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Robust Optimization in Data Envelopment Analysis

Extended theory and applications

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Abstract

Performance evaluation of decision-making units (DMUs) via the data envelopment analysis (DEA) is confronted with multi-conflicting objectives, complex alternatives and significant uncertainties. Visualizing the risk of uncertainties in the data used in the evaluation process is crucial to understanding the need for cutting edge solution techniques to organizational decisions. A greater management concern is to have techniques and practical models that can evaluate their operations and make decisions that are not only optimal but also consistent with the changing environment. Motivated by the myriad need to mitigate the risk of uncertainties in performance evaluations, this thesis focuses on finding robust and flexible evaluation strategies to the ranking and classification of DMUs. It studies performance measurement with the DEA tool and addresses the uncertainties in data via the robust optimization technique.

The thesis develops new models in robust data envelopment analysis with applications to management science, which are pursued in four research thrust. In the first thrust, a robust counterpart optimization with nonnegative decision variables is proposed which is then used to formulate new budget of uncertainty-based robust DEA models. The proposed model is shown to save the computational cost for robust optimization solutions to operations research problems involving only positive decision variables. The second research thrust studies the duality relations of models within the worst-case and best-case approach in the input – output orientation framework. A key contribution is the design of a classification scheme that utilizes the conservativeness and the risk preference of the decision maker. In the third thrust, a new robust DEA model based on ellipsoidal uncertainty sets is proposed which is further extended to the additive model and compared with imprecise additive models. The final thrust study the modelling techniques including goal programming, robust optimization and data envelopment to a transportation problem where the concern is on the efficiency of the transport network, uncertainties in the demand and supply of goods and a compromising solution to multiple conflicting objectives of the decision maker.

Several numerical examples and real-world applications are made to explore and demonstrate the applicability of the developed models and their essence to management decisions. Applications such as the robust evaluation of banking efficiency in Europe and in particular Germany and Italy are made. Considering the proposed models and their applications, efficiency analysis explored in this research will correspond to the practical framework of industrial and organizational decision making and will further advance the course of robust management decisions.

Acknowledgement

I would like to express my sincere gratitude to my supervisor, Professor Mehdi Toloo, for his continuous support and guidance towards this piece of research. His immerse knowledge and zeal for research publication motivated and helped me a lot in all the time of research and writing of this thesis. The exchange with him and the prompt response to queries is what every PhD candidate would wish for from a supervisor. I could not have imagined a better hardworking supervisor with lot of patience and time for my PhD study.

I cannot but heartily appreciate the encouragement and suggestions from my co-supervisor, Professor Matteo Rocca. My interest to work on robust optimization sparked from a research project with him and Prof Giovanni Crespi during my first year which later, I would combine the expertise of Prof. Toloo in Data Envelopment Analysis to produce this thesis. I am indeed grateful to all for this new defining research direction. I am also very grateful for the inspiration I had from Prof. Raffaello Seri and Prof. Ivan Moscati and all my PhD professors. In particular, I would like to appreciate the services of Ms. Gabriella Franchi for the administrative support I received at Insubria and Prof. Jana Hanclova for the hospitality during my visit at the Technical University of Ostrava, Czech Republic.

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All glory goes to God, my creator who has been the source of my strength, protection and hope throughout this 3-year PhD programme.

Dedication

To my mum, Faustina Kwofie and grandma, Martha Tanoah (blessed memory, 2018) for their unconditional love and support. None educated but got me educated to this level.

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"*If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties*". - Francis Bacon, The Advancement of Learning

Chapter 1: Introduction

1.1 Background

Data envelopment analysis (DEA) and robust optimization (RO) are somewhat two separate disciplines that have attracted management interest in the operations research (OR) and management science domain. Respectively, the two are methodologies that are frequently used to evaluate an organization's performance and robustness. The DEA, in particular, is a nonparametric efficiency measuring tool in operations research and economics that uses mathematical programming technique to evaluate the performance of peer units (e.g. universities, hospitals, bank branches) known as decision-making units (DMU) in terms of multiple inputs and multiple outputs. The evaluation of operations with DEA, i.e. how resources (inputs) are used to obtain products/services (outputs) is however masked in a complex real-world uncertain environment and so is the data for the evaluation. Uncertainty in data poses many challenges for management decisions. Indeed, the performance of the DMUs can be highly unstable and unreliable. The effect of uncertain data in the DEA models can lead the decision maker to extreme ranking decisions. In managerial applications, this also amounts to the difficulty in the objective of improving inefficient operations or benchmarking DMUs. The critical question one asks: is it possible to develop a model of uncertainty for the DEA that incorporates the randomness and uncertainties in inputs and outputs data, ensures less complex and tractable formulation and provide ranking strategies commensurable with the conservativeness of decision makers?

The purpose of this thesis is to find an approach in a direction that answers these questions. The general approach used is the RO technique for the DEA. The RO is a widely used optimization technique that addresses the issue of imperfect knowledge or uncertainty in data in mathematical programming problems from the perspective of computational tractability. The RO approach was originally introduced by Soyster (1973) and independently developed by Mulvey, Vanderbei, & Zenios (1995), Ben-Tal & Nemirovski (1998, 1999, 2000) and Bertsimas & Sim (2004). The modeling process is based on a scenario description of uncertainty or the design of uncertainty set from whence the uncertain data are immunized. The most common technique of the RO is perhaps, to consider the *worst-case scenario* and trade-off between performance and robustness as different scenario occurs. In the RO application in DEA, this is done with the formulation of models that yield efficiency solution guaranteed to be good for all or most possible realization of the uncertain inputs and outputs data in a pre-specified set: polyhedral, ellipsoidal uncertainty set or discrete set of scenarios. The RO was introduced in DEA by Sadjadi & Omrani (2008). However, the more popular models as developed in Sadjadi & Omrani (2008), Shokouhi et al (2010) and Omrani (2013) are only concerned with the robust ranking of DMUs and uncertainty in either inputs or outputs. In our literature survey of the few works done in the robust DEA, we observe that the general issue of constraint feasibility and complexity concerns for uncertainties in both inputs and outputs data as well as a robust frontier characterization for the production possibility set are yet to be attended to. Moreover, the basic duality relations in robust models and the relationship between the input- and output- oriented robust models have not been established. There is also a conspicuous lack of classification scheme for DMUs and a comparative analysis of robust DEA to the other techniques in the literature. We feel that for the robust DEA to have impact in theory and practice, these issues must be addressed. These concerns among others motivate the contributions of this research which are outlined in Section 1.3.

Structure of the chapter. In Section 1.2, we discuss the motivation of the RO in DEA. The research thrust, and contributions of the thesis are provided in Section 1.3. In Section 1.4, we outline the structure of this thesis.

1.2 Why robust optimization in data envelopment analysis?

In performance evaluation using the data envelopment analysis (DEA), the inputs and outputs data are assumed to be precisely known and as a result uncertainty in the data and its effect, subsequently in the efficiency scores and the ranking of the DMU is ignored. It is conceivable that measuring inputs utilization to outputs with data precision which nonetheless are contaminated with errors (measurement, prediction, etc.), noise and vagueness is tantamount to ineffective, wobbly and volatile decision towards efficiency.

The RO is one of the mathematical programming techniques which deal with uncertain optimization problems such as the DEA. The essence of the RO technique in DEA efficiency evaluation can be best understood from Ben-Tal & Nemirovski (2000) in their NETLIB case study:

"In real-world applications of Linear Programming, one cannot ignore the possibility that a small uncertainty in the data can make the usual optimal solution of the problem completely meaningless from a practical viewpoint"

Consequently,

"there exists a real need of a technique capable of detecting cases when data uncertainty can heavily affect the quality of the nominal solution, and in these cases to generate a "reliable" solution, one that is immunized against uncertainty".

Applying the RO technique in DEA thus, can overcome the effect of uncertainty in inputs and outputs data. More so, it can provide robust efficiency scores and stable performance ranking that are desirable and commensurate with the conservativeness of decision makers.

1.3 Research thrusts and contributions

The main contribution of this thesis is to advance the modeling of the robust DEA and provide an insightful framework for the ranking of DMUs. From the theoretical point of view, we develop robust DEA models with constraint feasibility for uncertainties in both input and output data. The first of such model is built on the budget of uncertainty set of Bertsimas & Sim (2004) and the second is built on the ellipsoidal uncertainty set of Ben-Tal & Nemirovski (1998, 2000). The models are formulated in their reduced form to ensure less computational difficulty. We provide a classification scheme for DMUs and then study the duality relations of the proposed models in the input – and output – orientation form. From the practical point of view, we apply these models to the efficiency analysis of banks in Europe in general and Germany and Italy in particular, where we compare and rank the operation strategies of banks under different robust approaches to data uncertainty. The final application is made to a transportation problem where the efficiency of the network is considered, and the goal programming technique is adopted with the RO to seek a desired compromising solution for the decision maker.

The objectives of this thesis are achieved by four research thrust which is pursued in Chapter 3 to Chapter 7. We conclude this section by describing each of the research thrust.

Robust optimization with nonnegative decision variables: A DEA approach

In this chapter, we propose robust counterparts with nonnegative decision variables – a reduced robust approach which attempts to minimize model complexity. This is an alternative robust formulation to the generally defined robust counterpart optimization with free-in-sign decision variables which to the best of our knowledge has not been considered before. The new framework is extended to the robust DEA with the aim of reducing the computational burden. In the DEA methodology, first we deal with the equality in the normalization constraint and then a robust DEA based on the reduced robust counterpart is proposed. The proposed model is examined with numerical data from 250 European banks operating across the globe. The results indicate that the proposed approach (*i*) reduces almost 50% of the computational burden required to solve DEA problems with nonnegative decision variables; (*ii*) retains only essential (non-redundant) constraints and decision variables without alerting the optimal value.

Duality, classification input – and output – orientations in robust DEA

The second research thrust is concerned with the extension to dual models of the robust DEA in Chapter 3. Duality relations is one of the basic but very important theory in DEA. However, the existing studies in the robust DEA have so far unscripted note on dual model formulations and their relationships. We develop robust models with the input – and output – orientation. For each orientation, we study the multiplier and envelopment models and establish a relationship between them. A key thrust of this research is the design of a classification scheme that utilizes the conservativeness of the decision maker. From the management perspective of robust efficiency interpretation, we classify DMUs into fully robust efficient, partially robust efficient and robust inefficient. Therefore, the robust DEA is able to provide effective ranking strategies analogous to managerial risk preference. Finally, an application is made to the banking industry, using a dataset from banks in Germany.

Robust efficiency measurement under ellipsoidal uncertainty sets

The third research thrust extends the robust DEA framework to the ellipsoidal uncertainty set. Some evaluations of DMUs studied previously (see Sadjadi & Omrani, 2008; Wang & Wei, 2010; Wu et al., 2017) consider an ellipsoidal set, however, they can only be applied to DEA models with uncertainty in either inputs or outputs data. We consider formulations where two uncertainty sets (*i*) a regular ellipsoid and (*ii*) interval–ellipsoid sets are designed for immunization of both uncertain inputs and outputs data. The insight into the connections and differences of the robust DEA to these uncertainties are provided. The chapter also studies the classification scheme for DMUs and as in the previous case provide a unifying framework for the ranking strategies of management evaluations. The last but not least, we extend the robust DEA model to the additive model and study its efficacy and classification of inefficient units with two imprecise additive models proposed in the literature.

