathbb{C}[p_0,\dots,p_4] \tag{4.2}$$

where the p_i are defined by

$$p_0 = \Theta[00]^4 + \Theta[01]^4 + \Theta[10]^4 + \Theta[11]^4, \qquad p_1 = 2(\Theta[00]^2\Theta[01]^2 + \Theta[10]^2\Theta[11]^2),$$
$$p_2 = 2(\Theta[00]^2\Theta[10]^2 + \Theta[01]^2\Theta[11]^2), \qquad p_3 = 2(\Theta[00]^2\Theta[11]^2 + \Theta[01]^2\Theta[10]^2),$$
$$p_4 = 4\Theta[00]\Theta[01]\Theta[10]\Theta[11]. \tag{4.3}$$

By means of the classical formula we can expand the ten $\theta[\delta]^4$ on this basis. We summarize the result in Table 4.2, for example:

$$\theta[\delta_7]^4 = 2p_2 + 2p_4. \tag{4.4}$$

The advantage of the selected basis is that it defines a map $\mathbb{P}^3 \to \mathbb{P}^4$, $(\Theta[00](\tau) :$

δ	$\theta^4[\delta]$	p_0	p_1	p_2	p_3	p_4
δ_1	$\theta^4\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	1	1	1	0
δ_2	$\theta^4\begin{bmatrix}0&0\\0&1\end{bmatrix}$	1	-1	1	-1	0
δ_3	$\theta^4\begin{bmatrix}0&0\\1&0\end{bmatrix}$	1	1	-1	-1	0
δ_4	$\theta^4\begin{bmatrix}0&0\\1&1\end{bmatrix}$	1	-1	-1	1	0
δ_5	$\theta^4\begin{bmatrix}0&1\\0&0\end{bmatrix}$	0	2	0	0	2
δ_6	$\theta^4\begin{bmatrix}0&1\\1&0\end{bmatrix}$	0	2	0	0	-2
δ_7	$\theta^4\begin{bmatrix}1&0\\0&0\end{bmatrix}$	0	0	2	0	2
δ_8	$\theta^4\begin{bmatrix}1&0\\0&1\end{bmatrix}$	0	0	2	0	-2
δ_9	$\theta^4\begin{bmatrix}1&1\\0&0\end{bmatrix}$	0	0	0	2	2
δ_{10}	$\theta^4\begin{bmatrix}1&1\\1&1\end{bmatrix}$	0	0	0	2	-2

Table 4.2: Expansion of $\theta^4[\delta]$ on the basis of p_i

$\Theta[01](\tau) : \Theta[10](\tau) : \Theta[11](\tau)) \to (p_0 : p_1 : p_2 : p_3 : p_4)$ and the image of $\mathbb{P}^3$ results to be defined by a quartic polynomial f_4 in 5 variables, the Igusa quartic, so that $I_4 = f_4(p_0, p_1, p_2, p_3, p_4)$ is identically zero. Its explicit expression is

$$I_4 = p_4^4 + p_4^2p_0^2 - p_4^2p_1^2 - p_4^2p_2^2 - p_4^2p_3^2 + p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 - 2p_0p_1p_2p_3. \tag{4.5}$$

Expressing the p_i in term of the four second order theta constant one verifies that this polynomial vanishes. We can also write I_4 using the ten classical theta constants obtaining:

$$I_4 = \frac{1}{192}\left[\left(\sum_\delta \theta^8[\delta]\right)^2 - 4\sum_\delta \theta^{16}[\delta]\right]. \tag{4.6}$$

It thus provides a relation of order 16 in the classical $\theta[\delta]$. From this we see that, as a graded ring,

$$\oplus_{k=0}^{\infty} M_{2k}(\Gamma_2(2)) \cong \mathbb{C}[y_0,\ldots,y_4]/(f_4(y_0,\ldots,y_4)), \tag{4.7}$$

and for $k \leq 3$ there are not constraints, so that the dimensions are $\dim M_2(\Gamma_2(2)) = 5$, $\dim M_4(\Gamma_2(2)) = \binom{5+1}{2} = 15$, $\dim M_6(\Gamma_2(2)) = \binom{5+2}{3} = 35$, $\dim M_8(\Gamma_2(2)) = \binom{5+3}{4} - 1 = 69$.

4.3.1 The Sp(4)-representations on the $M_{2k}(\Gamma_2(2))$

We will now study the representations of the symplectic group on the space of modular forms. We use the isomorphism $\mathrm{Sp}(4) \cong S_6$ and the irreducible representations will be labeled by partitions of 6. Table 4.3 collects the characters of the group S_6. In the second column are given the partitions of 6 and in the first one the names we have chosen for the corresponding representations. The symmetric group S_6 has eleven conjugacy

S_6	Partition	C_1	C_2	C_3	$C_{2,2}$	C_4	$C_{3,2}$	C_5	$C_{2,2,2}$	$C_{3,3}$	$C_{4,2}$	C_6
id_1	$[6]$	1	1	1	1	1	1	1	1	1	1	1
alt_1	$[1^6]$	1	−1	1	1	−1	−1	1	−1	1	1	−1
st_5	$[2^3]$	5	−1	−1	1	1	−1	0	3	2	−1	0
sta_5	$[3^2]$	5	1	−1	1	−1	1	0	−3	2	−1	0
rep_5	$[5\,1]$	5	3	2	1	1	0	0	−1	−1	−1	−1
repa_5	$[2\,1^4]$	5	−3	2	1	−1	0	0	1	−1	−1	1
n_9	$[4\,2]$	9	3	0	1	−1	0	−1	3	0	1	0
na_9	$[2^2\,1^2]$	9	−3	0	1	1	0	−1	−3	0	1	0
sw_{10}	$[3\,1^3]$	10	−2	1	−2	0	1	0	2	1	0	−1
swa_{10}	$[4\,1^2]$	10	2	1	−2	0	−1	0	−2	1	0	1
s_{16}	$[3\,2\,1]$	16	0	−2	0	0	0	1	0	−2	0	0

Table 4.3: Characters of the conjugacy classes of the eleven irreducible representations of S_6.

classes so that it has eleven irreducible representations. For example, the class $C_{3,2}$ consists of the product of a two-cycle and a three-cycle, and the character of the first ten dimensional representation, sw_{10} or $[3\,1^3]$ in standard notation, for this class is 1.

To find the correspondence among the symplectic group and the symmetric group we can relate the transformations of the theta constants under the action of the generators of Sp(4), given by the transformation formula for the theta constants (3.18), to the transformations of the same functions obtained by the action of S_6 on the branch points appearing in the Thomae formula. The generators of Sp(4) are:

$$M_i = \begin{pmatrix} I & B_i \\ 0 & I \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$
$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}; \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$
$$T = \begin{pmatrix} \tau_+ & 0 \\ 0 & \tau_- \end{pmatrix}, \quad \tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{4.8}$$

The phase factor $\epsilon(\delta, M)$, satisfying $\epsilon^8(\delta, M) = 1$, depends both on the characteristic δ and on the matrix M generating the transformation[2]. For the even characteristics $\delta = \left[{a \atop b}\right]$ the fourth powers of ϵ are given by:

$$\epsilon^4(\delta, M_i) = e^{\pi i {}^t a B_i a} \qquad i = 1, 2, \tag{4.9}$$
$$\epsilon^4(\delta, M_3) = \epsilon^4(\delta, S) = \epsilon^4(\delta, \Sigma) = \epsilon^4(\delta, T) = 1. \tag{4.10}$$

In Table 4.4 we report the relationship between the generators of the modular group and S_6.

M_1	M_2	M_3	S	Σ	T
(1 3)	(2 4)	(1 3)(2 4)(5 6)	(3 5)(4 6)	(1 2)(3 4)(5 6)	(1 3)(2 6)(4 5)

Table 4.4: Relationship between the generators of the modular group and S_6.

We now need to identify the representation $M_2(\Gamma_2(2))$. We know from the result of van Geemen in section 3.7.4 that this five dimensional representation is ρ_θ which is irreducible. The characters table of S_6 shows that there are four five dimensional irreducible representations. To find which one is supported by the $\theta[\delta]^4$'s we study the action of the permutation (12), that belongs to the conjugacy class C_2, on the basis $\theta[\delta_i]$, $i = 1, \cdots, 2$ of $M_2(\Gamma_2(2))$ (alternatively one can study the action of one generator of Sp(4), for example M_1, obtaining the same result). The matrix associated to this permutation is:

$$M_{1\,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{4.11}$$

[2]In ϵ there are both the contributes of $\kappa(M)$ and of $\Phi_\Delta(M)$ of (3.18)

The trace of this matrix is -1 and this is exactly the character of the representation spanned by the $\theta[\delta]^4$'s. Thus $\rho_\theta = \rho_{[2^3]}$, i.e. the representation is the st_5. More in general, using the theory of representations of S_6 we find:

$$\begin{aligned} M_2(\Gamma_2(2)) &\cong \rho_{[2^3]}, \\ M_4(\Gamma_2(2)) &\cong \mathrm{Sym}^2(\rho_{[2^3]}) = \mathbf{1} + \rho_{[42]} + \rho_{[2^3]}, \\ M_6(\Gamma_2(2)) &\cong \mathrm{Sym}^3(\rho_{[2^3]}) = \mathbf{1} + 2\rho_{[2^3]} + \rho_{[21^4]} + \rho_{[42]} + \rho_{[31^3]}, \\ M_8(\Gamma_2(2)) &\cong \mathrm{Sym}^4(\rho_{[2^3]}) - \mathbf{1} = \mathbf{1} + 3\rho_{[2^3]} + 3\rho_{[42]} + \rho_{[31^3]} + \rho_{[321]}. \end{aligned}$$

From these decompositions we can conclude that $\rho_r = \rho_{[21^4]}$ and $\sigma_\theta = \rho_{[42]}$ (from the next discussion or cf. [CD1] one can also deduce that $\rho_\theta + \rho_r$ is the representation on the $\theta[\delta]^{12}$, which is $\mathrm{Ind}_{O^+}^{Sp}(\epsilon)$ and $1 + \sigma_\theta$ is the permutation representation on the 10 even $\theta[\delta]^8$, hence its trace must be ≥ 0 on each conjugacy class). The computation of the symmetric powers of a representation can be done using a computer (for example with Magma or Mathematica), but also by hand. We expose the details for the computation of $\mathrm{Sym}^3(\rho_{[2^3]})$, the others being similar. To establish the characters of the representations on $\mathrm{Sym}^3(M_2(\Gamma_2(2)))$ we proceed as follows. If $\mathrm{Tr}(\rho^{V_\theta}(g)) = \sum_i \lambda_i$, where to shorten the notations we posed $V_\theta \equiv M_2(\Gamma_2(2))$, then for $g \in S_6$:

$$\mathrm{Tr}(\rho^{V_\theta}(g^2)) = \sum_i \lambda_i^2, \quad \mathrm{Tr}(\rho^{V_\theta}(g^3)) = \sum_i \lambda_i^3, \quad \mathrm{Tr}(\rho^{S^3 V_\theta}(g)) = \sum_{1 \leq i \leq j \leq k \leq 5} \lambda_i \lambda_j \lambda_k, \tag{4.12}$$

$$(\mathrm{Tr}(\rho^{V_\theta}))^3 = \sum_i \lambda_i^3 + 3 \sum_{i \neq j} \lambda_i^2 \lambda_j + 6 \sum_{1 \leq i < j < k \leq 5} \lambda_i \lambda_j \lambda_k. \tag{4.13}$$

Using the previous relations we finally obtain

$$\mathrm{Tr}(\rho^{S^3 V_\theta}(g)) = \frac{1}{6}(\mathrm{Tr}(\rho^{V_\theta}(g)))^3 + \frac{1}{2}\mathrm{Tr}(\rho^{V_\theta}(g))\mathrm{Tr}(\rho^{V_\theta}(g^2)) + \frac{1}{3}\mathrm{Tr}(\rho^{V_\theta}(g^3)). \tag{4.14}$$

To apply this formula we have to know in which conjugacy class are g^2 and g^3. For example, for the element $g = (12)$ its square is in the class of the identity, C_1, and its cube in the same class of g, C_2. Hence, we obtain that the character of the representation of $g = (12)$ on $\mathrm{Sym}^3 V_\theta = \mathrm{Sym}^3(M_2(\Gamma_2(2)))$ is -3. In the same way we can compute the characters χ of the other ten conjugacy classes:

$$\chi(\rho_{V_\theta}) = \{35, -3, 2, 3, 1, 0, 0, 13, 5, -1, 1\} \tag{4.15}$$

The representation on $\mathrm{Sym}^3 V_\theta$ will be a direct sum of the irreducible representations of S_6: $\mathrm{Sym}^3 V_\theta = \oplus_i m_i \rho_i$. Using the orthogonality of the characters we can compute the m_i as $m_i = \langle \chi_{\mathrm{Sym}^3 V_\theta}, \chi_i \rangle$. The inner product between characters is given by $\langle \chi, \psi \rangle = \frac{1}{|S_6|} \sum_{g \in S_6} \chi(g) \overline{\psi(g)} = {}^t\chi T \overline{\psi}$, where T is the matrix

$$T = \frac{1}{|S_6|} \mathrm{diag}\{|C_1|, \cdots, |C_6|\} = \frac{1}{720} \mathrm{diag}\{1, 15, 40, 45, 90, 120, 144, 15, 40, 90, 120\}. \tag{4.16}$$

On the diagonal there are the numbers of elements in each of the eleven conjugacy classes of S_6. Then, we get the complete decomposition

$$\mathrm{Sym}^3(\mathrm{st}_5) = \mathrm{id}_1 + \mathrm{n}_9 + \mathrm{repa}_5 + 2\mathrm{st}_5 + \mathrm{sw}_{10}, \tag{4.17}$$

which expresses exactly the desired result.
In the next subsection, in order to solve the constraints, we will construct the basis for the space of modular forms of weight eight in which there is the representations **1** and σ_θ, as well the space of modular form of weight 6 in which there is the representation ρ_θ. Here we conclude our analysis by looking at which representation is supported by some particular vector spaces. We will focus on the space $\mathrm{Sym}^3(M_2(\Gamma_2(2)))$, clarifying the origin of the functions Ξ_6 introduced by D'Hoker and Phong, and on some other spaces generated by polynomial of degree 12 in the theta constants.

Let us consider the matrix representation of S_6 in the space $\mathrm{Sym}^3 V_\theta$ and define the matrix $N_2 \in \mathrm{Mat}(35, \mathbb{Z})$ as the sum of the 15 matrices representing the elements in the conjugacy class C_2. If $N(\sigma)$ is the matrix representing any element $\sigma \in S_6$, clearly we have $N(\sigma)N_2N(\sigma)^{-1} = N_2$, as $N(\sigma)$ just changes the order of the addends in the sum. The eigenvalues of this matrix are

$$N_{2\,\mathrm{diag}} = \mathrm{diag}\{15, \underbrace{-9, \cdots, -9}_{5\text{ times}}, \underbrace{5, \cdots, 5}_{9\text{ times}}, \underbrace{-3 \cdots, -3}_{20\text{ times}}\}. \tag{4.18}$$

and its trace is -45. From Schur lemma, the matrix N_2 acts as a multiple of the identity over each subspace V_i supporting an irreducible representation, i.e. $N_2|_{V_i} = \lambda_i^{(2)} \,\mathrm{Id}\,|_{V_i}$ and from (3.39):

$$\lambda_i^{(2)} = \frac{15\mathrm{Tr}\rho_i(1\,2)}{\dim(V_i)}, \tag{4.19}$$

where we chosen $g = (1\,2)$ as two cocycle of S_6. We can deduce many interesting informations from this computation, even though this matrix is not enough to decompose the whole space $\mathrm{Sym}^3 V_\theta$: indeed, from (4.19) we deduce that the eigenvalue 15 corresponds to the representation id_1 of dimension one, the eigenvalue -9 to representation repa_5 of dimension five and the eigenvalue -5 to the representation n_9 of dimension nine. But for the last twenty eigenvalues we are not able to distinguish the spaces with the same eigenvalue (st_5 and sw_{10}). To recognize univocally all the subspace of $\mathrm{Sym}^3 V_\theta$ we then consider also the conjugacy class $C_{2,2,2}$ of the product of three 2-ciclyes. This class has 15 elements. As before we compute the matrix $N_{2,2,2} \in \mathrm{Mat}(35, \mathbb{Z})$, which for the Schur lemma acts as a multiple of identity on each subspace supporting an irreducible representation, and whose eignevalues are:

$$N_{2,2,2\,\mathrm{diag}} = \mathrm{diag}\{15, \underbrace{9, \cdots 9}_{10\text{ times}}, \underbrace{5, \cdots, 5}_{9\text{ times}}, \underbrace{3 \cdots, 3}_{15\text{ times}}\}. \tag{4.20}$$

In this case we find that the representation st_5 of dimension five has eigenvalue 9, the representation repa_5 of dimension five has eigenvalue 3 and the representation sw_{10} of dimension ten has eigenvalue 3. We collect all these informations in the Table 4.5.

	$\lambda_i^{(2)}$	$\lambda_i^{(2,2,2)}$	$\dim(V_i)$
id_1	15	15	1
st_5	-3	9	5
repa_5	-9	3	5
n_9	5	5	9
sw_{10}	-3	3	10

Table 4.5: Eigenvalues and dimensions of the eigenspaces appearing in the decomposition of $\mathrm{Sym}^3 V_\theta$.

Note that for each representation there is a different (ordered) couple of eigenvalues. Let W be a (modular invariant) space generated by some degree twelve polynomials in theta constants. Expanding all the polynomials generating W on a basis of $\mathrm{Sym}^3 V_\theta$ we obtains the coefficient matrix M_W with 35 rows and n columns, with $n = \dim W$. Thus, if W coincides with one of the V_i we get

$$^t(N_C - \lambda_i^C \,\mathrm{Id})M_W = 0, \tag{4.21}$$

where N_C stands for N_2 or $N_{2,2,2}$ and λ_i^C the corresponding eigenvalue[3].
The eigenvalue 15 with multiplicity one shows the presence of a subspace V_I of dimension one invariant under the action of S_6, i.e. there is an invariant cubic polynomial in p_i which we will call ψ_6. To find it, we can compute the kernel of the matrix $^tN - 15\,\mathrm{Id}_{35}$ (which is evidently one dimensional). We get

$$\Psi_6 = p_0^3 - 9p_0(p_1^2 + p_2^2 + p_3^2 - 4p_4^2) + 54p_1p_2p_3. \tag{4.22}$$

Using the classical formula we see that this polynomial coincides, up a multiplicative constant, with the modular form of weight six appearing in [DP4]. Let us now consider the space $V_\Xi = \langle \cdots, \Xi_6[\delta], \cdots \rangle$ of the forms introduced by D'Hoker and Phong in [DP4] to define the superstring measures in genus two. We find

$$^t(N_2 - (-3)\,\mathrm{Id})M_{V_\Xi} = 0, \tag{4.23}$$

$$^t(N_{2,2,2} - 9\,\mathrm{Id})M_{V_\Xi} = 0, \tag{4.24}$$

which shows that V_Ξ supports the representation st_5. The five dimensional subspace $V_S = \langle \theta^4[\delta] \sum_{\delta'} \theta^8[\delta'] \rangle$ provides the second representation st_5:

$$^t(N_{2,2,2} - 9\,\mathrm{Id})M_S = 0, \quad V_S \neq V_\Xi. \tag{4.25}$$

align Next, there is the space V_f supporting the representation repa_5, which is given by combinations of the functions $\Xi_6[\delta]$ and the derivatives of the Igusa quartic w.r.t. the

[3]The transpose appears because N_C transforms the basis of $\mathrm{Sym}^3 V_theta$ and M_W is the matrix of the coefficients.

$\theta[\delta]^4$, $V_f = \langle \cdots, 2\Xi_6[\delta] - \frac{\partial I_4}{\partial\theta[\delta]^4} \cdots \rangle$:

$$ {}^t(N_2 - (-9)\,\mathrm{Id})M_{V_f} = 0. \tag{4.26}$$

To find an expression for the representation spaces n_9 and sw_{10} we proceed as follows. Let us consider a space W (a priori it can be also reducible), compute the matrix M_W and define the product

$$\prod_{i,j} {}^t(N_2 - \lambda_i^{(2)}\,\mathrm{Id}_{35})\,{}^t(N_{2,2,2} - \lambda_j^{(2,2,2)}\,\mathrm{Id}_{35}). \tag{4.27}$$

The kernel of each factor is the space in which $N^C = N_2$ or $N_{2,2,2}$ has eigenvalue λ_i^C. If a vector does not belong to any representation for which the λ_i appears in the product, its image will be different from the null vector. If in the product (4.27) we omit a particular representation, say the k-th one, then the expression

$$\prod_{i,j} {}^t(N_2 - \lambda_i\,\mathrm{Id}_{35})\,{}^t(N_{2,2,2} - \lambda_j\,\mathrm{Id}_{35})M_W \tag{4.28}$$

is non vanishing if and only if the representation k belong to W. This follows from the fact that on a suitable basis the expression (4.27) assumes the diagonal form

$$\mathrm{diag}\{0, \cdots, 0, \mu_k, \cdots, \mu_k, 0, \cdots, 0\}, \tag{4.29}$$

$$\mu_k = \prod_i (\lambda_k^{(2)} - \lambda_i^{(2)}) \prod_j (\lambda_k^{(2,2,2)} - \lambda_j^{(2,2,2)}), \tag{4.30}$$

giving zero when multiplied by a vector belonging to any representation different from V_k. The multiplicity of μ_k is, clearly, equal to $n_k \dim(V_k)$, where $\mathrm{Sym}^3 = \oplus_k V_k^{n_k}$ and the n_k are computed as the coefficients appearing in the decomposition (4.17). The subspace W_k giving the representation k will be generated by the image of the product $\prod_{i,j} {}^t(N_2 - \lambda_i\,\mathrm{Id}_{35})\,{}^t(N_{2,2,2} - \lambda_j\,\mathrm{Id}_{35})$ acting on a basis of the whole $\mathrm{Sym}^3 V_\theta$ and one can extract a basis from it. Thus we can decompose the space $\mathrm{Sym}^3 V_\theta$ as:

$$S^3 V_\theta = V_I \oplus V_\Xi \oplus V_f \oplus V_S \oplus V_9 \oplus V_{10}, \tag{4.31}$$

where the spaces V_9 and V_{10} are suitable spaces of dimension nine and ten respectively and constructed as explained before. Applying the previous approach to some space of polynomials of degree twelve in theta constants we obtain the results reported in Table 4.6.

4.3.2 Construction of Ξ_8 at genus 2

We are now able to construct the superstring measure at $g = 2$ and prove its uniqueness. We recall, that $\dim M_8(\Gamma_2(2))^{O^+} = n_1 + n_\theta$, with n_1, n_θ the multiplicity of $\mathbf{1}$ and $\sigma_\theta = \rho_{[42]}$ in $M_8(\Gamma_2(2))$ respectively, see section 3.7.3. From the decomposition given there we get

$$\dim M_8(\Gamma_2(2))^{O^+} = 1 + 3 = 4.$$

Space	Dimension	Representations
$\langle \partial_{\delta_i} I_4 \rangle$	10	$\mathrm{st}_5 \oplus \mathrm{repa}_5$
$\langle \Xi_6[\delta][\delta_i] \rangle$	5	st_5
$\langle \theta^{12}[\delta_i] \rangle$	10	$\mathrm{st}_5 \oplus \mathrm{repa}_5$
$\langle \theta^4[\delta_i] \sum_{\delta'} \theta^8[\delta'] \rangle$	5	st_5
$\langle \theta^4[\delta_i] \theta^8[\delta_j] \rangle$	34	$2\mathrm{st}_5 \oplus \mathrm{repa}_5 \oplus \mathrm{n}_9 \oplus \mathrm{sw}_{10}$
$\langle \theta^{12}[\delta_i], \partial_{\delta_j} I_4 \rangle$	15	$\mathrm{st}_5 \oplus \mathrm{st}_5 \oplus \mathrm{repa}_5$
$\langle \theta^{12}[\delta_i], \Xi_6[\delta] \rangle$	15	$\mathrm{st}_5 \oplus \mathrm{repa}_5$
$\langle \theta^{12}[\delta_i], \theta^4[\delta_j] \sum_{\delta'} \theta^8[\delta'] \rangle$	15	$\mathrm{st}_5 \oplus \mathrm{repa}_5$
$\langle \theta^{12}[\delta_i], \theta^4[\delta_j] \sum_{\delta'} \theta^8[\delta'], \partial_{\delta_k} I_4 \rangle$	15	$\mathrm{st}_5 \oplus \mathrm{repa}_5$
$\langle \theta^4[\delta_i] \theta^4[\delta_j] \theta^4[\delta_k] \rangle_{\delta_i, \delta_j, \delta_k \text{ pari}}$	35	$S^3 V_\theta$
$\langle \theta^4[\delta_i] \theta^4[\delta_j] \theta^4[\delta_k] \rangle_{\delta_i + \delta_j + \delta_k \text{ pari}}$	35	$S^3 V_\theta$
$\langle \theta^4[\delta_i] \theta^4[\delta_j] \theta^4[\delta_k] \rangle_{\delta_i + \delta_j + \delta_k \text{ dispari}}$	20	$\mathrm{st}_5 \oplus \mathrm{repa}_5 \oplus \mathrm{sw}_{10}$

Table 4.6: Decomposition of some vectorial spaces. We intend: $\partial_{\delta_i} \equiv \frac{\partial}{\partial \theta[\delta_i]^4}$.

The subspace $M_8(\Gamma_2(2))^{O^+}$ contains the $Sp(4)$-invariant $\sum_\delta \theta[\delta]^{16} \in M_8(\Gamma_2(2))$ as well as the three dimensional subspace spanned by

$$F_1^{(2)} := \theta[{}^{00}_{00}]^{16}, \quad F_2^{(2)} := \theta[{}^{00}_{00}]^4 \sum_\delta \theta[\delta]^{12}, \quad F_3^{(2)} := \theta[{}^{00}_{00}]^8 \sum_\delta \theta[\delta]^8.$$

The apex indicates the genus and when it is clear from the context we will omit it. Using the classical theta formula, one easily sees that these four functions are linearly independent and thus are a basis of $M_8(\Gamma_2(2))^{O^+}$. The function $\Xi_8[0^{(2)}](\tau_2)$, to satisfy the third constraint, should restrict to $\Xi_8[0^{(1)}](\tau_1)\Xi_8[0^{(1)}](\tau_2)$ with $\Xi_8[0^{(1)}](\tau_1) = (\theta[{}^0_0]^4\eta^{12})(\tau_1)$ on $\mathbb{H}_1 \times \mathbb{H}_1 \subset \mathbb{H}_2$. This function is a multiple of $\theta[{}^0_0]^4(\tau_1)$. The restrictions of the F_i are also multiples of $\theta[{}^0_0]^4(\tau_1)$, but the restriction of $\sum \theta[\delta]^{16}$ is not. Hence $\Xi_8[0^{(2)}]$ should be linear combination of the three F_i only. We try to determine $a_i \in \mathbb{C}$ such that $\sum_i a_i F_i$ factors in this way for such period matrices. Note that

$$\theta[{}^{ab}_{cd}](\tau_{1,1}) \;=\; \theta[{}^a_c](\tau_1)\theta[{}^b_d](\tau_1'),$$

where $\tau_{1,1} = \mathrm{diag}(\tau_1, \tau_1')$ and $\tau_1, \tau_1' \in \mathbb{H}_1$. In particular, $\theta[{}^{ab}_{cd}](\tau_{1,1}) = 0$ if $ac = 1$. As $\theta[{}^{00}_{00}](\tau_{1,1})$ produces $\theta[{}^0_0]^4(\tau_1)\theta[{}^0_0]^4(\tau_1')$, it remains to find a_i such that

$$\left(\theta[{}^0_0]^4\eta^{12}\right)(\tau_1)\left(\theta[{}^0_0]^4\eta^{12}\right)(\tau_1') \;=\; (a_1F_1 + a_2F_2 + a_3F_3)(\tau_{1,1}).$$

Using the results from section 4.2, the restrictions of the F_i are

$$\begin{aligned}\theta[^{00}_{00}]^{16}(\tau_{1,1}) &= \theta[^0_0]^{16}(\tau_1)\theta[^0_0]^{16}(\tau_1') \\ &= \theta[^0_0]^4(\tau_1)(\tfrac{1}{3}f_{21}+\eta^{12})(\tau_1)\theta[^0_0]^4(\tau_1')(\tfrac{1}{3}f_{21}+\eta^{12})(\tau_1'),\end{aligned}$$

$$\begin{aligned}\big(\theta[^{00}_{00}]^4 \textstyle\sum_\delta \theta[\delta]^{12}\big)(\tau_{1,1}) &= \theta[^0_0]^4(\tau_1)(\theta[^0_0]^{12}+\theta[^0_1]^{12}+\theta[^1_0]^{12})(\tau_1)\theta[^0_0]^4(\tau_1')(\theta[^0_0]^{12}+\theta[^0_1]^{12}+\theta[^1_0]^{12})(\tau_1') \\ &= \theta[^0_0]^4(\tau_1)(\tfrac{2}{3}f_{21}-\eta^{12})(\tau_1)\theta[^0_0]^4(\tau_1')(\tfrac{2}{3}f_{21}-\eta^{12})(\tau_1'),\end{aligned}$$

$$\begin{aligned}\Big(\theta[^{00}_{00}]^8 \textstyle\sum_\delta \theta[\delta]^8\Big)(\tau_{1,1}) &= \Big(\theta[^0_0]^8(\theta[^0_0]^8+\theta[^0_1]^8+\theta[^1_0]^8\Big)(\tau_1)\Big(\theta[^0_0]^8(\theta[^0_0]^8+\theta[^0_1]^8+\theta[^1_0]^8)\Big)(\tau_1') \\ &= \theta[^0_0]^4(\tau_1)\tfrac{2}{3}f_{21}(\tau_1)\theta[^0_0]^4(\tau_1')\tfrac{2}{3}f_{21}(\tau_1').\end{aligned}$$

Next we require for the term $f_{21}(\tau_1)$ to disappear from the linear combination $(\sum a_iF_i)(\tau_{1,1})$, so that we must have

$$\Big(a_1(\tfrac{1}{3}f_{21}+\eta^{12})+2a_2(\tfrac{2}{3}f_{21}-\eta^{12})+2a_3\tfrac{2}{3}f_{21}\Big)(\tau_1') \;=\; 0$$

for all $\tau_1' \in \mathbb{H}_1$. This gives two linear equations for the a_i which have a unique solution, up to scalar multiple:

$$a_1+4a_2+4a_3=0, \qquad a_1-2a_2=0, \qquad \text{hence} \quad (a_1,a_2,a_3)=\lambda(-4,-2,3).$$

A computation shows that $(-4F_1-2F_2+3F_3)(\tau_{1,1}) = 6\theta[^0_0]^4(\tau_1)\eta^{12}(\tau_1)\theta[^0_0]^4(\tau_1')\eta^{12}(\tau_1')$. Thus we conclude that

$$\Xi_8[^{00}_{00}] \;:=\; \theta[^{00}_{00}]^4\big(-4\theta[^{00}_{00}]^{12}-2\sum_\delta\theta[\delta]^{12}+3\theta[^{00}_{00}]^4\sum_\delta\theta[\delta]^8\big)/6$$

satisfies the constraints. Because we use a basis for the O^+-invariants and the equations for the a_i's have an unique solution we conclude that the the $\Xi_8[^{00}_{00}]$ is the unique modular form on $\Gamma_2(1,2)$ satisfying the constraints.

As $\theta[^{00}_{00}]^4\Xi_6[^{00}_{00}]$ satisfies the same constraints (with $\Xi_6[^{00}_{00}]$ the modular form determined by D'Hoker and Phong in [DP1], [DP4]) we obtain from uniqueness that

$$\Xi_6[^{00}_{00}] \;=\; \Big(-4\theta[^{00}_{00}]^{12}-2\sum_\delta\theta[\delta]^{12}+3\theta[^{00}_{00}]^4\sum_\delta\theta[\delta]^8\Big)/6.$$

Another formula for this function is:

$$\Xi_6[^{00}_{00}] \;=\; -(\theta[^{00}_{11}]\theta[^{01}_{00}]\theta[^{10}_{01}])^4-(\theta[^{00}_{01}]\theta[^{01}_{10}]\theta[^{11}_{00}])^4-(\theta[^{00}_{10}]\theta[^{10}_{00}]\theta[^{11}_{11}])^4,$$

which is the one found by D'Hoker and Phong in [DP4]. To check the equality between the two expressions for $\Xi_6[^{00}_{00}]$ one can use the classical theta formula.

We observe that the five dimensional space generated by the $\Xi_6[\delta]$ is exactly the same as the one generated by the derivative of the Igusa quartic w.r.t. to the five p_i (cf. [CD1], Theorem 1, for details).

4.4 The case $g = 3$

In this case the decomposition of Sp(6)-representation on $M_{2k}(\Gamma_g(2))$ in irreducible representations is longer than in genus two. So, for sake of clarity, we first construct a modular form Ξ_8 satisfying the three constraints, then we will treat the algebraic proprieties of the representation on the space of modular forms and finally we will prove the uniqueness of Ξ_8.

4.4.1 Modular forms

In case $g = 3$, the 8 $\Theta[\sigma]$'s define a holomorphic map

$$\mathbb{H}_3 \longrightarrow \mathbb{P}^7, \qquad \tau \longmapsto (\Theta[000](\tau) : \ldots : \Theta[111](\tau)). \tag{4.32}$$

The closure of the image of this map is a 6-dimensional variety which is defined by a homogeneous polynomial[4] in eight variables of degree 16, as anticipated in section 3.5.3. In particular the holomorphic function $\tau \to F_{16}(\cdots, \Theta[\delta], \cdots)$ is identically zero on $\mathbb{H}_3$. It is interesting to review the details of the construction of F_{16} (see [vGvdG]), as we will use an analogous strategy to obtain a certain function, $G[\Delta]$, we will use to define the superstring measures. For all $\tau \in \mathbb{H}_3$, the following relation holds:

$$r_1 - r_2 = r_3, \qquad \text{with} \quad r_1 = \prod_{a,b\in\mathbb{F}_2} \theta[{}^{000}_{0ab}](\tau), \quad r_2 = \prod_{a,b\in\mathbb{F}_2} \theta[{}^{000}_{1ab}](\tau), \quad r_3 = \prod_{a,b\in\mathbb{F}_2} \theta[{}^{100}_{0ab}](\tau). \tag{4.33}$$

From these we deduce that $2r_1 r_2 = r_1^2 + r_2^2 - r_3^2$, and thus

$$r_1^4 + r_2^4 + r_3^4 - 2(r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2) \tag{4.34}$$

is zero, as function of τ, on $\mathbb{H}_3$. Let F_{16} be the homogeneous polynomial, of degree 16 in the $\Theta[\sigma]$'s, obtained (using the classical theta formula (3.33)) from this polynomial (of degree 8) in the $\theta[\Delta]^2$. In [vGvdG] it has been shown that F_{16} is not zero as a polynomial in the eight $\Theta[\sigma]$. Thus the polynomial F_{16} defines the image of $\mathbb{H}_3 \to \mathbb{P}^7$. The same polynomial can be written using the classical theta functions. A computer computation, using once again the classical formula, shows that F_{16} coincides, up to a scalar multiple, with the degree 16 polynomial in the $\Theta[\sigma]$ obtained from

$$8\sum_{\Delta} \theta[\Delta]^{16} - \Big(\sum_{\Delta} \theta[\Delta]^8\Big)^2 \tag{4.35}$$

by the classical theta formulas.

The polynomial F_{16} provides the only relation of degree 16 between the theta constants and the quotient ring is normal [R1, R2] so we get

$$M_{2k}(\Gamma_3(2)) = M^{\theta}_{2k}(\Gamma_3(2)) = (\mathbb{C}[\ldots, \Theta[\sigma], \ldots]_{4k})^{H_3} \tag{4.36}$$

[4] We use the notation F_{16} as in [vGvdG]. In the next chapter we will call $J^{(g)}$ the forms $J^{(g)} = 2^g \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{16} - (\sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^8)^2$, that in genus three reduces exactly to F_{16} and we will reserve the notation $F_{16}^{(g)}$ for the function $F_{16}^{(g)} = \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{16}$.

where

$$(\mathbb{C}[\ldots,\Theta[\sigma],\ldots]_{4k})^{H_3} = \begin{cases} (\mathbb{C}[\ldots,X_\sigma,\ldots]_{4k})^{H_3} & k\le 3, \\ (\mathbb{C}[\ldots,X_\sigma,\ldots]_{4k})^{H_3}/F_{16}(\mathbb{C}[\ldots,X_\sigma,\ldots]_{4k-16})^{H_3}, & k\ge 4. \end{cases} \tag{4.37}$$

4.4.2 The functions $F_i^{(3)}$

As in the genus two case, we want to find a modular form $\Xi_8[{}^{000}_{000}]$ of weight 8 on $\Gamma_3(2)$ which restricts to the 'diagonal' $\Delta_{1,2}$ as:

$$\Xi_8[{}^{000}_{000}](\tau_{1,2}) = \Xi_8[{}^{0}_{0}](\tau_1)\Xi_8[{}^{00}_{00}](\tau_2) = \Big(\theta[{}^{0}_{0}]^4\eta^{12}\Big)(\tau_1)\Big(\theta[{}^{00}_{00}]^4\Xi_6[{}^{00}_{00}]\Big)(\tau_2) \tag{4.38}$$

where $\tau_{1,2}\in\mathbb{H}_3$ is the block diagonal matrix with entries $\tau_1\in\mathbb{H}_1$ and $\tau_2\in\mathbb{H}_2$. Obvious generalizations of the functions $F_i^{(2)}$ which we considered in section 4.3.2 are

$$F_1^{(3)} := \theta[{}^{000}_{000}]^{16}, \qquad F_2^{(3)} := \theta[{}^{000}_{000}]^{4}\sum_\Delta \theta[\Delta]^{12}, \qquad F_3^{(3)} := \theta[{}^{000}_{000}]^{8}\sum_\Delta \theta[\Delta]^{8}, \tag{4.39}$$

where the sum is over the 36 even characteristics Δ in genus three. The functions F_i are modular forms of weight 8 on $\Gamma_3(1,2)$, see [CDG1]. We also have the Sp(6)-invariant $\sum_\Delta\theta[\Delta]^{16}$. However, there is no linear combination of these functions which has the desired restriction. Therefore we need another modular form $G[{}^{000}_{000}]$ of weight 8 on $\Gamma_3(1,2)$. To this end we need the notion of isotropic and Lagrangian subspaces.

4.4.3 Isotropic subspaces

In this section we introduce isotropic subspaces of a space $V\cong\mathbb{F}_2^{2g}$ for arbitrary g in a quite general approach, as we will use the same notions to tackle the genus four case. A subspace $W\subset V$ is isotropic if $E(w,w')=0$ for all $w,w'\in W$. Given a basis $e_1,\ldots,e_k$ of W it is not hard to see that one can extend it to a symplectic basis $e_1,\ldots e_{2g}$ of V (so $E(e_i,e_j)=0$ unless $|i-j|=g$ and then $E(e_i,e_j)=1$). In particular, the group $Sp(2g,\mathbb{Z})$ acts transitively on the isotropic subspaces of V of a given dimension. The number of k-dimensional isotropic subspaces of $V\cong\mathbb{F}_2^{2g}$ is given by

$$\begin{aligned} N_{iso}(g,k) &= \frac{(2^{2g}-1)(2^{2g-1}-2)(2^{2g-4}-4)\ldots(2^{2g-(k-1)}-2^{k-1})}{\sharp GL(k,\mathbb{F}_2)} \\ &= \frac{(2^{2g}-1)(2^{2g-1}-2)(2^{2g-2}-4)\ldots(2^{2g-(k-1)}-2^{k-1})}{(2^k-1)(2^k-2)\ldots(2^k-2^{k-1})} \\ &= \frac{(2^{2g}-1)(2^{2g-2}-1)(2^{2g-4}-1)(2^{2g-6}-1)\ldots(2^{2(g-k)+2}-1)}{(2^k-1)(2^{k-1}-1)\ldots(2-1)} \end{aligned} \tag{4.40}$$

where in the numerator we count the ordered k-tuples of independent elements $v_1,\ldots,v_k\in V$ with $E(v_i,v_j)=0$ for all i,j: for v_1 we can take any element in $V-\{0\}$, for v_2

we can take any element in $\langle v_1\rangle^\perp \cong \mathbb{F}_2^{2g-1}$ except $0, v_1$, so $v_2 \in \langle v_1\rangle^\perp - \langle v_1\rangle$, next $v_3 \in \langle v_1, v_2\rangle^\perp - \langle v_1, v_2\rangle$ and so on.
Let $W_1, \dots, W_N$ be the k-dimensional isotropic subspaces contained in an even quadric $Q \subset V$ defined by $q = 0$, where q is a quadratic form on V. Let $\sigma \in Sp(2g, \mathbb{Z})$ be a symplectic transformation. Then $\sigma(W_1), \dots, \sigma(W_N)$ are the k-dimensional isotropic subspaces in the even quadric $\sigma(Q) \subset V$ defined by $\sigma \cdot q = 0$, with $(\sigma \cdot q)(\sigma v) = q(v)$. In particular, all even quadrics in V contains the same number of isotropic subspaces of a given dimension. An even quadric contains a maximal isotropic subspace L: this is an isotropic subspace not contained in any higher dimensional nontrivial isotropic subspace. By trivial we mean the space containing the 0 only or the whole V. It follows that maximal isotropic subspaces are half dimensional (t.i. g dimensional) subspaces. For example, $L_0 = \{\left(\begin{smallmatrix} v_1 \dots v_g \\ 0 \dots 0\end{smallmatrix}\right) : v_i \in \mathbb{F}_2\}$ is contained in the even quadric Q corresponding to the characteristic $\left[\begin{smallmatrix} 0 \dots 0 \\ 0 \dots 0\end{smallmatrix}\right]$. Instead, odd quadrics do not contain maximal isotropic subspaces: if $L \subset Q$ were such a subspace, then, by transitivity, $\sigma(L) = L_0$ for a suitable $\sigma \in Sp(2g, \mathbb{Z})$. If $\sigma(Q)$ corresponds to the characteristic $\left[\begin{smallmatrix} a \\ b\end{smallmatrix}\right]$ then $L_0 \subset \sigma(Q)$ implies $a_1 = \dots = a_g = 0$, hence the characteristic must be even. A maximally isotropic subspace, L, is called a *Lagrangian.* For example, if $g = 3$, a subspace L of $V \cong \mathbb{F}_2^6$ is Lagrangian if $L \subset V$, $E(v, w) = 0$ for all $v, w \in L$ and $\dim L = 3$. The eight elements $\left(\begin{smallmatrix} abc \\ 000\end{smallmatrix}\right) \in V$ with $a, b, c \in \mathbb{F}_2$ form a Lagrangian subspace L_0 in V. Instead, the higher dimensional isotropic subspaces contained in an odd quadric are $(g-1)$-dimensional. For example, $W_0 = \{\left(\begin{smallmatrix} v_1 \dots v_{g-1} & 0 \\ 0 \dots 0 & 0\end{smallmatrix}\right) : v_i \in \mathbb{F}_2\}$ is contained in the odd quadric with characteristic $\left[\begin{smallmatrix} 0 \dots 01 \\ 0 \dots 01\end{smallmatrix}\right]$.

It is easy to count the number of even quadrics which contain a fixed k-dimensional isotropic subspace: we may assume that the subspace has basis $e_1, \dots, e_k$ so that the characteristic of an even quadric containing it is

$$\begin{bmatrix} 0 & \dots & 0 & a_{k+1} & \dots & a_g \\ b_1 & \dots & b_k & b_{k+1} & \dots & b_g \end{bmatrix} \qquad \text{with} \quad \sum_{i=k+1}^{g} a_{k+i} b_{k+i} = 0,$$

and the number of such even quadrics is

$$N_Q(k) = 2^k \cdot 2^{g-k-1}(2^{g-k} + 1). \tag{4.41}$$

Viceversa, to find the number of k-dimensional isotropic subspaces in an even quadric, we can count the pairs (W, Q) of isotropic subspaces W contained in arbitrary even quadrics Q in two ways: first as the product of the number of W with the number of even Q containing a fixed W and second as the product of the number of even quadrics with the number of k-dimensional isotropic subspaces in an even quadric. For example, let us consider maximal isotropic subspaces. Then $g = k$ and the total number of copies (W, Q) is in this case

$$N = N_{iso}(g, g) N_Q(g) = [(2^g + 1)(2^{g-1} + 1) \dots (2 + 1)]2^g.$$

On the other side we know that maximally isotropic spaces are contained in even quadrics only, which are in one-to-one correspondence with the set of even characteristics, which are $N_e = 2^{g-1}(2^g + 1)$. Thus, we conclude that the number of maximally isotropic subspaces contained in a given (even) quadric Q is

$$N^Q_{iso}(g) = N_{iso}(g, g)N_Q(g)/N_e = 2[(2^{g-1} + 1) \dots (2 + 1)]. \tag{4.42}$$

For example, the number of pairs (W, Q) of a maximally isotropic subspace in an even quadric in $\mathbb{F}_2^6$ is $135{\cdot}2^3$, and thus the number of such subspaces in a fixed Q is $135{\cdot}2^3/36 = 30$. The general idea to count the number of k-dimensional isotropic subspaces in an even quadric is graphically pictured in Figure 4.1: it is clear that in the first way we count the couples by columns, and in the second way by rows. For small g we list some

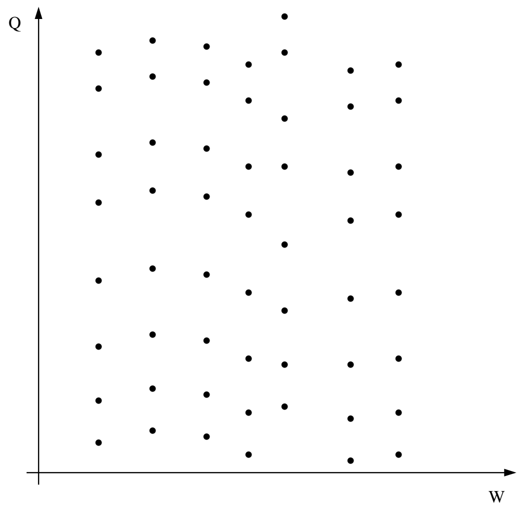

Figure 4.1: Counting of the couple $(Q\,W)$

of these dimensions in the table on the left below, in the table on the right we list the number of k-dimensional isotropic subspaces contained in an even quadric.

$g \setminus$ dimension	1	2	3	4
1	3			
2	15	15		
3	63	315	135	
4	255	5355	11475	2295

$g \setminus$ dimension	1	2	3	4
1	2			
2	9	6		
3	35	105	30	
4	135	1575	2025	270

4.4.4 The modular forms $G[\Delta]$

For each even characteristic Δ in $g = 3$ we define a modular form $G[\Delta]$ of weight 8 on $\Gamma_3(2)$. First, let us recall some facts and notations about characteristics and quadrics introduced in section 3.2, specializing to the case $g = 3$. An even characteristic Δ corresponds to a quadratic form

$$q_\Delta : V = \mathbb{F}_2^6 \longrightarrow \mathbb{F}_2$$

which satisfies $q_\Delta(v+w) = q_\Delta(v) + q_\Delta(w) + E(v,w)$, where $E(v,w) := \sum_{i=1}^{3}(v_i w_{3+i} + v_{3+i}w_i)$. If $\Delta = [^{abc}_{def}]$ then

$$q_\Delta(v) = v_1v_4 + v_2v_5 + v_3v_6 + av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6,$$

with $v = (v_1, \ldots, v_6) \in V$. We will also write $v = (^{v_1v_2v_3}_{v_4v_5v_6})$. Let $Q_\Delta = \{v \in V : q_\Delta(v) = 0\}$ be the corresponding quadric in V.

Let L be a Lagrangian subspace of V. For such a subspace L we define a modular form on a subgroup of $Sp(6,\mathbb{Z})$:

$$P_L := \prod_{Q \supset L} \theta[\Delta_Q]^2$$

here the product is over all even quadrics which contain L (there are eight such quadrics for each L, as explained in the previous Section) and Δ_Q is the even characteristic corresponding to Q. In case $L = L_0$ with

$$L_0 := \{(v_1, \ldots, v_6) \in V : v_4 = v_5 = v_6 = 0\},$$

we have

$$P_{L_0} = (2r_1r_2)^2 = \prod_{a,b,c \in \mathbb{F}_2} \theta[^{000}_{abc}]^2$$

with r_1, r_2 as in section 4.4.1. The action of $Sp(6,\mathbb{Z})$ on $V = \mathbb{Z}^6/2\mathbb{Z}^6$ permutes the Lagrangian subspaces L, and the subgroup $\Gamma_3(2)$ acts trivially on V. Similarly, the P_L are permuted by the action of $Sp(6,\mathbb{Z})$, see [CDG1], and as $\Gamma_3(2)$ fixes all L's, the P_L are modular forms on $\Gamma_3(2)$ of weight 8.

For an even characteristic Δ, the quadric Q_Δ contains 30 Lagrangian subspaces. The sum of the 30 P_L's, with L a Lagrangian subspace of Q_Δ, is a modular form $G[\Delta]$ of weight 8 on $\Gamma_3(2)$:

$$G[\Delta] := \sum_{L \subset Q_\Delta} P_L = \sum_{L \subset Q_\Delta} \prod_{Q' \supset L} \theta[\Delta_{Q'}]^2.$$

Note that $\theta[\Delta]^2$ is one of the factors in each of the 30 products. As the P_L are permuted by the action of $Sp(6,\mathbb{Z})$, also the $G[\Delta]$ are permuted:

$$G[M \cdot \Delta](M \cdot \tau) = \det(C\tau + D)^8 G[\Delta](\tau).$$

Since $\Gamma_3(1,2)$ fixes the characteristic $[^{000}_{000}]$, the function $G[^{000}_{000}]$ is a modular form on $\Gamma_3(1,2)$.

4.4.5 The restriction

Now, we can tackle the problem of finding a linear combination of the functions F_i, $i = 1,2,3$ and $G[^{000}_{000}]$, in order to satisfy the third constraint:

$$\left(\theta[^0_0]^4\eta^{12}\right)(\tau_1)\left(\theta[^{00}_{00}]^4\Xi_6[^{00}_{00}]\right)(\tau_2) = \left(b_1F_1 + b_2F_2 + b_3F_3 + b_4G[^{000}_{000}]\right)(\tau_{1,2}).$$

It is easy to see that the theta constants satisfy

$$\theta[^{abc}_{def}](\tau_{1,2}) \;=\; \theta[^{a}_{d}](\tau_1)\theta[^{bc}_{ef}](\tau_2),$$

so that, in particular, $\theta[^{abc}_{def}] \mapsto 0$ when $ad = 1$. As $[^{abc}_{def}]$ must be even, this happen for $[^{bc}_{ef}]$ odd, thus 6 of the 36 even theta constants map to zero. The other $30 = 3 \cdot 10$ are uniquely decomposed in the product of two even theta constants for $g = 1$ (3) and $g = 2$ (10) respectively. Using the results from 4.2, the functions $F_i(\tau_{1,2})$ are easily made explicit, and the function $G[^{000}_{000}](\tau_{1,2})$ has been determined in [CDG1]. The restrictions to $\Delta_{1,2} \cong \mathbb{H}_1 \times \mathbb{H}_2$ are:

$$\begin{aligned}
(\theta[^{000}_{000}]^{16})(\tau_{1,2}) &= \theta[^0_0]^{16}(\tau_1)\theta[^{00}_{00}]^{16}(\tau_2) = \left(\theta[^0_0]^4(\tfrac{1}{3}f_{21} + \eta^{12})\right)(\tau_1)\theta[^{00}_{00}]^{16}(\tau_2),\\
\theta[^{000}_{000}]^4(\textstyle\sum_\Delta \theta[\Delta]^{12})(\tau_{1,2}) &= \left(\theta[^0_0]^4(\theta[^0_0]^{12} + \theta[^0_1]^{12} + \theta[^1_0]^{12})\right)(\tau_1)\left(\theta[^{00}_{00}]^4(\textstyle\sum_\delta \theta[\delta]^{12})\right)(\tau_2)\\
&= \left(\theta[^0_0]^4(\tfrac{2}{3}f_{21} - \eta^{12})\right)(\tau_1)\left(\theta[^{00}_{00}]^4 \textstyle\sum_\delta \theta[\delta]^{12}\right)(\tau_2),\\
(\theta[^{000}_{000}]^8 \textstyle\sum_\Delta \theta[\Delta]^8)(\tau_{1,2}) &= \left(\theta[^0_0]^8(\theta[^0_0]^8 + \theta[^0_1]^8 + \theta[^1_0]^8)\right)(\tau_1)\left(\theta[^{00}_{00}]^8(\textstyle\sum_\delta \theta[\delta]^8)\right)(\tau_2)\\
&= \left(\theta[^0_0]^4 \tfrac{2}{3}f_{21}\right)(\tau_1)\left(\theta[^{00}_{00}]^8(\textstyle\sum_\delta \theta[\delta]^8)\right)(\tau_2),\\
G[^{000}_{000}](\tau_{1,2}) &= \left(\theta[^0_0]^4(\tfrac{1}{3}f_{21} - \eta^{12})\right)(\tau_1)\\
&\quad \left(\theta[^{00}_{00}]^4(\tfrac{1}{3}\theta[^{00}_{00}]^{12} + \tfrac{2}{3}\textstyle\sum_\delta \theta[\delta]^{12} - \tfrac{1}{2}\theta[^{00}_{00}]^4 \textstyle\sum_\delta \theta[\delta]^8)\right)(\tau_2).
\end{aligned}$$

In putting these expression in $(b_1F_1 + b_2F_2 + b_3F_3) + b_4G[^{000}_{000}]$, we note that the common function $\theta[^{000}_{000}]^4$ in front of the F_i gives the function $\theta[^0_0]^4(\tau_1)\theta[^{00}_{00}]^4(\tau_2)$. In particular, the restriction has then a factor $\theta[^0_0]^4\theta[^{00}_{00}]^4$. In order for this restriction to be a multiple of $\theta[^0_0]^4\eta^{12}$, the terms containing f_{21} must disappear. This leads to the equation

$$b_1\theta[^{00}_{00}]^{12} + 2b_2\sum_\delta \theta[\delta]^{12} + 2b_3\theta[^{00}_{00}]^4(\sum_\delta \theta[\delta]^8) + b_4(\tfrac{1}{3}\theta[^{00}_{00}]^{12} + \tfrac{2}{3}\sum_\delta \theta[\delta]^{12} - \tfrac{1}{2}\theta[^{00}_{00}]^4 \sum_\delta \theta[\delta]^8) = 0.$$

It has a unique solution (up to scalar multiples):

$$(b_1, b_2, b_3, b_4) \;=\; \mu(4, 4, -3, -12) \qquad (\mu \in \mathbb{C}).$$

Setting $\mu = 1$ and using the formula for $\Xi_6[^{00}_{00}]$ given in section 4.3.2 we get

$$\begin{aligned}
\left(4F_1 + 4F_2 - 3F_3 - 12G[^{000}_{000}]\right)(\tau_{1,2}) &=\\
\left(\theta[^0_0]^4\eta^{12}\right)(\tau_1)\left(\theta[^{00}_{00}]^4(8\theta[^{00}_{00}]^{12} + 4\textstyle\sum_\delta \theta[\delta]^{12} - 6\theta[^{00}_{00}]^4 \textstyle\sum_\delta \theta[\delta]^8)\right)(\tau_2) &=\\
12\left(\theta[^0_0]^4\eta^{12}\right)(\tau_1)\left(\theta[^{00}_{00}]^4\Xi_6[^{00}_{00}]\right)(\tau_2).
\end{aligned}$$

Hence the modular form $\Xi_8\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]$, of weight 8 on $\Gamma_3(1,2)$ defined by

$$\Xi_8\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right] := \left(4F_1 + 4F_2 - 3F_3 - 12G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]\right)/12$$

satisfies all the constraints except maybe $(\mathrm{iii}_0)(2)$. We have to check this last constraint separately. Let $M \in \mathrm{Sp}(6,\mathbb{Z})$ be such that $M \cdot \left[\begin{smallmatrix}000\\000\end{smallmatrix}\right] = \left[\begin{smallmatrix}abc\\def\end{smallmatrix}\right]$ with $a = d = 1$. As $G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right](\tau) = \theta^2\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right](\tau)G^\flat\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right](\tau)$ for a holomorphic function $G^\flat\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]$, $G\left[\begin{smallmatrix}abc\\def\end{smallmatrix}\right](M\cdot\tau)$ is the product of $\theta^2\left[\begin{smallmatrix}abc\\def\end{smallmatrix}\right](\tau)$ and a holomorphic function, hence $G\left[\begin{smallmatrix}abc\\def\end{smallmatrix}\right](M\cdot\tau_{1,2}) = 0$ because $\theta^2\left[\begin{smallmatrix}abc\\def\end{smallmatrix}\right](\tau_{1,2}) = \theta^2\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\tau_1)\theta^2\left[\begin{smallmatrix}bc\\ef\end{smallmatrix}\right](\tau_2) = 0$.

We conclude that $\Xi_8\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]$, defined as above, solves the problem.

Next, using the representation theory of finite group, we will show that, up to a constant, it is the only modular form of weight 8 on $\Gamma_3(1,2)$ which satisfies all constraints. As a by product, this implies that the desired functions $\Xi_6[\Delta]$ proposed by D'Hoker and Phong in [DP6] indeed do not exist because $G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]$ is not the product of $\theta\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]^4$ with a modular form of weight 6. As noticed by Morozov, it has been in part the convincement that the factorization of the term $\theta\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]^4$ was necessary to grant the vanishing of the cosmological constant to stop for a long time the success in finding higher loop candidates for the superstring measure. Instead, we will see that $\Xi_8\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]$ implies the vanishing of the cosmological constant.

4.4.6 Representations of Sp(6)

To prove uniqueness of the form Ξ_8, we need to come back to the study of the representations of Sp(6) on the space of modular forms.

Weight 2

From section 3.7.4 we know that $M_2((\Gamma_3(2))$, a fifteen dimensional vector space, is an irreducible $Sp(6)$-representation, denoted by ρ_θ. As $Sp(6)$ has a unique irreducible representation of dimension 15, denoted by $\mathbf{15}_a$ in [Fr1] and in Table 4.7, it follows that $\rho_\theta \cong \mathbf{15}_a$. In the spirit of the genus two case, this space of Heisenberg invariants has

Irrep./Class																														
$\mathbf{1}_a$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{7}_a$	7	3	-1	3	1	4	2	0	1	-1	-2	2	0	1	2	**-5**	-1	-3	1	-1	-2	1	-1	-2	0	-1	0	-1	1	0
$\mathbf{27}_a$	27	7	3	3	1	9	3	1	0	0	0	0	0	0	2	**15**	3	5	1	1	3	0	0	1	0	-1	-1	-1	-1	-1
$\mathbf{21}_a$	21	1	-3	5	-1	6	0	-2	0	0	3	3	-1	0	1	**9**	-3	3	-1	-1	0	0	0	2	-1	1	0	1	1	0
$\mathbf{35}_a$	35	-5	3	7	-1	5	-3	1	2	0	-1	3	1	-1	0	**-5**	3	-1	-1	1	1	-2	0	-1	0	0	-1	-1	1	0
$\mathbf{105}_a$	105	5	1	5	-1	15	1	-1	-3	1	-3	1	-1	0	0	**-35**	1	-5	-1	1	1	1	1	-1	0	0	1	1	-1	0
$\mathbf{189}_a$	189	-11	-3	9	1	9	-3	1	0	0	0	0	0	0	-1	**21**	-3	1	1	-1	-3	0	0	1	1	-1	1	1	-1	0
$\mathbf{21}_b$	21	5	5	1	1	6	2	2	0	2	3	-1	1	0	1	**-11**	-3	-3	-3	-1	-2	-2	0	0	-1	1	0	1	-1	0
$\mathbf{35}_b$	35	7	11	-1	1	5	-1	1	2	2	-1	-1	-1	-1	0	**15**	3	1	5	-1	3	0	0	-1	0	0	1	3	1	0
$\mathbf{189}_b$	189	13	-3	-3	1	9	-3	1	0	0	0	0	0	0	-1	**-51**	-3	1	1	1	-3	0	0	1	-1	-1	1	-3	1	0
$\mathbf{189}_c$	189	1	21	-3	-1	9	3	1	0	0	0	0	0	0	-1	**-39**	-3	-1	-5	-1	3	0	0	1	1	-1	-1	1	1	0
$\mathbf{15}_a$	15	3	7	-1	1	0	-2	0	3	1	-3	1	-1	0	0	**-5**	-1	1	-3	1	-2	1	-1	0	0	0	-2	3	-1	1
$\mathbf{105}_b$	105	9	-7	-3	1	0	-4	0	3	-1	6	2	0	0	0	**25**	1	-3	-3	-1	4	1	1	0	0	0	0	-3	-1	0
$\mathbf{105}_c$	105	-3	17	-3	-1	0	2	0	3	-1	6	2	0	0	0	**5**	-7	-1	3	1	2	-1	-1	0	0	0	2	1	-1	0
$\mathbf{315}_a$	315	3	-21	-5	-1	0	0	0	0	0	-9	3	1	0	0	**-45**	3	3	3	-1	0	0	0	0	0	0	0	3	-1	0
$\mathbf{405}_a$	405	-3	-27	-3	1	0	0	0	0	0	0	0	0	0	0	**45**	-3	-3	-3	1	0	0	0	0	0	0	0	5	1	-1
$\mathbf{168}_a$	168	8	8	0	0	6	2	2	-3	-1	6	2	0	0	-2	**40**	8	0	0	0	-2	1	-1	0	0	1	0	0	0	0
$\mathbf{56}_a$	56	8	-8	0	0	11	1	-1	2	-2	2	-2	0	-1	1	**-24**	0	-4	4	0	-3	0	0	1	1	1	-1	0	0	0
$\mathbf{120}_a$	120	8	-8	0	0	15	1	-1	0	-2	-6	-2	0	0	0	**40**	0	4	-4	0	1	-2	0	-1	0	0	1	0	0	1
$\mathbf{210}_a$	210	2	2	-2	-2	15	-1	-1	0	2	3	-1	1	0	0	**50**	-6	2	2	0	-1	2	0	-1	0	0	-1	-2	0	0
$\mathbf{280}_a$	280	-8	-8	0	0	10	-2	-2	1	1	10	-2	0	1	0	**-40**	8	0	0	0	2	-1	-1	0	0	0	0	0	0	0
$\mathbf{336}_a$	336	-16	16	0	0	6	-2	2	0	-2	-6	-2	0	0	1	**-16**	0	0	0	0	2	2	0	0	-1	1	0	0	0	0
$\mathbf{216}_a$	216	8	24	0	0	-9	-3	-1	0	0	0	0	0	0	1	**-24**	0	4	-4	0	-3	0	0	-1	1	1	1	0	0	-1
$\mathbf{512}_a$	512	0	0	0	0	-16	0	0	-4	0	8	0	0	-1	2	**0**	0	0	0	0	0	0	0	0	0	-1	0	0	0	1
$\mathbf{378}_a$	378	2	-6	6	2	-9	3	-1	0	0	0	0	0	0	-2	**-30**	-6	2	2	0	3	0	0	-1	0	1	-1	-2	0	0
$\mathbf{84}_a$	84	4	20	4	0	-6	2	-2	3	-1	3	-1	1	0	-1	**4**	4	0	0	0	-2	1	1	0	-1	-1	0	4	0	0
$\mathbf{420}_a$	420	-12	4	-4	0	0	4	0	3	1	-3	1	-1	0	0	**20**	4	0	0	0	-4	-1	1	0	0	0	0	-4	0	0
$\mathbf{280}_b$	280	8	24	0	0	-5	-3	-1	-2	0	-8	0	0	1	0	**40**	0	-4	4	0	1	-2	0	1	0	0	-1	0	0	0
$\mathbf{210}_b$	210	10	-14	6	-2	-15	1	1	3	1	-6	-2	0	0	0	**10**	2	-2	-2	0	1	1	-1	1	0	0	1	-2	0	0
$\mathbf{70}_a$	70	6	-10	2	-2	-5	-1	3	1	-1	7	-1	-1	1	0	**-10**	-2	2	2	0	-1	-1	1	-1	0	0	-1	2	0	0

Table 4.7: Table of characters of Sp(6). In the bold column there are the characters of the class of the transvections.

basis

$$P_0 = \Theta[000]^4 + \Theta[001]^4 + \Theta[010]^4 + \Theta[011]^4 + \Theta[100]^4 + \Theta[101]^4 + \Theta[110]^4 + \Theta[111]^4$$
$$P_1 = 2(\Theta[000]^2\Theta[001]^2 + \Theta[010]^2\Theta[011]^2 + \Theta[100]^2\Theta[101]^2 + \Theta[110]^2\Theta[111]^2)$$
$$P_2 = 2(\Theta[000]^2\Theta[010]^2 + \Theta[001]^2\Theta[011]^2 + \Theta[100]^2\Theta[110]^2 + \Theta[101]^2\Theta[111]^2)$$
$$P_3 = 2(\Theta[000]^2\Theta[011]^2 + \Theta[001]^2\Theta[010]^2 + \Theta[100]^2\Theta[111]^2 + \Theta[101]^2\Theta[110]^2)$$
$$P_4 = 2(\Theta[000]^2\Theta[100]^2 + \Theta[001]^2\Theta[101]^2 + \Theta[010]^2\Theta[110]^2 + \Theta[011]^2\Theta[111]^2)$$
$$P_5 = 2(\Theta[000]^2\Theta[101]^2 + \Theta[001]^2\Theta[100]^2 + \Theta[010]^2\Theta[111]^2 + \Theta[011]^2\Theta[110]^2)$$
$$P_6 = 2(\Theta[000]^2\Theta[110]^2 + \Theta[001]^2\Theta[111]^2 + \Theta[010]^2\Theta[100]^2 + \Theta[011]^2\Theta[101]^2)$$
$$P_7 = 2(\Theta[000]^2\Theta[111]^2 + \Theta[001]^2\Theta[110]^2 + \Theta[010]^2\Theta[101]^2 + \Theta[100]^2\Theta[011]^2)$$
$$P_8 = 4(\Theta[000]\Theta[001]\Theta[010]\Theta[011] + \Theta[100]\Theta[101]\Theta[110]\Theta[111])$$
$$P_9 = 4(\Theta[000]\Theta[001]\Theta[100]\Theta[101] + \Theta[010]\Theta[011]\Theta[110]\Theta[111])$$
$$P_{10} = 4(\Theta[000]\Theta[001]\Theta[110]\Theta[111] + \Theta[010]\Theta[011]\Theta[100]\Theta[101])$$
$$P_{11} = 4(\Theta[000]\Theta[010]\Theta[100]\Theta[110] + \Theta[001]\Theta[011]\Theta[101]\Theta[111])$$
$$P_{12} = 4(\Theta[000]\Theta[010]\Theta[101]\Theta[111] + \Theta[001]\Theta[011]\Theta[100]\Theta[110])$$
$$P_{13} = 4(\Theta[000]\Theta[011]\Theta[100]\Theta[111] + \Theta[001]\Theta[010]\Theta[101]\Theta[110])$$
$$P_{14} = 4(\Theta[000]\Theta[011]\Theta[101]\Theta[110] + \Theta[001]\Theta[010]\Theta[100]\Theta[111]).$$

This has been constructed following sec. 3.5: let us fix four elements $\sigma_1, \ldots, \sigma_4 \in \mathbb{Z}_2^3$ which sum up to 0. Then an invariant element is

$$P_{\{\sigma_1,\ldots,\sigma_4\}} = \sum_{x\in\mathbb{Z}^3} \Theta[\sigma_1 + x]\Theta[\sigma_2 + x]\Theta[\sigma_3 + x]\Theta[\sigma_4 + x].$$

Looking at all possible quadruples of numbers σ_i we easily recover the above basis. Using the classical formula one shows that this space is also spanned by the 36 $\theta[\Delta]^4$'s with Δ even.

Weight 4

From section 3.7.4 we know that $\mathrm{Sym}^2(M_2(\Gamma_3(2))) \subset M_4(\Gamma_3(2))$ and as an Sp(6)-representation we have, with an analogous computation as in section 4.3.1:

$$\mathrm{Sym}^2(M_2(\Gamma_3(2))) := \mathrm{Sym}^2(\mathbf{15}_a) \;=\; \mathbf{1} \,+\, \mathbf{35}_b \,+\, \mathbf{84}_a.$$

The invariant subspace is spanned by $\sum_\Delta \theta[\Delta]^8$ and the subrepresentation $\mathbf{1} + \mathbf{35}_b$ is spanned by the 36 $\theta[\Delta]^8$'s which are permuted by Sp(6).

We recall the relation, introduced in Section 4.4.1, which holds for all $\tau \in \mathbb{H}_3$:

$$r_1 - r_2 = r_3, \qquad \text{with} \quad r_1 = \prod_{a,b\in\mathbb{F}_2} \theta\left[\begin{smallmatrix}000\\0ab\end{smallmatrix}\right](\tau), \quad r_2 = \prod_{a,b\in\mathbb{F}_2} \theta\left[\begin{smallmatrix}000\\1ab\end{smallmatrix}\right](\tau), \quad r_3 = \prod_{a,b\in\mathbb{F}_2} \theta\left[\begin{smallmatrix}100\\0ab\end{smallmatrix}\right](\tau).$$

From this we deduce that $2r_1r_2 = r_1^2 + r_2^2 - r_3^2$. Thus r_1r_2, a product of 8 distinct $\theta[\Delta]$'s, is a linear combination of three products of four theta squares. The sum of the four characteristics in each product is zero, hence

$$r_1 r_2 \;=\; \prod_{a,b,c\in\mathbb{F}_2} \theta\left[\begin{smallmatrix}000\\abc\end{smallmatrix}\right] \in M_4(\Gamma_3(2)).$$

Using a computer and the classical theta formula, we verified that under the action of Sp(6) on r_1r_2 one obtains 135 functions which are a basis of $M_4(\Gamma_3(2))$ and which are permuted (without signs) by Sp(6).

Let $P \subset \mathrm{Sp}(6)$ be the stabilizer of r_1r_2, it consists of the matrices with blocks $A, \ldots, D$ with $C = 0$. There are no non-trivial homomorphisms $P \to GL_1(\mathbb{C}) = \mathbb{C}^*$, because these factor over $SL(3, \mathbb{Z}_2)$ (with P as before, one maps the matrix first to A) and this is a simple group (of order 168). Thus any $g \in P$ acts as the identity on r_1r_2. Acting with the whole group Sp(6) we generate the induced representation of the $\mathbf{1_P}$, by Proposition 3.8.1. We can then identify the representation of Sp(6) on $M_4(\Gamma_3(2))$ with $\mathrm{Ind}_P^{\mathrm{Sp}}(1_P)$, this representation is (cf. [Fr1], p. 113)

$$M_4(\Gamma_3(2)) \cong \mathrm{Ind}_P^{\mathrm{Sp}}(1_P) \cong \mathbf{1} + \mathbf{35}_b + \mathbf{84}_a + \mathbf{15}_a \cong \mathrm{Sym}^2(M_2(\Gamma_3(2))) + \mathbf{15}_a.$$

In particular, there is a unique complementary 15-dimensional subspace which is Sp(6)-invariant. Necessarily, the representation on this subspace must be $\mathbf{15}_a \cong \rho_\theta$. We can realize an isomorphism of representations, by using some geometry of quadrics, as follows.