Robust multi-objective transportation problem with network efficiency

The last thrust of the thesis deals with an application to a transportation problem. Transportation problem (TP) deals with shipping products from several sources to several destinations which either minimizes the total transportation cost (min-type) or maximizes the total transportation profit (max-type) under the intrinsic assumption of certain data. A network efficiency measurement of the TP arises when shipment arcs involve multiple min-type (inputs) and multiple max-type (outputs) factors. DEA method is an optimization approach which can measure the network efficiency by assigning weight to each min-type and max-type factors and then maximizes the ratio of the weighted sum of max-type factors over the weighted sum of min-type factors. Given that different conflicting objectives under unknown conditions exist concurrently in practice, this chapter analyses the TP with network efficiency focus under the multiple objective linear programming (MOLP) framework. The DEA and MOLP are integrated to minimize arc inefficiencies and other min-type factors while maximizing max-type factors. We study a linear programming robust model through goal programming (GP) approach in the presence of uncertain demands and supplies.

1.4 Outline

Each of the research thrusts listed above entails a chapter which has its own introduction, modeling framework, method, application and conclusion and stand independently on its own. The thesis contains 7 Chapters. The background of the study is given in Chapter 1. Chapter 2 provides the main concepts of the DEA efficiency and measurement as well as the RO technique to uncertain optimization. It also provides a review of the robust DEA and develops new models to different uncertainty sets.

Chapter 3 develops and analyses robust counterpart optimization for operations research and management problems with non-negative decision variables and study their comparative computational complexity. It follows up with a new robust DEA formulation under the budget of uncertainty called the *reduced robust* DEA model.

The thesis proceeds with the study of the duality relations of the reduced robust model in Chapter 4. Moreover, a classification scheme considering the conservativeness management decisions is further studied in the input – and output – orientation.

Chapter 5 is dedicated to the development of a robust DEA under ellipsoidal uncertainty sets, where a similar classification scheme for DMUs is provided.

Chapter 6 make application to transportation with multiple objectives, data uncertainty and focus on network efficiency.

Finally, in Chapter 7, we conclude this thesis with summary and future research directions.

Chapter 2: DEA and RO: An overview

Summary

This chapter provides a brief overview of the DEA and RO approaches. It details the concepts of efficiency and its measurement using the DEA in Section 2.1 as well as the modeling techniques with the RO in Section 2.2. The combination of the two which is known as the robust DEA is reviewed in Section 2.3. A significant part of this section is dedicated to the characterization of the DEA under alternative uncertainty sets.

2.1 Efficiency measurement with DEA

Efficiency measurement has economic production theory as its foundation. At the micro level, firms employ a set of inputs to produce outputs with the aim of maximizing profit. The concept of efficiency lay emphasis on the reduction of inputs or expansion of outputs of a production unit and supposes that firms do things right by aligning resources to operate at their most productive scale size in order to achieve profit objective. In its basic form, efficiency is measured much more like productivity as the ratio of outputs to input. It is expressed as:

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}}$$

More intuitively, by efficiency, what we have in mind is a comparison between observed and optimal values of outputs and inputs. By the latter, we mean the maximum potential output obtainable from observed input and the minimum potential input required to produce desired output respectively. The 'optimum' or 'maximum potential' is defined in terms of production possibilities or frontier line. This aptly measure of efficiency considering a vigorous analytical and frontier estimation was initiated in Koopmans (1951) and Debreu (1951). In that framework, instead of using a basic ratio definition above, one can measure efficiency with a frontier line (production possibility frontier) used as a production margin from which actual production are compared with. Such efficiency measure is defined by the "distance between the quantity of input and output, and the quantity of input and output that defines the frontier" (Daraio & Simar, 2007). The idea of a frontier efficiency method is to make comparison among firms or DMUs in order to measure how inputs are being utilized to produce outputs as well as to provide significant information concerning the identification of benchmarking policies, the estimation of optimal inputs and outputs and the effect of returns to scale. The literature expounds two competing modeling techniques for measuring frontier efficiency. They are the parametric and nonparametric approaches.

In the parametric approach (e.g. the stochastic frontier analysis (SFA)), the functional form of the input-output relationship is either known or estimated statistically. However, in most cases, the functional form cannot be determined. Instead, a facet of an efficient function called the *production possibility set* (PPS) derived from a set of observations is used to determine the input-output relationship. This latter approach is known as the nonparametric approach since it does not require any parameter estimation. Besides, the best practice function is computed empirically from observed inputs and outputs without any specification of the functional form. The DEA and the free disposal hull (FDH) are the main known nonparametric approaches in efficiency analysis of production and services activities (Daraio & Simar, 2007). The DEA has its basis from the Farrell's seminal paper on "the measurement of productive efficiency" (Farrell, 1957). Farrell's efficiency takes the form of uniform radial expansions or contractions from inefficient observations to a piecewise linear production frontier which is estimated based on the free disposability and the convexity of the inputs and outputs[1]. The FDH which was proposed by Deprins, Simar, & Tulkens (1984) is seen as a more general form of the DEA or the non-convex version of the DEA.

DEA as a linear programming tool: The pioneering work of Charnes, Cooper, & Rhodes (1978) forms the basis of the non-parametric DEA or the operationalization of Farrell's efficiency measurement concept as a linear programming (LP) tool. The acceptance of LP as a computational method for measuring efficiency in different economic decision-making problems, however, began with the work of Dorfman, Samuelson, & Solow (1958). Their text offered a clear, concise exposition of the relationship between LP and standard economic analysis. Early researchers such as Farrell & Fieldhouse (1962) and Boles (1971) utilized the LP to measure Farrell's efficiency in terms of multiple inputs – single output. However, it was Charnes et al. (1978) DEA approach that generalized the Farrell's measure to multiple inputs and outputs and more importantly, its duality relations. Charnes et al. (1978) approach provide an optimization framework for assessing the performance of a set of homogenous

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[1]Farrells efficiency provided the conceptual framework for both technical efficiency and allocative efficiency. These concepts were indeed influenced by Debreu (1951) decomposition of efficiency, hence the reference Farrell – Debreu efficiency as used in some text for technical efficiency. Allocative efficiency refers to whether inputs, for a given level of output and set of input prices, are chosen to minimize the cost of production, assuming that the organization being examined is already fully technically efficient. The definition of technical efficiency is provided in the text.

DMUs (a set of n peer DMUs, i.e. $\text{DMU}_j, j = 1, \dots, n$), which transform multiple inputs into multiple outputs. The starting point of the DEA via a fractional programming model known as the Charnes, Cooper, & Rhodes (CCR) model and proposed in Charnes et al. (1978) expresses a production index explicitly as[2]:

$$
\begin{array}{lll}
\max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} & & \\
\text{s.t.} & & \\
\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 & j = 1, \dots, n & \\
v_i \geq 0 & i = 1, \dots, m & \\
u_r \geq 0 & r = 1, \dots, s &
\end{array}
\tag{2.1}
$$

where x_{ij} and y_{rj} are observed non-negative input and output data and v_i and u_r are the weights assigned to the ith input and rth output respectively. The objective of the model is to obtain the ratio of weighted output to weighted input. Intuitively, the model maximizes the ratio of the DMU under evaluation ($\sum_{r=1}^{s} u_r y_{ro} / \sum_{i=1}^{m} v_i x_{io}$) written as DMU_o, subject to the fact that the ratio of all other DMUs ($\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}; j = 1, .., n$) is less than or equal to 1. In other

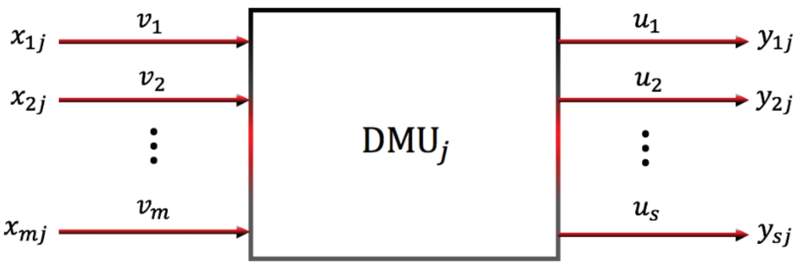


Figure 2.1. Structure of a DMU

words, the DEA drives weight directly from a given data and provides a positive efficiency score less than or equal to 1 by virtue of the constraint in (2.1). Under this paradigm, DMUs are classified into two mutually exclusive and collectively exhaustive groups, i.e. efficient (when the efficiency score is 1) and inefficient (when the efficiency score is less than 1). The DEA efficiency measure is 'technical' as opposed to *economic efficiency* which considers behavioral objectives (such as revenue maximization or cost minimization) by including market prices to determine whether services are worth the cost[3] (Lovell, 1993). *Technical*

[2] Note the fractional programming model is nonconvex and nonlinear

[3] Note that the DEA model can allow for the measurement of economic efficiency, as well as its breakdown into technical and allocative efficiency whenever there is available information on price.

efficiency (hereafter efficiency) refers to the maximization of a firm's feasible output from its given inputs or the minimization of feasible inputs that yield a desired level of output.

DEA orientation*:* A very important advantage of the DEA is the suggestion of explicit improvement targets for the inefficient DMUs. This property enables firms to sanction a better utilization of their resources to achieve efficiency. Firms are able to select efficient DMUs on the frontier as the *reference set* for the projection of the inefficient DMUs. The path projection to the efficient frontier is done with an orientation of the DEA model. The orientation holds the viewpoint of the improvement direction of the inefficient units, whether the goal is to expand output shortfalls or reduce input excesses, respectively in order to move the inefficient unit to the frontier (Paradi, Sherman, & Tam, 2018). Two orientations are defined for the traditional models: input and output. Figure 2.2 provides a projection path for the inefficient DMU A. In the input orientation, firms decrease their inputs level in order to be projected to the efficient frontier whiles maintaining the same output level. This is shown by the reduction of input from A to A_1. The output-oriented model seeks to maximize outputs with a given level of inputs in the direction of A to A_2. A third to the orientation dichotomy is the additive model proposed by Charnes, Cooper, Golany, Seiford, & Stutz (1985) which allows firms to decrease inputs and increase output simultaneously to reach the efficient frontier. This model is discussed later in Section 2.1.2.3.

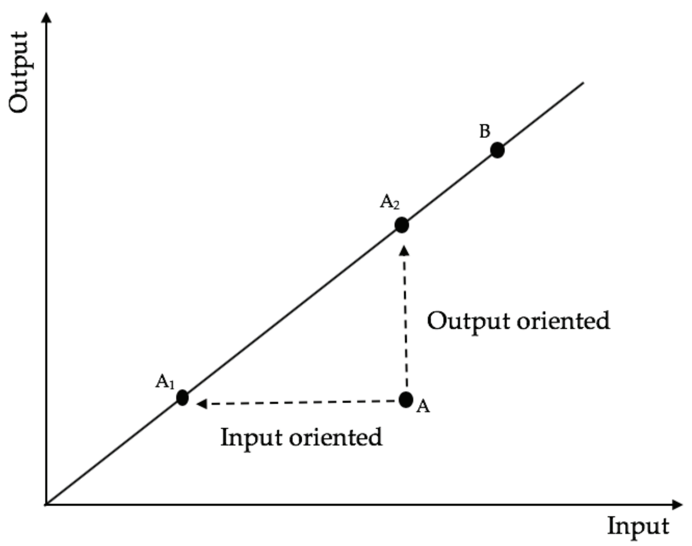


Figure 2.2. Input and output orientation

DEA strengths and weaknesses: One attracting feature of the DEA approach is that, the efficiency concept follows the condition of Pareto optimality for productive efficiency [A DMU is "Pareto efficient if no other unit or combination of units exists which can produce at least the same amount of outputs, with less for some resource(s) and no more for any other" (see Thanassoulis, 1997)] which position the DEA efficiency at the center of welfare economics and its associated efficient distribution of wealth in an economy. The DEA approach in measuring efficiency is flexible in allowing for the free and optimal selection of inputs and outputs weight of efficient DMUs from inefficient ones. The optimal weight in no doubt enables management in knowing the real importance of selected inputs and outputs. Moreover, by suggesting 'peers' as reference units for inefficient units, the DEA becomes a useful benchmarking tool in improving management operations. Not only does it suggest alternative ways of projecting inefficient units, but the DEA is also able to identify the sources of inefficiency, as to whether the unit is inefficient due to disadvantage conditions and/or actual inefficient operation. In fact, the DEA, per its easy implementation and user-friendliness of software applications[4], has received wide applications in various scientific and social science areas such as management science, operational research, engineering system, business analytics, decision sciences, economy etc. and it continues to be touted as an excellent data-oriented approach to efficiency measurement. Emrouznejad & Yang (2018) list over 10,000 research papers in the field. Today, the DEA is one of the key research strands in OR and MS field.

Notwithstanding its advantages, the traditional DEA models have some drawbacks that present some limitations that managers must be mindful of in its usage for performance decisions. Here we identify some main limitations that are given in Daraio & Simar (2007):

- Deterministic and non-statistical nature;
- Influence of outliers and extreme values;
- Unsatisfactory techniques for the introduction of environmental or external variables in the measurement (estimation) of the efficiency.

That is to say that, the DEA as a deterministic rather than statistical approach produces results that are sensitive to measurement errors and outliers. In other words, the DEA fails to capture the stochasticity or randomness in the data which in the SFA is dealt with by the error terms. It is also possible that the inputs and outputs data are inexactly defined such that the real values are uncertain. These challenges are at the heart of this thesis and they are addressed in the subsequent chapters via the robust DEA.

2.1.1 Production possibility set and efficient frontier

As aforementioned, instead of establishing inputs and outputs relationship through a functional form: $f: \mathbb{R}^m \rightarrow \mathbb{R}$ $(y = f(x))$, a PPS embodying the feasible alternatives of observed input-output correspondences is constructed. The boundary of the PPS known as the production possibility frontier (PPF) bounds all the feasible production plans. It is usually assumed that the production technology towards the efficiency measurement is known.

[4]A number of user-friendly software packages for DEA analysis include the DEA solver, DEA Frontier, LINGO, Max DEA and PIM-DEA, GAMS.

Assume that at a given technology, there are n DMUs (DMU_j; $j = 1, \ldots, n$) where each DMU_j is capable of producing s outputs, $\boldsymbol{y}_j = (\ldots, y_{rj}, \ldots); r = 1, \ldots, s$ from m inputs, $\boldsymbol{x}_j = (\ldots, x_{ij}, \ldots); i = 1, \ldots, m.$ All inputs and outputs for all DMUs are non-negative and a DMU has at least one positive input and one positive output. We refer to this as semipositive condition with mathematical characterization given by $x_{ij} \geq \mathbf{0}, x_{ij} \neq 0,$ and $y_{rj} \geq \mathbf{0}, y_{rj} \neq 0$ for $j = 1, \ldots, n$. The PPS comprises the set of all feasible production plan or input-output combinations available to DMU_j. The PPS is defined as the set:

$$\psi = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{m+s} | \; \boldsymbol{x} \text{ can produce } \boldsymbol{y}\} \tag{2.2}$$

Different ways of constructing the set ψ exist and are commonly based on the technology defined for all observed inputs and outputs which are feasible in principle. The following assumptions which are fairly weak, hold for any technology represented by a quasi-concave and weakly monotonic production function ψ:

Assumption 2.1. Properties of the set ψ

(A1): *Feasibility*: All observed activities are feasible. i.e. $\mathrm{DMU}_j = (\boldsymbol{x}_j, \boldsymbol{y}_j)$ for $j = 1, \ldots, n,$ then $\forall j, (\boldsymbol{x}_j, \boldsymbol{y}_j) \in \psi$.

(A2): *Free disposability*: If an input-output combination is a feasible activity, then any input-output combination where the input is larger and the output smaller is also a feasible activity. i.e. $(\boldsymbol{x}, \boldsymbol{y}) \in \psi \Longrightarrow \forall\, \overline{\boldsymbol{x}} \geq \boldsymbol{x}, \forall\, \overline{\boldsymbol{y}} \leq \boldsymbol{y}, (\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \in \psi$. The axiom denotes the dominance relation of feasible activities and it stipulates that we can freely dispose of unwanted inputs and outputs.

(A3): *Convexity*[5]: If two input-output combinations are feasible activity, then any mixture of the two or convex combination of the two is also a feasible activity. i.e. $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \psi \Longrightarrow \forall t \in [0,1], t(\boldsymbol{x}_1, \boldsymbol{y}_1) + (1 - t)(\boldsymbol{x}_2, \boldsymbol{y}_2) \in \psi$.

(A4): *Constant returns to scale*: Any ray scaled up or down from the origin with feasible activity generates a new activity which is feasible. i.e. $(\boldsymbol{x}, \boldsymbol{y}) \in \psi \Longrightarrow \forall \lambda > 0, (\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) \in \psi$.

We shall consider the two main technologies used in DEA: constant returns to scale (CRS) and variable returns to scale (VRS). The PPS based on the CRS technology, denoted as ψ_{CRS}, is built on assumptions (A1) – (A4) whiles the PPS based on the VRS technology, ψ_{VRS} is obtained by removing the CRS assumption (A4) and further assuming that no rescaling is possible. This involves considering $\lambda = 1$ which lead to the Banker, Charnes, & Cooper (1984)

[5] Note that convexity is a necessary assumption for establishing the duality between input and output sets and by extension cost and revenue functions. The FDH relaxes the assumption of the convexity of the DEA and so the computational technique requires mixed integer programming as compared to the linear programming of the DEA.

(BCC) model. The PPS which satisfies the set ψ_{CRS} and ψ_{VRS} can be written mathematically as follows[6]:

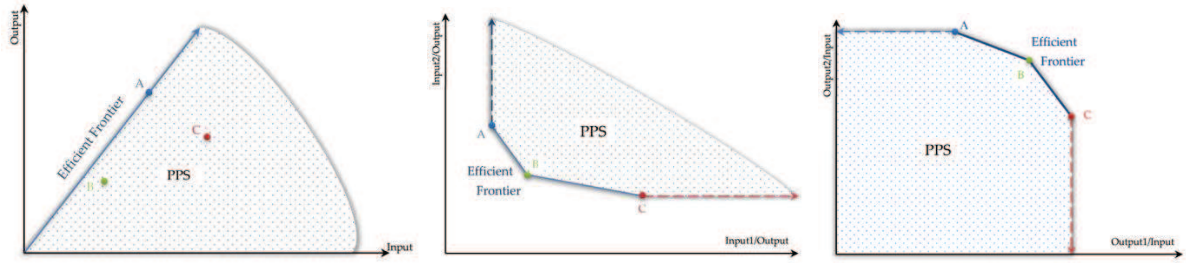
$$\psi_{CRS} = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq \boldsymbol{\lambda}\mathbf{X}, \boldsymbol{y} \leq \boldsymbol{\lambda}\mathbf{Y}, \boldsymbol{\lambda} \geq \mathbf{0}_n\} \tag{2.3}$$

$$\psi_{VRS} = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq \boldsymbol{\lambda}\mathbf{X}, \boldsymbol{y} \leq \boldsymbol{\lambda}\mathbf{Y}, \boldsymbol{\lambda}\mathbf{1}_n = 1, \boldsymbol{\lambda} \geq \mathbf{0}_n\} \tag{2.4}$$

where $\mathbf{X} \in \mathbb{R}^{m \times n}$ is an inputs matrix consisting of all input vectors and $\mathbf{Y} \in \mathbb{R}^{s \times n}$ represent outputs matrix containing all the input vectors. The matrix of $\mathbf{X}$ and $\mathbf{Y}$ are defined as follows:

$$\mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}_{m \times n}, \qquad \mathbf{Y} = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{s1} & \cdots & y_{sn} \end{bmatrix}_{s \times n}$$

Note that ψ_{CRS} and ψ_{VRS} envelop all data using a borderline known as the efficient frontier and it is constructed according to the *minimal extrapolation principle*. To illustrate the minimal set that satisfies the assumptions of ψ_{CRS}, we first define the efficient frontier.



a) 1 input – 1 output b) 2 inputs – 1 unitized output c) 1 unitized input – 2 outputs

Figure 2.3. Illustration of PPS with ψ_{CRS} for different inputs and outputs cases.

Efficient frontier: The efficient frontier (PPF) is a benchmark frontier line that spans all the best practice DMUs and envelops the non-best practicing DMUs. In other words, the efficient frontier is the non-dominated subset of the PPS.

Efficiency: A DMU with the pair (x, y) is efficient in ψ_{CRS} if it cannot be dominated by some DMU with $(\bar{x}, \bar{y}) \in \psi_{CRS}$. The efficient subset of ψ_{CRS}, ψ_{CRS}^E is written as:

$$\psi_{CRS}^E = \{(x, y) \in \psi_{CRS} | \ (x, y) \text{ is efficient in } \psi_{CRS}\} \tag{2.5}$$

The set of DMUs in ψ_{CRS} which do not denote belong to ψ_{CRS}^E are called the dominated DMUs or inefficient DMUs. Figure 2.3 illustrates the minimal subset that satisfies the CRS

[6] Note that $\mathbf{0}_n$ is a vector of zeros for the intensity variable.

technology, ψ_{CRS} for three DMUs (A, B and C) in different inputs and outputs cases. In Figure 2.3, only DMU A is on the efficient frontier and therefore efficient whereas DMU B and C are dominated or inefficient. On the other hand, DMU A, B, and C are all efficient in Figure 2.3. The minimal subset that satisfies the VRS technology, ψ_{VRS} is shown in Figure 2.4 where similarly, DMU A, B and C are on the efficient frontier.