Let $L \subset \mathbb{F}_2^6$ be a Lagrangian subspace: $L \cong \mathbb{F}_2^3$ and $E(v,w) = 0$ for all $v, w \in L$. Then we know that there are $N_Q(3) = 8$ even quadrics Q such that $L \subset Q$. When $L = L_0 = \{(v, v') \in \mathbb{Z}^3 \times \mathbb{Z}^3 : v' = 0\}$, the 8 even quadrics containing L_0 have the same characteristics as the eight theta constants in $r_1 r_2$ so that we can write

$$r_1 r_2 = \prod_{Q \supset L_0} \theta[\Delta_Q],$$

the product being on the quadrics containing L_0. Acting with the modular group Sp(6) will generate $N_{iso}(3,3) = 135$ elements of the orbit which are naturally identified, by construction, with the elements in the orbit of $r_1 r_2$, so that each one of them can be written as

$$P_L = \pm \prod_{Q \supset L} \theta[\Delta_Q],$$

for a unique Lagrangian subspace L, where the sign is determined by the condition that it must be $+1$ if $L = L_0$ and that $P_L = \rho(g) P_{L_0}$ for some $g \in \mathrm{Sp}(6)$, with ρ defining a representation.

We can exploit this description for the basis of $M_4(\Gamma_3(2))$ to write down the (unique) O^+-anti-invariant. Recall that O^+ is the stabilizer of the even characteristic [0] and let us call Q_0 the corresponding even quadric. An even (also called split) quadric Q in $\mathbb{F}_2^6$ contains $N_{iso}^Q(3) = 30$ Lagrangian subspaces. It is easy to see that the intersection of two distinct Lagrangian subspaces in the same quadric cannot have codimension smaller then $g+2$. For $g = 3$ this means that the intersection between two distinct Lagrangian subspaces must have dimension 0 or 1. We say that $L, L' \subset Q$ are in the same ruling if $L \cap L'$ is 1-dimensional. As fixing a point in Q one can easily count 15 Lagrangian subspaces containing it, we see that there are two rulings, each one containing 15 Lagrangian subspaces. Let us define $P[0]$ to be the sum of the 15 P_L's from one ruling minus the sum of the 15 P_L's from the other ruling of Q_0. Then $P[0]$ transforms with the representation ϵ of O^+. Indeed, the action of Sp(6) preserves intersection so that a given elements $h \in \mathrm{Sp}(6)$ or acts separately on each rule and then as the identity on $P[0]$, or mix the rules acting as -1 on $P[0]$ so that $P[0]$ supports a 1-dimensional non-trivial representation of O^+. But we have said in section 3.7 that ϵ is the unique non-trivial representation.

Thus the subrepresentation of $M_4(\Gamma_3(2))$ generated by $P[0]$ is contained in $\mathrm{Ind}_{O^+}^{\mathrm{Sp}}(\epsilon)$ and thus it must be $\rho_\theta = \mathbf{15}_a$, $\rho_r = \mathbf{21}_b$ or their direct sum. As only $\mathbf{15}_a$ is a component of the representation on $M_4(\Gamma_3(2))$, we conclude that the subrepresentation generated by $P[0]$ is isomorphic to $\mathbf{15}_a$ and thus is complementary to $\mathrm{Sym}^2(M_2(\Gamma_3(2)))$.

The Sp(6)-representation on $M_6(\Gamma_3(2))$

Starting from the basis of the polynomials P_j, it is easy to check by means of a computer that the 680 products $P_i P_j P_k$, $0 \le i \le j \le k \le 14$ are linearly independent in the 870-

dimensional vector space $(\mathbb{C}[\ldots,\Theta[\sigma],\ldots]_8)^{H_g}$. The spanned subspace is

$$\mathrm{Sym}^3(M_2(\Gamma_3(2))) := \langle\, P_iP_jP_k \,:\, P_i, P_j, P_k \in M_2(\Gamma_3(2))\,\rangle.$$

The identification $M_2(\Gamma_3(2)) \cong \mathbf{15}_a$, as $\mathrm{Sp}(6,\mathbb{F}_2)$-representation, implies

$$\mathrm{Sym}^3(M_2(\Gamma_3(2))) \cong 2\cdot\mathbf{15}_a + \mathbf{21}_b + \mathbf{35}_b + \mathbf{84}_a + \mathbf{105}_c + \mathbf{189}_c + \mathbf{216}_a.$$

To decompose all of $V := M_6(\Gamma_3(2))$ we use operator $C = C_V$ as in section 3.7.5. The eigenvalues λ of C on V and their multiplicities m_λ are easily computed by means of a computer, giving

$$(\lambda, m_\lambda) : (63,1), \quad (27,35), \quad (3,378), \quad (-7,216), \quad (-13,189), \quad (-21,30), \quad (-33,21).$$

$(63,1)$ corresponds to the one-dimensional trivial representation. We would now recognize $\mathrm{Sym}^3(M_2(\Gamma_3(2)))$ in $M_6(\Gamma_3(2))$. From the character table of Sp(6) in [Fr1], p.114–115 or in Table 4.7 (where t_v is in the 16^{th} conjugacy class labeled $1^{-5}2^6$), we see that the irreducible representations ρ of dimension d_λ such that C_ρ has eigenvalue λ in the cases $(\lambda, d_\lambda) = (-13,189)$, $(-33,21)$ are unique. Then, the irreducible representations $\mathbf{189}_c$ and $\mathbf{21}_b$ occur in $M_6(\Gamma_3(2))$, with multiplicity one. The irreducible representations ρ for which C_ρ has eigenvalue $\lambda = 27$ are $\mathbf{21}_a$ and $\mathbf{35}_b$. C has a 35-dimensional eigenspace with $\lambda = 27$, so that the representation $\mathbf{35}_b$ occurs with multiplicity one in $M_6(\Gamma_3(2))$. Also the irreducible representations for which C_ρ has eigenvalue -7 are two: $\mathbf{56}_a$ and $\mathbf{216}_a$. As before, for dimensional reasons, we can conclude that the representation $\mathbf{216}_a$ occurs with multiplicity one in $M_6(\Gamma_3(2))$. The irreducible representations ρ for which C_ρ has eigenvalue $\lambda = 3$ are $\mathbf{105}_c$, $\mathbf{84}_a$, $\mathbf{420}_a$, $\mathbf{210}_b$, then giving rise to two possible decompositions of the 378-dimensional eigenspace with $\lambda = 3$: $2\cdot\mathbf{105}_c + 2\cdot\mathbf{84}_a$ or $\mathbf{210}_b + 2\cdot\mathbf{84}_a$. As $\mathbf{105}_c$ is an irreducible component of $\mathrm{Sym}^3(\mathbf{15}_a)$, it must appear in $M_6(\Gamma_3(2))$, and then $2\cdot\mathbf{105}_c + 2\cdot\mathbf{84}_a$ is the right decomposition. The case when C_ρ has eigenvalue $\lambda = -21$ correspond to the representations: $\mathbf{105}_a$ and $\mathbf{15}_a$. As there is no more space for a 105 dimensional representation, the only possibility is that the 30-dimensional eigenspace of C with $\lambda = -21$ coincides with the representation $2\cdot\mathbf{15}_a$. We then conclude

$$M_6(\Gamma_3(2)) = \mathrm{Sym}^3(\mathbf{15}_a) + \mathbf{1} + \mathbf{84}_a + \mathbf{105}_c.$$

Asyzygous sextets

In the complement of $\mathrm{Sym}^3(\mathbf{15}_a)$ in $M_6(\Gamma_3(2))$, let us consider the subrepresentation space $\mathbf{105}_c$. This is exactly the same space which has been studied by D'Hoker and Phong in [DP6], as we will see in a moment. Following [DP6], let us consider sets $S = \{\Delta_1,\ldots,\Delta_6\}$ of six totally asyzygous (even) characteristics, which means that $\Delta_i+\Delta_j+\Delta_k$ is odd for distinct i,j,k. An example of such a sextet of even characteristics is

$$S_0 := \{\, \left[\begin{smallmatrix}110\\110\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}110\\111\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}111\\110\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}101\\101\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}101\\111\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}111\\101\end{smallmatrix}\right] \,\}.$$

These are of the form $\left[\begin{smallmatrix}1ab\\1cd\end{smallmatrix}\right]$ where $\left[\begin{smallmatrix}ab\\cd\end{smallmatrix}\right]$ runs over the six odd theta characteristics in genus 2. From the fact that the sum of any three odd characteristics in genus two is even, it follows that the sum of any three of these six characteristics for $g=3$ is indeed odd. There are 336 asyzygous sextets, on which $\mathrm{Sp}(6)$ acts transitively. The sum of the six characteristics each of these sextets is zero (in $\mathbb{F}_2$), hence to a sextet S we can associate a modular form F_S of weight 6 on $\Gamma_3(2)$:

$$F_S := \prod_{\Delta\in S}\theta[\Delta]^2 \qquad (\in M_6(\Gamma_3(2))).$$

Each F_S corresponds to a Heisenberg invariant which can be expressed as an homogeneous polynomial of degree 12 in the $\Theta[\sigma]$'s by means of the classical theta formula (3.33). A computer computation shows that the 336 functions F_S span a 105-dimensional vector space

$$W_{as} := \langle\, F_S \,:\, S \text{ totally asyzygous sextet}\,\rangle \qquad (\subset M_6(\Gamma_3(2))).$$

One can verify indeed that W_{as} has intersection $\{0\}$ with $\mathrm{Sym}^3(M_2(\Gamma_3(2)))$ and that C reduces itself as multiplication by 3 on W_{as}, so that $W_{as}\cong \mathbf{105}_c$, which is what we intended to prove.
In [DP6] it has been shown that W_{as} does not contain functions which transform as $\theta[0]^4$ under $O^+(6)$. This is clear in terms of our analysis, since such a function would generate a subrepresentation of $\mathrm{Ind}_{O^+}^{\mathrm{Sp}}(\epsilon) = \mathbf{15}_a\oplus\mathbf{21}_b$, then contradicting the identification $W_{as}\cong\mathbf{105}_c$.

The $\mathrm{Sp}(6)$-representation on $M_8(\Gamma_3(2))$

The space of modular forms of weight 8 on $\Gamma_3(2)$ is:

$$\mathbb{C}[\ldots,X_\sigma,\ldots]_{16}^{H_g} \cong M_8(\Gamma_3(2))\oplus\langle F_{16}\rangle,$$

where F_{16} is a homogeneous polynomial of degree 16 in the X_σ such that $F_{16}(\ldots,\Theta[\sigma](\tau),\ldots)=0$ for all $\tau\in\mathbb{H}_3$, see [vGvdG], [CDG1], and § 4.4.1:

$$F_{16} = 8\sum_\Delta\theta[\Delta]^{16} - \Big(\sum_\Delta\theta[\Delta]^8\Big)^2, \qquad \theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^2 = \sum_\sigma(-1)^{\sigma b}X_\sigma X_{\sigma+a}$$

so we substitute X_σ rather than $\Theta[\sigma]$ in the classical theta formula. In particular,

$$\dim M_8(\Gamma_3(2)) = 3993-1 = 3992.$$

The image $\mathrm{Sym}^4(M_2(\Gamma_3(2)))_0$ of $\mathrm{Sym}^4(M_2(\Gamma_3(2)))$ in this 3992-dimensional vector space is spanned by the products $P_iP_jP_kP_l$, for a basis $P_i\in\mathbb{C}[\ldots,X_\sigma,\ldots]_4^{H_g}$, $1\le i\le 15$, with 27 independent relations, and includes F_{16}. Thus, its dimension is

$$\dim\mathrm{Sym}^4(M_2(\Gamma_3(2)))_0 = \binom{15+4-1}{4} - 27 - 1 = 3032.$$

Starting from the identification $M_2(\Gamma_3(2)) = \mathbf{15}_a$, a computation with Sp(6)-representations shows that

$$\begin{aligned}\mathrm{Sym}^4(M_2(\Gamma_3(2))) &\cong 2\cdot\mathbf{1} + 2\cdot\mathbf{15}_a + \mathbf{27}_a + \mathbf{35}_a + 4\cdot\mathbf{35}_b + 4\cdot\mathbf{84}_a + \mathbf{105}_c + \\ &\quad +\mathbf{168}_a + \mathbf{189}_c + 2\cdot\mathbf{216} + 3\cdot\mathbf{280}_b + \mathbf{336}_a + \mathbf{420}_a.\end{aligned}$$

The $1+27$-dimensional kernel of the map $\mathrm{Sym}^4(M_2(\Gamma_3(2))) \to \mathrm{Sym}^4(M_2(\Gamma_3(2)))_0$ is an Sp(6)-representation, thus it must be $\mathbf{1} + \mathbf{27}_a$.
We then have to identify the complement W in the decomposition

$$M_8(\Gamma_3(2)) \cong \mathrm{Sym}^4(M_2(\Gamma_3(2)))_0 \oplus W, \qquad \dim W = 960.$$

By computing the operator C from section 3.7.5 on $\mathbb{C}[\ldots, X_\sigma, \ldots]_{16}^{H_g}$. we get the pairs given by the eigenvalues λ and their multiplicity m_λ which result to be

$$\begin{array}{llllll}(\lambda, m_\lambda): & (3,1050), & (9,840), & (15,168), & (27,140), & (63,2), \\ & (-3,672), & (-7,648), & (-9,35), & (-13,378), & (-21,60).\end{array}$$

Thus, we see that $\mathbf{27}_a$ cannot be a subrepresentation of $\mathbb{C}[\ldots, X_\sigma, \ldots]_{16}^{H_g}$ because the eigenvalue λ of C on $\mathbf{27}_a$ would have been 35, which is not an eigenvalue of C.
For the remaining eigenvalues we see that:

- the eigenvalue 63 corresponds to the subspace of invariants;
- the eigenvalues $\lambda = 9, -3, -7, -13$ occur only on the irreducible representations $\mathbf{280}_b$, $\mathbf{336}_a$, $\mathbf{216}_a$, $\mathbf{189}_c$ respectively, hence $3\cdot\mathbf{280}_b + 2\cdot\mathbf{336}_a + 3\cdot\mathbf{216}_a + 2\cdot\mathbf{189}_c$ is a summand of $M_8(\Gamma_3(2))$, and W has a summand $\mathbf{189}_c + \mathbf{216}_a + \mathbf{336}_a$;
- the eigenvalue 3 occurs only on $\mathbf{84}_a$, $\mathbf{105}_c$, $\mathbf{210}_b$ and $\mathbf{420}_a$. Now, $4\cdot\mathbf{84}_a + \mathbf{105}_c + \mathbf{420}_a$ is a summand of $\mathrm{Sym}^4(M_2(\Gamma_3(2)))_0$, so that there remains a subrepresentation of dimension $1050 - 861 = 189$ in W with the same eigenvalue. Thus $\mathbf{84}_a + \mathbf{105}_c$ is a summand of W;
- the eigenvalue 15 occurs only on $\mathbf{105}_b$, $\mathbf{168}_a$ and $\mathbf{210}_a$. The eigenspace of C for this eigenvalue has dimension 168 so that $\mathbf{168}_a$ is a summand of $M_8(\Gamma_3(2))$ which lies in $\mathrm{Sym}^4(\mathbf{15}_a)$;
- the eigenvalue 27 occurs only on $\mathbf{21}_a$ and $\mathbf{35}_b$. $4\cdot\mathbf{35}_b$ is a summand of $\mathrm{Sym}^4(M_2(\Gamma_3(2)))_0$ and the dimension of this eigenspace of C is 140. Thus, none of these two representations occurs in W;
- the lowest dimensional representation where the eigenvalue -9 occurs is $\mathbf{35}_a$. As the eigenspace for $\lambda = -9$ has dimension 35, we conclude that $\mathbf{35}_a$ is a summand of $M_8(\Gamma_3(2))$, which lies in $\mathrm{Sym}^4(\mathbf{15}_a)$;
- the eigenvalue -21 occurs only on $\mathbf{15}_a$ and $\mathbf{105}_a$. The eigenspace of C for this eigenvalue has dimension 60, then $4\cdot\mathbf{15}_a$ is a summand of $M_8(\Gamma_3(2))$ and the summand $2\cdot\mathbf{15}_a$ is contained in W.

These considerations lead us to the conclusion that

$$W = 2\cdot \mathbf{15}_a + \mathbf{84}_a + \mathbf{105}_c + \mathbf{189}_c + \mathbf{216}_a + \mathbf{336}_a.$$

4.4.7 The uniqueness of $\Xi_8[0^{(3)}]$

In section 4.4.5 we have determined a modular form $\Xi_8[{}^{000}_{000}]$ which satisfies the three reduced constraints of section 4.1.
Such constraints imply that $\Xi_8[0^{(3)}]$ is a modular form on $\Gamma_3(2)$ of weight 8 and should be O^+-invariant (equivalently, it is a modular form on $\Gamma_3(1,2)$), so it must lie in $M_8(\Gamma_3(2))^{O^+}$. As we explained in section 3.7.3, the only Sp(6)-representations which have $O^+(6)$-invariants are $\mathbf{1}$ and $\sigma_\theta = \mathbf{35}_b$. Hence, the decomposition of $M_8(\Gamma_3(2))$ obtained in section 4.4.6 implies that

$$\dim M_8(\Gamma_3(2))^{O^+} = 1 + 4 = 5.$$

The subspace $M_8(\Gamma_3(2))^{O^+}$ contains the Sp(6)-invariant $\sum_\Delta \theta[\Delta]^{16} \in M_8(\Gamma_3(2))$ as well as the three dimensional subspace

$$\theta[{}^{000}_{000}]^4 M_6(\Gamma_3(2))^\epsilon := \{\theta[{}^{000}_{000}]^4 f : \ f \in M_6(\Gamma_3(2))^\epsilon\},$$

since both $\theta[{}^{000}_{000}]^4$ and such f are O^+-anti-invariant. A basis of this space is furnished[5], for example, by the three functions $F_i^{(3)}$, with $i = 1, 2, 3$, of section 4.4.2. Also we defined the functions $G[\Delta]$ that are modular forms of weight 8 on $\Gamma_3(2)$ and we proved [CDG1] that $G[0] \in M_8(\Gamma_3(2))^{O^+}$.
Using the classical theta formulas, one can check that these functions span the O^+-invariants

$$M_8(\Gamma_4(2))^{O^+} = \theta[{}^{000}_{000}]^4 M_6(\Gamma_4(2))^\epsilon \oplus \langle \sum_\Delta \theta[\Delta]^{16},\ G[0] \rangle.$$

Moreover, the modular form $\Xi_8[0^{(0)}]$ should restrict to $(\theta[{}^0_0]^4\eta^{12})(\tau_1)(\theta[{}^{00}_{00}]^4\Xi_6[{}^{00}_{00}])(\tau_2)$ on $\mathbb{H}_1 \times \mathbb{H}_2 \subset \mathbb{H}_3$. Note that this restriction is a multiple of $\theta[{}^0_0](\tau_1)$. Differently from the other four functions, the restriction of $\sum \theta[\Delta]^{16}$ to the diagonal is not a multiple of $\theta[{}^0_0](\tau_1)$. Thus $\Xi_8[0^{(3)}]$ should be linear combination of just these four functions. The explicit computations done in Section 4.4.5 showed that there is a unique linear combination satisfying the restriction constraint. This verifies the uniqueness of $\Xi_8[0^{(3)}]$. As we said, as a corollary this implies the non existence of the form $\Xi_6[0^{(3)}]$ proposed by D'Hoker and Phong. It is however interesting to look better at this fact as it gives some new interesting information. The modular form $\Xi_6[0^{(3)}]$ should live in $M_6(\Gamma_g(2))^\epsilon$. Now, the only Sp(6)-representations with O^+-anti-invariants are $\rho_\theta = \mathbf{15}_a$ and $\rho_r = \mathbf{21}_b$, which contain a unique such anti-invariant (cf. 3.7.3). Thus, from the decomposition of $M_6(\Gamma_3(2))$ given in 4.4.6, it follows that $\dim M_6(\Gamma_3(2))^\epsilon = 3$. One can verify that the

[5] It is clear that a basis for the space $M_6(\Gamma_3(2))^\epsilon$ of $O^+(6)$-anti-invariants is given by $\theta[{}^{000}_{000}]^{12}$, $\theta[{}^{000}_{000}]^4 \sum_\Delta \theta[\Delta]^{12}$ and $\theta[{}^{000}_{000}]^8 \sum_\Delta \theta[\Delta]^8$.

following functions are a basis

$$M_6(\Gamma_g(2))^\epsilon = \langle\ \theta\left[{000 \atop 000}\right]^{12},\quad \sum_\Delta \theta[\Delta]^{12},\quad \theta\left[{000 \atop 000}\right]^4 \sum_\Delta \theta[\Delta]^8\ \rangle.$$

The function $\Xi_6[0^{(3)}]$ should restrict to $\Xi_6[0^{(1)}](\tau_1)\Xi_6[0^{(2)}](\tau_2)$ for $(\tau_1,\tau_2) \in \mathbb{H}_1 \times \mathbb{H}_2 \subset \mathbb{H}_3$, with $\Xi_6[0^{(1)}](\tau_1) = \eta^{12}(\tau_1)$.

The linear combination $a\theta\left[{000 \atop 000}\right]^{12} + b\sum_\Delta \theta[\Delta]^{12} + c\theta\left[{000 \atop 000}\right]^4(\sum_\Delta \theta[\Delta]^8)$ restricts, using the results from 5.10.1 and the restriction of section 4.4.5 (without the common factor $\theta\left[{000 \atop 000}\right]^4$), to a function of the form $\eta^{12}(\tau_1)g(\tau_2)$ iff

$$a\theta\left[{00 \atop 00}\right]^{12} + 2b\sum_\delta \theta[\delta]^{12} + 2c\theta\left[{00 \atop 00}\right]^4 \sum_\delta \theta[\delta]^8 = 0$$

on $\mathbb{H}_2$. However, it follows from the results in Table 4.6 that the ten even $\theta[\delta]^{12}$'s span a ten dimensional space which does not contains $\theta\left[{00 \atop 00}\right]^4 \sum_\delta \theta[\delta]^8$. Hence we must have $a = b = c = 0$, which proves that there is no function $\Xi_6[0^{(3)}]$ satisfying the three constraints imposed in [DP6].

4.5 The case $g = 4$

Finally, we will consider here the case of genus four, where we will again be able to find a suitable modular form Ξ_8 satisfying the four constraints of section 2.3. Thus, this function turn out to be a good candidate for the superstring measure. We will prove the uniqueness in a weaker form w.r.t. the previous cases (cf. § 4.5.3), in [OPSY] the general case is considered. The construction procedure is the same as in the previous cases: we search functions that are O^+-invariant and then look for a linear combination satisfying the factorization constraint. However, in this case we make use of a further assumption: we require for the forms $\Xi_8[\Delta]$ to be polynomials in the theta constants. This new assumption is motivated by the non normality of the ring of modular forms in genus four, thus it could contain modular forms that are not polynomial in theta constants. The uniqueness (in this sense) will follow, as before, from the fact we use a basis for the space of (polynomial) modular forms on $\Gamma_4(1,2)$, but is then restricted to the polynomial subspace. The factorization constraints which must be imposed to the forms Ξ_8 when the period matrix $\tau_4 \in \mathbb{H}_4$ becomes reducible, are $\Xi_8[0^{(4)}](\tau_{k,4-k}) = \Xi_8[0^{(k)}](\tau_k)\Xi_8[0^{(4-k)}](\tau_{4-k})$, with $\tau_{k,4-k} := \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_{4-k} \end{pmatrix} \in \mathbb{H}_4$. As in genus three, first we construct the forms Ξ_8, then we tackle the problem of the uniqueness.

4.5.1 The modular forms $F_i^{(4)}$, $G_1^{(4)}[0^{(4)}]$ and $G_2^{(4)}[0^{(4)}]$

It is easy to write down some of the O^+-invariants in genus four as extensions of the functions defined in genus two and three. An obvious generalization of the functions F_i introduced in sections 4.3.2and 4.4.2 is:

$$F_1^{(4)} := \theta\left[{0000 \atop 0000}\right]^{16},\qquad F_2^{(4)} := \theta\left[{0000 \atop 0000}\right]^4 \sum_\Delta \theta[\Delta]^{12},\qquad F_3^{(4)} := \theta\left[{0000 \atop 0000}\right]^8 \sum_\Delta \theta[\Delta]^8,$$

where the sum is over the 136 even characteristics Δ in genus four. These functions are modular forms of weight 8 on $\Gamma_4(1,2)$. When clear from the contest we omit the apex indicating the genus. The restriction of these forms to $\mathbb{H}_1 \times \mathbb{H}_3$ and $\mathbb{H}_2 \times \mathbb{H}_2$ are

$$\begin{aligned} F_1^{(4)}(\tau_{2,2}) &= \theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]^{16}(\tau_2)\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]^{16}(\tau_2') \\ &= F_1^{(2)}(\tau_2)F_1^{(2)}(\tau_2'), \\ F_2^{(4)}(\tau_{2,2}) &= \Big(\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]^4 \textstyle\sum_\delta \theta[\delta]^{12}\Big)(\tau_2)\Big(\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]^4 \textstyle\sum_\delta \theta[\delta]^{12}\Big)(\tau_2') \\ &= F_2^{(2)}(\tau_2)F_2^{(2)}(\tau_2'), \\ F_3^{(4)}(\tau_{2,2}) &= \theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]^8(\tau_2)\theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]^8(\tau_2')\Big(\textstyle\sum_\delta \theta[\delta]^8\Big)(\tau_2)\Big(\textstyle\sum_\delta \theta[\delta]^8\Big)(\tau_2') \\ &= F_3^{(2)}(\tau_2)F_3^{(2)}(\tau_2'), \end{aligned}$$

whereas the restrictions to $\mathbb{H}_1 \times \mathbb{H}_3$ are

$$F_1^{(4)}(\tau_{1,3}) = \Big(\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4(\tfrac{1}{3}f_{21} + \eta^{12})\Big)(\tau_1)F_1^{(3)},(\tau_3)$$

$$F_2^{(4)}(\tau_{1,3}) = \Big(\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4(\tfrac{2}{3}f_{21} - \eta^{12})\Big)(\tau_1)F_2^{(3)}(\tau_3),$$

$$F_3^{(4)}(\tau_{1,3}) = \Big(\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4\tfrac{2}{3}f_{21}\Big)(\tau_1)F_3^{(3)}(\tau_3).$$

The modular form $G^{(3)}$ of section 4.4.4 can be generalized in two ways. The first one is to stick to three dimensional isotropic subspaces W in $\mathbb{F}_2^8$. Given such a W, there are $3 \cdot 8 = 24$ even quadrics Q_Δ such that $W \subset Q_\Delta$. Let $Q_0 \subset \mathbb{F}_2^8$ be the even quadric with characteristic $\Delta_0 = [0^{(4)}]$. We can use the octets of quadrics which contain Q_0 to define a modular form $G_1[0]$:

$$G_1\left[\begin{smallmatrix}0000\\0000\end{smallmatrix}\right] := \sum_{W\subset Q_0} \prod_{w\in W} \theta[\Delta_0 + w]^2,$$

where the sum is extended to all the 2025 three dimensional isotropic subspaces $W \subset Q_0$, and for each such subspace we take the product of the eight even $\theta[\Delta_0+w]^2$. As $0 \in W$, for any subspace W, the function $G_1[0]$ is a multiple of $\theta[\Delta_0]^2$. The function $G_1[0]$ is a modular form on $\Gamma_4(1,2)$ of weight 8, as can be shown using the explicit transformation theory of theta functions as in Section 3.5 for genus three or in the Appendices of [CDG1] (or cf. [I2] or see [Gr], Proposition 13).
Using methods similar to those in Appendix C of [CDG1] for the case $g = 3$ we find the restriction of $G_1[0]$ to $\mathbb{H}_1 \times \mathbb{H}_3$:

$$\begin{aligned} G_1\left[\begin{smallmatrix}0000\\0000\end{smallmatrix}\right](\tau_{1,3}) = {}& \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^{16}(\tau_1)G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right](\tau_3) + \Big(\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^8(\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^8 + \theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^8)\Big)(\tau_1)\Big(H\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right] + 7G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]\Big)(\tau_3) \\ = {}& \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4(\tau_1)\Big(\tfrac{1}{3}f_{21}(\tau_1)(H\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right] + 8G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right])(\tau_3) - \eta^{12}(\tau_1)(H\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right] + 6G\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right])(\tau_3)\Big) \end{aligned}$$

where

$$H[^{000}_{000}] := \sum_{W'\subset Q_0} \prod_{w\in W} \theta[\Delta_0^{(3)} + w]^4,$$

and f_{21} as in Section 4.2. The sum in H is over the 105 isotropic 2-dimensional subspaces W' contained in Q_0, where now $Q_0 \subset \mathbb{F}_2^3$.
Similarly, the restriction of $G_1[0]$ to $\mathbb{H}_2 \times \mathbb{H}_2$ is

$$\begin{aligned} G_1[^{0000}_{0000}](\tau_{2,2}) &= \theta[^{00}_{00}]^4(\tau_2)g(\tau_2)F_3^{(2)}(\tau_2') + F_3^{(2)}(\tau_2)\theta[^{00}_{00}]^4(\tau_2')g(\tau_2') \\ &+ 9\theta[^{00}_{00}]^4(\tau_2)g(\tau_2)\theta[^{00}_{00}]^4(\tau_2')g(\tau_2'), \end{aligned}$$

with $F_3^{(2)}$ as in section 4.3.2,

$$g(\tau_2) = \sum_{W'\subset Q_0} \prod_{w\in W'-\{0\}} \theta[\Delta_0^{(2)} + w]^4(\tau_2),$$

where the sum is over the 6 isotropic 2-dimensional subspaces W' of $Q_0 \subset \mathbb{F}_2^4$, and we take a product of only three terms (the factor $\theta[0^{(2)}]^4$ for $w = 0$ is taken out in the formula for $G_1[0](\tau_{2,2})$).
Another generalization of G makes use of Lagrangian subspaces $L \cong \mathbb{F}_2^4$ of $V = \mathbb{F}_2^8$. For each L there are 16 even quadrics Q_Δ with $L \subset Q_\Delta$. For an even characteristic Δ we define

$$G_2[\Delta] = \sum_{L\subset Q_\Delta} \prod_{Q\supset L} \theta[\Delta_Q],$$

the sum being over the 270 Lagrangian subspaces L of $V = \mathbb{F}_2^8$ which are contained in Q_Δ. Again, $L \subset Q_\Delta$ implies that $G_2[\Delta]$ is a multiple of $\theta[\Delta]$. The function $G_2[0]$ is also modular form on $\Gamma_4(1,2)$ of weight 8.
The restriction of $G_2[0]$ to $\mathbb{H}_1 \times \mathbb{H}_3$ is

$$\begin{aligned} G_2[^{0000}_{0000}](\tau_{1,3}) &= \Big(\theta[^0_0]^8(\theta[^0_1]^8 + \theta[^1_0]^8)\Big)(\tau_1)G[^{000}_{000}](\tau_3) \\ &= \Big(\theta[^0_0]^4(\tfrac{1}{3}f_{21} - \eta^{12})\Big)(\tau_1)G[^{000}_{000}](\tau_3), \end{aligned}$$

whereas its to $\mathbb{H}_2 \times \mathbb{H}_2$ is as follows:

$$G_2[^{0000}_{0000}](\tau_{2,2}) = \theta[^{00}_{00}]^4(\tau_2)\theta[^{00}_{00}]^4(\tau_2')g(\tau_2)g(\tau_2'),$$

with g as above.
Before considering the restriction to $\mathbb{H}_1 \times \mathbb{H}_3$ of the form $\Xi_8^{(4)}$ let us give a couple of identities which we need to express the restrictions of the $G_i^{(4)}[0]$'s as linear combinations of the $F_i^{(3)}$'s and $G^{(3)}[\Delta]$ in genus three and the $F_i^{(2)}$'s in genus two:

$$H[^{000}_{000}] = (2F_1^{(3)} + 8F_2^{(3)} - 3F_3^{(3)})/6, \qquad \theta[^{00}_{00}]^4 g = (2F_1^{(2)} + 4F_2^{(2)} - 3F_3^{(2)})/6. \quad (4.43)$$

These identities can be verified using the classical theta formula (it is helpful to use a computer as well).
Let us now consider a general linear combination of the 5 functions

$$\Xi_8[^{0000}_{0000}] = a_1F_1^{(4)} + \ldots + a_4G_2[0] + a_5G_1[0],$$

and try first to impose the restriction to $\mathbb{H}_1 \times \mathbb{H}_3$:

$$\Xi_8[{}^{0000}_{0000}](\tau_{1,3}) \;=\; \Xi_8[{}^{0}_{0}](\tau_1)\Xi_8[{}^{000}_{000}](\tau_3) \;=\; \Big(\theta[{}^{0}_{0}]^4\eta^{12}\Big)(\tau_1)\Xi_8[{}^{000}_{000}](\tau_3). \qquad (4.44)$$

As function of $\tau_1 \in \mathbb{H}_1$, this restriction is a linear combination of $\Xi_8[0^{(1)}]$ and $\theta[0^{(1)}]^4 f_{21}$. The second type terms however must disappear in the correct factorization so that $\Xi_8[{}^{0000}_{0000}](\tau_{1,3})$ is a multiple of $\eta^{12}(\tau_1)$ iff

$$a_1F_1 + 2a_2F_2 + 2a_3F_3 + a_4G[{}^{000}_{000}] + a_5(H[{}^{000}_{000}] + 8G[{}^{000}_{000}]) = 0.$$

Using the formula for H given in (4.43) we get:

$$(a_1 + \frac{1}{3}a_5)F_1 + (2a_2 + \frac{4}{3}a_5)F_2 + (2a_3 - \frac{1}{2}a_5)F_3 + (a_4 + 8a_5)G[{}^{000}_{000}] = 0.$$

As the four functions here are independent (cf. [DvG]), we get the solutions

$$(a_1, a_2, a_3, a_4, a_5) \;=\; \lambda(-2, -4, 3/2, -48, 6), \qquad (\lambda \in \mathbb{C}).$$

For such a_i the linear combination $a_1\theta[0^{(4)}]^4F_1 + \ldots + a_4G_2[0] + a_5G_1[0]$ restricts to:

$$\theta[{}^{0}_{0}]^4\eta^{12}\Big(a_1F_1 - a_2F_2 - a_4G[{}^{000}_{000}] - a_5(H[{}^{000}_{000}] + 6G[{}^{000}_{000}])\Big)$$

which, using again the formula for H gives a genus three factor

$$(a_1 - \frac{1}{3}a_5)F_1 - (a_2 + \frac{4}{3}a_5) + \frac{1}{2}a_5F_3 - (a_4 + 6a_5)G[{}^{000}_{000}].$$

Setting $\lambda = -2$, so that $(a_1, \ldots, a_5) = (4, 8, -3, 96, -12)$, we get (cf. 4.4.5)

$$8F_1 + 8F_2 - 6F_3 - 24G[{}^{000}_{000}] \;=\; 24\Xi_8[{}^{000}_{000}].$$

Thus, the function

$$\Xi_8[{}^{0000}_{0000}] \;:=\; \Big(4F_1 + 8F_2 - 3F_3 + 96G_2[{}^{0000}_{0000}] - 12G_1[{}^{0000}_{0000}]\Big)/24 \qquad (4.45)$$

satisfies correctly the constraint on the restriction to $\mathbb{H}_1 \times \mathbb{H}_3$.
However, having found the solution, we must check that it satisfies correctly the restriction to $\mathbb{H}_2 \times \mathbb{H}_2$ also. It is useful to observe that:

$$\begin{aligned} 24\Xi_8[{}^{0000}_{0000}](\tau_{2,2}) \;&=\; \Big(4F_1 + 8F_2 - 3F_3 + 96G_2 - 12G_1\Big)(\tau_{2,2}) \\ &=\; \theta[{}^{00}_{00}]^4(\tau_2)\theta[{}^{00}_{00}]^4(\tau_2')h(\tau_2, \tau_2'), \end{aligned}$$

with h a holomorphic function which we will not write down explicitly. Taking out the factor $\theta[0^{(2)}]^4(\tau_2)\theta[0^{(2)}]^4(\tau_2')$, which also occurs in $\Xi_8[0^{(2)}](\tau_2)\Xi_8[0^{(2)}](\tau_2')$, simplifies the computation. One finds, using the classical theta formula and a computer, that $h(\tau_2, \tau_2') = \Xi_6[0^{(2)}](\tau_2)\Xi_6[0^{(2)}](\tau_2')$ and thus:

$$\Xi_8[{}^{0000}_{0000}](\tau_{2,2}) \;=\; \Xi_8[{}^{00}_{00}](\tau_2)\Xi_8[{}^{00}_{00}](\tau_2').$$

Therefore the modular form $\Xi_8[0^{(4)}]$ on $\Gamma_4(1,2)$ of weight 8, defined in (4.45), satisfies all the factorization constraints in genus four. Using the representation theory of group we will be also able to prove the uniqueness of this modular form.