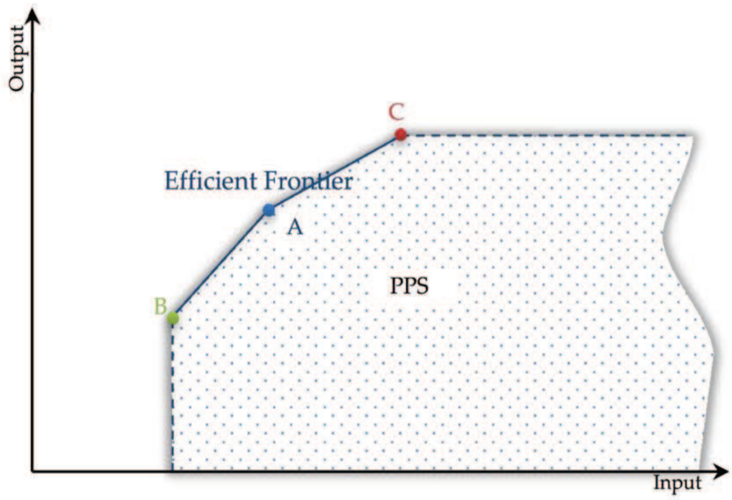


Figure 2.4. Illustration of PPS with ψ_{VRS}

2.1.2 Basic DEA models

There are several extended and specific DEA models given in literature that deal with specific problems. The most representative DEA models include the CCR model by Charnes et al. (1978), the BCC model by Banker, et al. (1984), the Färe and Grosskopf (FG) model by Fare & Grosskopf (1985), and the Seiford and Thrall (ST) model by Seiford & Thrall (1990)[7]. However, given that we defined the PPS in section 2.1.2 for technology set ψ_{CRS} and ψ_{VRS}, we limit ourselves to the CCR and BCC models, their oriented models and a third, the additive model. Essentially these are the required models that will be used for the robust analysis in this thesis Interested readers can refer to Cooper, Seiford, & Tone (2006), Zhu & Cook (2007) and Toloo (2014)b for other advanced models that address specific issues.

2.1.2.1 CCR Model

The CCR model named after its developers, Charnes et al. (1978) measure the technical and scale efficiency based on the minimal extrapolation of the set ψ_{CRS}. The model is an

[7] These models can be solved by available commercial software such as DEA solver, LINGO, Max DEA, GAMS etc.

optimization formulation of the set ψ_{CRS} known as the envelopment form of the CCR model. Mathematically, we solve the following linear programming problem:

$$\begin{array}{ll} \min \theta & \\ \text{s.t} & \\ \sum_{j=1}^{n} \lambda_j \, x_{ij} \leq \theta x_{io} & i = 1, \ldots, m \\ \sum_{j=1}^{n} \lambda_j \, y_{rj} \geq y_{ro} & r = 1, \ldots, s \\ \lambda_j \geq 0 & j = 1, \ldots, n \end{array} \tag{2.6}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2 \ldots, \lambda_n)^T$ is a nonnegative weight or intensity vector. Model (2.6) will be called the Input-oriented Envelopment CCR (IECCR) model since the model sought to decrease the input vector $\boldsymbol{x}_o$ radially to $\theta \boldsymbol{x}_o$ in order to obtain an activity level $(\boldsymbol{\lambda x}, \boldsymbol{\lambda y})$ that optimizes the efficiency of a DMU under evaluation, $\text{DMU}_o = (\boldsymbol{x}_o, \boldsymbol{y}_o)$. Model (2.6) has the following properties (Toloo, 2014):

(P1): The model entails $m + s$ constraints and n decision variables.

(P2) The model is solved n times to obtain the relative efficiency for all DMUs.

(P3): $(\theta, \boldsymbol{\lambda}) = (1, \boldsymbol{e}_o)$[8] is a feasible solution of the model all the time.

(P4): Given the feasibility $(1, \boldsymbol{e}_o)$, we obtain a bounded a solution $\theta^* \leq 1$.

(P5): The optimal objective value is positive. i.e. $\theta^* > 0$.

(P6): From (P4) and (P6), the efficiency of every DMU is between $0 < \theta^* \leq 1$.

Putting (P6) into perspective, it is possible to define the efficiency concept by model (2.6) as:

Definition 2.1 DMU_o is CCR-efficient if $\theta^* = 1$ otherwise it is CCR-inefficient.

Note that Definition 2.1 signifies a weak efficiency concept since it is possible that at the optimal solution, some alternate optima may contain nonzero intensity vector whiles other may not, e.g. it is possible to obtain the solution $\theta^* = 1, \lambda_o^* = 1, \lambda_j^* = 0 \ (j \neq 0)$. This implies that while a DMU may lie on the efficient frontier, it is still possible to increase (decrease) the production of some outputs (inputs) which intuitively indicate technical inefficiency. Suppose we associate model (2.6) with slack variables $\boldsymbol{s}^- = \{s_i^-\}^T \in \mathcal{R}^m$ and $\boldsymbol{s}^+ = \{s_r^+\}^T \in \mathcal{R}^s$ where the input constraint $s^- = \theta x_{i0} - \sum_{j=1}^{n} x_{ij} \lambda_j \in \mathbb{R}^m$ and the output constraint $s^+ = \sum_{j=1}^{n} y_{rj} \lambda_j - y_{r0} \in \mathbb{R}^m$ present input excesses and output shortfalls respectively. A sufficient condition for the efficiency of DMU_o is achieved if all inputs and outputs are utilized and no excess or shortage exists. This entails solving the following max – slack model in which slacks are taken to their maximal values:

[8] $\boldsymbol{e}_j, j = 1, \ldots, n$ is the j^{th} unit vector. i.e. a vector with zero components, except for a 1 in the j^{th} position.

$$
\begin{aligned}
&\min \textstyle\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
&\text{s.t} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta^* x_{io} && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} && r = 1, \dots, s \\
&\lambda_j \geq 0 && j = 1, \dots, n \\
&s_i^- \geq 0 && i = 1, \dots, m \\
&s_r^+ \geq 0 && r = 1, \dots, s
\end{aligned}
\tag{2.7}
$$

Using model (2.7) we make two definitions of efficiency:

Definition 2.2. DMU_o is CCR-efficient if and only if

(i) $\theta^* = 1$

(ii) all slacks $\boldsymbol{s}^{-*} = \boldsymbol{0}_m$, $\boldsymbol{s}^{+*} = \boldsymbol{0}_s$

otherwise, DMU_o is CCR-efficient.

Definition 2.3. DMU_o is CCR- weakly-efficient if and only if

(i) $\theta^* = 1$

(ii) all slacks $\boldsymbol{s}^{-*} \neq \boldsymbol{0}_m$ and/or $\boldsymbol{s}^{+*} \neq \boldsymbol{0}_s$ for some i or r in some alternate optima.

Definition 2.2 depicts full (100%) efficiency since for all alternate optimal solutions, efficiency is achieved if and only if there is no inputs excesses or output shortfalls. In other words, efficiency restricted to conditions (*i*) and (*ii*) exemplifies the DEA technical efficiency, strong efficiency or Pareto-Koopman efficiency in economics because the zero slacks imply that no additional improvement in input or output is possible without worsening any other input or output. This is formalized by the definition below:

Definition 2.4. (*Pareto - Koopmans efficiency*): DMU_o is technically efficient if any reduction in input requires an increase in at least one other input or a reduction in at least one output, and if any increase in an output requires a reduction in at least one other output or an increase in at least one input.

The Pareto Koopmans efficiency distinguishes technical efficiency from the Farrell efficiency. The latter identifies itself with Definition 2.3 and it is sometimes referred to as *mix inefficiency* because of its nonzero slack. Figure 2.5 shows DMUs identified under weak and strong efficiency in one unitized input – two output space. As observed on the frontier P_1P_2, DMU B, C and E are the technically (strong) efficient units. DMU F indicates the special case of weak efficiency since it is not on the efficient part of the frontier. For instance, although DMU A, D, and G are inefficient, the shortfall in D outputs can be improved or projected to D′ by increasing its outputs 1 and 2 without altering their proportions. As also in the case of DMU G, these inefficiencies are called technical inefficiency. On the other hand, the improvement of DMU A requires first a radial measure of A to A′ followed by a projection to

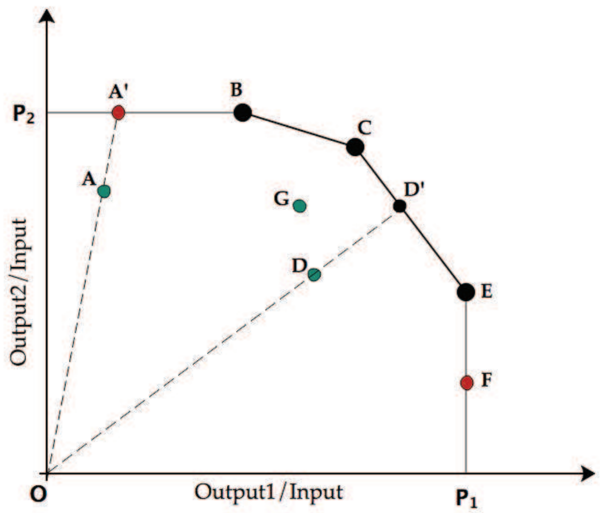


Figure 2.5. Illustration of weak and strong efficiency

B in order to remove the shortfall in output 1. The inefficiency exhibited by DMU A is known as mix inefficiency. We now look at the dual of model (2.6) which is given by the following model[9]

$$
\begin{array}{ll}
\eta^* = \max \sum_{r=1}^{s} u_r y_{ro} & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} = 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 & j = 1, \dots, n \\
v_i \ge 0 & i = 1, \dots, m \\
u_r \ge 0 & r = 1, \dots, s
\end{array} \qquad (2.8)
$$

In this model, the weights, v_i and u_r are referred to as multipliers, hence the model name, Input-oriented Multiplier CCR (IMCCR) model. The first constraint called the normalization constraint ensures the relative efficiency of the DMU_o to the other DMUs. The second constraint is a common constraint to all the DMUs. The multiplier CCR model measures efficiency by maximizing the ratio of the weighted sum of outputs to a weighted sum of inputs of a unit subject to the condition that, the same ratio of all other units is less than or equal to one. The efficiency obtained by model (2.8) follows the definition below:

Definition 2.5. DMU_o is CCR-efficient if and only if

(i) $\eta^* = 1$

(ii) There exists at least one strictly positive optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$

[9]Notice that this model is also the LP transformation of the nonconvex and nonlinear fractional programming model (2.1). See Toloo (2014) for further discussion.

It is important the two conditions provided for the *strong efficiency* of DMUs. The efficiency definition restricted to only condition (i) even though known as *weak efficiency* is characterized as inefficiency. This implies that DMU_o is CCR-inefficient if either $\eta^* < 1$ or $\eta^* = 1$ and for *every* optimal solution there exists at least one zero weights. On the other hand, Definition 2.5 represents a strong efficiency or the Pareto-Koopmans efficiency which like the max–slack condition in model (2.7) permits no improvement in any input or output without worsening at least one other input or output.