4.5.2 The Sp(8)-representation on $M_{2k}^{\theta}(\Gamma_4(2))$

In case $g = 4$ we are no longer sure if $M_{2k}^{\theta}(\Gamma_4(2))$, the space of modular forms of weight of $2k$ which are (Heisenberg-invariant) polynomials in the $\Theta[\sigma]$'s, is equal to all of $M_{2k}(\Gamma_4(2))$ (cf. [OSM]).

The Sp(8)-representation on $M_2^{\theta}(\Gamma_4(2))$, which we denoted by ρ_θ in section 3.7.3, is the unique 51-dimensional irreducible representation of Sp(8) (a table of the 81 irreducible representations of Sp(8) can be easily generated with the computer algebra program 'Magma').

The complement of $\mathrm{Sym}^2(M_2^{\theta}(\Gamma_4(2))$ in $M_4^{\theta}(\Gamma_4(2))$ (of Sp(8)-representations) now has codimension 918:

$$\dim M_4^{\theta}(\Gamma_4(2)) \,-\, \dim \mathrm{Sym}^2(M_2^{\theta}(\Gamma_4(2))) \,=\, 2244 - \binom{51+1}{2} \,=\, 2244 - 1326 \,=\, 918.$$

We computed the operator C from section 3.7.5 on $M_4^{\theta}(\Gamma_4(2))$. The resulting pairs of its eigenvalues λ with multiplicity m_λ are:

$$(\lambda, m_\lambda): \qquad (-25, 918), \quad (39, 1190), \quad (119, 135), \quad (255, 1).$$

The last three eigenspaces of C correspond to the irreducible representations σ_c, σ_θ and $\mathbf{1}$ respectively and their direct sum is $\mathrm{Sym}^2(\rho_\theta)$, cf. 3.7.4. The character table shows that there are only 10 irreducible representations of Sp(8) with dimension less then 918 and there is a unique irreducible representation with dimension 918. However, of these eleven irreducible representations, the map C has eigenvalue $\lambda = -25$ only on the one of dimension 918. Thus we conclude that $M_4^{\theta}(\Gamma_4(2))$ is the sum of just four irreducible representations (like $M_4^{\theta}(\Gamma_3(2))$, cf. section 4.4.6). In principle, in this way, one could decompose also the representations on the space of modular forms of weight six and eight, as in the case of Sp(6), but the computation of the Casimir C_ρ is very time and memory consuming.

4.5.3 The uniqueness of $\Xi_8[0^{(4)}]$

Using the same methods as in section 4.4.7 for the case $g = 3$ and the observation in section 4.3.2 for the case $g = 2$, we can show that the three constraints characterize the form $\Xi_8[0^{(4)}]$ up to an additive term λJ, where $J = 0$ defines the Jacobi locus J_4 (the locus of period matrices of Riemann surfaces in $\mathbb{H}_4$) and $\lambda \in \mathbb{C}$. In fact, J is a modular form of weight 8 on Γ_4, so it can be added to $\Xi_8[0^{(4)}]$ without changing its $\Gamma_g(1,2)$-invariance. Moreover $\mathbb{H}_1 \times \mathbb{H}_3$ and $\mathbb{H}_2 \times \mathbb{H}_2$ are contained in the closure of the Jacobi locus, so J is zero on these loci.

The key point is the determination of the dimension of $M_8^{\theta}(\Gamma_4(2))^{O^+}$, which could be done by computer. M. Oura determined this dimension using the methods from [R1], [R2]: $\dim M_8^{\theta}(\Gamma_4(2))^{O^+} = 7$. We already know 7 independent functions in $M_8^{\theta}(\Gamma_4(2))^{O^+}$. It follows that $M_8(\Gamma_4(2))^{O^+} = M_8^{\theta}(\Gamma_4(2))^{O^+}$ and that

$$M_8(\Gamma_4(2))^{O^+} \,=\, \langle \sum_\Delta \theta[\Delta]^{16},\ (\sum_\Delta \theta[\Delta]^8)^2,\ F_1,\ F_2,\ F_3,\ G_1[0^{(4)}],\ G_2[0^{(4)}]\,\rangle.$$

The modular form J vanishing on the Jacobi locus is

$$J = 16\sum \theta[\Delta]^{16} - (\sum \theta[\Delta]^8)^2$$

(cf. [I3]). Now the proof of the uniqueness of $\Xi_8[0^{(4)}]$ in $M_8^\theta(\Gamma_g(2))$ can be obtained with arguments similar to those in sections 4.4.7 and 4.3.2.

4.6 The cosmological constant in $g=3$, $g=4$

Supersymmetry impose the same number of fermionic and bosonic states. This leads to the non renormalization theorems [Ma1, Ma2, Mo4] which in particular require for the zero points function, t.i. the cosmological constant, to vanishes identically. We will prove that our solution for the chiral superstring measure gives a vanishing contribution to the cosmological constant for both the case $g=3$ and $g=4$. Independent proofs of this result in $g=4$, which make use of different techniques, can be found in [Gr] or in [SM2]. The GSO projection for type II superstring gives for the phases in (2.15) the value $c_{\Delta,\Delta'}=1$ so that we will prove that $\sum_\Delta \mathrm{d}\mu[\Delta]=0$ or, equivalently, $(\sum_\Delta \Xi_8[\Delta])(\Omega)=0$.

4.6.1 The case $g=3$

The sum of the 36 functions[6] $\Xi_8[\Delta]$ is invariant under Sp(6), hence it is a modular form of weight 8 on $Sp(6,\mathbb{Z})$. In the decomposition of $M_8(\Gamma_3(2))$, see section 4.4.6, the representation **1** has multiplicity one, thus there is a unique, up to a scalar multiple, Sp(6)-invariant on $M_8(\Gamma_3(2))$. This invariant is $\sum_\Delta \theta[\Delta]^{16}$. Hence, the sum of the functions $\Xi_8[\Delta]$ must be a scalar multiple of this invariant:

$$\Big(\sum_\Delta \Xi_8[\Delta]\Big)(\tau) = \mu\Big(\sum_\Delta \theta[\Delta]^{16}\Big)(\tau). \tag{4.46}$$

Note that the function $\sum_\Delta \Xi_8[\Delta]$, obtained from the $\Xi_8[0]$ of section 4.4.5 with the transformation formula of the theta constants, is given by

$$-4\sum_\Delta \theta[\Delta]^{16} - 4\sum_\Delta \theta[\Delta]^4\Big(\sum_{\Delta'} \epsilon_{\Delta,\Delta'}\theta[\Delta']^{12}\Big) + 3\Big(\sum_\Delta \theta[\Delta]^8\Big)^2 + 12\sum_\Delta G[\Delta],$$

where, again, the constants $\epsilon_{\Delta,\Delta'}=\pm 1$ are determined by the transformation theory. We will now show that $\mu=0$ by looking first at diagonal form period matrices $\tau = \mathrm{diag}(\tau_1,\tau_2,\tau_3)$ and then setting $\tau_1,\tau_2,\tau_3 \to i\infty$. On the theta constants this gives

$$\theta\big[\begin{smallmatrix} abc \\ def \end{smallmatrix}\big] \longmapsto \begin{cases} 1 & \text{if } a=b=c=0, \\ 0 & \text{else,} \end{cases}$$

hence $\sum_\Delta \theta[\Delta]^{16} \mapsto 8$ and $\sum_\Delta \theta[\Delta]^8 \mapsto 8$. In the summand $\sum_\Delta \theta[\Delta]^4(\sum_{\Delta'} \epsilon_{\Delta,\Delta'}\theta[\Delta']^{12})$ we thus only need to consider the terms with[7] $\Delta = \big[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\big]$, $\Delta' = \big[\begin{smallmatrix} 0 \\ b' \end{smallmatrix}\big]$. The terms with $\Delta = \big[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\big]$

[6] This is the number of the even characteristics in $g=3$.

[7] Here we mean $0=(0,0,0)$ and $b=(b_1,b_2,b_3)$

are summands of $\Xi_8[{0 \atop b}](\tau)$. Let M be the symplectic matrix

$$M = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \qquad B = \mathrm{diag}(b_1, b_2, b_3), \qquad \text{so} \quad M \cdot [{0 \atop b'}] = [{0 \atop b+b'}].$$

In particular, $M \cdot [{0 \atop 0}] = [{0 \atop b}]$ and thus $\Xi_8[{0 \atop b}](\tau) = \Xi_8[{0 \atop 0}](M^{-1}\tau)$ (note that $\gamma(M, M^{-1} \cdot \tau) = 1$). From the definition of the theta constants as series in 3.3.1 it is obvious that

$$\theta[{0 \atop b'}]^4(M^{-1} \cdot \tau) \;=\; \theta[{0 \atop b+b'}]^4(\tau)$$

hence $\epsilon_{\Delta,\Delta'} = +1$ if $\Delta = [{0 \atop b}]$, $\Delta' = [{0 \atop b'}]$. Thus we get

$$\sum_{\Delta} \theta[\Delta]^4 (\sum_{\Delta'} \epsilon_{\Delta,\Delta'} \theta[\Delta']^{12}) \longmapsto \sum_{b} \theta[{0 \atop b}]^4 (\sum_{b'} \epsilon_{[{0 \atop b}],[{0 \atop b'}]} \theta[{0 \atop b'}]^{12}) \longmapsto 8 \cdot 8 = 64.$$

Finally, each $G[\Delta]$ is a sum of P_L's and each P_L is a product of eight distinct theta constants. Thus all P_L's map to zero except for $P_{L_0} := \prod_{d,e,f} \theta[{000 \atop def}]$ which maps to 1. Note that $L_0 = \{ ({abc \atop 000}) \}$ and that $L_0 \subset Q_\Delta$ iff $\Delta = [{000 \atop def}]$. Thus exactly 8 of the $G[\Delta]$ map to one, and the others map to zero. The constant μ can now be determined

$$-4 \cdot 8 - 4 \cdot 8^2 + 3 \cdot 64 + 12 \cdot 8 = \mu \cdot 8 \qquad \Longrightarrow \qquad \mu = 0,$$

hence the cosmological constant is zero.

4.6.2 The case $g = 4$

In this case we have not the full decomposition in irreducible of the space $M_8(\Gamma_4(2))$. Nevertheless, we are able to determine, using a computer, that the dimension of the space of Sp(8)-invariant in $\mathbb{C}[\cdots, X_\sigma, \cdots]^{H_g}]$ is two. The space $M_8(\Gamma_4(2))$ is huge, so one starts with finding invariants for the transvections which acts 'diagonally' on the X_σ to reduce the computation to a smaller space. These two invariants correspond to the modular forms Ψ_8 and Ψ_4^2, with $\Psi_{4k}(\tau) := \sum_\Delta \theta[\Delta^{(4)}]^{8k}(\tau)$. The combination $J = 16\Psi_8 - \Psi_4^2$ vanishes on the Jacobi locus, i.e. the space of the matrices $\tau \in H_g$ that are, also, a period matrix of some Riemann surfaces. In genus four there are 136 even characteristics, thus we have to consider the sum of the 136 functions $\Xi_8[\Delta]$. As $\sum_\Delta \Xi_8[\Delta^{(4)}]$ is an $Sp(8)$-invariant of weight 8, there are constants λ, μ such that

$$\sum_{\Delta} \Xi_8[\Delta^{(4)}] \;=\; \lambda \Psi_8 \;+\; \mu \Psi_4^2 \tag{4.47}$$

and it suffices to show that $\lambda + 16\mu = 0$. For this one can specialize τ to a diagonal matrix $\tau = \mathrm{diag}(\tau_1, \ldots, \tau_4)$ and then let $\tau_j \mapsto \infty$ for $j = 1, \ldots, 4$, similar to the computation in genus three of the previous section.

Chapter 5

Genus five

In this chapter we construct the measure for the case $g = 5$. The genus five case is more complicate due to the increasing dimensions of the O^+-invariants and to the lack of a complete decomposition of the space $M_8(\Gamma_5(2))$ in irreducible representations. Moreover, due to the non normality of the ring of modular forms not all the modular forms can be expressed as polynomial in the classical theta functions. In addition, some relations, not known, among theta functions could exist. Nevertheless these difficulty, we will construct the forms $\Xi_8[\Delta]$ satisfying the three constraints. To this aim first we rewrite the measure for the lower genus cases using a different basis for the O^+-invariants as in the previous chapters [D] and then we extend the construction to genus five. In this basis the theta functions will appear with greater power than before. In [OPSY] another candidate for the superstring measure was proposed. The authors defined the functions $\Xi_8[\Delta]$ using a different formalism: they exploit the notion of lattice theta functions instead of the classical theta functions. In Section 5.7 we will review their construction. Then we relate the two constructions expanding both the functions Ξ_8 in Fourier series and we prove the substantial equivalence of the two formalism. This result is the content of the Theorem 5.7.1. Actually, the two construction leads to the same measure, provided we add to the three constraints the request of the vanishing of the cosmological constant. Indeed, in genus five the three constraints alone no longer characterize uniquely the measure, as stated in theorem 5.1.1.

5.1 Review of the construction

In Chapter 4 we have constructed a candidate for the three and four loop superstring vacuum-to-vacuum amplitude. As explained, our procedure was inspired by the valuable series of papers of D'Hoker and Phong in which they determined an expression for the two loop superstring measure. In all these cases the measure is expressed in terms of suitable polynomials in the theta constants. The latter are defined on the Siegel upper half space $\mathbb{H}_g$ and not just on the subvariety $J_g \subseteq \mathbb{H}_g$ of period matrices of genus g Riemann surfaces. This makes the superstring measure, for $g \leq 4$, a function over the whole $\mathbb{H}_g$. For the genus three case, this fact is not completely surprising because, for $g \leq 3$, $\mathbb{H}_g$ and J_g have the same dimension and J_g is an open set of $\mathbb{H}_g$. In genus four, instead, this is a remarkable fact, the dimension of the two varieties being no longer the same, see Table 5.1. In [Gr] a candidate was proposed for the superstring measure for

g	$\dim \mathbb{H}_g$	$\dim J_g$	$\operatorname{codim}_{\mathbb{H}_g} J_g$
2	3	3	0
3	6	6	0
4	10	9	1
5	15	12	3
g	$\frac{1}{2}g(g+1)$	$3g-3$	$\frac{1}{2}(g-2)(g-3)$

Table 5.1: Dimensions of the varieties $\mathbb{H}_g$ and J_g.

any genus g. However, these are not a priori well defined for $g \geq 5$ due to the presence of roots. In [SM2], it was proved that for $g = 5$ this measure is well defined, at least on J_g. From these facts it is quite natural to investigate if the measure for $g = 5$ could be extended, using the classical theta constants, over all $\mathbb{H}_5$ and what happens for $g > 5$. Recently in [OPSY] a candidate for the superstring measure for $g = 5$ was proposed employing the notion of the lattice theta series. This formalism is almost equivalent to the one of the classical theta constants. Actually, the spaces spanned by theta series and the ones generated by the bases for the O^+-invariants defined in Sections 5.3.1, 5.3.2 and 5.3.3 are the same [DGdC]. In genus five both formalisms lead to the same solutions, providing we add to the constraints also the request of the vanishing of the cosmological constant, as we will show in Section 5.10.5, see also [DGdC].

In the previous chapters we explained that the proposal for the g-loop superstring measure rests on the ansatz (not yet proved) of D'Hoker and Phong [DP5] that the genus g vacuum to vacuum amplitude takes the form of an integral over the moduli space of genus g Riemann surfaces of a suitable differential form that splits into a holomorphic and anti-holomorphic part. Moreover, the measure $d\mu[\Delta^{(g)}]$ should satisfy certain reasonable constraints. We proved that this characterises it uniquely for $g \leq 4$. In this chapter we prove that, assuming these features for the amplitude, the superstring measure can be defined on the whole $\mathbb{H}_g$ for $g \leq 5$, but for $g = 5$ the correct restriction (see point 3 of theorem 5.1.1) holds true just on J_4. This result is stated by the following:

Theorem 5.1.1. *If the genus g vacuum to vacuum amplitude takes the general form:*

$$\mathcal{A} = \int_{\mathcal{M}_g} (\det \operatorname{Im} \tau)^{-5} \sum_{\Delta, \bar{\Delta}} c_{\Delta,\bar{\Delta}} d\mu[\Delta^{(g)}](\tau) \wedge \overline{d\mu[\bar{\Delta}^{(g)}](\tau)}, \tag{5.1}$$

where the form $d\mu[\Delta^{(g)}]$ can be written as:

$$d\mu[\Delta^{(g)}] = c_g \Xi_8^{(g)}[\Delta^{(g)}](\tau^{(g)}) d\mu_B^{(g)}$$

and the functions $\Xi_8^{(g)}[\Delta^{(g)}]$ *satisfy the following ansätze:*

1. *they are holomorphic functions on* J_g*;*
2. *under the action of* $\Gamma_g = \mathrm{Sp}(2g,\mathbb{Z})$ *they should transform as* $\Xi_8^{(g)}[M\cdot\Delta^{(g)}](M\cdot\tau) = \det(C\tau+D)^8\Xi_8^{(g)}[\Delta^{(g)}](\tau)$*, for all* $M\in\mathrm{Sp}(2g,\mathbb{Z})$*;*
3. *the restriction of these functions to 'reducible' period matrices is a product of the corresponding functions in lower genus;*

then $\Xi_8^{(g)}[\Delta^{(g)}]$*, and so* $d\mu[\Delta^{(g)}]$*, are defined everywhere on* $\mathbb{H}_g$*, they can be expressed in terms of polynomials in square of theta constants and they are unique if* $g\leq 4$*. For* $g=5$*, for every even characteristic* $\Delta^{(5)}$*, at least one (actually many) forms* $\Xi_8^{(5)}[\Delta^{(5)}]$ *exist, it is defined on* $\mathbb{H}_5$*, it can be written as a polynomial in the theta constants and the restriction requested is satisfied just on* J_4*. The three constraints do not characterize it uniquely (at least on* $\mathbb{H}_5$*).*

In the theorem the uniqueness for $g=4$ must be understood as uniqueness up to a multiple of $J^{(4)}$, that vanishes on the Jacobi locus[1] J_4 (see below). In genus five, instead, the uniqueness is completely lost. One can find many forms $\Xi_8^{(5)}[0^{(5)}]$ that differ not just for something vanishing on the Jacobi locus. Actually, starting from a form $\Xi_8^{(5)}[0^{(5)}]$ satisfying the three constraints and adding a multiple of $J^{(5)}$, one obtains again a form satisfying the constraints. In genus five $J^{(5)}$ does not vanish on the Jacobi locus, see [GS]. It is still an open problem whether, adding to the constraints the request of the vanishing of the cosmological constant, the uniqueness of the measure is guaranteed. However, the vanishing of the cosmological constant should be automatic for a supersymmetric theory and not imposed by hand. At the moment it is not known if could exist some function satisfying the three constraints and differing from a $\Xi_8^{(5)}$ not just for a multiple of $J^{(5)}$. Moreover, nothing we can say if we consider also the non normal part of the ring of genus five modular forms.

As usual, $\Delta^{(g)}$ and $\overline{\Delta^{(g)}}$ denote two even genus g theta characteristics, $c_{\Delta,\bar{\Delta}}$ are suitable constant phases depending on the details of the string model, $d\mu[\Delta^{(g)}](\tau)$ $(\overline{d\mu[\Delta^{(g)}](\tau)})$ is a holomorphic (anti-holomorphic) form and $d\mu_B^{(g)}$ is the well defined genus g bosonic measure. However, there is not an explicit form for $d\mu_B^{(g)}$ in higher genus. We observe that the transformation request for the form $\Xi_8^{(g)}[\Delta^{(g)}]$ is automatic for the integral to make sense, but, as usual, we prefer to emphasize this property for its crucial role in what follows.

For the genus two and three cases we have shown the uniqueness of the forms $\Xi_8^{(g)}[\Delta^{(g)}]$ in Chapter 4 (see also [DvG]). Assuming that the measure is a polynomial in the theta constants, we also show the uniqueness (up to a term proportional to $J^{(4)}$) for the genus four case and in [OPSY] the general case is considered (see Section 5.5.2 below). In genus five the uniqueness can not be longer assured. Actually,

[1] Note that we indicate, as in literature, with $J^{(g)}$ the modular form $J^{(g)}=2^gF_{16}^{(g)}-F_8^{(g)}$, see Section 5.3.4, and with J_g the Jacobi locus.

in [OPSY] a candidate for the genus five superstring measure is proposed. The authors make use of the notion of the lattice theta series. First, we will construct the functions $\Xi_8^{(5)}[\Delta^{(5)}]$ using the classical theta functions then we introduce the formalism of the the lattice theta functions and we compare the two solutions to the constraints. An analysis of the different expressions for the chiral superstring measure can also be found in [DbMS, MV2]. In Section 5.10.5 we will prove that the form $\Xi_8^{(5)}[0^{(5)}]$ defined using the lattice theta series and the one defined in Section 5.5.3 are different on $\mathbb{H}_5$ and also on J_5, see [DGdC]. Actually, the difference is proportional to $J^{(5)}$. The supplementary request of the vanishing of the cosmological constant makes equivalent the two forms.

From the result of Salvati Manni [SM2], we know that the square root appearing in Grushevsky expression of the five loop measure (in the function $G_5^{(5)}[0^{(5)}]$, built up with five dimensional isotropic spaces) is well defined on the moduli space of curves J_5. Further investigations are needed to understand if, at least, on the locus of curves, it is polynomial in the classical theta constants.

An indication that the three constraints cannot define the forms $\Xi_8^{(g)}[\Delta^{(g)}]$ defined over the whole $\mathbb{H}_g$ and are sufficient to assure their uniqueness comes from the increasing difference between the dimensions of $\mathbb{H}_g$ and J_g. The dimension of $\mathbb{H}_g$ is quadratic in g, instead the dimension of J_g has a linear growth in g and their difference is quadratic in g, see Table 5.1. Thus, it is not surprising that the constraints for the $\Xi_8^{(g)}[\Delta^{(g)}]$ are not strong enough to characterize it uniquely.

5.2 The strategy

In this section we briefly recall the strategy we used in Chapter 4 to define the form $\Xi_8^{(g)}[\Delta^{(g)}]$. Inspired by the factorisation of the superstring chiral measure at lower genus, a modification of the ansätze of D'Hoker and Phong was proposed for the superstring measure. Accordingly, the measure should be written as $d\mu[\Delta^{(g)}] = c_g \Xi_8^{(g)}[\Delta^{(g)}](\tau^{(g)}) d\mu_B^{(g)}$, where $d\mu_B^{(g)}$ is the bosonic measure at genus g and $\Xi_8^{(g)}[\Delta^{(g)}]$ are suitable functions, g is the genus of the Riemann surfaces considered and $\Delta^{(g)}$ is an even characteristic at genus g. The functions $\Xi_8^{(g)}[\Delta^{(g)}]$ are required to satisfy suitable regularity, transformation and factorisation constraints, as explained in Section 2.3. Here we emphasise that the first constraint requires that the forms $\Xi_8^{(g)}[\Delta^{(g)}]$ are defined on the subvariety $J_g \subset \mathbb{H}_g$ of period matrices of Riemann surfaces of genus g and not on the whole Siegel upper half space. In fact, $\dim \mathbb{H}_g = g(g+1)/2$ and $\dim J_g = 3g-3$ so these two spaces are the same just for $g \leq 3$. Since we are interested in arbitrary genus, we write J_g instead of $\mathbb{H}_g$. Actually, for $g \leq 4$ the superstring measure can be extended to the Siegel upper half space, instead for $g = 5$ the forms constructed using the classical theta constants, although well defined over the whole $\mathbb{H}_5$, have the correct factorization just on the Jacobi locus J_4, see Section 5.5.3.

In the previous chapter, it was pointed out that the $\Xi_8^{(g)}[\Delta^{(g)}]$ are modular forms with respect to the normal subgroup $\Gamma_g(2)$ of $\mathrm{Sp}(2g,\mathbb{Z})$. Furthermore we can restrict our attention on a single function (see, [CDG1] Section 2.7) say $\Xi_8^{(g)}[0^{(g)}]$, where $[0^{(g)}] :=$

$\left[\begin{smallmatrix}0\cdots 0\\0\cdots 0\end{smallmatrix}\right]$, which has to be a modular form of weight 8 on $\Gamma_g(1,2)$, where $\Gamma_g(1,2)$ is the subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ which fixes the characteristic $[0^{(g)}]$. Thus, the $\Xi_8^{(g)}[\Delta^{(g)}]$ are obtained by employing the transitive action of $\mathrm{Sp}(2g,\mathbb{Z})$ on the even characteristics. The action of $M \in \mathrm{Sp}(2g) := \mathrm{Sp}(2g,\mathbb{F}_2) \cong \Gamma_g/\Gamma_g(2)$ on a characteristic $\Delta^{(g)}$ is given by (2.23). As explained in Chapter 3 (cf. also [DvG]), the group Γ_g acts on the 2^{2g} points of $\mathbb{F}_2^{2g}$ and on the characteristics through its quotient $\mathrm{Sp}(2g) \cong \Gamma_g/\Gamma_g(2)$. We defined the subgroup $\Gamma_g(1,2)$ of $\Gamma_g \equiv \Gamma_g(1)$ as the stabiliser of $[0^{(g)}]$ and the image of this subgroup in $\mathrm{Sp}(2g)$ was called $O^+(2g) := \Gamma_g(1,2)/\Gamma_g(2) \subset \mathrm{Sp}(2g)$.

The three requests, which the function $\Xi_8^{(g)}[0^{(g)}]$ should satisfy, imply that it must belong to the subspace of O^+-invariants of weight 8:

$$M_8(\Gamma_g(2))^{O^+} := \{f \in M_8(\Gamma_g(2)) \,:\, \rho(h)f = f \quad \forall h \in O^+(2g)\,\}.$$

Here $M_k(\Gamma_g(2))$ is the finite dimensional complex vector space of the Siegel modular forms of genus g, weight k and level 2 and ρ is the representation of the finite group $\mathrm{Sp}(2g)$ on this space defined by:

$$(\rho(h^{-1})f)(\tau) := \det(C\tau + D)^{-k} f(M\cdot\tau),$$

where $M \in \Gamma_g$ is a representative of $h \in \mathrm{Sp}(2g)$ and $f \in M_k(\Gamma_g(2))$. The action of Γ_g on $\tau \in \mathbb{H}_g$ was defined in Section 3.3. Among the functions in $M_8(\Gamma_g(2))$, we will search for the ones satisfying the three constraints. This is a general procedure, but for $g > 4$ there are some subtleties due to the loss of the uniqueness of the form $\Xi_8^{(g)}[0^{(g)}]$. By means of the 2^g second order theta constant we are able to built the O^+-invariants, as explained in detail in Chapter 4, see also [DvG] The dimension of the space of O^+-invariants can be determined from the decomposition of the $\mathrm{Sp}(2g)$-representation into irreducible representations and using the Frobenius reciprocity. We obtained this decomposition for $g \leq 4$ in the previous chapter. Thus, the dimension is given by the multiplicity of the trivial representation $\mathbf{1}$ of O^+ in the O^+-representation $\mathrm{Res}_{O^+}^{\mathrm{Sp}(2g)}(V)$:

$$\dim V^{O^+} = \langle\, \mathrm{Res}_{O^+}^{Sp}(V), \mathbf{1}\,\rangle_{O^+} = \langle\, V, \mathrm{Ind}_{O^+}^{Sp}(\mathbf{1})\,\rangle_{Sp}.$$

As before, $\mathrm{Res}_{O^+}^{Sp}(V)$ is the restriction of the representation from $Sp(2g)$ to $O^+(2g)$, $\mathrm{Ind}_{O^+}^{Sp}(\mathbf{1})$ is the induced representation of the representation $\mathbf{1}$ of $O^+(2g)$ to the whole $Sp(2g)$ and the second identity is the Frobenius identity, see [DvG,CD2,Sa]. Frame [Fr2] showed that $\mathrm{Ind}_{O^+}^{Sp}(\mathbf{1}) = \mathbf{1} + \sigma_\theta$, where $\mathbf{1}$ is the trivial representation and σ_θ is an irreducible representation of dimension $2^{g-1}(2^g+1)-1$, so that if the multiplicities of $\mathbf{1}$ and σ_θ in V are n_1 and n_{σ_θ} respectively, the dimension of the space of O^+-invariants is $\dim V^{O^+} = n_1 + n_{\sigma_\theta}$.

Like in Chapter 4 we will label the irreducible representations of $\mathrm{Sp}(2g)$ with the partitions of 3 and 6 for genus one and two respectively (recall that $\mathrm{Sp}(2) \cong S_3$ and $\mathrm{Sp}(4) \cong S_6$), as in [CD1]; we will follow Frame's notation [Fr1] for genus three and indicate them just with their dimensions[2] for $g \geq 4$. In Table 5.2 are reported the

[2]If they are not unique at the given size, we will indicate also the character of the second conjugacy class, the one of the non zero transvections (which has 255 and 1023 elements for genus four and five respectively). Transvections are analogous to reflections in orthogonal groups (cf. [J], § 6.9 or [DvG]).

g	σ_θ	$\dim(\sigma_\theta)$
1	$\rho_{[21]}$	2
2	$\rho_{[42]}$	9
3	$\mathbf{35}_b$	35
4	$\mathbf{135}$	135
5	$\mathbf{527}$	527

Table 5.2: The σ_θ representations for the low genus cases and their dimensions.

σ_θ representations for the lower genus cases. In case $g = 1$, $\rho[21]$ is the unique two dimensional representation of $S_3 \cong \mathrm{Sp}(2)$ and [21] is the partition of 3 labelling it. For $g = 2$, $\rho[42]$, or n_9 in the notations of [CD1], is the nine dimensional representation of $S_6 \cong \mathrm{Sp}(4)$ for which the character of $\mathbf{1} + \sigma_\theta$ is positive ([42] is the partition of six labelling this irreducible representation; see [DvG] Section 4.2 and [CD2] Section 5.2.1 for the explanation of why the character of the representation must be positive). For $g = 3$, the $\mathbf{35}_b$ is the unique 35 dimensional representation of Sp(6), as reported in [Fr1] or as can be computed using, for example, the software Magma. For $g = 4$, $\mathbf{135}$ is the unique 135 dimensional irreducible representation of Sp(8) and for $g = 5$, $\mathbf{527}$ is the unique 527 dimensional irreducible representation of Sp(10), as can be computed using Magma.

In the previous chapter (see also [DvG]), the representations of $\mathrm{Sp}(2g)$ on the vector space $M_k(\Gamma_g(2))$ were studied for small genus g, and decomposed into irreducible representations. For the applications in string theory, we are interested in the representations of $\mathrm{Sp}(2g)$ on the space $M_8(\Gamma_g(2))$, the modular forms of weight eight with respect to the group $\Gamma_g(2)$. Let us recall the decomposition of these representations for $g \leq 3$:

$$\begin{aligned}
M_8(\Gamma_1(2)) &\cong \mathrm{Sym}^4(\rho_{[21]}) = \mathbf{1} + 2\rho_{[21]},\\
M_8(\Gamma_2(2)) &\cong \mathrm{Sym}^4(\rho_{[2^3]}) - \mathbf{1} = \mathbf{1} + 3\rho_{[2^3]} + 3\rho_{[42]} + \rho_{[31^3]} + \rho_{[321]},\\
M_8(\Gamma_3(2)) &= \mathbf{1} + 4\cdot\mathbf{15}_a + \mathbf{35}_a + 4\cdot\mathbf{35}_b + 5\cdot\mathbf{84}_a + 2\cdot\mathbf{105}_c + \mathbf{168}_a +\\
&\qquad 2\cdot\mathbf{189}_c + 3\cdot\mathbf{216} + 3\cdot\mathbf{280}_b + 2\cdot\mathbf{336}_a + \mathbf{420}_a.
\end{aligned}$$

We have explained as from the Frobenius identity it follows that for $g = 1$ the dimension of the space of O^+-invariants is three, for $g = 2$ is four and for $g = 3$ is five. For genus four and five we do not know the decomposition of the whole $M_8(\Gamma_g(2))$ and, moreover, the ring of modular forms is not understood in terms of Heisenberg invariant polynomials in theta constants. However, in Sections 5.5.2 and 5.5.3, we will restrict our attention to the space $M_8^\theta(\Gamma_g(2))$, searching the O^+-invariants there.

5.3 The construction of the O^+-invariants

Once the dimension of $M_8(\Gamma_g(2))^{O^+}$ is known, the main problem is to find an explicit expression for a basis of this space in terms of theta constants, if possible. We employed the notion of isotropic subspaces to find such bases for the genus three and four cases. Recall that, if V is provided with a symplectic form, $W \subset V$ is an isotropic subspace if on every pair of vectors in W the symplectic form vanishes. This way to determine the invariants makes use of the geometry underlying the theta characteristics and the corresponding action of the symplectic group on them. For example, the condition for a subspace to be isotropic is preserved under the action of $\mathrm{Sp}(2g)$. Moreover, it is quite simple to determine the restriction on a block diagonal period matrix of the O^+-invariant built in this way, despite to the huge number of terms appearing in these functions, cf. the discussion in Appendix C of [CDG1]. The knowledge of a basis for these spaces allows to find, for $g \leq 5$, a linear combination of the O^+-invariants such that its restrictions fits all requests in the ansätze discussed in Section 2.3. For $g = 1, 2, 3$ the fact that any modular form of weight $2k$ can be expressed as a polynomial of degree $4k$ in theta constants in a unique way (unique up to a multiple of $J^{(3)}$ if $g = 3$ and $k > 4$) allows us to prove the uniqueness for the expression of the superstring measure. In genus four the ring of Siegel modular forms is not normal. This means that in general there could be some modular forms that cannot be expressed as polynomials in theta constants. In this case the uniqueness was proved in a weakened form, assuming the polynomiality for the amplitude, i.e. considering O^+-invariants contained in the space $M_8^\theta(\Gamma_4(2))$ only. In [OPSY] the proof is also extended to the general case. As anticipated in Section 5.1, in genus five the three constraints are not strong enough to assure the uniqueness of the superstring measure neither if we restrict to the normal part of the ring of modular forms as we will prove in Section 5.5.3 and in [DGdC]. Thus, in genus five the loss of the uniqueness is not due just to the non normality of the ring of modular forms.

In [Gr] a generalisation for the expression of the chiral measure at any genus g was proposed. In the approach used there, the action of the symplectic group underlying that expression is not manifest although the correct factorisation is obtained. The author restricts the search for the g loop amplitudes to a suitable vector space of dimension $g + 1$ then finding there a unique solution of the constraints. However, for genus three (four and more) the vector space defined by the transformation constraint has dimension five (≥ 7), see §7.4 and 7.5, which is larger than the dimensions of the starting spaces selected in [Gr]. Moreover, his expression might be not well-defined for $g > 5$ (Salvati Manni in [SM2] discusses the case $g = 5$) due to the presence of some roots.

We will now provide new expressions for $\Xi_8^{(g)}[\Delta^{(g)}]$ at lower genus. In these new formulas, the theta constants appear at higher power than in the expressions given in the previous chapter.

5.3.1 O^+-invariants for the genus 3 case

For genus $g = 3$ the only Sp(6)-representations that have an O^+-invariant are $\mathbf{1}$ and $\sigma_\theta = \mathbf{35}_b$ and we know that there are five such linearly independent invariants, see [DvG]. The representation $\mathbf{1}$ provides the Sp(6)-invariant and $\mathbf{35}_b$ the representation on the $\theta[\Delta^{(3)}]^8$. A natural question is about the number of linearly independent O^+-invariants of degree 16 that can be written as quadratic polynomials in $\theta[\Delta^{(3)}]^8$. From the decomposition of the tensor products in irreducible representations we find that:

$$\begin{aligned}\mathrm{Sym}^2(\mathbf{1}+\mathbf{35}_b) &= \mathbf{1}+\mathbf{35}_b+\mathrm{Sym}^2(\mathbf{35}_b)\\ &= \mathbf{1}+\mathbf{35}_b+\mathbf{1}+\mathbf{27}_a+2\cdot\mathbf{35}_b+\mathbf{84}_a+\mathbf{168}_a+\mathbf{280}_b\\ &= 2\cdot\mathbf{1}+\mathbf{27}_a+3\cdot\mathbf{35}_b+\mathbf{84}_a+\mathbf{168}_a+\mathbf{280}_b,\end{aligned}$$

so that we get the two Sp(6)-invariants $\sum_\Delta \theta[\Delta^{(3)}]^{16}$ and $(\sum_\Delta \theta[\Delta^{(3)}]^8)^2$ and three O^+-invariants (but not Sp(6)), two of which are $\theta[0^{(3)}]^{16}$ and $\theta[0^{(3)}]^8\sum_\Delta \theta[\Delta^{(3)}]^8$. In order to find the third invariant quadratic in $\theta[\Delta^{(3)}]^8$ we can adopt a general method that allows us to generate many O^+-invariants (clearly not all independent). This consists in starting from a certain monomial of degree sixteen which contains the theta constants to the power at least four, and imposing some suitable condition on the corresponding characteristics. For example, in the spirit of [DP5] and [DP6], we can take $\theta[\Delta_1^{(3)}]^4\theta[\Delta_2^{(3)}]^4\theta[\Delta_3^{(3)}]^4\theta[\Delta_4^{(3)}]^4$ with the conditions $\Delta_1+\Delta_2+\Delta_3+\Delta_4 = 0^{(3)}$. There are 1611 such monomials which summed up give an O^+-invariant. In fact, this is redundant because there are "sub-polynomials" that are orbits for O^+ and then are themselves invariant. In [DvG] Section 8.5, the generators of $O^+ \cong S_8$ are given in terms of transvections acting on the theta constants. Thus, an orbit can be determined acting on a single monomial with these transvections until the number of terms of the generated polynomial stops to grow. Next, one considers a second monomial (not in the orbit of the first) and repeats the procedure. In this way we can recognise eight O^+-invariants inside the big polynomial, as shown in Table 5.3. It is clear that the searched invariant could be the sixth or the seventh. Using a computer or (quite lengthy!) by hand and the classical theta formula (cf. [CDG1], Section 3.2), we verify that each of them is linearly independent from the other invariants. We then choose the sixth, which we will call $F_{88}^{(3)}$ (and $F_{88}^{(g)}$ for arbitrary genus g). Thus, each monomial in $F_{88}^{(3)}$ is the product of two theta constants at the eighth power, with the conditions that the sum of their characteristics is odd (even, if we choose the seventh), the two characteristics are not equal and both are not zero. Note the following equality between the O^+-invariants:

$$\begin{aligned}\left(\sum_\Delta \theta[\Delta^{(3)}]^8\right)^2 - \sum_\Delta \theta[\Delta^{(3)}]^{16} &= 2\sum_{(\Delta_i,\Delta_j)_e} \theta[\Delta_i^{(3)}]^8\theta[\Delta_j^{(3)}]^8 + 2\sum_{(\Delta_i,\Delta_j)_o} \theta[\Delta_i^{(3)}]^8\theta[\Delta_j^{(3)}]^8\\ &\quad + 2(\theta[0^{(g)}]^8\sum_\Delta \theta[\Delta^{(3)}]^8 - \theta[0^{(3)}]^{16}),\end{aligned}$$

where the first two functions on the r.h.s. are the O^+-invariants of lines six and seven of the Table 5.3 and the "e" and "o" stand for even sum and odd sum of the two

orbit	general expression	condition	num. elem.
1	$\theta[0]^{16}$		1
2	$\theta[\Delta]^{16}$	$\Delta \neq 0$	35
3	$\theta[0]^8 \sum_{\Delta} \theta[\Delta]^8$	$\Delta \neq 0$	35
4	$\theta[0]^4 \sum_{\Delta_i,\Delta_j,\Delta_k} \theta[\Delta_i]^4\theta[\Delta_j]^4\theta[\Delta_k]^4$	$\Delta_i + \Delta_j + \Delta_k = 0$ $\Delta_i, \Delta_j, \Delta_k \neq 0$ $\Delta_i \neq \Delta_j \neq \Delta_k$	105
5	$\sum_{\Delta_i,\Delta_j,\Delta_k,\Delta_l} \theta[\Delta_i]^4\theta[\Delta_j]^4\theta[\Delta_k]^4\theta[\Delta_l]^4$	$\Delta_i + \Delta_j + \Delta_k + \Delta_l = 0$ $\Delta_i + \Delta_j + \Delta_k$ even $\Delta_i + \Delta_j$ even $\Delta_i, \Delta_j, \Delta_k, \Delta_l \neq 0$ $\Delta_i \neq \Delta_j \neq \Delta_k \neq \Delta_l$	210
6	$\sum_{\Delta_i,\Delta_j} \theta[\Delta_i]^8\theta[\Delta_j]^8$	$\Delta_i + \Delta_j$ odd $\Delta_i, \Delta_j \neq 0$ $\Delta_i \neq \Delta_j$	280
7	$\sum_{\Delta_i,\Delta_j} \theta[\Delta_i]^8\theta[\Delta_j]^8$	$\Delta_i + \Delta_j$ even $\Delta_i, \Delta_j \neq 0$ $\Delta_i \neq \Delta_j$	315
8	$\sum_{\Delta_i,\Delta_j,\Delta_k,\Delta_l} \theta[\Delta_i]^4\theta[\Delta_j]^4\theta[\Delta_k]^4\theta[\Delta_l]^4$	$\Delta_i + \Delta_j + \Delta_k + \Delta_l = 0$ $\Delta_i + \Delta_j + \Delta_k$ even $\Delta_i + \Delta_j$ even $\Delta_k + \Delta_l$ odd $\Delta_i, \Delta_j, \Delta_k, \Delta_l \neq 0$ $\Delta_i \neq \Delta_j \neq \Delta_k \neq \Delta_l$	630

Table 5.3: Orbits under the action of O^+ (genus three case).

characteristics respectively. The two functions on the l.h.s. are the two Sp(6)-invariants, $F_8^{(3)}$ and $F_{16}^{(3)}$ as we will call them in the following. In fact, for genus three the two Sp(6)-invariants are not linearly independent but there is a relation between them (the $J^{(3)}$, see below, or F_{16} in the notation of [DvG]). Therefore, to find a basis we need to look for another invariant which cannot be expressed as a quadratic polynomial in $\theta[\Delta^{(3)}]^8$. We can take $\theta[0^{(3)}]^4 \sum_\Delta \theta[\Delta^{(3)}]^{12}$.

In Section 5.5.1 we will show how to build the chiral measure from these functions.

5.3.2 O^+-invariants for the genus 4 case.

For genus $g = 4$, the only Sp(8)-representations containing an O^+-invariant are **1** and **135**. Now, it is not known if $M^\theta_{2k}(\Gamma_4(2))$, the space of modular forms of weight $2k$ which are (Heisenberg-invariants) polynomial in $\Theta[\sigma]$'s, coincides with $M_{2k}(\Gamma_4(2))$. Recently, Oura determined the dimension of $M_{2k}^{\theta}(\Gamma_g(2))^{O^+}$ obtaining 7 for the $g=4$ case. In principle these dimensions could also be computed using a method similar to those we used for $g<4$, i.e. searching for the decomposition of $M_8^{\theta}(\Gamma_g(2))$ in irreducible representations, but it is very time and memory consuming for increasing g. As for the genus three case, we want to find a basis for the O^+-invariants in which the theta constants appear with the highest possible degree. Let us start by determining the decomposition of the symmetric product $\mathrm{Sym}^2(\mathbf{1}+\mathbf{135})$ in irreducible representations. This can be done using Magma or, by hand, with the character table of Sp(8) and the character inner product (see [CD2], Section 5.2.1, for the case $g=2$). We obtain the decomposition:

$$\begin{aligned}\mathrm{Sym}^2(\mathbf{1}+\mathbf{135}) &= \mathbf{1}+\mathbf{135}+\mathrm{Sym}^2(\mathbf{135})\\ &= 2\cdot\mathbf{1}+\mathbf{119}+3\cdot\mathbf{135}+\mathbf{1190}+\mathbf{3400}+\mathbf{4200}.\end{aligned}$$

This means that we can find five O^+-invariants that are quadratic polynomials in the theta constants at the eighth power. Two of them are the Sp(8)-invariants $\sum_\Delta \theta[\Delta^{(4)}]^{16}$ and $(\sum_\Delta \theta[\Delta^{(4)}]^8)^2$ which are now linearly independent because the Schottky relation $J^{(4)}$, the analogous of $J^{(3)}$ for genus four (see Section 5.3.4 for the definition), vanishes just on J_4 and not identically on the whole $\mathbb{H}_4$. The remaining three invariants are $\theta[0^{(4)}]^{16}$, $\theta[0^{(4)}]^8\sum_\Delta \theta[\Delta^{(4)}]^8$ and the generalisation of the O^+-invariant found in Section 5.3.1 to the genus four case, $F_{88}^{(4)}$ (the construction of such a function for $g\geq 4$ is straightforward).

We now check how many O^+-invariants can be written as polynomials of degree four in the $\theta[\Delta^{(4)}]^4$. This can be done decomposing the symmetric product $\mathrm{Sym}^4(\rho_\theta)$ in irreducible representations[3], and counting the multiplicity of the representations $\mathbf{1}$ and σ_θ ($\sigma_\theta=\mathbf{135}$ in this case). In [vG2] it was shown that the representation of $\mathrm{Sp}(2g)$ on the subspace $M_2^{\theta}(\Gamma_g(2))\subset M_2(\Gamma_g(2))$, that is spanned by the $\theta[\Delta^{(g)}]^4$, is isomorphic to the representation ρ_θ found by Frame [Fr2] that supports O^+-anti-invariants. This representation has dimension $\dim\rho_\theta=(2^g+1)(2^{g-1}+1)/3$, so for $g=4$ one finds $\rho_\theta=\mathbf{51}$; see [DvG] for details. Thus, a function belonging to $\mathrm{Sym}^{2n}(\rho_\theta)$ is an O^+-invariant of degree $2n$ in $\theta[\Delta^{(g)}]^4$, $n\in\mathbb{N}$. We have:

$$\begin{aligned}\mathrm{Sym}^4(\mathbf{51}) = {}& 2\cdot\mathbf{1}+\mathbf{51}+\mathbf{119}+4\cdot\mathbf{135}+\mathbf{510}+2\cdot\mathbf{918}+5\cdot\mathbf{1190}+\mathbf{1275}\\ &+\mathbf{2856}_{-504}+2\cdot\mathbf{3400}+3\cdot\mathbf{4200}+\mathbf{5712}+\mathbf{5950}_{-210}+\mathbf{7140}\\ &+\mathbf{8160}+\mathbf{11900}_{700}+3\cdot\mathbf{13600}+\mathbf{18360}+2\cdot\mathbf{19040}+\mathbf{23800}_{-1960}\\ &+\mathbf{32130}_{2898}+\mathbf{34560}+\mathbf{57120},\end{aligned}$$

so we get six O^+-invariants, the five found before and $\theta[0^{(4)}]^4\sum_\Delta\theta[\Delta^{(4)}]^{12}$. The seventh invariant cannot be written in this way, but we can search for it as a polynomial in the $\theta[\Delta^{(2)}]^2$. Due to the signs appearing on the transformation formula of the theta constants, there is not a representation of $\mathrm{Sp}(2g,\mathbb{F}_2)$ on the space generated by the

[3]Here ρ_θ is the representation of $\mathrm{Sp}(2g)$ on the $\theta[\Delta^{(g)}]^4$ ($2g=8$ in this case).

$\theta[\Delta^{(g)}]^2$. So we can not determine the number of the O^+-invariants that are polynomial in $\theta[\Delta^{(g)}]^2$ using representation theory of finite group (i.e. we cannot repeat the previous method using something like $\mathrm{Sym}^8(\cdots)$). However, we already know at least one O^+-invariant linearly independent from the others that can be written as a polynomial in $\theta[\Delta^{(4)}]^2$: the invariant $G_1[0^{(4)}]$ defined in Section 4.5.1 (see also [CDG2] and [Gr]), which, for later convenience, will be renamed[4] $G_3^{(4)}[0^{(4)}]$. Using a computer, we have verified that it is linearly independent from the other six. The same conclusion can be achieved using the approach of [DGdC].

Having found seven linearly independent O^+-invariants, according to Oura's result, we have a basis for $M_8^{\theta}(\Gamma_4(2))^{O^+}$ and in Section 5.10.4 we will search for a linear combination of them to build the function $\Xi_8^{(4)}[0^{(4)}]$ which restricts correctly.

5.3.3 O^+-invariants for the genus 5 case

For genus five, the ring of modular forms, as for $g = 4$, is not normal. Moreover, there may exist many relations satisfied by the second order theta constants, but in any case they are not known. Finally, the Schottky relation does not vanish on the Jacobi locus (although this was conjectured by Belavin and Knizhnik [BK2], Conjecture 3, by Morozov and Perelemov [BKMP, Mo3] and by D'Hoker and Phong in [DP6], Section 4.1 and it was shown that it vanishes for any genus on the hyperelliptic locus by Poor [P]): a very recent result [GS], Corollary 18, shows that the zero locus of this form is the locus of trigonal curves.

Despite these difficulties, starting from the seven functions and mimicking the $g = 4$ invariants, we can try to add a further linearly independent O^+-invariant polynomial and look for a linear combination (possibly unique) which factorises in the right way. Indeed, for genus five it is known that

$$\dim M_8^{\theta^2}(\Gamma_5(2))^{O^+} \leq \dim M_8^{\theta}(\Gamma_5(2))^{O^+} \leq \dim M_8^{\theta_S}(\Gamma_5(2))^{O^+} = 8, \tag{5.2}$$

where the first term is the space of modular forms with respect to the group $\Gamma_5(1,2)$ of weight eight which are polynomial in $\theta[\Delta^{(5)}]^2$, the second is the space of modular forms polynomial in $\theta[\Delta^{(5)}]$ (w.r.t. the same group and of same weight as before), and the third is the space of the theta series associated to quadratic forms, see [AZ]. Note that it is not clear that $M_8^{\theta}(\Gamma_5(2))^{O^+}$ is a subset of $M_8^{\theta_S}(\Gamma_5(2))^{O^+}$, in fact in paper [DGdC] we have shown that these two spaces are the same.

In order to construct a basis for $M_8^{\theta}(\Gamma_5(2))^{O^+}$ we generalise the form $G_3^{(4)}[0^{(4)}]$ used before and the inequalities (5.2) show that we can find at most one O^+-invariant polynomial in $\theta[\Delta^{(5)}]$. To this aim we consider the form $G_4^{(4)}[0^{(4)}]$ introduced in Section 4.5.1 (see [CDG2] and [Gr]) and define it also for the $g = 5$ case. This goes straightforward with the notion of isotropic subspaces and in principle we can define similar forms for arbitrary genus g using isotropic subspaces of dimension at most g (see e.g. [Gr]).

[4] We made a change of notation with respect our previous works: all the forms built using the isotropic space will be indicated by $G_d^{(g)}[0^{(g)}]$, where d is the dimension of the isotropic subspace and g the genus we are considering. For example the form $H[0^{(3)}]$ of [CDG2] becomes $G_2^{(3)}[0^{(3)}]$ in the new notation.

In the next two sections we give the genus five extension of the definition of the two functions $G_3^{(4)}[\Delta^{(4)}]$ and $G_4^{(4)}[0^{(4)}]$ defined before for $g = 4$.

The form $G_3^{(5)}[0^{(5)}]$

We will follow the definitions of [CDG2]. Let $W \subset \mathbb{F}_2^{10}$ be a three dimensional isotropic subspace. Given such a W, there are $10 \cdot 8 = 80$ even quadrics Q_Δ such that $W \subset Q_\Delta$. Let $Q_0 \subset \mathbb{F}_2^{10}$ be the even quadric with characteristic $\Delta_0^{(5)} = [0^{(5)}]$. We will only use the octets of quadrics which contain Q_0 to define the modular form $G_3^{(5)}[0^{(5)}]$, or $P_{3,2}^{(5)}$ in the notations of [Gr]:

$$G_3^{(5)}[0^{(5)}] := \sum_{W \subset Q_0} \prod_{w \in W} \theta[\Delta_0^{(5)} + w]^2,$$

where we sum over the 118575 three dimensional isotropic subspaces $W \subset Q_0$, and for each such subspace we take the product of the eight $\theta[\Delta_0^{(5)} + w]^2$.

The form $G_4^{(5)}[0^{(5)}]$

Let $W \subset \mathbb{F}_2^{10}$ be a four dimensional isotropic subspace. Given such a W, there are 48 even quadrics Q_Δ such that $W \subset Q_\Delta$. Let $Q_0 \subset \mathbb{F}_2^{10}$ be the even quadric with characteristic $\Delta_0^{(5)} = [0^{(5)}]$. We will only use the sets of quadrics which contain Q_0 to define the modular form $G_4^{(5)}[0^{(5)}]$, or $P_{4,1}^{(5)}$ as in [Gr]:

$$G_4^{(5)}[0^{(5)}] := \sum_{W \subset Q_0} \prod_{w \in W} \theta[\Delta_0^{(5)} + w],$$

where we sum over the 71145 four dimensional isotropic subspaces $W \subset Q_0$, and for each such subspace we take the product of the sixteen $\theta[\Delta_0^{(5)} + w]$.

Remark

The eight functions $F_1^{(5)}$, $F_2^{(5)}$, $F_3^{(5)}$, $F_8^{(5)}$, $F_{18}^{(5)}$, $F_{16}^{(5)}$, $G_3^{(5)}[0^{(5)}]$, $G_4^{(5)}[0^{(5)}]$ are linearly independent, as can be checked by a computer or by the restriction we will deduce in Section 5.5.3 or using the technique of [DGdC]. Thus, it follows that in (5.2) an equality must hold between the second and the third term:

$$\dim M_8^{\theta}(\Gamma_5(2))^{O^+} = \dim M_8^{\theta_S}(\Gamma_5(2))^{O^+} = 8$$

From the computations at genus four, given in Section 5.10.4, and from a result of Nebe [N], it also follows that at genus five:

$$\dim M_8^{\theta^2}(\Gamma_5(2))^{O^+} = 7. \tag{5.3}$$

Indeed, we obtain seven linear independent functions in the space $M_8^{\theta^2}(\Gamma_5(2))^{O^+}$ so its dimension is greater or equal than seven. In [N] it was determined seven as an upper limit for the dimension of the space of theta square series associated to quadratic forms. As this space contains $M_8^{\theta^2}(\Gamma_5(2))^{O^+}$, it follows the equality (5.3). This result can also be checked using the method of [DGdC]. This fixes all the dimensions of the spaces appearing in the previous inequality (5.2).

5.3.4 Genus g expressions for O^+-invariants

In this Section we recall the six O^+-invariants belonging to $\mathrm{Sym}^4\,\rho_\theta$, found for the lower genus, and we generalise them for arbitrary g. The functions $F_8^{(g)}$ and $F_{16}^{(g)}$ are the two $\mathrm{Sp}(2g)$-invariants, $F_{88}^{(g)}$ is the generalised O^+-invariant introduced in Section 5.3.1 and the modular form $J^{(g)} := 2^g F_{16}^{(g)} - F_8^{(g)}$ vanishes identically for genus three, as explained in [CDG1]. These are:

$$
\begin{aligned}
F_1^{(g)} &:= \theta[0^{(g)}]^{16},\\
F_2^{(g)} &:= \theta[0^{(g)}]^4 \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{12},\\
F_3^{(g)} &:= \theta[0^{(g)}]^8 \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{8},\\
F_8^{(g)} &:= (\sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{8})^2,\\
F_{88}^{(g)} &:= \sum_{(\Delta_i^{(g)},\Delta_j^{(g)})_o} \theta[\Delta_i^{(g)}]^8\theta[\Delta_j^{(g)}]^8,\\
F_{16}^{(g)} &:= \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{16},\\
J^{(g)} &:= 2^g \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{16} - (\sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{8})^2 = 2^g F_{16}^{(g)} - F_8^{(g)},
\end{aligned}
$$

where $(\Delta_i^{(g)},\Delta_j^{(g)})_o$ stands for all the pairs of distinct even characteristics such that their sum is odd. Behind these, we also introduced the forms $G_3^{(g)}[0^{(g)}]$ for $g = 4, 5$ and $G_4^{(g)}[0^{(g)}]$ for $g = 5$. However $G_3^{(g)}[0^{(g)}]$ ($G_4^{(g)}[0^{(g)}]$) could be defined for every genus[5] $g \geq 3$ ($g \geq 4$) considering three (four) dimensional isotropic subspace of $\mathbb{F}_2^{2g}$. In the same way, we can consider two dimensional isotropic subspaces of $\mathbb{F}_2^{2g}$, for $g \geq 2$ and introduce another O^+-invariant, $G_2^{(g)}[0^{(g)}]$ (clearly not linear independent from the others), that is a sum of suitable products of four theta constants at the fourth power. This form will appear in the factorisation of some O^+-invariants.

5.4 Factorization of the O^+-invariants

5.4.1 Genus one formulae

For the construction of the forms $\Xi_8^{(g)}[0^{(g)}]$ defining the chiral measure and to check that they have the correct restriction on $\mathbb{H}_1 \times \mathbb{H}_{g-1}$, it will be useful to recall some identities between theta constants at genus one. Again we will use the Dedekind function η for which the classical formula $\eta^3 = \theta[^0_0]\theta[^0_1]\theta[^1_0]$ holds[6], so $3\eta^{12} = \theta[^0_0]^{12} - \theta[^0_1]^{12} - \theta[^1_0]^{12}$. Also

[5] In [CDG1], where it was considered the case $g = 3$, this form is called $G[0]$.

[6] Note that our definition of the Dedekind function differs from the classical ones, cf. [RL], for a factor $\frac{1}{2}$. This explains the difference for a global factor 2^{4g} between our definition of the forms $\Xi_8^{(g)}[0^{(g)}]$ and the ones in [OPSY].

we recall the definition of the function $f_{21} = 2\theta[{}^0_0]^{12} + \theta[{}^0_1]^{12} + \theta[{}^1_0]^{12}$. In Section 5.10.1 we found that the two functions η^{12} and f_{21} are a basis for the genus one O^+-anti-invariants and so we can expand the anti-invariants on this basis and the O^+-invariants on the basis $\theta[{}^0_0]^4\eta^{21}$, $\theta[{}^0_0]^4 f_{21}$ and $F^{(1)}_{16} = \theta[{}^0_0]^{16} + \theta[{}^1_0]^{16} + \theta[{}^0_1]^{16}$. Another useful basis for the genus one O^+-invariants is given by $F^{(1)}_1$, $F^{(1)}_2$ and $F^{(1)}_{16}$. The proof of these identities is straightforward using the Jacobi identity $\theta[{}^0_0]^4 = \theta[{}^0_1]^4 + \theta[{}^1_0]^4$:

$$
\begin{aligned}
&F^{(1)}_1 = \theta^{16}[{}^0_0] = \theta[{}^0_0]^4\left(\frac{1}{3}f_{21} + \eta^{12}\right),\\
&F^{(1)}_3 = \theta[{}^0_0]^8(\theta[{}^0_0]^8 + \theta[{}^0_1]^8 + \theta[{}^1_0]^8) = \theta[{}^0_0]^4\frac{2}{3}f_{21} = \frac{2}{3}F^{(1)}_1 + \frac{2}{3}F^{(2)}_2,\\
&F^{(1)}_2 = \theta[{}^0_0]^4\left(\theta[{}^0_0]^{12} + \theta[{}^0_1]^{12} + \theta[{}^1_0]^{12}\right) = \theta[{}^0_0]^4\left(\frac{2}{3}f_{21} - \eta^{12}\right),\\
&\theta[{}^0_0]^8\theta[{}^0_1]^8 + \theta[{}^0_0]^8\theta[{}^1_0]^8 = \theta[{}^0_0]^4\left(\frac{1}{3}f_{21} - \eta^{12}\right) = -\frac{1}{3}F^{(1)}_1 + \frac{2}{3}F^{(1)}_2,\\
&\theta[{}^0_0]^{16} + 2\theta[{}^0_0]^8\theta[{}^0_1]^8 + 2\theta[{}^0_0]^8\theta[{}^1_0]^8 = \theta[{}^0_0]^4\left(\theta[{}^0_0]^4(f_{21} - \eta^{12})\right) = \frac{1}{3}F^{(1)}_1 + \frac{4}{3}F^{(1)}_2,\\
&\frac{1}{2}\theta[{}^0_0]^8\theta[{}^0_1]^8 + \frac{1}{2}\theta[{}^0_0]^8\theta[{}^1_0]^8 - \theta[{}^0_0]^4\theta[{}^0_1]^{12} - \theta[{}^0_0]^4\theta[{}^1_0]^{12} = \theta[{}^0_0]^4\left(\frac{3}{2}\eta^{12} - \frac{1}{6}f_{21}\right) = \frac{5}{6}F^{(1)}_1 - \frac{2}{3}F^{(1)}_2,\\
&\theta[{}^0_0]^{16} + 3\theta[{}^0_0]^8\theta[{}^0_1]^8 + 3\theta[{}^0_0]^8\theta[{}^1_0]^8 - 2\theta[{}^0_0]^4\theta[{}^0_1]^{12} - 2\theta[{}^0_0]^4\theta[{}^1_0]^{12} = \theta[{}^0_0]^4\left(\frac{2}{3}f_{21} + 2\eta^{12}\right) = 2F^{(1)}_1,\\
&\theta[{}^0_1]^8\theta[{}^1_0]^8 = \frac{1}{2}\theta[{}^0_1]^{16} + \frac{1}{2}\theta[{}^1_0]^{16} + \frac{1}{2}\theta[{}^0_0]^8\theta[{}^0_1]^8 + \frac{1}{2}\theta[{}^0_0]^8\theta[{}^1_0]^8 - \theta[{}^0_0]^4\theta[{}^0_1]^{12} - \theta[{}^0_0]^4\theta[{}^1_0]^{12}\\
&\qquad = \frac{1}{2}F^{(1)}_{16} + \theta[{}^0_0]^4(\eta^{12} - \frac{1}{3}f_{21}) = \frac{1}{3}F^{(1)}_1 - \frac{2}{3}F^{(1)}_2 + \frac{1}{2}F^{(1)}_{16}.
\end{aligned}
$$

5.4.2 The restrictions on $\mathbb{H}_1 \times \mathbb{H}_{g-1}$

Let us report here the factorisation of the six O^+-invariants found before for a reducible period matrix:

$$\begin{aligned}
F^{(g)}_{1\,|\Delta_{1,g-1}} =&\theta[^0_0]^4(\tfrac{1}{3}f_{21}+\eta^{12})F^{(g-1)}_1 = F^{(1)}_1F^{(g-1)}_1,\\
F^{(g)}_{2\,|\Delta_{1,g-1}} =&\theta[^0_0]^4(\tfrac{2}{3}f_{21}-\eta^{12})F^{(g-1)}_2 = F^{(1)}_2F^{(g-1)}_2,\\
F^{(g)}_{3\,|\Delta_{1,g-1}} =&\tfrac{2}{3}\theta[^0_0]^4f_{21}F^{(g-1)}_3 = F^{(1)}_3F^{(g-1)}_3,\\
F^{(g)}_{8\,|\Delta_{1,g-1}} =&(\theta[^0_0]^{16}+\theta[^0_1]^{16}+\theta[^1_0]^{16}+2\theta[^0_0]^8\theta[^0_1]^8+2\theta[^0_0]^8\theta[^1_0]^8+2\theta[^0_1]^8\theta[^1_0]^8)F^{(g-1)}_8\\
=&2F^{(1)}_{16}F^{(g-1)}_8,\\
F^{(g)}_{88\,|\Delta_{1,g-1}} =&(\theta[^0_0]^{16}+\theta[^0_1]^{16}+\theta[^1_0]^{16}+2\theta[^0_0]^8\theta[^0_1]^8+2\theta[^0_0]^8\theta[^1_0]^8-2\theta[^0_1]^8\theta[^1_0]^8)F^{(g-1)}_{88}\\
&+\theta[^0_1]^8\theta[^1_0]^8F^{(g-1)}_8\\
=&\theta[^0_1]^4f_{21}(\frac{4}{3}F^{(g-1)}_{88}-\frac{1}{3}F^{(g-1)}_8)+\theta[^0_1]^4\eta^{12}(-4F^{(g-1)}_{88}+F^{(g-1)}_8)+\frac{1}{2}F^{(1)}_{16}F^{(g-1)}_8\\
=&\left(\frac{2}{3}F^{(1)}_2-\frac{1}{3}F^{(1)}_1\right)\left(4F^{(g-1)}_{88}-F^{(g-1)}_8\right)+\frac{1}{2}F^{(1)}_{16}F^{(g-1)}_8,\\
F^{(g)}_{16\,|\Delta_{1,g-1}} =&(\theta[^0_0]^{16}+\theta[^0_1]^{16}+\theta[^1_0]^{16})F^{(g-1)}_{16} = F^{(1)}_{16}F^{(g-1)}_{16}.
\end{aligned}$$

The factorisation of the forms $G^{(g)}_3[0^{(g)}]$ and $G^{(g)}_4[0^{(g)}]$ can be determined for any g using Theorem 15 of [Gr] and we report the result for the cases $g=4,5$ in Section 5.5.2 and 5.5.3 respectively.

5.5 Solution of the constraints for $g \le 5$

In this section we find a solution for the three constraints, using the basis for the O^+-invariants just constructed, and we obtain the functions $\Xi^{(g)}_8[\Delta^{(g)}]$. These solutions are equivalent to the ones found in Chapter 4 for $g=3,4$, but written in a different basis. The advantage is that in this way the theta constants appear at higher power than before. Moreover, here we full exploit the insight given by the group theory.

5.5.1 Genus three case

In Section 5.3.1 we found a basis for the five dimensional space of O^+-invariants satisfying the transformation constraints. Let us search a linear combination satisfying the factorisation constraints. We will follow the strategy of Chapter 4: write the more general vector in this space,

$$\Xi^{(3)}_8[0^{(3)}] = a_1F^{(3)}_1+a_2F^{(3)}_2+a_3F^{(3)}_3+a_4F^{(3)}_8+a_5F^{(3)}_{88}, \tag{5.4}$$

and then impose for it to factorise as the product of the genus one form $\Xi^{(1)}_8[0^{(1)}] = \theta[^0_0]^4\eta^{12}$ and the form $\Xi^{(2)}_8[0^{(2)}] = \frac{2}{3}F^{(2)}_1+\frac{1}{3}F^{(2)}_2-\frac{1}{2}F^{(2)}_3$ at genus two. In this way we obtain a linear equation in the five coefficients a_i.

The restriction of $\Xi_8^{(3)}[0^{(3)}]$ on $\mathbb{H}_1 \times \mathbb{H}_2$

The factorisation of the expression (5.4) for a reducible period matrix of the form $\tau_{1,2} = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$ is:

$$\begin{aligned}
&\left(a_1 F_1^{(3)} + a_2 F_2^{(3)} + a_3 F_3^{(3)} + a_4 F_8^{(3)} + a_5 F_{88}^{(3)}\right)(\tau_{1,2}) \\
&= \theta[{}^0_0]^4 \left(a_1(\tfrac{1}{3} f_{21} + \eta^{12}) F_1^{(2)} + a_2(\tfrac{2}{3} f_{21} - \eta^{12}) F_2^{(2)} + a_3 \tfrac{2}{3} f_{21} F_3^{(2)}\right) \\
&+ a_4 2 F_{16}^{(1)} F_8^{(2)} + a_5 \left[\theta[{}^0_1]^4 f_{21}(\frac{4}{3} F_{88}^{(2)} - \frac{1}{3} F_8^{(2)}) + \theta[{}^0_1]^4 \eta^{12}(-4F_{88}^{(2)} + F_8^{(2)}) + \frac{1}{2} F_{16}^{(1)} F_8^{(2)}\right]
\end{aligned}$$

Necessary condition for the restriction to take the form:

$$\begin{aligned}
&(\Xi_8^{(3)}[0^{(3)}])(\tau_{1,2}) = \\
&\left(\theta[{}^0_0]^4 \eta^{12}\right)(\tau_1) \Xi_8^{(2)}[0^{(2)}](\tau_2) \equiv \left(\theta[{}^0_0]^4 \eta^{12}\right)(\tau_1) \left(\theta[0^{(2)}]^4 \Xi_6[0^{(2)}]\right)(\tau_2)
\end{aligned}$$

is that the terms proportional to f_{21} and to $F_{16}^{(1)}$ disappear. Here $\Xi_6[0^{(2)}]$ is the function found by D'Hoker and Phong in [DP4]. First, let us impose the condition to get rid of the terms proportional to f_{21}. This condition is satisfied if:

$$a_1 \frac{1}{3} F_1^{(2)} + a_2 \frac{2}{3} F_2^{(2)} + a_3 \frac{2}{3} F_3^{(2)} + a_5 \left(\frac{4}{3} F_{88}^{(2)} - \frac{1}{3} F_8^{(2)}\right) = 0.$$

This equation has a unique solution up to a scalar multiple:

$$(a_1, a_2, a_3, a_5) = \lambda \left(\frac{16}{3}, \frac{16}{3}, -4, 1\right), \qquad \lambda \in \mathbb{C}.$$

To eliminate the terms proportional to $F_{16}^{(1)}$ the expression $(2a_4 + \frac{1}{2} a_5) F_8^{(2)}$ must vanish, so, from the solution 5.5.1, we obtain

$$a_4 = -\frac{1}{4} a_5 = -\lambda \frac{1}{4}.$$

Thus the expression for the factorised measure is:

$$\theta[{}^0_0]^4 \eta^{12}\, \lambda \left[\frac{16}{3} F_1^{(3)} + \frac{16}{3}(-F_2^{(3)}) + \left(F_8^{(3)} - 4F_{88}^{(3)}\right)\right],$$

and it is of the form $\Xi_8^{(3)}[0^{(3)}](\tau_{1,2}) = (\theta[{}^0_0]^4 \eta^{12})(\tau_1)(\theta[0^{(2)}]^4 \Xi_6[0^{(2)}])$ if $\lambda = \frac{1}{16}$, as can be verified with a computer or using the classical theta formula. This solution for the form $\Xi_8^{(3)}[0^{(3)}]$ is, up to a term proportional to $J^{(3)}$, the same found in Section 5.5.1. Using this basis, the theta constants in the function $\Xi_8^{(3)}[0^{(3)}]$ appear with higher power than using the one of Section 5.5.1: the four functions $F_1^{(3)}$, $F_3^{(3)}$, $F_8^{(3)}$ and $F_{88}^{(3)}$ are polynomials in $\theta[\Delta^{(3)}]^8$ and they belong to $\mathrm{Sym}^2(\mathbf{1} + \mathbf{35}_b)$ and the $F_2^{(2)}$ is a polynomial in $\theta[\Delta^{(3)}]^4$ and belongs to $\mathrm{Sym}^4(\mathbf{15}_a)$. The final expression for the form $\Xi_8^{(3)}[0^{(3)}]$ is:

$$\Xi_8^{(3)}[0^{(3)}] = \frac{1}{3} F_1^{(3)} + \frac{1}{3} F_2^{(3)} - \frac{1}{4} F_3^{(3)} - \frac{1}{64} F_8^{(3)} + \frac{1}{16} F_{88}^{(3)},$$

and it is completely equivalent to the one determined before:

$$\Xi_8^{(3)}[0^{(3)}] = \frac{1}{3}F_1^{(3)} + \frac{1}{3}F_2^{(3)} - \frac{1}{4}F_3^{(3)} - G_3^{(3)}[0^{(3)}].$$

In fact, they differ for a multiple of the form $J^{(3)}$ (that vanishes on the whole $\mathbb{H}_3$), precisely the new expression is equal the old one $-\frac{5}{448}J^{(3)}$, as it can be computed using the results of Table 5.4. This procedure will be explained in detail for the genus four case in the next section. So we can use this two expressions to show that the form $G_3^{(3)}[0^{(3)}]$ is polynomial in $\theta[\Delta^{(3)}]^8$:

$$G_3^{(3)}[0^{(3)}] = \frac{1}{64}F_8^{(3)} - \frac{1}{16}F_{88}^{(3)} - \frac{5}{448}(8F_{16}^{(3)} - F_8^{(3)}). \tag{5.5}$$

We include also the form $J^{(3)}$ because it is zero as a function of $\tau \in \mathbb{H}_3$ and using a computer to perform the computations one has to add explicitly this fact.