Note that models (2.6) and (2.8) are equivalent by the strong duality theorem[10]. i.e. $\theta^* = \eta^*$. Now suppose a DMU projection is made in the output directions, the output-oriented models that necessitate the technical efficiency is given by the following dual pairs:

$$\begin{aligned} &\max \varphi \\ &\mathrm{s.t.} \\ &\textstyle\sum_{j=1}^{n} \gamma_j \, x_{ij} \leq x_{io} && i = 1, \dots, m \\ &\textstyle\sum_{j=1}^{n} \gamma_j \, y_{rj} \geq \varphi y_{ro} && r = 1, \dots, s \\ &\lambda_j \geq 0 && j = 1, \dots, n \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\phi^* = \min \textstyle\sum_{i=1}^{m} v_i x_{io} \\ &\mathrm{s.t.} \\ &\textstyle\sum_{r=1}^{s} \varpi_r y_{ro} = 1 \\ &\textstyle\sum_{r=1}^{s} \varpi_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 && j = 1, \dots, n \\ &v_i \geq 0 && i = 1, \dots, m \\ &u_r \geq 0 && r = 1, \dots, s \end{aligned} \tag{2.10}$$

We refer to model (2.9) and (2.10) as the Output-oriented Envelopment CCR (OECCR) and Output-oriented Multiplier CCR (OECCR) models respectively. The relationship between these models and the earlier input-oriented models given by the theorem below:

Theorem 2.1. Let $(\theta^*, \boldsymbol{\lambda}^*) \in \mathbb{R}^{1+n}$ or $(\boldsymbol{v}^*, \boldsymbol{\varpi}^*) \in \mathbb{R}^{m+s}$ be the optimal solution of the input-oriented model (2.6) and model (2.8), respectively. Then $\left(\frac{1}{\theta^*}, \frac{\boldsymbol{\lambda}^*}{\theta^*}\right) = (\varphi^*, \boldsymbol{\varpi}^*)$ or $\left(\frac{1}{\theta^*}, \frac{\boldsymbol{u}^*}{\theta^*}\right) = (\phi^*, \boldsymbol{\varpi}^*)$ is an optimal solution for the output-oriented model (2.9) and (2.10) respectively and vice versa.

Proof. See Cooper, Seiford, & Tone (2006).

[10] We refer interested readers to Appendix A where duality in linear programming and its exposition in DEA is discussed thoroughly.

The above relation implies a DMU under an input-oriented CCR model is efficient if and only if it is also efficient when the output-oriented CCR model is used to evaluate its performance. Furthermore, since $\theta^* \in (0,1]$, it implies that $\varphi^* \in [1, \infty)$.

2.1.2.2 BCC Model

Banker, Charnes, & Cooper (1984) introduced the BCC model with the view that not all units operate at the optimal scale and so the most productive scale size (MPSS) may not be attainable for a unit operating at other scales. They therefore proposed a piecewise linear and concave frontier which examines DMUs that are not operating at the optimal scale. This includes, as indicated in section 2.1.2, imposing a convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ to replace the more restrictive assumption of ray rescaling in the possibility set ψ_{CRS}. The BCC model induced by the set ψ_{VRS} is the following:

$$
\begin{array}{lll}
\min \vartheta & & \\
\text{s.t} & & \\
\sum_{j=1}^{n} \lambda_j x_{ij} \leq \vartheta x_{io} & i = 1, \dots, m & \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} & r = 1, \dots, s & \\
\sum_{j=1}^{n} \lambda_j = 1 & & \\
\lambda_j \geq 0 & j = 1, \dots, n & \\
\vartheta \text{ is free in sign} & &
\end{array}
\qquad (2.11)
$$

The dual of model (2.11) (multiplier model of the BCC) is formulated as follows:

$$
\begin{array}{ll}
\vartheta^* = \max \sum_{r=1}^{s} u_r y_{ro} + u_o & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} = 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0 & j = 1, \dots, n \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s \\
u_o \text{ is free} &
\end{array}
\qquad (2.12)
$$

where u_o is the returns to scale variable. The efficiency definition for the BCC model exudes similarly from the CCR model. Here we provide a definition for the BCC model (2.12) as the following:

Definition 2.6. DMU_o is BCC-efficient if $\vartheta^* = 1$ and there exist at least one strictly positive optimal solution (i.e. $\forall i, v_i^* > 0, \forall r, u_r^* > 0$) otherwise it is BCC-inefficient.

The BCC measures pure technical efficiency (PTE) unlike the overall technical efficiency of the CCR model. A DMU that is efficient in the CCR sense is also efficient by the BCC. In other words, decomposing technical efficiency (CCR efficiency) into purely technical (BCC efficiency) and scale efficiencies, a DMU which is purely technical efficient and scale efficient is seen to be operating at the MPSS. In Figure 2.6, DMUs B, C, and D are the purely technical efficient units lying on the VRS frontier while DMU A represents an inefficient unit. DMU C which is lying also on the CRS frontier represent the optimal or maximum productivity for a given mix of inputs and outputs, hence called the MPSS. The BCC -efficiency score of DMU

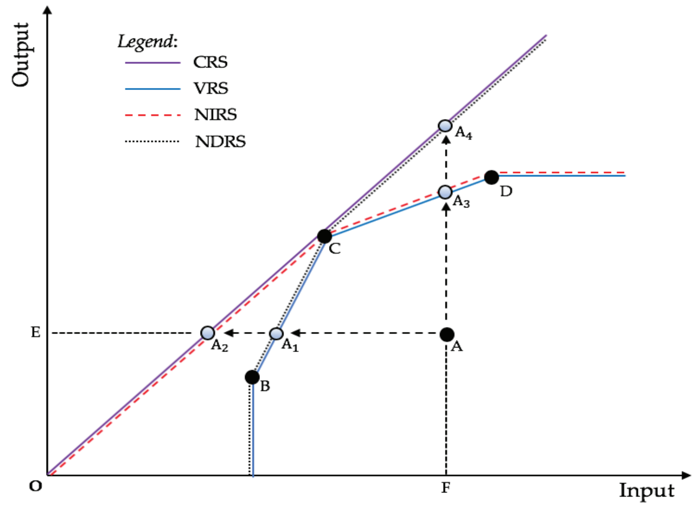


Figure 2.6. Various returns to scale technologies

A is obtained as a radial projection from the envelopment surface (arrow AD and arrow AB in Figure 2.6). Here, the envelopment surface is a piecewise linear and concave which result in decreasing, constant or increasing returns to scale. The identification of returns to scale (RTS) is put forward in Banker & Thrall (1992) by the following theorem:

Theorem 2.2. *Let* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *be on the efficient frontier of* ψ_{VRS}*. The following conditions identify the RTS for the BCC models (2.11) and (2.12):*

1. Increasing returns to scale: *This prevails at* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *if and only if* $\sum_{j=1}^{n} \lambda_j^* < 1$ *(or* $u_o^* > 0$*) for all alternate optimal solutions.*
2. Decreasing returns to scale: *This prevails at* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *if and only if* $\sum_{j=1}^{n} \lambda_j^* > 1$ *(or* $u_o^* < 0$*) for all alternate optimal solutions*.
3. Constant returns to scale: *This prevails at* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *if and only if* $\sum_{j=1}^{n} \lambda_j^* = 1$ *(or* $u_o^* = 0$*) for any alternate optimal solution.*

In addition, the production process could exhibit non-increasing RTS (NIRS) and non-decreasing RTS (NDRS), obtained when the convexity constraint in (2.4) is relaxed by changing the equality to inequality. The frontier in ψ_{VRS} exhibit NIRS if $\sum_{j=1}^{n} \lambda_j^* \leq 1$ and NDRS if $\sum_{j=1}^{n} \lambda_j^* \geq 1$ are modified for $\sum_{j=1}^{n} \lambda_j = 1$ in model (2.11) respectively. Figure 2.6 shows the NIRS frontier (dotted lines) and the CRS and VRS frontiers. Suppose ϑ_{NIRS} is the efficiency of DMU_k obtained under the NIRS model. Then with θ_{VRS}, if $\vartheta_{NIRS} = \vartheta_{VRS}$ as in the case of A_3 in Figure 2.6, DRS exist for DMU_k; if, however, $\vartheta_{NIRS} \neq \vartheta_{VRS}$ as in the case of A_1, then IRS is present for DMU_k. Note that the RTS above holds for DMUs belonging to the set ψ^E_{VRS}. It, therefore, requires that ϑ^* is estimated in model (2.11) for all alternate optima. For discussion including RTS identification from inefficient DMUs (CCR and BCC models), see Banker, Chang, & Cooper (1996).

2.1.2.3 Additive Model

The CCR and BCC models are focused on either minimizing inputs (input oriented) or maximizing output (output oriented) using radial projection to the frontiers. The additive model, on the other hand, combines both orientations in a single model. It utilizes an l_1- distance projection to simultaneously decrease inputs (eliminating input excesses, s_i^-) and increase output (eliminating output shortfalls, s_r^+). Proposed by Charnes, Cooper, Golany, Seiford, & Stutz, (1985), the additive model is given by the following LP:

$$
\begin{aligned}
& \mathrm{s}_o^* = \max \textstyle\sum_{j=1}^{m} s_i^- + \sum_{j=1}^{s} s_r^+ \\
& \mathrm{s.t} \\
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} && i = 1, \dots, m \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} && r = 1, \dots, s \\
& \lambda_j \geq 0 && j = 1, \dots, n \\
& s_i^- \geq 0 && i = 1, \dots, m \\
& s_r^+ \geq 0 && r = 1, \dots, s
\end{aligned}
\tag{2.13}
$$

The efficiency of the additive model is obtained by simultaneously considering the inputs and outputs slacks. That is DMU_o is ADD-efficient if and only if $s_o^* = 0$ $(i.e.\ s_i^- = s_r^+ = 0, \forall i, \forall r)$ otherwise, DMU_o is called additive-inefficient.

Theorem 2.3. DMU_o is ADD-efficient if and only if it is CCR-efficient.

Proof. See Ahn, Charnes, & Cooper (1988)

Theorem 2.3 holds similarly for the BCC model. i.e. DMU_o is ADD-efficient if and only if it is BCC-efficient. The additive model is partly significant because of its very important property called the *translation invariance.* This property is defined below.

Definition 2.7. A DEA model is said to be *translation invariant* if translating the original input and/or output data values results in a new problem that has the same optimal solution for the envelopment form as the old one.

Practically, the property allows the additive model to translate negative data and solve them as if they were positive data. Moreover, it allows the envelopment form of many DEA models to deal with negative data. Pastor & Ruiz (2007) list a few of these envelopment models that have the translation invariance property[11].

1. The BCC envelopment model (2.11) (also the output-oriented model) is translation invariant with respect *only* to outputs (inputs)
2. The envelopment form of the output-oriented NIRS radial model is translation invariant with respect *only* to inputs.
3. The envelopment form of the input-oriented NDRS radial model is translation invariant with respect *only* to outputs.

Practically, however, the CCR model (input- or output-oriented) is neither translation invariant with respect to outputs nor inputs and so the CCR model cannot be used with any type of negative data.

2.1.3 Sensitivity issues in DEA

We look at sensitivity in DEA on two fronts. The first is the sensitivity of the efficiency result to input-output specification and the size of the sample. The DEA estimates the efficient frontier relative to the sample input and output data, hence its discriminating power become sensitive to the proportionate selection of DMUs and data. In other words, having too many DMUs can reduce the discriminating power of the DEA model. The issue is addressed by following a general rule of thumb to ensure a statistical balance between the number DMUs and the number of performance measures. The rule of thumb is given as (for more details see Toloo et al., 2015, Paradi et al., 2018)

$$n \geq \max\,\{m \times s, 3(m + s)\} \tag{2.14}$$

where n is the total number of DMUs (observations), m is the number of inputs and s is the number of outputs. The rule states that the number of DMUs must exceed at least thrice the sum of inputs and outputs or the products of inputs and outputs. For insights on how this rule is embedded in DEA models including management imposing their opinion or predetermined performance measures, see Toloo et al. (2015).

The other issue of sensitivity in DEA is the sensitivity of the efficiency results to data perturbation which is a problem as old as the DEA. The concern of how to preserve the efficiency of DMUs to small data perturbation was looked at by early researchers; Charnes, Cooper, Lewin, Morey, & Rousseau (1984); Charnes, Roussea, & Semple (1996) and others. These researchers conducted a sensitivity analysis that defines a stability region with ''radius of stability'' within which data variations will not alter a DMUs classification. Organizations efficiencies are then called robust once they remain unchanged within the stability region. For instance, Charnes, Roussea, & Semple (1996) analyzed the additive model and its sensitivity

[11] Note that the multiplier form of the DEA models does not have the same translation invariance properties as the envelopment form and so the focus here is on the envelopment models

to data perturbation by embedding l_p −norms such as the ∞ −norm and 1 −norm as the radius of stability for the DMUs. The radius can be best interpreted as a measure of the DMUs classification stability and robustness, especially with respect to errors in the data. Thus, within the radius, the sensitivity analysis focuses on how much the efficiency score to a perturbed data or observations differs from the actual efficiency scores. The RO approach adopted in this thesis seeks a similar objective. Sensitivity analysis and RO are both mathematical approaches to data uncertainty. In the RO, rather than quantifying locally the stability of efficiency scores with respect to infinitesimal data perturbations, in contrast, we seek to identify "how much the optimal solution to the nominal problem can violate the constraints of the perturbed problem" (Ben-Tal & Nemirovski, 2000). The next section describes various approaches used together with the RO technique to deal with data uncertainty or inexactness in DEA data.

2.1.4 Inexactness in DEA data

The traditional DEA models that are so far given in Section 2.1.3 suppose all the inputs and outputs to be "crisp" or "exact" data. This implies that a fixed measure of efficiency can be obtained with the efficient frontier from the exact/precise amount of inputs and outputs. However, in real life situations such as in banking, manufacturing and production process, the data are volatile, and it is quite difficult to know the exact values. Ideally, observed input and outputs data are saturated with noise, sometimes imprecise, vague and mostly uncertain. Assuming crisp values for uncertain values can, therefore, lead to infeasible or suboptimal efficiency decisions. Besides, the DEA efficiency could be wobbly and sensitive to parameter perturbation.

The traditional stalwart approach to deal with inexact and noise in DEA data is the stochastic DEA, including the chance-constrained DEA (see, Land, Lovell, Thore, Land, & Lovell, 1993; Olesen & Petersen, 1995; Olesen & Petersen, 2016). Stochastic DEA models are generally built on probabilistic assumptions of the randomness in the input and output data and mostly are statistical. The very common statistical stochastic approaches are due to Banker (1993) and Simar & Wilson (1998, 2007). These researches offer a similar solution to randomness in data but differ on the assumptions required to obtain a random reference technology. Simar & Wilson (2000) advocate for distributional assumptions on the data. In a two-stage analysis where the first stage determines the efficiency scores, and the second stage regresses contextual variables affecting productivity, Simar & Wilson (2000) consider a truncated regression, combined with bootstrapping via confidence intervals on the efficiency scores as a re-sampling technique to correct biased estimates and serial correlation. On the other hand, Banker (1993) suggest imposing statistical axioms and use maximum likelihood or ordinary least squares estimation for the second stage analysis. See also Daraio & Simar (2007) and Bogetoft & Otto (2011) for general discussion on statistical and robust nonparametric approaches to the influence of outliers or noise in efficiency measurement.

Undoubtedly, the stochastic approaches are conceptually intuitive and quite impressive in handling DMU-specific distributions of noise and inefficiency since they involve appropriate assumptions and specification of the data generating process (DGP). Nonetheless, their implementation is sometimes problematic due to the difficulty in obtaining historical data that determine the distribution of the random variable.

Some researchers prefer to use deterministic methods which to a larger extent prevent the problem of probability estimations. One of such early methods suggests the treatment of vagueness and ambiguity in data with fuzzy logic through fuzzy set theory. The fuzzy DEA, for instance, was introduced by Sengupta (1992) to characterize imprecise inputs and outputs by fuzzy numbers and membership functions. In the fuzzy DEA, inexact and imprecise input and output data are represented as linguistic variables characterized by fuzzy numbers reflecting the general feeling or experience of the decision maker (Guo & Tanaka, 2001). 'Fuzzification' of unknown crisp values are justified to provide good approximation and sensitivity minimization via the representation of uncertainty as fuzzy data. Generally, the fuzzy DEA hinges on the theory of fuzzy set[12] where the $\alpha -$ cut approach is the main technique used. Other techniques used include the fuzzy ranking, possibility, tolerance, fuzzy arithmetic and fuzzy random/type – 2 approach (Hatami-Marbini, Emrouznejad, & Tavana, 2011). Kao & Liu (2000) proposed a transformation of fuzzy DEA to a family of crisp DEA. They adopt the $\alpha -$ cut technique and use membership function to represent the fuzzy data. Guo & Tanaka (2001) considered an extension of the fuzzy DEA to handle general crisp, fuzzy and hybrid data. The recent survey on the development of the subject is summarized in Hatami-Marbini et al. (2011) for further reading.

Cooper, Park, & Yu (1999) addressed imprecision in data in its general form by proposing an imprecise DEA (IDEA) model that deals with bounded/interval data, ordinal data and ratio-bounded data, including also a mix of imprecise and exact data. However, incorporating these imprecise data into the DEA model result in a nonlinear and convex-programming problem. Zhu (2003) showed that the IDEA model can be transformed into a standard linear DEA model by using scale transformation and variable alternations or procedures that convert the imprecise data into exact data. The transformation approach leads to solving the IDEA model in the standard linear CCR model and therefore permit the standard analysis of performance benchmarking, RTS identification, etc. One of the popular approaches that has emanated from the IDEA is the interval DEA which seek to provide efficiency of DMUs with their lower and upper bound values. Despotis & Smirlis (2002) propose an approach which treats interval DEA as a peculiar case of DEA with exact data following a transformation of interval variables. The authors define lower and upper bound for interval efficiency scores and further discriminate DMUs into fully efficient, efficient and inefficient units. Incorporating decision makers preference in determining the bounds of the

[12] The fuzzy theory is based on the fuzzy set algebra developed by Zadeh (1965)

interval efficiency, similarly, Entani, Maeda, & Tanaka (2002) adopt the interval DEA to determine the optimistic and pessimistic ranking of DMUs with fuzzy approach. Optimistic measures are obtained from the upper limit of the interval efficiency and the lower limit of the inefficiency while the lower limit of interval efficiency and upper limit of the interval inefficiency determines the pessimistic measures.

Table 2.1. A taxonomy of robust approaches in DEA

Deterministic	*Stochastic*
Fuzzy DEA Sengupta (1992), Kao & Liu (2000), Guo & Tanaka (2001), Hatami-Marbini et al. (2011), Lertworasirikul et al. (2003)	**Chance constrained DEA** Land et al. (1993), Olesen & Petersen (1995), Cooper et al. (1998), T. Y. Chen (2002), Cooper et al. (2004), Talluri et al. (2006), Azadi & Saen (2012), Tavana et al. (2013)
Imprecise DEA Cooper et al.(1999), Zhu (2003), Despotis & Smirlis (2002), Entani et al. (2002), Wang, et al. (2005), Park (2007) Toloo & Nalchigar (2011), Toloo et al. (2018), **Robust DEA** Sadjadi et al (2008, 2011a), Shokouhi et al (2010), Hafezalkotob et al. (2015), Salahi et al. (2016), Toloo & Mensah (2018), Salahi, et al (2018)	**Statistical stochastic DEA** - *Maximum likelihood/OLS* Banker (1993, 1996), Banker & Natarajan (2008), McDonald (2009), Ramalho et al. (2010), - *Bootstrapping* Simar & Wilson (1998,2000,2007), Alexander et al. (2010), Wanke & Barros (2014) - *Robust nonparametric estimation* Cazals, Florens, & Simar (2002), Daraio & Simar (2006), Daraio & Simar (2007)

A new stream of research in inexact DEA concerns uncertainty in inputs and outputs data which is addressed in the lens of the RO. This approach, deterministic in nature is the main focus of this thesis and its discussion is reserved for Sections 2.3. We provide in Table 2.1 a taxonomy of the main robust approaches adopted in DEA. Note that the stochastic approaches might contain some lesser known techniques in literature which are not captured in Table 2.1. So far, there are three main deterministic approaches used in handling imprecision and uncertainty in DEA: fuzzy DEA, imprecise/interval DEA and robust DEA. It is important to mention that, although the analysis with "imprecise data" and "uncertain data" both lead to robust efficiency scores, a common mistake is to perceive and use the two interchangeable since the modelling approach appropriate for one might not be appropriate for the other. For instance, whiles it is possible to use the RO approach for imprecise data such as in interval DEA (see Shokouhi et al., 2010; Aghayi & Maleki, 2016), it is quite difficult to model same for ordinal data or ratio-bounded data.

2.1.4.1 Characterization of uncertainty in DEA

Lio & Liu (2017) adopted the description "uncertain variable" as a better choice for describing the imprecise inputs and outputs of a DEA model. The authors developed a new uncertain DEA model based on uncertainty theory of Wen (2015). Ehrgott, Holder, & Nohadani (2018) recently proposed an uncertain DEA (uDEA) model which determines the configuration and minimal amount of uncertainty that suffices to render a DMU efficient. Uncertainty and imprecision are understood to be portmanteau words that are used contextually and sometimes interchangeably. They are sometimes confused with each other, nonetheless, the two concepts are distinct. In fact, "uncertainty in data" generally connotes the various form of uncertainty that may arise from imprecision, ambiguity or lack of clarity in quantifying the exact values of data. See French (1995). Different ways exist in handling uncertainty, more importantly when the source of uncertainty is known. Here, we describe two main types of uncertainty including randomness inherent in the input and output data of DMUs.

b) *Uncertainty arising from lack of knowledge, miscalculation or computational errors*. This includes the entire spectrum of the different degrees of knowledge including lack of information or whilst information is available, the difficulty to quantify the exact values of data due to vagueness or imprecision of the data. Organizational data such as banks data, hospitals data, etc. which are used for evaluation are usually obtained through computations, predictions or by some statistical computation. Even at the power of modern computers or an efficient algorithm, one cannot accept the result 100% as they may be errors resulting from measurement, computation, statistical approximations or truncations.

b) *Uncertainty arising from physical randomness of the inputs and outputs data*. This type of uncertainty pertains to data that are randomly generated and whose actual values are unknown. The variability in input and output data could result from uncaptured noise or natural stochasticity of the data.

The uncertainties described above can be generally termed as epistemic uncertainty and aleatory (variability) uncertainty. Epistemic uncertainty arises due to limited knowledge and refers to the uncertainty description in (a). Aleatory uncertainty occurs as a result of natural randomness in data as described by (b). Aleatory uncertainty in particular is mostly modeled using mathematical probability. In DEA, this has been extensively discussed in Wen (2015).

Table 2.2. Exact and uncertain data for two inputs – one input case.