5.5.2 Genus four case

For the genus four case we repeat the method used for the genus three in the previous section. In Section 5.3.2 we found seven linear independent O^+-invariants that form a basis. We can now search a linear combination that also satisfies the factorisation constraints:

$$\Xi_8^{(4)}[0^{(4)}] = a_1F_1^{(4)} + a_2F_2^{(4)} + a_3F_3^{(4)} + a_4F_8^{(4)} + a_5F_{88}^{(4)} + a_6F_{16}^{(4)} + a_7G_3^{(4)}[0^{(4)}], \tag{5.6}$$

where $G_3^{(4)}[0^{(4)}]$ is the function defined in 5.3.2, or in the notations of Grushevsky $P_{3,2}^{(4)}$. In this case we also use the $F_{16}^{(4)}$ because in $g = 4$ it is independent from $F_8^{(4)}$, i.e. the expression $J^{(4)}$ is not identically zero on the whole $\mathbb{H}_4$, but just on the Jacobi locus.

The restriction of $\Xi_8^{(4)}[0^{(4)}]$ on $\mathbb{H}_1 \times \mathbb{H}_3$

The restriction on $\mathbb{H}_1 \times \mathbb{H}_3$ of the function $G_3^{(4)}[0^{(4)}]$ was found in Section 4.5.1 (cf. [CDG2]):

$$\begin{aligned} G_3^{(4)}[0^{(4)}](\tau_{1,3}) &= \theta[^0_0]^4(\tau_1)\left[\frac{1}{3}f_{21}(\tau_1)\left(G_2^{(3)}[0^{(3)}] + 8G_3^{(3)}[0^{(3)}]\right)(\tau_3)\right.\\ &\quad \left. -\eta^{12}(\tau_1)\left(G_2^{(3)}[0^{(3)}] + 6G_3^{(3)}[0^{(3)}]\right)(\tau_3)\right], \end{aligned}$$

this follows also from Theorem 15 of [Gr]. The modular forms $G_3^{(3)}[0^{(3)}]$ and $G_2^{(3)}[0^{(3)}]$ are defined, as usual, using three and two dimensional isotropic subspaces respectively (see also [CDG1, CDG2, Gr]). Thus the factorisation of the expression (5.6) for a reducible period matrix of the form $\tau_{1,3} = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{smallmatrix}\right)$ is:

$$\begin{aligned}
&\left(a_1 F_1^{(4)} + a_2 F_2^{(4)} + a_3 F_3^{(4)} + a_4 F_8^{(4)} + a_5 F_{88}^{(4)} + a_6 F_{16}^{(4)} + a_7 G_3^{(4)}[0^{(4)}]\right)(\tau_{1,3}) \\
&= \theta[\begin{smallmatrix}0\\0\end{smallmatrix}]^4 \left[a_1(\tfrac{1}{3} f_{21} + \eta^{12}) F_1^{(3)} + a_2(\tfrac{2}{3} f_{21} - \eta^{12}) F_2^{(3)} + a_3 \tfrac{2}{3} f_{21} F_3^{(3)}\right] + a_4 2 F_{16}^{(1)} F_8^{(3)} \\
&+ a_5 \left[\theta[\begin{smallmatrix}0\\1\end{smallmatrix}]^4 f_{21}(\frac{4}{3} F_{88}^{(3)} - \frac{1}{3} F_8^{(3)}) + \theta[\begin{smallmatrix}0\\1\end{smallmatrix}]^4 \eta^{12}(-4 F_{88}^{(3)} + F_8^{(3)}) + \frac{1}{2} F_{16}^{(1)} F_8^{(3)}\right] \\
&+ a_6 F_{16}^{(1)} F_{16}^{(3)} + a_7 \left[\theta[\begin{smallmatrix}0\\0\end{smallmatrix}]^4 f_{21}(\frac{1}{3} G_2^{(3)}[0^{(3)}] + \frac{8}{3} G_3^{(3)}[0^{(3)}]) \right.\\
&\left. + \theta[\begin{smallmatrix}0\\0\end{smallmatrix}]^4 \eta^{12}(-G_2^{(3)}[0^{(3)}] - 6 G_3^{(3)}[0^{(3)}])\right].
\end{aligned}$$

The terms proportional to f_{21} disappear if:

$$\begin{aligned}
&a_1 \frac{1}{3} F_1^{(3)} + a_2 \frac{2}{3} F_2^{(3)} + a_3 \frac{2}{3} F_3^{(3)} + a_5 \left(\frac{4}{3} F_{88}^{(3)} - \frac{1}{3} F_8^{(3)}\right) \\
&+ a_7 \frac{1}{3}(G_2^{(3)}[0^{(3)}] + 8 G_3^{(3)}[0^{(3)}]) = 0.
\end{aligned}$$

This equation has a unique solution up to a scalar multiple:

$$(a_1, a_2, a_3, a_5, a_7) = \lambda\left(-\frac{56}{5}, -\frac{112}{5}, \frac{42}{5}, -\frac{21}{5}, \frac{168}{5}\right), \qquad \lambda \in \mathbb{C}.$$

The term proportional to $F_{16}^{(1)}$ vanishes if:

$$a_4 2 F_8^{(3)} + a_5 \frac{1}{2} F_8^{(3)} + a_6 F_{16}^{(3)} = 0.$$

This equation has infinitely many solutions. Due to the vanishing of $J^{(3)}$ on the whole Siegel upper half space we can rewrite the previous equation as $(2a_4 + \frac{1}{2}a_5 + \frac{1}{8}a_6)F_8 = 0$, which has solution $a_4 = -\frac{a_5}{4} - \frac{a_6}{16}$ with $a_6 \in \mathbb{C}$. For any choice of a_6 an additive term proportional to $J^{(4)}$ appears in the expression of $\Xi_8^{(4)}[0^{(4)}]$ and precisely it is $\frac{a_6}{16}(16 F_{16}^{(4)} - F_8^{(4)})$. In this sense the form $\Xi_8^{(4)}[0^{(4)}]$ is unique up to a term proportional to $J^{(4)}$, as proved in Chapter 4. Thus, we can choose $a_6 = 0$ and $a_4 = -\frac{1}{4}a_5$. The request for the restriction to be of the form $\Xi_8^{(4)}[0^{(4)}](\tau_{1,3}) = \left(\theta[\begin{smallmatrix}0\\0\end{smallmatrix}]^4 \eta^{12}\right)(\tau_1)\Xi_8^{(3)}[0^{(3)}](\tau_3)$ fixes the value of $\lambda = -\frac{5}{336}$. This follow from the condition:

$$\begin{aligned}
&\theta[\begin{smallmatrix}0\\0\end{smallmatrix}]^4 \eta^{12} \lambda \left[-\frac{56}{5} F_1^{(3)} - \frac{112}{5}(-F_2^{(3)}) + \frac{21}{5}\left(F_8^{(3)} - 4 F_{88}^{(3)}\right)\right. \\
&\qquad \left. + \frac{168}{5}\left[-\left(G_2^{(3)}[0^{(3)}] + 6 G_3^{(3)}[0^{(3)}]\right)\right]\right] = \theta[\begin{smallmatrix}0\\0\end{smallmatrix}]^4 \eta^{12} \Xi_8^{(3)}[0^{(3)}],
\end{aligned}$$

and, using again the fact that $J^{(3)}$ identically vanishes, we obtain $\lambda = -\frac{5}{336}$.

The above discussion shows that the form $\Xi_8^{(4)}[0^{(4)}]$ is:

$$\Xi_8^{(4)}[0^{(4)}] = \frac{1}{6} F_1^{(4)} + \frac{1}{3} F_2^{(4)} - \frac{1}{8} F_3^{(4)} + \frac{1}{64} F_8^{(4)} - \frac{1}{16} F_{88}^{(4)} - \frac{1}{2} G_3^{(4)}[0^{(4)}], \tag{5.7}$$

which, for the uniqueness (up to a multiple of $J^{(4)}$) of the form $\Xi_8^{(4)}[0^{(4)}]$, is equivalent to the one found in Section 4.45:

$$\Xi_8^{(4)}[0^{(4)}] = \frac{1}{6}F_1^{(4)} + \frac{1}{3}F_2^{(4)} - \frac{1}{8}F_3^{(4)} - \frac{1}{2}G_3^{(4)}[0^{(4)}] + 4G_4^{(4)}[0^{(4)}]. \tag{5.8}$$

This two expressions must be equated and they could differ just for a multiple of the Schottky relation $J^{(4)}$. Calling the form $\Xi_8^{(4)}[0^{(4)}]$ (5.7) $\Xi_8^{(4)}[0^{(4)}]_{DP}$ and the (5.8) $\Xi_8^{(4)}[0^{(4)}]_{CDG}$ we can write:

$$\Xi_8^{(4)}[0^{(4)}]_{DP} = \Xi_8^{(4)}[0^{(4)}]_{CDG} + aJ^{(4)}.$$

Summing over the 136 even theta characteristics and using the results of Table 5.5 we find:

$$\frac{45}{56}J^{(4)} = \frac{12}{7}J^{(4)} + 136aJ^{(4)},$$

from which it follows that $a = -3/448$. This shows that the modular form $G_4^{(4)}[0^{(4)}]$ is, actually, polynomial in $\theta[\Delta^{(4)}]^8$:

$$\begin{aligned} G_4^{(4)}[0^{(4)}] &= \frac{1}{256}F_8^{(4)} - \frac{1}{64}F_{88}^{(4)} + \frac{3}{1792}J^{(4)} \\ &= \frac{1}{448}F_8^{(4)} - \frac{1}{64}F_{88}^{(4)} + \frac{3}{112}F_{16}^{(4)}. \end{aligned} \tag{5.9}$$

Remark

Recently, Oura has proved, as a consequence of the results in [N], that the space of modular forms with respect to the subgroup $\Gamma_4(1,2)$ of weight 8, quadratic in the theta constants, has dimension no bigger than 7, $\dim M_8^{\theta^2}(\Gamma_4(2))^{O^+} \leq 7$. The computations in the previous section show that this dimension is precisely seven:

$$\dim M_8^{\theta^2}(\Gamma_4(2))^{O^+} = 7.$$

Moreover, in [OPSY] it is proved that the space of cusp forms $[\Gamma_4(1,2), 8]_0$, in which $\Xi_8^{(4)}[0^{(4)}]$ lies, has dimension two. From this, it follows the uniqueness (up to a multiple of $J^{(4)}$) of the form $\Xi_8^{(4)}[0^{(4)}]$ (as explained in [OPSY] and [GS]) and not just in a weakened form, i.e. assuming polynomiality in the theta constants, as in previous chapter (cf. [DvG]).

5.5.3 Genus five case

In Section 5.3.3 we found eight linear independent O^+-invariants that form a basis. Their general linear combination is:

$$\begin{aligned} \Xi_8^{(5)}[0^{(5)}] = {} & a_1F_1^{(5)} + a_2F_2^{(5)} + a_3F_3^{(5)} + a_4F_8^{(5)} + a_5F_{88}^{(5)} \\ & + a_6F_{16}^{(5)} + a_7G_3^{(5)}[0^{(5)}] + a_8G_4^{(5)}[0^{(5)}]. \end{aligned} \tag{5.10}$$

We will search for eight coefficients a_i such that this expression satisfies the right factorisation.

The restriction of $G_3^{(5)}[0^{(5)}]$ and $G_4^{(5)}[0^{(5)}]$ on $\mathbb{H}_1 \times \mathbb{H}_4$

These restrictions follow quite directly from the Theorem of Grushevsky, identifying $G_4^{(5)}[0^{(5)}]$ with $P_{4,1}^{(5)}$ and $G_3^{(5)}[0^{(5)}]$ with $P_{3,2}^{(5)}$. For the function $G_3^{(5)}[0^{(5)}]$ we get:

$$\begin{aligned}P_{3,2}^{(5)}(\tau_{1,4}) &= P_{0,16}^{(1)}(\tau_1)P_{3,2}^{(4)}(\tau_4) + P_{1,8}^{(1)}(\tau_1)P_{2,4}^{(4)}(\tau_4) + 7P_{1,8}^{(1)}(\tau_1)P_{3,2}^{(4)}(\tau_4)\\ &= \theta[^0_0]^4(\frac{8}{3}f_{21} - 6\eta^{12})(\tau_1)G_3^{(4)}[0^{(4)}](\tau_4) + \theta[^0_0]^4(\frac{1}{3}f_{21} - \eta^{12})(\tau_1)G_2^{(4)}[0^{(4)}](\tau_4),\end{aligned}$$

where we used $P_{0,16}^{(1)} = \theta[0^{(1)}]^4\left(\frac{1}{3}f_{21} + \eta^{12}\right)$, $P_{1,8}^{(1)} = \theta[0^{(1)}]^4\left(\frac{1}{3}f_{21} - \eta^{12}\right)$ and $P_{2,4}^{(4)} = G_2^{(4)}[0^{(4)}]$. For $G_4^{(5)}[0^{(5)}]$ we get:

$$\begin{aligned}P_{4,1}^{(5)}(\tau_{1,4}) &= P_{0,16}^{(1)}(\tau_1)P_{4,1}^{(4)}(\tau_4) + P_{1,8}^{(1)}(\tau_1)P_{3,2}^{(4)}(\tau_4) + 15P_{1,8}^{(1)}(\tau_1)P_{4,1}^{(4)}(\tau_4)\\ &= \theta[^0_0]^4(\frac{16}{3}f_{21} - 14\eta^{12})(\tau_1)G_4^{(4)}[0^{(4)}](\tau_4) + \theta[^0_0]^4(\frac{1}{3}f_{21} - \eta^{12})(\tau_1)G_3^{(4)}[0^{(4)}](\tau_4),\end{aligned}$$

where, as before, $P_{0,16}^{(1)} = \theta[0^{(1)}]^4\left(\frac{1}{3}f_{21} + \eta^{12}\right)$ and $P_{1,8}^{(1)} = \theta[0^{(1)}]^4\left(\frac{1}{3}f_{21} - \eta^{12}\right)$. These restrictions could be also determined using the method of isotropic subspaces, as in [CDG1], or by direct computation using a computer. In the next section it will be useful to use also $G_2^{(4)}[0^{(4)}] = \left(2F_1^{(4)} + 16F_2^{(4)} - 3F_3^{(4)}\right)/6$ (instead, at genus three we have $G_2^{(3)}[0^{(3)}] = \left(2F_1^{(3)} + 8F_2^{(3)} - 3F_3^{(3)}\right)/6$).

The restriction of $\Xi_8^{(5)}[0^{(5)}]$ on $\mathbb{H}_1 \times \mathbb{H}_4$

Using the results of the previous section and of Section 5.4 the factorisation of the expression (5.10) for a reducible period matrix of the form $\tau_{1,4} = \left(\begin{smallmatrix}\tau_1 & 0\\ 0 & \tau_4\end{smallmatrix}\right)$ is:

$$\begin{aligned}&\Big(a_1F_1^{(5)} + a_2F_2^{(5)} + a_3F_3^{(5)} + a_4F_8^{(5)} + a_5F_{88}^{(5)}\\ &\qquad +a_6F_{16}^{(5)} + a_7G_3^{(5)}[0^{(5)}] + a_8G_4^{(5)}[0^{(5)}]\Big)(\tau_{1,4}) =\\ &\Big[a_1\theta[^0_0]^4(\tfrac{1}{3}f_{21} + \eta^{12})F_1^{(4)} + a_2\theta[^0_0]^4(\tfrac{2}{3}f_{21} - \eta^{12})F_2^{(4)} + a_3\tfrac{2}{3}\theta[^0_0]^4f_{21}F_3^{(4)}\Big] +\\ &a_42F_{16}^{(1)}F_8^{(4)} + a_5\Big[\theta[^0_1]^4f_{21}(\frac{4}{3}F_{88}^{(4)} - \frac{1}{3}F_8^{(4)}) + \theta[^0_1]^4\eta^{12}(-4F_{88}^{(4)} + F_8^{(4)})\\ &+\frac{1}{2}F_{16}^{(1)}F_8^{(4)}\Big] + a_6F_{16}^{(1)}F_{16}^{(4)} +\\ &a_7\theta[^0_0]^4\Big[\frac{1}{3}f_{21}(G_2^{(4)}[0^{(4)}] + 8G_3^{(4)}[0^{(4)}]) + \eta^{12}(-G_2^{(4)}[0^{(4)}] - 6G_3^{(4)}[0^{(4)}])\Big] +\\ &a_8\theta[^0_0]^4\Big[f_{21}\Big(\frac{16}{3}G_4^{(4)}[0^{(4)}] + \frac{1}{3}G_3^{(4)}[0^{(4)}]\Big) + \eta^{12}\Big(-14G_4^{(4)}[0^{(4)}] - G_3^{(4)}[0^{(4)}]\Big)\Big].\end{aligned}$$

From the relations of paragraph 5.4.1 we obtain the condition for the vanishing of the terms proportional to f_{21}:

$$a_1\frac{1}{3}F_1^{(4)}+a_2\frac{2}{3}F_2^{(4)}+a_3\frac{2}{3}F_3^{(4)}+a_5\left(\frac{4}{3}F_{88}^{(4)}-\frac{1}{3}F_8^{(4)}\right)+$$
$$+a_7\frac{1}{3}(G_2^{(4)}[0^{(4)}]+8G_3^{(4)}[0^{(4)}])+a_8\frac{1}{3}(G_3^{(4)}[0^{(4)}]+16G_4^{(4)}[0^{(4)}])=0.$$

Again, this equation, using the fact that $J^{(4)}$ vanishes on the Jacobi locus (actually, one solves the equation modulo the Schottky relation $J^{(4)}$), has an unique solution, up to a scalar multiple:

$$(a_1,a_2,a_3,a_5,a_7,a_8)=\lambda\left(-\frac{14}{3},-\frac{56}{3},\frac{7}{2},-7,14,-112\right),\qquad \lambda\in\mathbb{C}.$$

The term proportional to $F_{16}^{(1)}$ vanishes if:

$$a_4 2F_8^{(4)}+a_5\frac{1}{2}F_8^{(4)}+a_6F_{16}^{(4)}=0.$$

As for $g=4$ this equation has infinitely many solutions and using again the fact that $J^{(4)}$ vanishes on the Jacobi locus we obtain $(2a_4+\frac{1}{2}a_5+\frac{1}{16}a_6)F_8^{(4)}=0$, which has solution $a_4=-\frac{a_5}{4}-\frac{a_6}{32}$, with $a_6\in\mathbb{C}$. For any choice of the coefficient a_6 the additive term $\frac{a_6}{32}(32F_{16}^{(5)}-F_8^{(5)})$ appears in $\Xi_8^{(5)}[0^{(5)}]$. However, in genus five this term vanishes just on the locus of trigonal curves and not on the whole Jacobi locus. This shows how the uniqueness of the form $\Xi_8^{(5)}[0^{(5)}]$ can not be longer assured by the three constraints of Section 5.2 on J_5, as also pointed out in [OPSY]. Thus, we can choose $a_6=0$ and $a_4=-\frac{1}{4}a_5$. The request for the restriction to be of the form $\Xi_8^{(5)}[0^{(5)}](\tau_{1,4})=(\theta[^0_0]^4\eta^{12})(\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4)$ means that:

$$\begin{aligned}(\theta[^0_0]^4\eta^{12})(\tau_1)\lambda\Big[&a_1F_1^{(4)}-a_2F_2^{(4)}+a_5\left(F_8^{(4)}-4F_{88}^{(4)}\right)\\&+a_7(-G_2^{(4)}[0^{(4)}]-6G_3^{(4)}[0^{(4)}])+a_8(-14G_4^{(4)}[0^{(4)}]-G_3^{(4)}[0^{(4)}])\Big]\\=(\theta[^0_0]^4\eta^{12})(\tau_1)\Xi_8^{(4)}[0^{(4)}],&\end{aligned}$$

where $\Xi_8^{(4)}[0^{(4)}]$ is the function found in Section 5.10.4, and this should fix the constant λ. Therefore, we impose:

$$\begin{aligned}\theta[^0_0]^4\eta^{12}\,\lambda\Big[&-\frac{14}{3}F_1^{(4)}-\frac{56}{3}(-F_2^{(4)})-7(F_8^{(4)}-4F_{88}^{(4)})+14(-G_2^{(4)}[0^{(4)}]-6G_3^{(4)}[0^{(4)}])\\&-112(-14G_4^{(4)}-G_3^{(4)})\Big]=\theta[^0_0]^4\eta^{12}\left(\Xi_8^{(4)}[0^{(4)}]+\Lambda J^{(4)}\right),\end{aligned}$$

this equation has solution $\lambda=-\frac{1}{56}$ and $\Lambda=-\frac{3}{64}$. Actually, using $\lambda=-\frac{1}{56}$ and summing over all the even characteristics one finds that the expression in the square brackets on the left and the $\Xi_8^{(4)}[0^{(4)}]$ on the right sides of the previous equation differ by $-\frac{3}{64}J^{(4)}$.

Thus, in genus five the function $\Xi_8^{(5)}[0^{(5)}]$ satisfying the three constraints on the Jacobi locus is:

$$\Xi_8^{(5)}[0^{(5)}]=\frac{1}{12}F_1^{(5)}+\frac{1}{3}F_2^{(5)}-\frac{1}{16}F_3^{(5)}-\frac{1}{32}F_8^{(5)}+\frac{1}{8}F_{88}^{(5)}-\frac{1}{4}G_3^{(5)}[0^{(5)}]+2G_4^{(5)}[0^{(5)}]. \quad (5.11)$$

Note that also for the solution found in [OPSY] the correct restriction holds if one restrict to J_4.

The constraint on $\mathbb{H}_2\times\mathbb{H}_3$

Now we consider the restriction of the function $\Xi_8^{(5)}[0^{(5)}]$ to $\mathbb{H}_2\times\mathbb{H}_3$ and this, to satisfy the factorization constraint of Section 5.2, must be equal to the product $\Xi_8^{(2)}[0^{(2)}]\Xi_8^{(3)}[0^{(3)}]$ i.e. the genus two times the genus three measure.

In order to obtain the restriction of $\Xi_8^{(5)}[0^{(5)}]$ we need the restriction of the eight basis functions. We have:

$$\begin{aligned}
F^{(5)}_{1\,|\Delta_{2,3}} &= F_1^{(2)}F_1^{(3)},\\
F^{(5)}_{2\,|\Delta_{2,3}} &= F_2^{(2)}F_2^{(3)},\\
F^{(5)}_{3\,|\Delta_{2,3}} &= F_3^{(2)}F_3^{(3)},\\
F^{(5)}_{8\,|\Delta_{2,3}} &= F_8^{(2)}F_8^{(3)},\\
F^{(5)}_{16\,|\Delta_{2,3}} &= F_{16}^{(2)}F_{16}^{(3)},\\
F^{(5)}_{88\,|\Delta_{2,3}} &= F_1^{(2)}(\frac{16}{3}F_{88}^{(3)}-\frac{4}{3}F_8^{(3)})+F_2^{(2)}(\frac{32}{3}F_{88}^{(3)}-\frac{8}{3}F_8^{(3)})\\
&\quad +F_3^{(2)}(-8F_{88}^{(3)}+2F_8^{(3)})+F_{16}^{(2)}F_8^{(3)},\\
G^{(5)}_{3\,|\Delta_{2,3}} &= G_0^{(2)}G_3^{(3)}+7G_1^{(2)}G_3^{(3)}+G_1^{(2)}G_2^{(3)}+42G_2^{(2)}G_3^{(3)}+9G_2^{(2)}G_2^{(3)}+G_2^{(2)}G_1^{(3)}\\
&= F_1^{(2)}(\frac{1}{3}F_1^{(3)}+\frac{8}{3}F_2^{(3)}-\frac{2}{3}F_3^{(3)}+\frac{1}{8}F_8^{(3)}-\frac{1}{2}F_{88}^{(3)})\\
&\quad +F_2^{(2)}(\frac{4}{3}F_1^{(3)}+8F_2^{(3)}-\frac{7}{3}F_3^{(3)}+\frac{7}{16}F_8^{(3)}-\frac{7}{4}F_{88}^{(3)})\\
&\quad +F_3^{(2)}(-\frac{2}{3}F_1^{(3)}-\frac{14}{3}F_2^{(3)}+\frac{5}{4}F_3^{(3)}-\frac{7}{32}F_8^{(3)}+\frac{7}{8}F_{88}^{(3)}),\\
G^{(5)}_{4\,|\Delta_{2,3}} &= G_1^{(2)}G_3^{(3)}+21G_2^{(2)}G_3^{(3)}+G_2^{(2)}G_2^{(3)}\\
&= F_1^{(2)}(\frac{1}{9}F_1^{(3)}+\frac{4}{9}F_2^{(3)}-\frac{1}{6}F_3^{(3)}+\frac{3}{32}F_8^{(3)}-\frac{3}{8}F_{88}^{(3)})\\
&\quad +F_2^{(2)}(\frac{2}{9}F_1^{(3)}+\frac{8}{9}F_2^{(3)}-\frac{1}{3}F_3^{(3)}+\frac{7}{32}F_8^{(3)}-\frac{7}{8}F_{88}^{(3)})\\
&\quad +F_3^{(2)}(-\frac{1}{6}F_1^{(3)}-\frac{2}{3}F_2^{(3)}+\frac{1}{4}F_3^{(3)}-\frac{19}{128}F_8^{(3)}+\frac{19}{32}F_{88}^{(3)}).
\end{aligned}$$

The first five relations follow quite easily from the definitions and the classical theta formula. The sixth is longer to prove in the same manner and it can be obtained using software like Mathematica. The last two follow from Theorem 15 of [Gr]. Using the linear relations between the lattice theta series and the classical theta constants of Section 5.10.5, another proof of the restrictions can be given.

We can now obtain the restriction of the form $\Xi_8^{(5)}[0^{(5)}]$:

$$\begin{aligned}\Xi_8^{(5)}[0^{(5)}](\tau_{2,3}) &= \left(\frac{2}{3}F_1^{(2)} + \frac{1}{3}F_2^{(2)} - \frac{1}{2}F_3^{(2)}\right)\\ &\cdot\left(\frac{1}{3}F_1^{(3)} + \frac{1}{3}F_2^{(3)} - \frac{1}{4}F_3^{(3)} - \frac{1}{64}F_8^{(3)} + \frac{1}{16}F_{88}^{(3)}\right)\\ &= \Xi_8^{(2)}[0^{(2)}](\tau_2)\Xi_8^{(3)}[0^{(3)}](\tau_3).\end{aligned}$$

Therefore the modular form $\Xi_8^{(5)}[0^{(5)}]$ on $\Gamma_5(1,2)$ of weight 8, defined in 5.5.3, satisfies all the factorization constraints in genus five.

Remark

It is interesting to investigate the possibility to apply a similar procedure to the genus six case. However, in this case it is hard to think that the procedure will work. Indeed, in Section 5.7 we will show that the dimension of the space of the O^+-invariants polynomial in the theta constants is eight for all $g \geq 5$. Moreover, at the moment we have no indication wether in genus $g \geq 6$ a solution for the constraints can be find if one consider also the non part of the ring of modular forms.

5.5.4 On the dimensions of certain space of modular forms

In Sections 5.5.2 and 5.5.3 we considered the space of the modular forms of weight 8 with respect the group $\Gamma_g(1,2)$. In particular we focused on the modular forms polynomial in theta constants. In order to find the forms $\Xi_8^{(g)}[0^{(g)}]$ that factorise in the right way we searched for a basis for these spaces and this allowed us to find the dimensions of the spaces. We summarise these results in the following (cf. Remark 5.3.3):

Proposition 5.5.1. *For the space $M_8^{\theta^2}(\Gamma_4(2))^{O^+}$, $M_8^{\theta^2}(\Gamma_5(2))^{O^+}$ and $M_8^{\theta}(\Gamma_5(2))^{O^+}$ the following equalities hold:*

$$\begin{aligned}\dim M_8^{\theta^2}(\Gamma_4(2))^{O^+} &= 7,\\ \dim M_8^{\theta^2}(\Gamma_5(2))^{O^+} &= 7,\\ \dim M_8^{\theta}(\Gamma_5(2))^{O^+} &= 8.\end{aligned}$$

5.6 The vanishing of the cosmological constant

In this section we reinterpret the vanishing of the cosmological constant on the light of the group representation theory. In Section 5.2 we pointed out that the O^+-invariants belong to the $\mathbf{1}$ and σ_θ representations. For the case $g \leq 5$ we know that the only $\mathrm{Sp}(2g)$ invariants are $F_{16}^{(g)}$ and $F_8^{(g)}$ (they are not independent for $g = 3$) and they form a basis for the **1** part of the space of the O^+-invarints. Let $\{e_{\sigma_i}\}_{i=1,\cdots,n_{\sigma_\theta}}$ be the basis for the σ_θ part. Then, an O^+-invariant decomposes in two parts: the first one lying in the representation **1** and the second one in the σ_θ. Thus, if $f[0^{(g)}] \in M_8(\Gamma_g(2))^{O^+}$, we can write $f[0^{(g)}] = aF_8^{(g)} + bF_{16}^{(g)} + \sum_i^{n_{\sigma_\theta}} c_i e_{\sigma_i}$, for $g \leq 5$. Acting on these functions with all the generators of the group $\mathrm{Sp}(2g)$ and summing up the result at each step we obtain a $\mathrm{Sp}(2g)$-invariant. We know that the unique $\mathrm{Sp}(2g)$-invariants are the two functions $F_8^{(g)}$ and $F_{16}^{(g)}$ so that the σ_θ representation part gives no contribution to the sum. Therefore, if the function $f[0^{(g)}]$ contains a non trivial part proportional to $F_8^{(g)}$ or $F_{16}^{(g)}$, the result of the sum will be non zero.

The cosmological constant is the sum of the functions $\Xi_8^{(g)}[\Delta^{(g)}]$ over all the even characteristics. This sum is a $\mathrm{Sp}(2g)$-invariant and it must then be proportional to a combination of $F_8^{(g)}$ and $F_{16}^{(g)}$. Thus the cosmological constant vanishes if this sum is zero. We now verify this for the genus three, four and five cases.

5.6.1 Genus three

In Table 5.4 we report the sums of each term appearing in the form $\Xi_8^{(3)}[0^{(3)}]$. These show that for the expression of the measure in the three bases (the one of Chapter 4 (CDG), the one of this Chapter (DP) and the basis of [Gr] (Gr)) for the space of O^+-invariants we always obtain the vanishing of the cosmological constant (as expected) due to the vanishing of the form $J^{(3)}$.

Function	Sum	CDG	DP	Gr
$F_1^{(3)}$	$F_{16}^{(3)}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{8}$
$F_2^{(3)}$	$8F_{16}^{(3)}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$F_3^{(3)}$	$F_8^{(3)}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0
$F_8^{(3)}$	$36F_8^{(3)}$	0	$-\frac{1}{64}$	0
$F_{88}^{(3)}$	$8F_8^{(3)} - 8F_{16}^{(3)}$	0	$\frac{1}{16}$	0
$G_1^{(3)}[0^{(3)}]$	$F_8^{(3)} - F_{16}^{(3)}$	0	0	$-\frac{1}{8}$
$G_2^{(3)}[0^{(3)}]$	$11F_{16}^{(3)} - \frac{1}{2}F_8^{(3)}$	0	0	$\frac{1}{4}$
$G_3^{(3)}[0^{(3)}]$	$\frac{1}{28}(13F_8^{(3)} - 76F_{16}^{(3)})$	-1	0	-1
Total		$\frac{5}{7}(8F_{16}^{(3)} - F_8^{(3)})$	$\frac{5}{16}(8F_{16}^{(3)} - F_8^{(3)})$	$\frac{5}{7}(8F_{16}^{(3)} - F_8^{(3)})$

Table 5.4: Sums of the terms appearing in $\Xi_8^{(3)}[0^{(3)}]$. In the third, fourth and fifth columns we report the coefficients of the O^+-invariants appearing in the expression of $\Xi_8^{(3)}[0^{(3)}]$ in the three basis.

5.6.2 Genus four

As for the genus three case, we report in Table 5.5 the sums of each term appearing in the form $\Xi_8^{(4)}[0^{(4)}]$. Again, for the three equivalent bases of the space of O^+-invariants, the cosmological constant vanishes on the Jacobi locus due to the vanishing of the form

$J^{(4)}$. It should be noted that the cosmological constant vanishes just on the moduli space of curves even if the forms $\Xi_8^{(4)}[\Delta^{(4)}]$ are well defined on the whole $\mathbb{H}_4$.

Function	Sum	CDG	DP	Gr
$F_1^{(4)}$	$F_{16}^{(4)}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{16}$
$F_2^{(4)}$	$16F_{16}^{(4)}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$F_3^{(4)}$	$F_8^{(4)}$	$-\frac{1}{8}$	$-\frac{1}{8}$	0
$F_8^{(4)}$	$136F_8^{(4)}$	0	$\frac{1}{64}$	0
$F_{88}^{(4)}$	$32F_8^{(4)} - 32F_{16}^{(4)}$	0	$-\frac{1}{16}$	0
$F_{16}^{(4)}$	$136F_{16}^{(4)}$	0	0	0
$G_1^{(4)}[0^{(4)}]$	$F_8^{(4)} - F_{16}^{(4)}$	0	0	$-\frac{1}{16}$
$G_2^{(4)}[0^{(4)}]$	$43F_{16}^{(4)} - \frac{1}{2}F_8^{(4)}$	0	0	$\frac{1}{8}$
$G_3^{(4)}[0^{(4)}]$	$\frac{15}{7}(\frac{3}{4}F_8^{(4)} - 5F_{16}^{(4)})$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$G_4^{(4)}[0^{(4)}]$	$\frac{29}{7}F_{16}^{(4)} - \frac{11}{56}F_8^{(4)}$	4	0	4
Total		$\frac{12}{7}(16F_{16}^{(4)} - F_8^{(4)})$	$\frac{45}{56}(16F_{16}^{(4)} - F_8^{(4)})$	$\frac{12}{7}(16F_{16}^{(4)} - F_8^{(4)})$

Table 5.5: Sums of the terms appearing in $\Xi_8^{(4)}[0^{(4)}]$. In the third, fourth and fifth columns we report the coefficients of the O^+-invariants appearing in the expression of $\Xi_8^{(4)}[0^{(4)}]$ in the three basis.

5.6.3 Genus five

As for the two previous cases we report in Table 5.6 the sums of each term appearing in the form $\Xi_8^{(5)}[0^{(5)}]$. In the Table the functions $G_i^{(5)}[0^{(5)}]$, $i = 0, \cdots, 5$, with $G_0^{(5)}[0^{(5)}] \equiv F_1^{(5)}$, are the same as in [Gr]. In the genus five case the cosmological constant no longer vanishes neither on J_5. Actually, it was shown in [GS] that the zero locus of $J^{(5)}$ is the locus of trigonal curves. Following [OPSY], if we subtract from the forms $\Xi_8^{(5)}[0^{(5)}]$ the value of the cosmological constants divided by 528, the number of the even characteristics in genus five, we obtain again a function satisfying the three constraints and, moreover, having zero cosmological constant. The correct factorization is due to the fact that the form $J^{(5)}$ vanishes when restrict both on $\mathbb{H}_1 \times \mathbb{H}_4$ and on $\mathbb{H}_2 \times \mathbb{H}_3$. Moreover, this consideration shows that in genus five the three constraints no longer assure the uniqueness of the form $\Xi_8^{(5)}[0^{(5)}]$ because we can always add a multiple of the Schottky relation that is not zero on J_5 obtaining another forms with the correct behaviour.