DMUs	*Exact data*			*Uncertain data*		
	x_1	x_2	y	x_1'	x_2'	y
A	1	5	1	1	5.2	1
B	1.2	3	1	1.6	3	1
C	2	2	1	2.6	1.6	1
D	4.2	1.2	1	4	1.2	1
E	5	1	1	5.1	0.6	1
F	7	1	1	7	0.5	1
G	3	3	1	2.6	3	1
H	5.2	1.6	1	6	1.5	1

2.1.4.2 The effect of uncertainty in DEA data

Uncertain data has effect on the discrimination of DMUs and the right decision on their performance. For instance, since DMUs are compared with each other to determine their relative efficiency, the question arises whether or not a particular DMU has been selected as efficient as a result of its uncertain data and if so, then there is a reasonable argument against its perceived performance (Ehrgott, Holder, & Nohadani, 2018). The uncertainty in DEA data has two main effects: Firstly, the uncertain data dislocate the efficient frontier. Figure 2.7 shows two efficient frontiers; one depicting the true data (solid lines) and the other uncertain

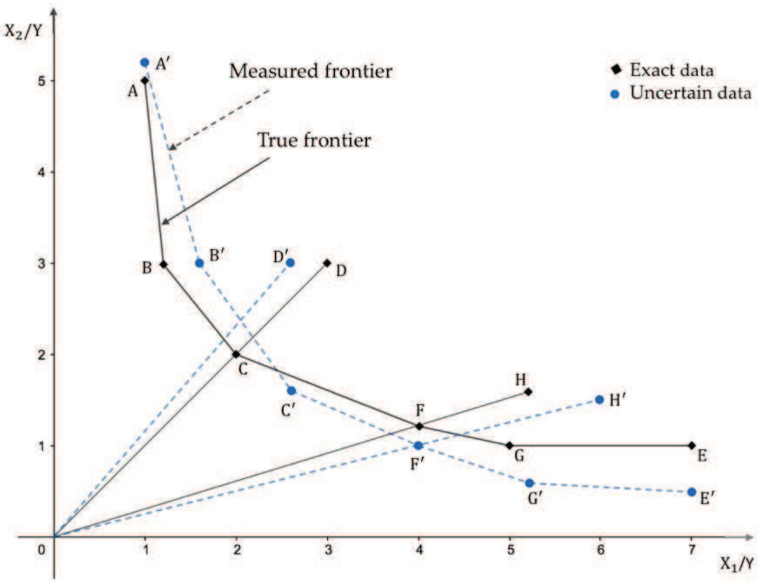


Figure 2.7. Frontiers of exact and uncertain data

/measured data (dashed lines) from the data points in Table 2.2. DMUs A, B, C, E, F, G from the actual data and their corresponding uncertain data efficient are on both frontiers although the frontier from the uncertain data is distorted. Observe that the actual reference unit for the projection of the inefficient DMU D is DMU C. On the other hand, DMU D' would be projected to the frontier (dashed lines) using DMUs B' and C'. Moreover, the radial measurement of uncertain data points is quite different from what their actual data would suggest. In such cases, the identification of the inefficient units from the best practice units and the right amount of potential improvement possible of an inefficient unit becomes quite difficult to ascertain. Secondly, it is obvious that from the DEA model becomes sensitive to small data perturbation and fail to preserve the efficiency of the DMUs. The effect of data uncertainty on DEA models when neglected can affect the reliability of the efficiency scores as well make the nominal DEA model highly infeasible following a small perturbation in the uncertain data. The erroneous selection of an under-performing DMU as a benchmark unit for others, for example, DMU C' has the potential of rendering the decision on DMUs performance useless. To overcome this drawback, the RO which immune data to a prescribed uncertainty set is introduced.

2.2 Robust optimization

Robust optimization is a field in optimization that deals with uncertainty in the data of optimization problems such as the DEA. RO addresses uncertainty based on an uncertainty set which is centered around the nominal values of the uncertain parameters. In its appealing form to practitioners, the RO focuses on searching for an acceptable performance or the best solution that is feasible under all possible realization of the uncertain parameters in a small "realistic set" (i.e. the uncertainty set) defined by the practitioner. The uncertainty set – induced RO began with the work of Soyster (1973). He sought to obtain an optimal solution for an inexact linear optimization such that the constraints are satisfied under all possible perturbations of the data in an interval set. Although Soyster's model is feasible, the resulting robust counterpart often produces result which is considered aggressively conservative in that too much of the optimality has to be scarified for robustness. To reduce the level of conservatism of the robust counterpart, several concepts and model formulations have been proposed. The reference to the modern literature on this relates to the work of Ben-tal, Ghaoui, & Nemirovski (2009). Specifically, the concept of reliability of the robust solution (Ben-Tal & Nemirovski, 2000), the control of the price of robustness (Bertsimas & Sim, 2004), and the adjustable robustness (Ben-Tal, Goryashko, Guslitzer, & Nemirovski, 2004) among others have all been proposed. Broadly speaking and in accordance with these authors, one of the major modelling concerns is to design a tractable[13] robust formulation for the nominal problem and the guarantee that the constraints will not be violated or will be feasible with

[13] By tractability, we mean the existence of an explicit polynomial time algorithm to an equivalent formulation of the nominal optimization problem

high probability for the realization of the uncertain parameters in the uncertainty set. This is very important since the optimization may be tractable while its robust version may not or may be very complex and difficult to solve. In the sections that follow, we show how the uncertain optimization is turned to a tractable robust counterpart.

2.2.1 Solving the uncertain optimization problem

Consider the uncertain LP below:

$$\begin{aligned} &\max \textstyle\sum_{j=1}^{n} c_j x_j \\ &\text{s.t.} \\ &\textstyle\sum_{j=1}^{m} \tilde{a}_{ij} x_j \leq b_i \quad && i = 1, \dots, m \\ &x_j \geq 0 && j = 1, \dots, n \end{aligned} \tag{2.15}$$

where x represent a vector of decision variables and $\tilde{a}_{ij}$ is a technological coefficient with entries $\tilde{a}_{ij} = [\tilde{a}_{1j}, \dots, \tilde{a}_{nj}]$. For simplicity, only the element $\tilde{a}_{ij}, j \in J_i$ where J_i represent the set of coefficients in row i that are subjected to the uncertainty. i.e. $\tilde{a}_{ij} \in \mathcal{U}$[14]. In RO, the true value $\tilde{a}_{ij}$, of an uncertain parameter is modeled as:

$$\tilde{a}_{ij} = a_{ij} + \eta_{ij} \hat{a}_{ij} \quad \forall i \tag{2.16}$$

where $\eta_{ij} = [\eta_{1j}, \dots, u_{nj}]$ is the random part of $\tilde{a}_{ij}$ (sometimes assumed to have symmetric distribution in the interval [-1, 1] and, a_{ij} and $\hat{a}_{ij}$ are the nominal and estimate of the maximum deviation from a_i respectively. The RO approach deals with finding a solution to problem (2.15) such that the constraint feasibility for any realization of the uncertain parameter in $\mathcal{U}$. To ensure such feasibility, the constraint is rewritten as:

$$a_i x_j + \max_{\eta_{ij} \in \mathcal{U}} \textstyle\sum_{j \in J_i} \hat{a}_{ij} x_j \eta_{ij} \leq b_i \quad \forall i \tag{2.17}$$

Constraint (2.17) is called the *robust counterpart* of the uncertain constraint in (2.15). The explicit form of the robust counterpart of the later depends on the specific uncertainty $\mathcal{U}$ used for the former.

2.2.2 Two alternative representation of uncertainty in RO

Two main alternative approaches exist for characterizing the uncertainty. The first one is the continuous-based description of the uncertain data for a range of values and the second is the discrete-based scenario set description for each uncertain parameter.

[14] The set $\mathcal{U}$ is assumed to be closed and convex without loss of generality.

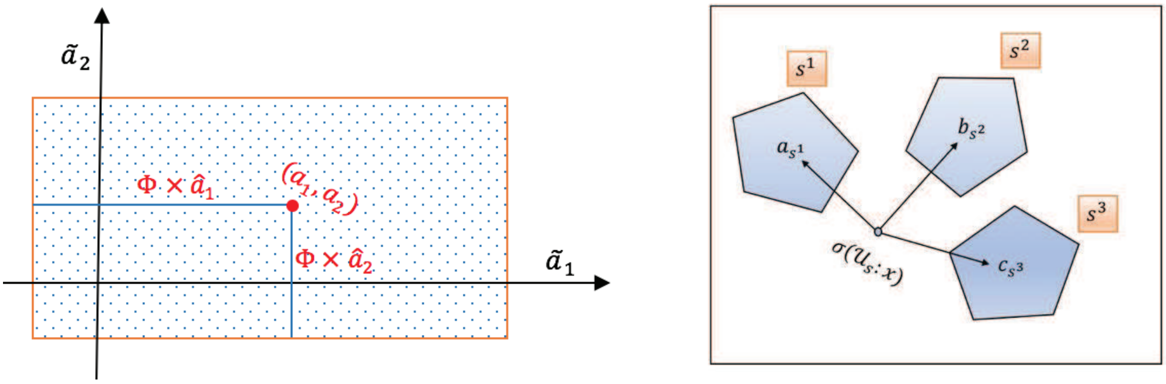


Figure 2.8. Uncertainty description. *Left*: Box uncertainty set, *Right*: Discrete scenario set

2.2.2.1 Continuous-based uncertainty set

In this approach, the uncertain situation is characterized by an uncertainty set as used in Soyster (1973). The simplest uncertainty set to have is the box (interval) uncertainty set which is given as:

$$\mathcal{U}_\Phi = \{\tilde{a}_{ij} = a_{ij} + \eta_{ij}\hat{a}_{ij} | \parallel \boldsymbol{\eta} \parallel_\infty \leq \Phi\}$$

for $\eta_{ij} = [-1, 1]$ where Φ is an adjustable parameter. Other basic uncertainty sets are the norm uncertainty, polyhedral uncertainty, ellipsoidal uncertainty sets. Figure 2.8 (left) illustrates the box uncertainty set for the realized values of the uncertainty parameters. The optimization model with respect to $\boldsymbol{\eta}$ in the set $\mathcal{U}_\Phi$ is discussed in Chapter 3.

2.2.2.2 Discrete scenario-based set

This approach characterizes the realizable values of uncertain data by a discrete scenario set with occurrence probability p_s. Suppose that the parameters (a, b, c) in (2.15) are all uncertain. We define a finite set of scenarios $\mathcal{S} = \{s^1, s^2, s^3\}$ and for each scenario $s^i \in \mathcal{S}$, we associate the set $\mathcal{U}_s = \{a_{s^1}, b_{s^2}, c_{s^3}\}$ of realization for the parameters where $\sum_s^{\mathcal{S}} p_s = 1$. Figure 2.8 (right) shows the scenario set $\mathcal{S} = \{s^1, s^2, s^3\}$ for the parameters. $\sigma(\mathcal{U}_s, x)$ aim to optimize model (2.15) with respect to each scenario. The RO model to the discrete scenario set of scenarios was proposed in Mulvey, Vanderbei, & Zenios (1995). Note that large scenario set entails specifying each scenario which may be cumbersome a task. Most often, the discrete optimization problem becomes NP-hard in their robust version, as in the case of the assignment problem, shortest path problem, the resources allocation problem etc. (Kouvelis & Yu, 1997). Nonetheless, the discrete scenario description of the uncertainty has its own merit in the following (Yu, 1997):

1. useful when the parameters assume only discrete values;
2. accurate description of the uncertainty finite if historical data is available and can be used to form the discrete scenarios

3. correlation among parameters can be categorized by the discrete scenario specification without needing probability distributions.