Remark

The sums reported in Table 5.4, 5.5, and 5.6 can be computed using a computer and a software (for example, Mathematica), and in any case follow directly from Lemma 9 in [SM2].

Function	Sum	DP	Gr
$F_1^{(5)}$	$F_{16}^{(5)}$	$\frac{1}{12}$	$\frac{1}{32}$
$F_2^{(5)}$	$32F_{16}^{(5)}$	$\frac{1}{3}$	0
$F_3^{(5)}$	$F_8^{(5)}$	$-\frac{1}{16}$	0
$F_8^{(5)}$	$528F_8^{(5)}$	$-\frac{1}{32}$	0
$F_{88}^{(5)}$	$128F_8^{(5)} - 128F_{16}^{(5)}$	$\frac{1}{8}$	0
$F_{16}^{(5)}$	$528F_{16}^{(5)}$	0	0
$G_1^{(5)}[0^{(5)}]$	$F_8^{(5)} - F_{16}^{(5)}$	0	$-\frac{1}{32}$
$G_2^{(5)}[0^{(5)}]$	$171F_{16}^{(5)} - \frac{1}{2}F_8^{(5)}$	0	$\frac{1}{16}$
$G_3^{(5)}[0^{(5)}]$	$\frac{173}{28}F_8^{(5)} - \frac{299}{7}F_{16}^{(5)}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$G_4^{(5)}[0^{(5)}]$	$\frac{389}{7}F_{16}^{(5)} - \frac{43}{56}F_8^{(5)}$	2	2
$G_5^{(5)}[0^{(5)}]$	$-\frac{733}{217}F_{16}^{(5)} + \frac{475}{3472}F_8^{(5)}$	0	-32
Total		$\frac{51}{14}(32F_{16}^{(5)} - F_8^{(5)})$	$\frac{1632}{217}(32F_{16}^{(5)} - F_8^{(5)})$

Table 5.6: Sums of the terms appearing in $\Xi_8^{(5)}[0^{(5)}]$. In the third, fourth and fifth columns we report the coefficients of the O^+-invariants appearing in the expression of $\Xi_8^{(5)}[0^{(5)}]$ in the three basis.

5.7 Superstring amplitude and lattice theta series

In [OPSY] another candidate for the genus five superstring measure has been proposed. The authors have made use of the notion of lattice theta series. The forms $\Xi_8[\Delta]$ defined there to build up the measure, just like the ones obtained using the classical theta constants, satisfies all the constraints. The same tools has been used to obtain the expressions of the measures for $g \leq 4$. It is not clear if the two constructions are equivalent and lead to the same forms $\Xi_8[\Delta]$, thereby to the same measure $d\mu[\Delta]$. Obviously, this is the case for $g \leq 4$, as a consequence of the uniqueness theorems in lower genus. The goal of the last part of this chapter is to show that also in genus five the two constructions are equivalent, and the forms obtained are equal on the whole Siegel half upper plane $\mathbb{H}_5$, provided we add to the three constraints the supplementary request of vanishing cosmological constant. Otherwise they could differ for a multiple of the Schottky form $J^{(5)}$ (that vanishes on the locus of trigonal curves, cf. [GS]). Actually, adding a scalar multiple of $J^{(5)}$ to a form satisfying the three constraints one obtains a function again satisfying the same constraints: the Schottky $J^{(5)}$ is a modular form of weight eight and the restriction to $\mathbb{H}_1 \times \mathbb{H}_4$ is proportional to $F_{16}^{(1)}$ times $J^{(4)}$ and this product vanishes on the Jacobi locus. This is a remarkable fact because, differently from the genus four case, the zero locus $J^{(5)}$ is not the whole Jacobi locus, but the space of trigonal curves. Thus, the three constraints do not characterize uniquely the superstring measure, see [GS, DbMS, MV2]. This freedom can be fixed requiring the vanishing of the cosmological constant. Nevertheless, this should be a prediction of the theory and it should not be imposed by hand. This is a remarkable result both for the viewpoint of physics and of mathematics. Indeed, this shows that there are an infinity of different forms satisfying the three constraints on $\mathbb{H}_5$, actually on J_5. Thus, the constraints, without the additional request on the cosmological constant, do not suffice to characterize the measure uniquely in any genus. Furthermore, a deeper question arises about the conjecture by D'Hoker and Phong on the general expression (2.11) for the superstring chiral measure and about the procedure leading to it. These issues are at the basis of the mathematical correct formulation of the string theory in the perturbative approach. To solve these problems some more insight in the physics leading to the (conjectured) ansatz (2.11) is necessary.

Mathematically, to prove the equivalence of the forms $\Xi_8^{(5)}$, one has to show that the space spanned by the lattice theta series and the one spanned by the eight functions defined in Section 5.8 (that are a basis for $M_8^\theta(\Gamma_5(2))$, cf. [D]) are the same space of dimension eight. This is the content of the following theorem:

Theorem 5.7.1. *The spaces* $M_8^\theta(\Gamma_5(2))^{O^+}$ *and* $M_8^{\Theta_S}(\Gamma_5(2))$ *coincide.*

Here $M_8^\theta(\Gamma_5(2))^{O^+}$ is the space of genus five modular forms of weight eight with respect to the group $\Gamma_5(2)$ that are O^+-invariant polynomials in the classical theta constants, $M_8^{\Theta_S}(\Gamma_5(2))$ is the space of modular forms of weight eight spanned by the lattice theta series ($[\Gamma_5^{\Theta_S}(1,2),8]$ in the notation of [OPSY]), and $M_8(\Gamma_5(2))^{O^+}$ is the space of genus five modular forms of weight eight with respect to the group $\Gamma_5(2)$, which are O^+-invariant, cf. [CD2,DvG,D,OPSY,MV2] for details. The theorem follows from a result of Salvati Manni [SM3,SM4,SM5] in which it was proved that the space generated by the lattice theta series contains the subspace generated by classical theta constants that are Γ_g invariant whenever 4 divides the weight (see also [Fre], theorem VI.1.5). The result applies also for the $\Gamma_g(1,2)$ case and, as a consequence, one has:

$$M_{4k}^\theta(\Gamma_g(2))^{O^+} \subset M^{\Theta_S}(\Gamma_g(2)), \tag{5.12}$$

for integer k. In genus five the dimensions of both spaces is eight, see [OPSY] for the $M^{\Theta_S}(\Gamma_g(2))$ case, and Section 5.3.3 and Proposition 5.5.1 (cf. [D]) for the $M_8^\theta(\Gamma_g(2))^{O^+}$ one where also a basis for this space has been constructed. Thus, the theorem follows from the equality of the dimensions of the spaces.

In this chapter we exhibit a complete map between the two spaces obtaining all the linear relations between the lattice theta series and the basis functions of the space $M_8^\theta(\Gamma_5(2))^{O^+}$ defined in Section 5.8 by means of the classical theta constants. To obtain the map we compute certain Fourier coefficients of the functions appearing in the definition of the superstring measure. Since the spaces $M_8^\theta(\Gamma_5(2))^{O^+}$ and $M_8^{\Theta_S}(\Gamma_5(2))$ have dimension eight (see [D, OPSY]) we need at least eight suitable Fourier coefficients to get linear isomorphisms between these spaces and two copies of $\mathbb{C}^8$. In particular, being the two spaces the same, there must be linear relations among the Fourier coefficients of the elements of the two bases, which obviously extend to the complete series. In

Section 5.11.2 we also give an analytic proof of the equivalence between the functions $\Xi_8[\Delta]$ constructed employing the three constraints and the supplementary request of the vanishing of the cosmological constant. In addition, the Fourier coefficients method will permit to obtain, for $g \leq 4$, the complete set of linear relations between the lattice theta series and the basis functions of $M_8^\theta(\Gamma_g(2))^{O^+}$. We will also check the well known linear relations among the lattice theta series themselves [DbMS, OPSY].

5.7.1 Lattices and theta series

A powerfull tool for constructing in a general way modular forms is the theta series constructed using lattices. In this section we introduct the notion of lattices, quadratic forms associated with them and lattice theta series, see [AZ, CS] for details. An n dimensional lattice in $\mathbb{R}^n$ has the form $\Lambda = \{\sum_{i=1}^n a_i v_i \text{ s.t. } a_i \in \mathbb{Z}\}$, where v_i are the elements of a basis of $\mathbb{R}^n$ and are called basis for the lattice. A fundamental region is a building block which when repeated many times fills the whole space with just one lattice point in each copy. Different basis vector could define the same lattice, but the volume of the fundamental region is uniquely determined by Λ. The square of this volume is called the determinant or discriminant of the lattice. The matrix

$$M = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix}, \tag{5.13}$$

where $v_i = (v_{i1}, \cdots, v_{im})$ are the basis vectors is called generator matrix for the lattice. The matrix $A = M^tM$ is called Gram matrix and the entry (i, j) of A is the inner product $v_i \cdot v_j$. The determinant of Λ is the determinant of A. A generic vector $x = (x_1, \cdots, x_n)$ of the lattice can be written as $x = \zeta M = \zeta_1 v_1 + \cdots + \zeta_n v_n$, where $\zeta = (\zeta_1, \cdots, \zeta_n)$ is an arbitrary vector with integer components. Its norm is $N(x) = x \cdot x = \zeta A^t\zeta$. This is a quadratic form associated with the lattice in the integer variables $\zeta_1, \cdots, \zeta_n$. Any n-dimensional lattice Λ has a dual lattice, Λ^*, given by:

$$\Lambda^* = \{x \in \mathbb{R}^n \ s.t.: \ x \cdot u \in \mathbb{Z} \text{ for all } u \in \Lambda\}. \tag{5.14}$$

If a lattice can be obtained from another one by a rotation, reflection and change of scale we say that the two lattices are equivalent (or similar). Two generators matrices define equivalent lattices if and only if they are related by $M' = cUMB$, where c is a non zero constant, U is a matrix with integer entries and determinant ± 1, and B is a real orthogonal matrix. Then, the corresponding Gram matrices are related by $A' = c^2 U A {}^t U$. If $c = 1$ the two lattices are congruent and if also $\det U = 1$ they are directly congruent. Quadratic forms corresponding to congruent lattices are called integrally equivalent, so there is a one to one correspondence between congruence classes of lattice and integral equivalence classes of quadratic forms. If Λ is a lattice in n-dimensional space that is spanned by n independent vectors (i.e. a full rank lattice), then M has rank n, A is a positive definite matrix, and the associated quadratic form is called a positive definite form. A lattice or a quadratic form is called integral if the inner product of any two lattice vectors is an integer or, equivalently, if the Gram matrix A has integer entries. One can prove that a lattice is integral if and only if $\Lambda \subseteq \Lambda^*$. An integral lattice with $\det \Lambda = 1$, or equivalently with $\Lambda = \Lambda^*$ is called unimodular or self-dual. If Λ is integral then the inner product $x \cdot x$ is necessarily an integer for all points x of the lattice. If $x \cdot x$ is an even integer for all $x \in \Lambda$ then the lattice is called even, otherwise odd. Even unimodular lattices exist if and only if the dimension is a multiple of 8, while odd unimodular lattices exist in all dimensions.

For a lattice Λ let N_m be the number of vectors $x \in \Lambda$ of norm $m = x \cdot x$. Thus, N_m is also the number of integral vectors ζ that are solutions of the Diophantine equation

$$\zeta A^t\zeta = m \tag{5.15}$$

or, in other words, the number of times that the quadratic form associated with Λ represents the number m. The (genus one) theta series of a lattice Λ is a holomorphic function on the Siegel upper half space $\mathbb{H}_1$, defined by

$$\Theta_\Lambda(\tau) = \sum_{x\in\Lambda} q^{x\cdot x} = \sum_{m=0}^{\infty} N_m q^m, \tag{5.16}$$

where $q = e^{\pi i\tau}$ and $\tau \in \mathbb{H}_1$. For example, the theta series associated to the lattice $\mathbb{Z}$ is the classical Jacobi theta constant $\Theta_\mathbb{Z}(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots \equiv \theta[^0_0](\tau)$, see Section 5.8. This definition generalizes to theta series of arbitrary genus g. In this case the vector ζ becomes a $g \times n$ matrix $\underline{\zeta}$ with integer entries. In addition, one also introduces a $g \times n$ array $\underline{x}$ whose rows are the vectors of the lattice Λ. It can be written as $\underline{x} = \underline{\zeta}M$. Let $N_{\underline{m}} \in \mathbb{Z}$ be the number of integral matrix solutions of the Diophantine system

$$\underline{\zeta}A^t\underline{\zeta} = \underline{m}, \tag{5.17}$$

where $\underline{m}$ is a $g \times g$ symmetric matrix with integer entries. The component (i, j) of $\underline{m}$ represents the scalar product between the vectors $x_i \in \Lambda$ and $x_j \in \Lambda$ of $\underline{x}$. Thus, $N_{\underline{m}}$ is also the number of the sets $\underline{x}$ of g-vectors such that $x_i \cdot x_j = m_{ij}$. In the same spirit of the genus one case, the genus g theta series associated to a lattice Λ is a holomorphic function on the Siegel upper half space $\mathbb{H}_g$, defined by

$$\Theta^{(g)}_\Lambda(\tau) = \sum_{x\in\Lambda^{(g)}} e^{\pi i \operatorname{Tr}(x\cdot x\tau)} = \sum_{\underline{\zeta}\in\mathbb{Z}^{g,n}} e^{\pi i \operatorname{Tr}(\underline{\zeta}A^t\underline{\zeta}\tau)} = \sum_{\underline{m}} N_{\underline{m}} \prod_{i\le j} e^{\pi i m_{ij}\tau_{ij}}, \tag{5.18}$$

and $\tau \in \mathbb{H}_g$. Lattice theta series corresponding to a self-dual n-dimensional lattice, with n divisible by 8, is a modular form of weight $\frac{n}{2}$ with respect to the group $\Gamma_g(1, 2)$ if the lattice is odd and with respect to Γ_g if the lattice is even. Thus, lattice theta series associated to 16-dimensional self-dual lattices are modular forms of weight 8. There are eight 16-dimensional self-dual lattice [CS], two even and six odd, and they can be obtained from the root lattice of some Lie algebra. See also [DbMS, OPSY, MV2]. In what follows we will use a nice property of lattice theta series when restricted to block diagonal period matrices: indeed, they factorize in a very simple way when $\tau \in \mathbb{H}_k \times \mathbb{H}_{g-k}$:

$$\Theta_\Lambda^{(g)} \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_{g-k} \end{pmatrix} = \Theta_\Lambda^{(k)}(\tau_k)\Theta_\Lambda^{(g-k)}(\tau_{g-k}). \tag{5.19}$$

5.7.2 Fourier coefficients of lattice theta series

In order to express the relations between lattice theta series and the classical theta constants, we first expand in Fourier series the lattice theta constants. We just need the coefficient $N_{\underline{m}}$ of the series (5.18) for some integer matrix $\underline{m}$. It is known (cf. [OPSY]) that in genus five the eight theta series are all independent, whereas for lower genus there are linear relations among them. Thus, we have to choose at least eight $\underline{m}$ in such a way that the matrix of the Fourier coefficients $N_{\underline{m}}$ of the eight theta series has rank 8. In Table 5.7 are shown the Fourier coefficients for the eight theta series up to $g = 5$. We computed the coefficients for the matrices:

$$m_1 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \quad m_2 = \begin{pmatrix} 2&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \quad m_3 = \begin{pmatrix} 3&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \quad m_4 = \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}$$

$$m_5 = \begin{pmatrix} 2&0&0&0&0\\0&2&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \quad m_6 = \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \quad m_7 = \begin{pmatrix} 2&0&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \quad m_8 = \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0 \end{pmatrix}$$

$$m_9 = \begin{pmatrix} 2&0&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&0 \end{pmatrix} \quad m_{10} = \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}.$$

Appealing to the geometric interpretation for the matrices $\underline{m}_k$, with $k = 1, \cdots, 10$, for each of the eight 16-dimensional self-dual lattices, we are looking for the number of integer solutions of the Diophantine equation $\underline{\zeta} A_\Lambda {}^t\underline{\zeta} = \underline{m}_k$. In other terms, we are counting the number of sets $\underline{x}$ of five vectors in the lattice Λ such that the vector x_i has norm $(m_k)_{ii}$ and the inner product with the vector x_j is $x_i \cdot x_j = (m_k)_{ij}$. It is clear that the Fourier coefficients corresponding, for example, to the matrix m_4 can be interpreted as the Fourier coefficients for the genus two theta series in which the two orthogonal vectors x_1 and x_2 have both norm 1, but also as the coefficients of the theta series of genus $g > 2$ in which the vectors x_i with $i > 2$ have null norm. It is not hard to perform this computation using a software like Magma, although the computation of the coefficients corresponding to the matrix $\mathrm{diag}(2, 2, 2, 2, 0)$ may take some hours.

The notation of the Table 5.7 is the same as in [OPSY]. The rows contain the Fourier coefficients of the theta series corresponding to the eight lattices $(D_8 \oplus D_8)^+$, $\mathbb{Z} \oplus A_{15}^+$, $\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$, $\mathbb{Z}^4 \oplus D_{12}^+$, $\mathbb{Z}^8 \oplus E_8$, $\mathbb{Z}^{16}$, $E_8 \oplus E_8$, and D_{16}^+, where the last two lattices are the even ones. At the top of the columns we just indicated the diagonal elements of the matrices $\underline{m}_k$, the other elements being zero. As anticipated, the rank of the full matrix of the coefficient is eight, thus no linear relations between genus five theta series exist. However, considering the same matrix for genus less than five one can obtain the relations between theta series, as we will show in the following, for every $g \leq 4$. In the top of the table we write in bold the matrices strictly necessary for the computation, whereas some other columns are added as a check. The same convention will be used throughout in the chapter.

	$(\mathbf{1,0,0,0,0})$	$(\mathbf{2,0,0,0,0})$	$(\mathbf{3,0,0,0,0})$	$(\mathbf{1,1,0,0,0})$	$(2,2,0,0,0)$	$(\mathbf{1,1,1,0,0})$	$(2,2,2,0,0)$	$(\mathbf{1,1,1,1,0})$	$(\mathbf{2,2,2,2,0})$	$(\mathbf{1,1,1,1,1})$
$\Theta_{(D_8\oplus D_8)^+}$	0	224	4096	0	38976	0	5069568	0	475270656	0
$\Theta_{\mathbb{Z}\oplus A_{15}^+}$	2	240	4120	0	43680	0	5765760	0	518918400	0
$\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}$	4	256	4144	8	48896	0	6676992	0	644668416	0
$\Theta_{\mathbb{Z}^4\oplus D_{12}^+}$	8	288	4192	48	60864	192	9181440	384	964200960	0
$\Theta_{\mathbb{Z}^8\oplus E_8}$	16	352	4288	224	90944	2688	17176320	26880	2316142080	215040
$\Theta_{\mathbb{Z}^{16}}$	32	480	4480	960	175680	26880	47174400	698880	8858304000	16773120
$\Theta_{E_8\oplus E_8}$	0	480	0	0	175680	0	47174400	0	9064742400	0
$\Theta_{D_{16}^+}$	0	480	0	0	175680	0	47174400	0	8858304000	0

Table 5.7: Fourier coefficients for the lattice theta series.

5.8 Riemann theta constants and the forms $\Xi_8^{(g)}$

The form $\Xi_8^{(g)}[0^{(g)}]$, appearing in the expression for the superstring chiral measure, belongs to $M_8(\Gamma_g(2))^{O^+}$, the space of modular forms of weight eight with respect to the group $\Gamma_g(2)$, and invariant under the action of $O^+ := \Gamma_g(1,2)/\Gamma_g(2)$. In Section 5.3.3 a basis for these spaces has been found for $g \leq 5$ and a suitable linear combination among these basis vectors has been obtained by imposing the constraints of Section 2.3.

Before starting the computation of the Fourier coefficients of the functions defined in Section 5.3.3 we recall briefly the definition, introduced in Chapter 3, of theta constants with characteristics, which are a powerful tool for constructing modular forms on $\Gamma_g(2)$. An even characteristic is a $2 \times g$ matrix $\Delta = [^a_b]$, with a, $b \in \{0,1\}$ and $\sum a_i b_i \equiv 0 \bmod 2$. Let $\tau \in \mathbb{H}_g$, the Siegel upper half space, then we define the theta constants with characteristic:

$$\theta[^a_b](\tau) := \sum_{^t m \in \mathbb{Z}^g} e^{\pi i((m+a/2)\tau{}^t(m+a/2)+(m+a/2){}^t b)}, \tag{5.20}$$

where m is a row vector. Thus, theta constants are holomorphic functions on $\mathbb{H}_g$. One can build modular forms of weight eight as suitable polynomials of degree sixteen in the theta constants. Defining the $g \times g$ symmetric matrix M with entries $M_{ii} = m_i^2 + a_i m_i + \frac{a_i^2}{4}$, $i = 1, \cdots, g$ and $M_{ij} = m_i m_j + \frac{a_j}{2} m_i + \frac{a_i}{2} m_j + \frac{a_i a_j}{4}$, $1 \leq i < j \leq g$, the definition of theta constant can be rewritten as

$$\begin{aligned} \theta[^a_b](\tau) := & \sum_{m \in \mathbb{Z}^g} (-)^{\frac{a_1 b_1}{2}+\cdots+\frac{a_g b_g}{2}} (-)^{b_1 m_1 + \cdots + b_g m_g} e^{\pi i \operatorname{Tr}(M\tau)} \\ = & (-)^{\frac{a_1 b_1}{2}+\cdots+\frac{a_g b_g}{2}} \sum_{m \in \mathbb{Z}^g} (-)^{b_1 m_1 + \cdots + b_g m_g} \prod_{i \leq j} e^{\pi i (2-\delta_{ij}) M_{ij} \tau_{ij}} \\ = & \sum_{A \in \frac{\mathbb{Z}^{g,g}}{4}, {}^t A = A} N_A \prod_{i \leq j} e^{\pi i A_{ij} \tau_{ij}}, \end{aligned} \tag{5.21}$$

where A is a symmetric $g \times g$ matrix with entries in $\frac{1}{4}\mathbb{Z}$ and N_A is an integer coefficient. In particular N_A is the number of times[7] that the particular matrix A appears in the sum (5.21). Note that the factor $(-)^{\frac{a_1 b_1}{2}+\cdots+\frac{a_g b_g}{2}}$ is a global sign depending only on the characteristic Δ and the coefficient $(-)^{b_1 m_1+\cdots+b_g m_g}$ is a sign depending on the second row of the theta characteristic and on the matrix M.

In previous sections, we have computed the dimensions of the spaces of O^+-invariants for $g \leq 5$. It turned out that these dimensions are 3, 4, 5, 7 and 8 for $g = 1, 2, 3, 4$ and 5 respectively, see Proposition 5.5.1. Moreover, a basis has been provided for each of these genera, by means of the classical Riemann theta constants. For each genus $g \leq 5$ we chose the bases reported in Table 5.8, where the symbol $\surd$ means that the same function as in lower genus has been taken as element of the basis (with obvious modifications). We will indicate generically with $e_i^{(g)}$ the elements of the genus g basis. Each function

Basis/g	1	2	3	4	5
F_1	$\theta[0]^{16}$	$\surd$	$\surd$	$\surd$	$\surd$
F_2	$\theta[0]^4 \sum_\Delta \theta[\Delta]^{12}$	$\surd$	$\surd$	$\surd$	$\surd$
F_{16}	$\sum_\Delta \theta[\Delta]^{16}$	$\surd$	$\surd$	$\surd$	$\surd$
F_3		$\theta[0]^8 \sum_\Delta \theta[\Delta]^8$	$\surd$	$\surd$	$\surd$
F_{88}			$\sum_{(\Delta_i,\Delta_j)_o} \theta[\Delta_i]^8 \theta[\Delta_j]^8$	$\surd$	$\surd$
F_8				$(\sum_\Delta \theta[\Delta]^8)^2$	$\surd$
$G_3[0]$				$G_3[0]$	$\surd$
$G_4[0]$					$G_4[0]$

Table 5.8: Basis for the O^+-invariants

in Table 5.8 is a suitable polynomial of degree sixteen in the theta constants and the forms $\Xi_8^{(g)}[0^{(g)}]$ are suitable linear combinations of them. In order to compare the two expressions of the proposed superstring chiral measure for $g \leq 5$ we also need the Fourier coefficients of the basis of the O^+-invariants. In general, given two series $\sum_n a_n q^n$ and $\sum_m b_m q^m$, their product is $\sum_n a_n q^n \sum_m b_m q^m = \sum_k c_k q^k$, with $c_k = \sum_{m+n=k} a_n b_m$. In this way one computes the Fourier coefficients of the eight functions starting from the ones of the theta constants. However, for increasing g the computation becomes extremely lengthy, due to the huge number of monomials appearing in the definition of the $e_i^{(g)}$. Thus, although in principle possible by hand, we perform the computation using a computer and the $C++$ language, see Section 5.12.

5.9 CDG ansätze and OPSY ansätze for $\Xi_8^{(g)}$

Before starting the computation of the Fourier coefficients we review the expressions of the forms $\Xi_8^{(g)}[0^{(g)}]$ for $g \leq 5$ in both formalisms. Again we will call $\Xi_8^{(g)}[0]_{CDG}$ the forms

[7] Counted with signs given by the factor multiplying the product of exponentials.

we have defined and $\Xi_8^{(g)}[0]_{OPSY}$ the forms of [OPSY]. The expressions of the forms $\Xi_8^{(g)}[0^{(g)}]_{CDG}$ constructed using the classical theta constants, see also [CDG1, CDG2] for the case $g \leq 5$, are:

$$
\begin{aligned}
\Xi_8^{(1)}[0]_{CDG} &= \frac{2}{3}F_1^{(1)} - \frac{1}{3}F_2^{(1)},\\
\Xi_8^{(2)}[0]_{CDG} &= \frac{2}{3}F_1^{(2)} + \frac{1}{3}F_2^{(2)} - \frac{1}{2}F_3^{(2)},\\
\Xi_8^{(3)}[0]_{CDG} &= \frac{1}{3}F_1^{(3)} + \frac{1}{3}F_2^{(3)} - \frac{1}{4}F_3^{(3)} - \frac{1}{64}F_8^{(3)} + \frac{1}{16}F_{88}^{(3)},\\
\Xi_8^{(4)}[0]_{CDG} &= \frac{1}{6}F_1^{(4)} + \frac{1}{3}F_2^{(4)} - \frac{1}{8}F_3^{(4)} + \frac{1}{64}F_8^{(4)} - \frac{1}{16}F_{88}^{(4)} - \frac{1}{2}G_3^{(4)}[0^{(4)}] - c_4 J^{(4)},\\
\Xi_8^{(5)}[0]_{CDG} &= \frac{1}{12}F_1^{(5)} + \frac{1}{3}F_2^{(5)} - \frac{1}{16}F_3^{(5)} - \frac{1}{32}F_8^{(5)} + \frac{1}{8}F_{88}^{(5)} - \frac{1}{4}G_3^{(5)}[0^{(5)}]\\
&\quad + 2G_4^{(5)}[0^{(5)}] - c_5 J^{(5)}.
\end{aligned}
$$

Here we have included the terms $-c_4 J^{(4)}$ and $-c_5 J^{(5)}$ to have vanishing cosmological constant on the whole $\mathbb{H}_4$ and $\mathbb{H}_5$ and to compare these functions to the ones of [OPSY]. In particular, $c_4 = \frac{3^2\cdot 5}{2^6\cdot 7\cdot 17}$ and $c_5 = \frac{17}{2^5\cdot 7\cdot 11}$ (see Section 5.6). The forms $\Xi_8^{(g)}[0^{(g)}]_{OPSY}$ defined in [OPSY] by means of the lattice theta series are:

$$
\begin{aligned}
\Xi_8^{(1)}[0]_{OPSY} &= -\frac{31}{32}\Theta_{(D_8\oplus D_8)^+} + \frac{512}{315}\Theta_{\mathbb{Z}\oplus A_{15}^+} - \frac{16}{21}\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} + \frac{1}{9}\Theta_{\mathbb{Z}^4\oplus D_{12}^+}\\
&\quad - \frac{1}{168}\Theta_{\mathbb{Z}^8\oplus E_8} + \frac{1}{10080}\Theta_{\mathbb{Z}^{16}},\\
\Xi_8^{(2)}[0]_{OPSY} &= \frac{155}{512}\Theta_{(D_8\oplus D_8)^+} - \frac{16}{21}\Theta_{\mathbb{Z}\oplus A_{15}^+} + \frac{23}{42}\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} - \frac{3}{32}\Theta_{\mathbb{Z}^4\oplus D_{12}^+}\\
&\quad + \frac{29}{5376}\Theta_{\mathbb{Z}^8\oplus E_8} - \frac{1}{10752}\Theta_{\mathbb{Z}^{16}},\\
\Xi_8^{(3)}[0]_{OPSY} &= -\frac{155}{4096}\Theta_{(D_8\oplus D_8)^+} + \frac{1}{9}\Theta_{\mathbb{Z}\oplus A_{15}^+} - \frac{3}{32}\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} + \frac{101}{4608}\Theta_{\mathbb{Z}^4\oplus D_{12}^+}\\
&\quad - \frac{3}{2048}\Theta_{\mathbb{Z}^8\oplus E_8} + \frac{1}{36864}\Theta_{\mathbb{Z}^{16}},\\
\Xi_8^{(4)}[0]_{OPSY} &= \frac{31}{16384}\Theta_{(D_8\oplus D_8)^+} - \frac{1}{168}\Theta_{\mathbb{Z}\oplus A_{15}^+} + \frac{29}{5376}\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}\\
&\quad - \frac{3}{2048}\Theta_{\mathbb{Z}^4\oplus D_{12}^+} + \frac{23}{172032}\Theta_{\mathbb{Z}^8\oplus E_8} - \frac{1}{344064}\Theta_{\mathbb{Z}^{16}} - b_4\left(\Theta_{E_8\oplus E_8} - \Theta_{D_{16}^+}\right),\\
\Xi_8^{(5)}[0]_{OPSY} &= -\frac{1}{32768}\Theta_{(D_8\oplus D_8)^+} + \frac{1}{10080}\Theta_{\mathbb{Z}\oplus A_{15}^+} - \frac{1}{10752}\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}\\
&\quad + \frac{1}{36864}\Theta_{\mathbb{Z}^4\oplus D_{12}^+} - \frac{1}{344064}\Theta_{\mathbb{Z}^8\oplus E_8} + \frac{1}{10321920}\Theta_{\mathbb{Z}^{16}} - b_5\left(\Theta_{E_8\oplus E_8} - \Theta_{D_{16}^+}\right).
\end{aligned}
$$

Here $b_4 = \frac{2^2\cdot 3^3\cdot 5\cdot 11}{7\cdot 17}$ and $b_5 = -\frac{2^5\cdot 17}{7\cdot 11}$ (see [MV2]) make the cosmological constant vanishing on the whole $\mathbb{H}_4$ and $\mathbb{H}_5$ respectively. One of the goals of this chapter is to show that up to genus five the two expressions for the superstring chiral measure coincide. For $g \leq 4$ this was expected from the uniqueness theorems. Instead, for $g = 5$ the formalism of the classical theta constants and the one of the lattice theta series lead to distinct functions both satisfying the three constraints of Section 2.3. Actually, this indetermination could appear for each choice for the basis of the spaces $M_8^{\theta}(\Gamma_5(2))^{O^+}$ or $M_8^{\theta_S}(\Gamma_5(2))$. Moreover, their difference is proportional to the Schottky form $J^{(5)}$ and the two forms become equivalent if one requires also the vanishing of the cosmological constants, i.e. the vanishing of their sum over all the even characteristics, $\sum_\Delta \Xi_8^{(g)}[\Delta^{(g)}] = 0$.

5.10 Change of basis

In this section we search the relations between the functions defined in Section 5.8 and the lattice theta series. For $g \leq 3$ one can proceed in several way, but for $g \geq 4$ the knowledge of the Fourier coefficients becomes necessary.

5.10.1 The case g=1

In genus one we can expand the eight lattice theta series on the basis of O^+-invariants $F_1^{(1)}$, $F_2^{(1)}$, $F_{16}^{(1)}$ using the Table 2 in [OPSY], page 491, that we reproduce in Table 5.9. There, Λ_i, $i = 0, \cdots, 7$ label the eight lattices and τ_i, b_i and c_i are the coefficients

i	Λ_i	τ_i	b_i	c_i
0	$(D_8 \oplus D_8)^+$	0	1	0
1	$\mathbb{Z} \oplus A_{15}^+$	2	1	0
2	$\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$	4	1	0
3	$\mathbb{Z}^4 \oplus D_{12}^+$	8	1	0
4	$\mathbb{Z}^8 \oplus E_8$	16	1	0
5	$\mathbb{Z}^{16}$	32	1	0
6	$E_8 \oplus E_8$	0	0	1
7	D_{16}^+	0	0	1

Table 5.9: Linear relation between lattice theta series.

of the linear expansions of the series Θ_{Λ_i} on the basis $\Xi_8^{(1)}[0^{(1)}]_{OPSY}$, $\Theta_{\Lambda_0}^{(1)}$, $\Theta_{\Lambda_6}^{(1)}$ for the space $[\Gamma_1(1,2), 8]$. Thus, $\Theta_{\Lambda_i}^{(1)} = \tau_i \Xi_8^{(1)}[0^{(1)}] + b_i \Theta_{\Lambda_0}^{(1)} + c_i \Theta_{\Lambda_6}^{(1)}$. It is easy to show that the relations $\Xi_8^{(1)}[0^{(1)}]_{OPSY} = \frac{1}{16}\theta[^0_0]^4\eta^{12} = \frac{1}{24}F_1^{(1)} - \frac{1}{48}F_2^{(1)}$ (cf. [D], section 4.1), $\Theta_{\mathbb{Z}^{16}} = \theta[^0_0]^{16} \equiv F_1^{(1)}$ (cf. [CS], first formula, page 46), $\Theta_{(D_8\oplus D_8)^+} = -\frac{1}{3}F_1^{(1)} + \frac{2}{3}F_2^{(1)}$ (by the fifth line of Table 5.9) and $\Theta_{E_8\oplus E_8} = \frac{1}{2}F_{16}^{(1)}$ (cf. [CS], last formula, page 47) hold. Thus, the linear relations of Table 5.10 follow immediatly.

Moreover, the lattice theta series in genus one are not all linear independent, but they generate a three dimensional vector space. Therefore, they must satisfy some linear relations, which can be obtained studying the five dimensional kernel of the first three bold columns of Table 5.7 computed with Magma. This give the following relations

Theta series/Basis	F_1	F_2	F_{16}
$\Theta_{(D_8\oplus D_8)^+}$	-1/3	2/3	0
$\Theta_{\mathbb{Z}\oplus A_{15}^+}$	-1/4	15/24	0
$\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}$	-1/6	7/12	0
$\Theta_{\mathbb{Z}^4\oplus D_{12}^+}$	0	1/2	0
$\Theta_{\mathbb{Z}^8\oplus E_8}$	1/3	1/3	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0
$\Theta_{E_8\oplus E_8}$	0	0	1/2
$\Theta_{D_{16}^+}$	0	0	1/2

Table 5.10: Theta series on the basis F_1, F_2 and F_{16}.

among theta series:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 15 & -16 & 0 & 0 & 0 & 1 & 0 & 0 \\ 7 & -8 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8\oplus D_8)^+} \\ \Theta_{\mathbb{Z}\oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4\oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8\oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8\oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

As a check, one can show that these relations are in complete agreement with those that can be computed using Table 5.10. For example, from the second line one reads $15\Theta_{(D_8\oplus D_8)^+} - 16\Theta_{\mathbb{Z}\oplus A_{15}^+} + \Theta_{\mathbb{Z}^8\oplus E_8} = 0$. From the Fourier coefficients of the eight theta series and from their expansion on the basis of the O^+-invariants we can also find the Fourier coefficients for the three functions $F_1^{(1)}$, $F_2^{(1)}$ and $F_{16}^{(1)}$ expressed as polynomials of degree sixteen in the classical theta constants as showed in Table 5.11. As this space is three dimensional, we just need three coefficients and we choose the ones corresponding to the matrices (that in $g = 1$ are just numbers) 1, 2 and 3. Using the $C++$ program (cf. Section 5.12) we also checked the correctness of the coefficients and further we computed the coefficient corresponding to the matrix 0. Actually, for lower genus this computation can be performed easily by hand.

5.10.2 The case g=2

Using the factorization properties of the classical theta constants one obtains the factorization of the basis of the space of $O^{(+)}$ invariants, whereas for the theta series one can apply property (5.19). Thus, we can find the expansions of the $g = 2$ theta series on the basis of the four O^+-invariants as follows (sometimes for brevity we will indicate

Functions/m	0	**1**	**2**	**3**
F_1	1	32	480	4480
F_2	2	16	576	8384
F_{16}	2	0	960	0

Table 5.11: Fourier coefficients for the F_1, F_2 and F_{16} in genus one.

this space as O_g). In general we have

$$\Theta^{(g)}_{\Lambda_i}(\tau) = \sum_{j=1}^{\dim O_g} k_i^{(g)\,j} e_j^{(g)}, \tag{5.22}$$

where $e_j^{(g)}$ are the basis for the genus g O^+-invariants, written as polynomials in the classical theta constants, and $k_i^{(g)j}$ are the constants we want to determine. The restriction on $\mathbb{H}_1 \times \mathbb{H}_{g-1}$ of the theta series is

$$\begin{aligned}\Theta^{(g)}_{\Lambda_i}(\tau_{1,g-1}) = \Theta^{(g)}_{\Lambda_i}(\tau_1)\Theta^{(g-1)}_{\Lambda_i}(\tau_{g-1}) &= \sum_{j=1}^{\dim O_1} k_i^{(1)\,j} e_j^{(1)} \sum_{m=1}^{\dim O_{g-1}} k_i^{(g-1)\,m} e_m^{(g-1)} \\ &= \sum_{j=1}^{\dim O_1} \sum_{m=1}^{\dim O_{g-1}} k_i^{(1)\,j} k_i^{(g-1)\,m} e_j^{(1)} e_m^{(g-1)},\end{aligned} \tag{5.23}$$

but also

$$\begin{aligned}\Theta^{(g)}_{\Lambda_i}(\tau_{1,g-1}) &= \sum_{j=1}^{\dim O_g} k_i^{(g)\,j} e_j^{(g)}(\tau_{1,g-1}) = \sum_{j=1}^{\dim O_g} k_i^{(g)\,j} \Big(\sum_{l=1}^{\dim O_1} a_j^{(1)l} e_l^{(1)}\Big)\Big(\sum_{m=1}^{\dim O_{g-1}} a_j^{(g-1)m} e_m^{(g-1)}\Big) \\ &= \sum_{j=1}^{\dim O_g} \sum_{l=1}^{\dim O_1} \sum_{m=1}^{\dim O_{g-1}} k_i^{(g)\,j} a_j^{(1)l} a_j^{(g-1)m} e_l^{(1)} e_m^{(g-1)}.\end{aligned} \tag{5.24}$$

The expressions (5.23) and (5.24) must be equal. Thus, for every fixed choice of l and m we obtain a linear equation in $k_i^{(g)\,j}$. The solution of this linear system gives the coefficients in the change of basis. We give the result for the case $g = 2$ in Table 5.12.

As expected (cf. [DvG,D,OPSY]), the matrix of the coefficients has rank four, which is then also the dimension of the kernel and we can determine the linear relations among

Theta series/Basis	F_1	F_2	F_3	F_{16}
$\Theta_{(D_8\oplus D_8)^+}$	1/3	2/3	-1/2	0
$\Theta_{\mathbb{Z}\oplus A_{15}^+}$	7/32	35/64	-45/128	0
$\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}$	1/8	7/16	-7/32	0
$\Theta_{\mathbb{Z}^4\oplus D_{12}^+}$	0	1/4	0	0
$\Theta_{\mathbb{Z}^8\oplus E_8}$	0	0	1/4	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0	0
$\Theta_{E_8\oplus E_8}$	0	0	0	1/4
$\Theta_{D_{16}^+}$	0	0	0	1/4

Table 5.12: Theta series on the basis F_1, F_2, F_3 and F_{16}.

the theta series

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -105 & 224 & -120 & 0 & 0 & 1 & 0 & 0 \\ -21 & 48 & -28 & 0 & 1 & 0 & 0 & 0 \\ -3 & 8 & -6 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8\oplus D_8)^+} \\ \Theta_{\mathbb{Z}\oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4\oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8\oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8\oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

For example, from the third line, we have $-21\ \Theta_{(D_8\oplus D_8)^+} + 48\ \Theta_{\mathbb{Z}\oplus A_{15}^+} - 28\ \Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} + \Theta_{\mathbb{Z}^8\oplus E_8} = 0$. One can verify that the same relations result by the study of the kernel of the first four bold columns of the Table 5.7 of the Fourier coefficients for the lattice theta series.

As for the genus one case, we compute the Fourier coefficients for the four functions $F_1^{(2)}$, $F_2^{(2)}$, $F_3^{(2)}$ and $F_{16}^{(2)}$ both using the previous results and the $C++$ program. The Table 5.13 shows the result.

Functions/m	$(0,0)$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{2},\mathbf{0})$	$(\mathbf{3},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$	$(2,2)$
$\mathbf{F_1}$	1	32	480	4480	960	175680
$\mathbf{F_2}$	4	32	1152	16768	192	243456
$\mathbf{F_3}$	4	64	1408	17152	896	363776
$\mathbf{F_{16}}$	4	0	1920	0	0	702720
F_8	16	0	7680	0	0	2810880
F_{88}	0	0	1024	-16384	0	546816

Table 5.13: Fourier coefficients for the F_1, F_2, F_3 and F_{16} in genus two.

5.10.3 The case $g = 3$

In genus three we can obtain the expansion of the theta series on the basis $e_i^{(3)}$ with the method of factorization explained in the previous section. We report the result in Table 5.14. As expected, the matrix of the coefficients has rank five, thus its kernel has

Theta series/Basis	F_1	F_2	F_3	F_{16}	F_{88}
$\Theta_{(D_8\oplus D_8)^+}$	0	0	0	1/8	-1/16
$\Theta_{\mathbb{Z}\oplus A_{15}^+}$	7/512	35/512	-45/2048	315/4096	-315/8192
$\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}$	1/64	7/64	-7/256	21/512	-21/1024
$\Theta_{\mathbb{Z}^4\oplus D_{12}^+}$	0	1/8	0	0	0
$\Theta_{\mathbb{Z}^8\oplus E_8}$	0	0	1/8	0	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0	0	0
$\Theta_{E_8\oplus E_8}$	0	0	0	1/8	0
$\Theta_{D_{16}^+}$	0	0	0	1/8	0

Table 5.14: Theta series on the basis F_1, F_2, F_3, F_{16} and F_{88} in genus three.

dimension three. Again we find the linear relations studying the kernel of the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 315 & -896 & 720 & -140 & 0 & 1 & 0 & 0 \\ 21 & -64 & 56 & -14 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8\oplus D_8)^+} \\ \Theta_{\mathbb{Z}\oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4\oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8\oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8\oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

As in the two previous cases, the same linear relations follow from the Table 5.7 of the Fourier coefficients of the lattice theta series considering the first five bold columns.

As for genus one and two we compute the Fourier coefficients for the functions $F_1^{(3)}$, $F_2^{(3)}$, $F_3^{(3)}$, $F_{16}^{(3)}$ and $F_{88}^{(3)}$ and we control the result using the computer. In Table 5.15 we show the result. We also compute the Fourier coefficients of the functions $F_8^{(3)}$ and $G_3^{(3)}[0^{(3)}]$. Thus, we get another proof of the relation (5.5):

$$G_3^{(3)}[0^{(3)}] = \frac{1}{64}F_8^{(3)} - \frac{1}{16}F_{88}^{(3)} - \frac{5}{448}(8F_{16}^{(3)} - F_8^{(3)}), \qquad (5.25)$$

as can be check inserting in the previous equation the Fourier coefficients.

5.10.4 The case $g = 4$

The genus four case is the first interesting case because the factorization approach does no more work. The failure of this method is due to the fact that the space of moduli of

Functions/m	$(0,0,0)$	$(\mathbf{1,0,0})$	$(\mathbf{2,0,0})$	$(\mathbf{3,0,0})$	$(\mathbf{1,1,0})$	$(2,2,0)$	$(\mathbf{1,1,1})$	$(2,2,2)$
$\mathbf{F_1}$	1	32	480	4480	960	175680	26880	47174400
$\mathbf{F_2}$	8	64	2304	33536	384	486912	1536	73451520
$\mathbf{F_3}$	8	128	2816	34304	1792	727552	21504	137410560
$\mathbf{F_{16}}$	8	0	3840	0	0	1405440	0	377395200
F_8	64	0	30720	0	0	11243520	0	3019161600
$\mathbf{F_{88}}$	0	0	4096	-65536	0	2187264	0	673677312
$G_3[0]$	1	0	224	4096	0	38976	0	5069568

Table 5.15: Fourier coefficients for the F_1, F_2, F_3, F_{16} and F_{88} in genus three.

curves is not the whole Siegel upper half plane. Indeed, the two theta series defined by the lattice D_{16}^+ and $E_8 \oplus E_8$ are no longer the same function and the differences among this two functions are lost by restricting on the boundary of $\mathbb{H}_4$.

Thus, in order to find the relations between the lattice theta series and the functions $e_i^{(4)}$ we need the Fourier coefficients of the functions $e_i^{(4)}$. We have computed them with the $C++$ program. The results are reported in Table 5.16. Adding the rows of this

	$(\mathbf{0,0,0,0,0})$	$(\mathbf{1,0,0,0,0})$	$(\mathbf{2,0,0,0,0})$	$(\mathbf{3,0,0,0,0})$	$(\mathbf{1,1,0,0,0})$	$(2,2,0,0,0)$	$(\mathbf{1,1,1,0,0})$	$(2,2,2,0,0)$	$(\mathbf{1,1,1,1,0})$	$(\mathbf{2,2,2,2,0})$
$\mathbf{F_1}$	1	32	480	4480	960	175680	26880	47174400	698880	8858304000
$\mathbf{F_2}$	16	128	4608	67072	768	973824	3072	146903040	6144	15427215360
$\mathbf{F_3}$	16	256	5632	68608	3584	1455104	43008	274821120	430080	37058273280
$\mathbf{F_{16}}$	16	0	7680	0	0	2810880	0	754790400	0	141732864000
$\mathbf{F_8}$	256	0	122880	0	0	44974080	0	12076646400	0	2320574054400
$\mathbf{F_{88}}$	0	0	16384	-262144	0	8749056	0	2694709248	0	549726191616
$\mathbf{G_3[0]}$	15	32	3616	61824	-64	655808	256	85511424	-1536	8099185152
$G_4[0]$	1	0	224	4096	0	38976	0	5069568	0	386797056
$J^{(4)}$	0	0	0	0	0	0	0	0	0	-52848230400

Table 5.16: Fourier coefficients for the basis F_1, F_2, F_3, F_{16}, F_{88}, F_8 and $G_3[0]$ in genus four. In addition we compute the coefficients of $G_4[0]$ and of $J^{(4)}$.

table to the ones of Table 5.7 and considering the first seven bold columns, one finds, as expected, that the complete matrix has rank seven. Again, we get the expansions of the lattice theta series on the basis $e_i^{(4)}$. The result is shown in Table 5.17. These Fourier coefficients also provide a proof of the relation (5.9):

$$\begin{aligned} G_4^{(4)}[0^{(4)}] &= \frac{1}{256} F_8^{(4)} - \frac{1}{64} F_{88}^{(4)} + \frac{3}{1792} J^{(4)} \\ &= \frac{1}{448} F_8^{(4)} - \frac{1}{64} F_{88}^{(4)} + \frac{3}{112} F_{16}^{(4)}. \end{aligned}$$

Theta series/Basis	F_1	F_2	F_3	F_{16}	F_{88}	F_8	$G_3[0]$
$\Theta_{(D_8\oplus D_8)^+}$	0	0	0	0	-1/64	1/256	0
$\Theta_{\mathbb{Z}\oplus A_{15}^+}$	7/8192	35/4096	-45/32768	135/16384	-315/65536	45/65536	315/8192
$\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}$	1/512	7/256	-7/2048	0	0	0	21/512
$\Theta_{\mathbb{Z}^4\oplus D_{12}^+}$	0	1/16	0	0	0	0	0
$\Theta_{\mathbb{Z}^8\oplus E_8}$	0	0	1/16	0	0	0	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0	0	0	0	0
$\Theta_{E_8\oplus E_8}$	0	0	0	0	0	1/256	0
$\Theta_{D_{16}^+}$	0	0	0	1/16	0	0	0

Table 5.17: Theta series on the basis F_1, F_2, F_3, F_{16}, F_{88}, F_8 and $G_3[0]$ in genus four.

Moreover, we obtain a linear relation between the lattice theta series

$$\begin{pmatrix} 1 & -1024/315 & 64/21 & -8/9 & 2/21 & -1/315 & -3/7 & 3/7 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8\oplus D_8)^+} \\ \Theta_{\mathbb{Z}\oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4\oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8\oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8\oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

5.10.5 The case $g = 5$

In genus five, we consider the eight columns of Table 5.7. This matrix has rank eight, so all the theta series are linearly independent. As in genus four, to study the relations between the Riemann theta constants and the lattice theta series we need the Fourier coefficients of the functions $e_i^{(5)}$. We have computed them by the computer and we report the result in Table 5.18, that also has rank eight. Gluing this table to the one of

	$(\mathbf{0,0,0,0,0})$	$(\mathbf{1,0,0,0,0})$	$(\mathbf{2,0,0,0,0})$	$(\mathbf{3,0,0,0,0})$	$(\mathbf{1,1,0,0,0})$	$(2,2,0,0,0)$	$(\mathbf{1,1,1,0,0})$	$(2,2,2,0,0)$	$(\mathbf{1,1,1,1,0})$	$(\mathbf{2,2,2,2,0})$	$(\mathbf{1,1,1,1,1})$
$\mathbf{F_1}$	1	32	480	4480	960	175680	26880	47174400	698880	8858304000	16773120
$\mathbf{F_2}$	32	256	9216	134144	1536	1947648	6144	293806080	12288	30854430720	0
$\mathbf{F_3}$	32	512	11264	137216	7168	2910208	86016	549642240	860160	74116546560	6881280
$\mathbf{F_{16}}$	32	0	15360	0	0	5621760	0	1509580800	0	283465728000	0
$\mathbf{F_8}$	1024	0	491520	0	0	179896320	0	48306585600	0	9282296217600	0
$\mathbf{F_{88}}$	0	0	65536	-1048576	0	34996224	0	10778836992	0	2198904766464	0
$\mathbf{G_3[0]}$	155	480	38560	640640	64	7174336	-2304	954147072	22016	90356353536	-225280
$\mathbf{G_4[0]}$	31	32	7200	127360	-64	1279424	256	166624512	-1536	14287938048	12288
$J^{(5)}$	0	0	0	0	0	0	0	0	0	-211392921600	0

Table 5.18: Fourier coefficients for the F_1, F_2, F_3, F_{16}, F_{88}, F_8, $G_3[0]$ and $G_4[0]$ in genus five.

the Fourier coefficients for the lattice theta series we obtain a matrix of rank eight. *So, all the lattice theta series can be expressed as linear combination of $e_i^{(5)}$ and vice versa!* Indeed we can be more precise. As the rank of the whole set of coefficients is 8, we get

8 linear relations among the two bases:

$$F_{16} = 2^5\Theta_{D_{16}^+}, \qquad F_8 = 2^{10}\Theta_{E_8\oplus E_8}, \qquad F_1 = \Theta_{\mathbb{Z}^{16}}, \tag{5.26}$$

$$F_3 = 2^5\Theta_{\mathbb{Z}^8\oplus E_8}, \qquad F_2 = 2^5\Theta_{\mathbb{Z}^4\oplus D_{12}^+}, \qquad F_8 - 4F_{88} = 2^{10}\Theta_{(D_8\oplus D_8)^+}, \tag{5.27}$$

$$-4F_1 - 112F_2 + 7F_3 - 84G_3 = -16384\Theta_{\mathbb{Z}^2\oplus(E_7\oplus E_7)^+}, \tag{5.28}$$

$$-28F_1 - 560F_2 + 45F_3 - 1260G_3 - 10080G_4 = -524288\Theta_{\mathbb{Z}\oplus A_{15}^+}. \tag{5.29}$$

Note that the relations (5.26) can be directly checked. The relations (5.27) also are simply a generalization of the lower genus ones. However, for all the relations we can also give some consistency cheks. Summing each side of the eight equalities over the 528 characteristics we obtain eight identities. For example for the (5.29) we obtain $-524288F_{16}^{(5)} = -524288 \cdot 2^5\Theta_{D_{16}^+}$ and $2^5\Theta_{D_{16}^+}$ is exactly the $F_{16}^{(5)}$. These sums can be performed using Table 5.6 and Table 1 and Appendix B.2 of [MV2]. Moreover, one verifies that also the restriction to $\mathbb{H}_1 \times \mathbb{H}_4$ of each equality is an identity.

5.11 Equivalence of the CDG and the OPSY construction

In this section we prove the equivalence of the two functions constructed using the classical theta functions and the lattice theta series. They at most differ by a multiple of the Schottky form and become identical if one fixes the value of the cosmological constant to zero. We first study the Fourier coefficients of the two $\Xi_8^{(g)}[0^{(g)}]$, then we give an analytic proof of their equivalence.

5.11.1 Fourier coefficients for the partition function

Inserting the Fourier coefficients of the basis $e_i^{(g)}$ and of the lattice theta series in the definition of the functions $\Xi_8^{(g)}[0^{(g)}]$ of Section 5.9 we can compute, for every genus $g \leq 5$, the Fourier expansions of the $\Xi_8^{(g)}[0^{(g)}]$. Table 5.19 shows these coefficients for the two expressions of the forms Ξ_8. We also add 0 in the first column for the functions $\Xi_8^{(g)}[0]_{OPSY}$, because, from the geometric discussion of Section 5.7, it is clear that there are no vectors in the lattice of null norm. We conclude that the two functions are the same up to genus five, apart for an unessential global factor 2^{4g} due to the different definition of the Dedekind function used here and in [OPSY] (cf. also the footnote 6 of this chapter or the footnote 7 in [D], page 17).

5.11.2 Analytic proof of the equivalence of the CDG and the OPSY construction

In this section we give an analytic proof of the equivalence of the two constructions of the forms $\Xi_8[\Delta]$ through the study of their restriction to $\mathbb{H}_1 \times \mathbb{H}_4$. We will show that $(\Xi_8^{(5)}[0^{(5)}]_{CDG} - \Xi_8^{(5)}[0^{(5)}]_{OPSY})(\tau_{1,4}) = 0$ on the whole $\mathbb{H}_1 \times \mathbb{H}_4$. To compare the two expressions of the forms Ξ_8 one has to get rid of the factor 2^{4g}. We choose to multiply

	$(\mathbf{0,0,0,0,0})$	$(\mathbf{1,0,0,0,0})$	$(\mathbf{2,0,0,0,0})$	$(\mathbf{3,0,0,0,0})$	$(\mathbf{1,1,0,0,0})$	$(2,2,0,0,0)$	$(\mathbf{1,1,1,0,0})$	$(2,2,2,0,0)$	$(\mathbf{1,1,1,1,0})$	$(\mathbf{2,2,2,2,0})$	$(\mathbf{1,1,1,1,1})$
$\Xi_8^{(1)}[0]_{OPSY}$	0	1	8	12							
$\Xi_8^{(1)}[0]_{CDG}$	0	16	128	192							
$\Xi_8^{(2)}[0]_{OPSY}$	0	0	0	0	1	64					
$\Xi_8^{(2)}[0]_{CDG}$	0	0	0	0	256	16384					
$\Xi_8^{(3)}[0]_{OPSY}$	0	0	0	0	0	0	1	192			
$\Xi_8^{(3)}[0]_{CDG}$	0	0	0	0	0	0	4096	786432			
$\Xi_8^{(4)}[0]_{OPSY}$	0	0	0	0	0	0	0	0	1	$\frac{38976}{17}$	
$\Xi_8^{(4)}[0]_{CDG}$	0	0	0	0	0	0	0	0	65536	$\frac{2554331136}{17}$	
$\Xi_8^{(5)}[0]_{OPSY}$	0	0	0	0	0	0	0	0	0	$\frac{16043183100}{11}$	1
$\Xi_8^{(5)}[0]_{CDG}$	0	0	0	0	0	0	0	0	0	$\frac{16822496762265600}{11}$	1048576

Table 5.19: Fourier coefficients for the two expressions of the form Ξ_8. In the first line of each genus are the coefficients of the OPSY forms and in the second line the ones of the CDG forms.

$\Xi_8^{(g)}[0]_{OPSY}$ by 2^{4g} that implies that the constants b_4 and b_5 of Section 5.9 become $b_4 = -\frac{2^7 \cdot 3}{7 \cdot 17}$ and $b_5 = -\frac{2^5 \cdot 17}{7 \cdot 11}$. Indeed, using the expressions of Section 5.9:

$$\begin{aligned}
\Xi_8^{(5)}[0^{(5)}]_{OPSY}(\tau_{1,4}) &= \Xi_8^{(1)}[0^{(1)}](\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&+ \left(\frac{2^5 \cdot 3 \cdot 13}{7 \cdot 17}\Theta^{(1)}_{\mathbb{Z}^8 \oplus E_8} - \frac{2^6 \cdot 3^2 \cdot 5}{7 \cdot 17}\Theta^{(1)}_{\mathbb{Z}_{16}} + \frac{2^5 \cdot 17}{7 \cdot 11}\Theta^{(1)}_{E_8 \oplus E_8}\right)\left(\Theta^{(4)}_{E_8 \oplus E_8} - \Theta^{(4)}_{D_{16}^+}\right) \\
&= \Xi_8^{(1)}[0^{(1)}](\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&+ \left[\frac{3}{2^2 \cdot 7 \cdot 17}\left(-\frac{2}{3}F_1^{(1)} + 5F_2^{(1)}\right) - \frac{17}{2^4 \cdot 7 \cdot 11}F_{16}^{(1)}\right] J^{(4)},
\end{aligned} \tag{5.30}$$

where we have used the linear relation among the genus four lattice theta series found in 5.10.4, the genus one relations among the lattice theta series and the basis functions $e_i^{(4)}$ of Section 5.10.1, and the fact that[8] $J^{(4)} = -2^8(\Theta^{(4)}_{E_8 \oplus E_8} - \Theta^{(4)}_{D_{16}^+})$. With a similar computation we obtain for the form $\Xi_8^{(5)}[0^{(5)}]_{CDG}$:

$$\begin{aligned}
\Xi_8^{(5)}[0^{(5)}]_{CDG}(\tau_{1,4}) &= \Xi_8^{(1)}[0^{(1)}](\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&+ \left(-\frac{3^2}{2^3 \cdot 7 \cdot 17}F_1^{(1)} + \frac{3 \cdot 5}{2^2 \cdot 7 \cdot 17}F_2^{(1)} - \frac{17}{2^4 \cdot 7 \cdot 11}F_{16}^{(1)}\right) J^{(4)} \\
&= \Xi_8^{(1)}[0^{(1)}](\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&+ \left[\frac{3}{2^2 \cdot 7 \cdot 17}\left(-\frac{2}{3}F_1^{(1)} + 5F_2^{(1)}\right) - \frac{17}{2^4 \cdot 7 \cdot 11}F_{16}^{(1)}\right] J^{(4)},
\end{aligned} \tag{5.31}$$

that is exactly the same as (5.30). This and the fact that the sum over the 528 genus five even characteristics of both the forms $\Xi_8^{(5)}[0^{(5)}]$ is a multiple of the Schottky form show the equivalence of the two construcions. Fixing the value of the cosmological constant and getting rid of the factor 2^{4g}, they do not differ neither for a multiple of $J^{(5)}$ because, if so, a term proportional to $F_{16}^{(1)}J^{(4)}$ should appear in the difference of their restrictions due to the fact that $J^{(5)}(\tau_{1,4}) = 2F_{16}^{(1)}J^{(4)}$. The factorizations can be obtained using the

[8] In general $J^{(g)} = -2^{2g}(\Theta^{(g)}_{E_8 \oplus E_8} - \Theta^{(g)}_{D_{16}^+})$.

properties of the lattice theta series (see Section 5.7) and the restrictions properties of the functions $e_i^{(5)}$. Alternatively, one can employs the linear relations found in Section 5.10.5. Indeed changing the basis with those relations one obtains $\Xi_8^{(5)}[0^{(5)}]_{CDG}$ from $\Xi_8^{(5)}[0^{(5)}]_{OPSY}$ and vice versa. This is another check for the computation leading to relations (5.26), (5.27), (5.28) and (5.29).

5.12 The program

In this section we briefly present the structure of the program we used to compute the Fourier coefficients of the functions $e_i^{(g)}$. The code is available on `http://www.dfm.uninsubria.it/thetac/`

An element of $\mathbb{H}_g$ has the generic form:

$$\tau = \begin{pmatrix} \tau_1 & \tau_{g+1} & \cdots & \cdots & \tau_{2g-1} \\ \tau_{g+1} & \tau_2 & \tau_{2g} & \cdots & \tau_{3g-3} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \tau_{2g-2} & \tau_{3g-4} & \cdots & \tau_{g-1} & \tau_{g(g+1)/2} \\ \tau_{2g-1} & \tau_{3g-3} & \cdots & \cdots & \tau_g \end{pmatrix}. \tag{5.32}$$

Thus, from the definition of theta constant (5.21) it is clear that truncating the series we obtain a polynomial in $g(g+1)/2$ variables $q_{ij} = e^{\pi i \tau_{ij}}$, with $1 \leq i \leq j \leq g$ and the same holds true for the functions $e_i^{(g)}$. It will be useful to rewrite the definition (5.21) as:

$$\theta[^b_a](\tau) = (-)^{\frac{1}{2}\sum_i a_i b_i} \sum_{m \in \mathbb{Z}^g} (-)^{\sum_i m_i b_i} \left(\prod_i p_{ii}^{(2m_i+a_i)^2} \right) \left(\prod_{i<j} p_{ij}^{2(2m_i+a_i)(2m_j+a_j)} \right), \tag{5.33}$$

with $p_{ij} = q_{ij}^{1/4}$, so the exponents are integer numbers. This renders faster the computations with the computer. The previous expansion may be thought as a polynomial in p_{ii} with coefficients that are polynomials in p_{ij}, $i < j$ (this observation will be useful later).

To perform the computation we have defined some `C++` classes. First, we have defined the generic class `Polynomial`, defined as `template <typename CffType, typename ExpType> class Polynomial`, which accepts two types as parameters, `CffType` and `ExpType`. `CffType` represents the type of the coefficient of a single monomial in `Polynomial` and `ExpType` the type of the exponent. In order to perform the elementary operations with polynomials, we have introduced the operators of addition, multiplication and raising to power for the `Polynomial` class. Then, we have defined a simple polynomial with integer coefficients: `typedef Polynomial<cln::cl_I, short> IntPol`[9]. This type will be the coefficient for the `ThetaPol` polynomial, which will be used to

[9] To manage long integer coefficients we use the `cl_I` class from CLN library, `http://www.ginac.de/CLN/`

represent the series expansion of the theta constants: `typedef Polynomial<IntPol, unsigned short> ThetaPol`.

In order to compute the Fourier coefficients corresponding to the ten diagonal matrices of Section 5.7.2 we proceed as follows. For each even theta constant[10] we "fill up" the `ThetaPol` by computing the (finite) sums (5.33) in which each component of $m \in \mathbb{Z}^g$ is no bigger than three. Using the operations on the polynomials we just defined, the `ThetaPol`'s are the bricks to build up the functions $e_i^{(g)}$ from their definition. Therefore, the function $e_i^{(g)}$ has the generic form:

$$e_i^{(g)}(\tau) = \sum_{n_1,\cdots,n_g \in \mathbb{N}_0} (\cdots) p_{11}^{n_1} \cdots p_{gg}^{n_g}, \tag{5.34}$$

where in $(\cdots)$ there are the non diagonal or constant terms. Note that the exponents of the diagonal terms p_{ii} are always positive, hence multiplying the polynomials of the theta constants the exponents cannot decrease. Due to our choice for the ten matrices, we can introduce a sort of "filter" for the value of the exponents. Roughly speaking, in the expansion (5.34) we neglect the terms with exponent of p_{ii} "bigger than the ones appearing in the diagonal of the ten matrices". This allows us to make the computations very fast. Thus, the Fourier coefficients of the matrix $m = \mathrm{diag}(m_1, \cdots, m_g)$ is the constant term in $(\cdots)$ of the monomial with $n_1 = 4m_1, \cdots, n_g = 4m_g$.

[10]Recall that the number of even theta constants is $2^{g-1}(2^{g+1})$.

Chapter 6

Conclusions and perspectives

In this thesis we have considered the problem of the computation of superstring amplitudes. Such topic directly leads to the analysis of the perturbative formulation of superstring theory. If the bosonic case is well understood and finds solid basis in theorems of algebraic geometry, the supersymmetric case is more delicate and includes certain problems not yet solved. Actually, it is a well known fact that the path integral formulation of superstring theory at higher genus is affected by ambiguities, mainly due to the difficulty in finding a supercovariant formulation. Indeed, even though the super moduli space of super Riemann surfaces can be locally split in even and odd part, this does not work globally and the result comes out to depend on the choice of a bosonic slice in a non covariant way. For these reasons, the path integral computation of amplitudes from first principles is highly nontrivial and it is not clear how it can be performed for genus higher than two. In a series of papers, D'Hoker and Phong have been able to determine the genus two amplitudes by direct calculation. This is a remarkable result. Their solution is expressed in terms of some suitable equivariant modular forms. As a byproduct they formulated a set of ansätze that should be satisfied by the amplitudes at all genera. However, they were not able to obtain an analog expression for the amplitudes neither for the genus three case. This must be imputed to the too much restrictive assumptions for the expression of the measure. In this thesis we have proposed a slightly modification of their ansätze and we have shown that a solution exists and for low genus it turns out to be unique. We have provided a detailed explanation of the results and the methods adopted to determine a good candidate for the superstring measure. Our method is based on the representation theory of finite groups. Indeed, this approach makes clear the transformation properties that the superstring measure is required to satisy. We have used the action of the finite symplectic group on modular forms, and we have recovered that the representation space of interest for the computation of superstring measure is the space of modular forms of weight eight that are left invariant by a suitable subgroup of the symplectic group. In the construction of the measures we have token advantage of the theory of induced representation. This approach is similar to the method used by Wigner to classify the irreducible representations of the Poincaré group induced from the representation of the little group. Indeed, the representation

furnished by the space of forms is built up from the representation given by a suitable subspace invariant under the action of subgroup of the entire modular group. Thus, we have devoted a good part of this thesis to the study of the representations of the finite symplectic group on the space of modular forms of various weight and genus. At low genus one can decompose the space of modular forms in irreducible representation subspaces in a systematic way. In principle this technique can be extended to higher genus but the complexity of the computations increase rapidly for growing g.
Our axiomatic strategy, yet adopted by D'Hoker and Phong and by many other authors earlier (see [Mo1] and references therein), is not to provide a direct computation from first principles, but consists in looking for reasonable ansätze, inspired by first principles, which should lead to a unique solution that must then satisfy a number of tests. The ansatz we have chosen is very closed to the one of D'Hoker and Phong, but the very slight modification has been proved to be crucial in providing a solution. This ansatz is the more general one after the assumption of the validity of the relation (2.15). We have seen that such assumption is highly criticizable, but the fact that for low genus it gives rise to the existence of a unique solution is quite encouraging. For genus four, the result seem to be weakened by the fact that the Shottky set has strict positive codimension in the Siegel upper half plane, and, indeed, we have proved the uniqueness in a restricted form. However, in [OPSY] the authors show the uniqueness of the solution without any restriction.
We have also checked that, as in general predicted by supersymmetry, the cosmological constant computed with our solutions vanishes up to genus four. In [Mo2] it has been shown that also the two-point and the three-point functions vanish, according to the non renormalization theorems (cf. [Ma1,Ma2,Mo4]), but the proof is restricted to hyperelliptic surfaces, which are a zero measure subset for genus higher than 2. A complete proof of the vanishing of the two-point function at genus $g = 3$ for our solution has been provided in [GSM]. The check of the same condition for the three point function and for the $g = 4$ case have yet to be provided beyond the hyperelliptic case.
Recently other papers appeared providing new expressions and generalizations of our results, see for example [Gr, SM1, MV1, OPSY]. In particular, the remarkable paper of Grushevsky [Gr] provided an elegant formal expressions for the solution at any genus g. Unfortunately, such expressions involve square and higher order roots of modular forms, which are not well defined in general. In [SM1] it has been proved that the Grushevsky's expression is well defined for $g = 5$. In Chapter 5 we have considered the genus five case and we have obtained a candidate for the superstring measure constructed by means of the classical theta constants in which no roots appear. Another candidate for the $g = 5$ case is determined in [OPSY]. In this paper the authors started from a basis of the lattice theta series of weight eight $M_8^{\theta_S}(\Gamma_5(2))$, whereas we start from a basis of the genus five modular forms of weight eight $M_8^{\theta}(\Gamma_5(2))^{O^+}$. In each case it has been determined a unique solution modulo $J^{(5)}$. By computing the Fourier coefficients of the functions of both bases we have shown that these two solution are equivalent modulo $J^{(5)}$ and coincide if we impose the vanishing of the cosmological constant. It is not yet

clear if these two expressions are equivalent to the solution of Grushevsky. In order to prove the equivalence of our solution and the one found in [OPSY] we also determined an explicit identification of the spaces $M_8^{\theta_S}(\Gamma_g(2))$ and $M_8^{\theta}(\Gamma_g(2))^{O^+}$ for $g = 1, \ldots, 5$. This identification is based on the result of Salvati Manni, asserting that in any genus $M_8^{\theta}(\Gamma_g(2))^{O^+} \subseteq M_8^{\theta_S}(\Gamma_g(2))$ (see Proposition 5.5.1). As the dimension of the space of the lattice theta series for $g \geq 5$ is eight, it is clear that in genus greater than five it cannot exist a solution polynomial in the theta constants. To search for a solution, if it exists, one must include the non normal part of the ring of modular forms. Indeed, for $g \geq 5$ there might exist modular forms that are not polynomial in theta constants and that satisfy the constraints. These considerations also lead to the question wether for $g > 5$ the ambiguity left open by the constraints is again an indetermination of the Schottky form contribution or has a stronger nature. Moreover, the trick of fixing the value of the cosmological constant does not work for $g > 5$, as pointed out in [DbMS]. The answer to this kind of questions would lead to a generalization of the uniqueness theorems proved up to genus four.

In any case, as remarked in [DbMS] the solutions at genus 5 are no more uniquely determined by the ansatz and the vanishing of the cosmological constant is indeed added as a further condition. Also, it is not clear whether the three ansätze and the condition on the cosmological constant imply the uniqueness of the solution. This means that the ansatz does not definitively encode the whole physical requirements and one is led to turn back to a direct analysis by first principles, as done by D'Hoker and Phong for the genus two case. Only such a kind of analysis could give a mathematical proof of the starting assumption on the general form for the amplitudes (2.15) or improve it.

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