2.2.3 From uncertain optimization to robust optimization

We explain the modeling technique used in the RO to attain the robust solution. The approach used throughout this thesis is based on the uncertainty set adaptation. From the uncertain problem (2.15), we describe three main steps that can be thought of as the modeling process leading to a robust solution: (i) the construction of the uncertainty set, (ii) the formulation of a robust counterpart optimization and (iii) the probability bound for constraint violation of the uncertain parameter. Figure 2.9 shows the steps. The next sections explain the steps.

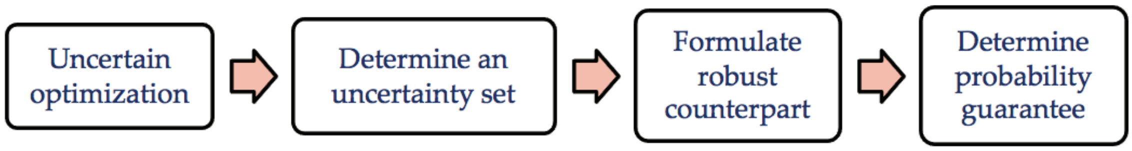


Figure 2.9. Modeling process in robust optimization

2.2.3.1 Choosing the uncertainty set

Depending on the physical realization of the uncertain parameters or the distribution of the uncertainty, there are subtle ways of constructing/choosing the uncertainty set uncertainty set $\mathcal{U}$. The construction of $\mathcal{U}$ start with the raw data which is then processed to meet the preferences of the decision maker and the assumptions on the structure and scale of the uncertainty set[15]. There are several ways of designing $\mathcal{U}$, largely driven by the data. Bertsimas, Gupta, & Kallus, (2018) show concrete procedures for choosing an appropriate uncertainty set for a given application using historical data and statistical estimates. Bertsimas & Brown (2009) provide insights into the construction of $\mathcal{U}$ from the risk preference of the decision maker such as the *coherent risk measure.* In the literature, the more specific representation of $\mathcal{U}$ proposed relates to the structure of the set. Table 2.3 list few classes of the uncertainty sets that are frequently used. These are sets which have known tractable robust counterpart and deemed useful for robust analysis in DEA.

Each uncertainty set is equipped with a size parameter $(\Phi, \Delta, \Omega, \Gamma, \gamma)$ which defines the robust level preferred by the decision maker. These parameters control the trade-off between the probability of constraint violation and the impact of the objective function of the nominal

[15] The structure of $\mathcal{U}$ refers to the shape or geometry of the set whiles the scale refers to the size of the structure defining $\mathcal{U}$. For instance, the structure of $\mathcal{U}$ must be convex and on the other hand, the deviation of the uncertain parameters (scale) must be properly scaled in order for the robust counterpart to be computationally tractable (Gregory, Darby-Dowman, & Mitra, 2011).

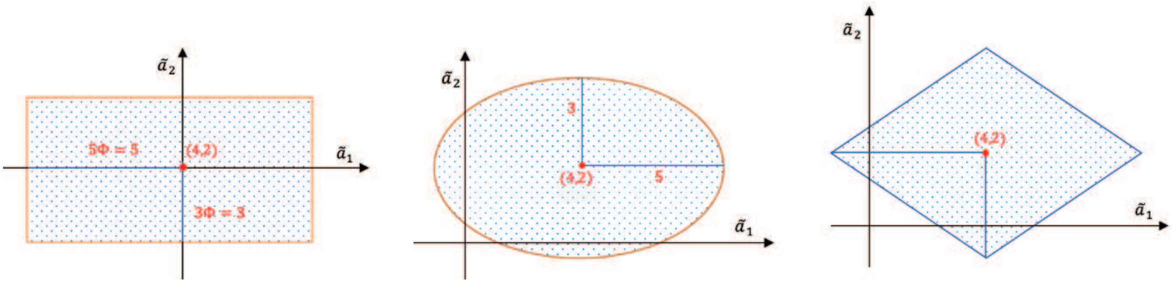
Table 2.3. Examples of uncertainty set

Uncertainty type	*Uncertainty set* $\mathcal{U}$	*Author*
Box/Interval	$\mathcal{U}_{(\Phi)} = \{\tilde{a}_{ij} = a_{ij} + \eta_{ij}\hat{a}_{ij} \mid \ \parallel \boldsymbol{\eta} \parallel_{\infty} \leq \Phi\}$	Soyster, 1973
Norm-based[16]	$\mathcal{U}_{(\Delta)} = \{\mathrm{A} \mid \lVert M(vec(\tilde{A}) - vec(A)) \rVert \leq \Delta\}$	Bertsimas et al. (2004)
Ellipsoid	$\mathcal{U}_{(\Omega)} = \{\tilde{a}_{ij} \mid \ \parallel \boldsymbol{\eta} \parallel_2 \leq \Omega\}$	Ben-Tal & Nemirovski (1998)
Interval+ellipsoid	$\mathcal{U}_{(\Omega)} = \{\tilde{a}_{ij} \mid \ \parallel \boldsymbol{\eta} \parallel_{\infty} \leq 1, \parallel \boldsymbol{\eta} \parallel_2 \leq \Omega\}$	Ben-Tal& Nemirovski (2000)
Polyhedral	$\mathcal{U}_{(\Gamma)} = \{\tilde{a}_{ij} \mid \ \parallel \boldsymbol{\eta} \parallel_1 \leq \Gamma\}$	---
Interval+polyhedral	$\mathcal{U}_{(\Gamma)} = \{\tilde{a}_{ij} \mid \ \parallel \boldsymbol{\eta} \parallel_{\infty} \leq 1, \parallel \boldsymbol{\eta} \parallel_1 \leq \Gamma\}$	Bertsimas &Sim(2004)
CLT – based set[17]	$\mathcal{U}_{(\gamma)} = \{\tilde{a}_{ij} \mid \lvert \sum_{i=1}^{n} \eta_{ij} - n\mu \rvert \leq \gamma\sigma\sqrt{n}\}$	Bandi & Bertsimas (2012)

problem. Bertsimas & Sim (2004) termed the trade-off as the *price of robustness.* The price paid to gain a robust solution is seen as the difference between the objective function value of the nominal model and the case for the specific value of the robust model at Γ_i. In other words, as the parameter (Γ_i) increases, there is more protection and robustness and high price to pay in terms of the performance of the robust model. Notice that $\mathcal{U}$ is constructed to meet a probability guarantee expectation of the modeler and should, therefore, take cognizance of the type of $\mathcal{U}$ since it determines the computational complexity of the robust counterpart (Bertsimas, Brown, & Caramanis, 2011). Figure 2.10 illustrates the distribution of the random variable η_{ij} defined with the uncertainty dynamics $\mathcal{U} = \{(4 + 5\eta_1, 2 + 3\eta_2) \mid -1 \leq \eta_1, \eta_2 \leq 1\}$. The uncertainty sets which takes the shape of a box, ellipsoid and polyhedron with unit parameter (i.e. $\Phi_i = \Omega_i = \Gamma_i = 1$) as shown below are demonstrated in Appendix A. In Chapter 3, these uncertainty sets combined with the interval uncertainty set and their corresponding robust counterparts are reviewed. Subsequently, we provide a *reduced robust counterpart* for these sets. For each type of uncertainty set that is chosen, the goal of the modeler is to ensure that the constraint remains satisfied for any possible realization of the uncertain parameters. Choosing $\mathcal{U}$ such that the constraints are not violated when the uncertain parameters take their worst – case values is therefore very important since the computational tractability, the complexity of the robust counterpart and conservativeness of the solution all depend on $\mathcal{U}$.

[16] M is an invertible matrix and $vec(\tilde{A})$ denote the vector obtained by stacking its rows of matrix A on top of one another.

[17] μ and σ are the mean and variance of the iid variable η_{ij}.



(a) Box uncertainty set (b) Ellipsoidal uncertainty set (c) Polyhedral uncertainty set

Figure 2.10. Plot of basic uncertainty sets

2.2.3.2 Formulating the robust counterpart

Under the robust counterpart formulation, we are concerned with solutions which have the best performance under most realizations of the uncertain technological coefficients rather than a usual optimal solution (Ben-Tal, El Ghaoui, & Nemirovski, 2009). The concern, therefore, is to select a computationally tractable uncertainty set such that the gap between the optimal solution in the nominal case and the robust optimal value is as close as possible. We outline three steps for obtaining the robust counterpart for a specific $\mathcal{U}$. Here we use the budget of uncertainty of Bertsimas & Sim (2004) for the purpose of its linear tractability.

Step 1 (Worst-case reformulation): The worst-case formulation of the uncertain constraint in model (2.15) is given by:

$$a_{ij}x_j + \max_{\eta_{ij}\in\mathcal{U}} \textstyle\sum_{j\in J_i} \hat{a}_{ij}x_j\eta_{ij} \le b_i \qquad \forall i \tag{2.18}$$

where $\mathcal{U}_{(\Gamma_i)} = \tilde{a}_{ij} | \parallel \boldsymbol{\eta} \parallel_\infty \le 1, \parallel \boldsymbol{\eta} \parallel_1 \le \Gamma_i$ is used as the uncertainty set. The subproblem (inner maximization problem) is equivalently expressed as the following:

$$\max_{\eta_{ij}\in\mathcal{U}} \textstyle\sum_{j\in J_i} \hat{a}_{ij}x_j\eta_{ij} = \begin{cases} \max \sum_{j\in J_i} \hat{a}_{ij}x_j\eta_{ij} \\ s.\mathrm{t.} \\ \sum_{j\in J_i} |\eta_{ij}| \le \Gamma_i \\ 0 \le \eta_{ij} \le 1 \qquad\qquad \forall j \in J_i \end{cases} \tag{2.19}$$

Step 2 (Duality): We take the dual of the subproblem. Note that the subproblem and its dual yield the same optimal objective value by the strong duality theorem. Hence, model (2.19) is equivalent to the following:

$$\begin{aligned} &\min \textstyle\sum_{j\in J_i} q_{ij} + p_i\Gamma_i \\ &\mathrm{s.t} \\ &p_i + q_{ij} \ge \hat{a}_{ij}x_j && \forall j \in J_i \\ &q_{ij} \ge 0 && j \in J_i \\ &p_i \ge 0 \end{aligned} \tag{2.20}$$

Step 3 (Robust counterpart): We can now write the robust counterpart for uncertain constraint in model (2.15) noting that model (2.20) is still feasible for at least one q_{ij} and p_i if we omit the minimization term. Now adding the objective function $\max c^T x$ of the LP, we arrive at the following robust counterpart.

$$\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + p_i \Gamma_i + \sum_{j=1}^{n} q_{ij} \le b_i & i = 1, \dots, m \\
p_i + q_{ij} \ge \hat{a}_{ij} y_j & \forall i, \forall j \in J_i \\
q_{ij} \ge 0 & \forall i, j \in J_i \\
p_i \ge 0 & i = 1, \dots, m \\
x_j \ge 0 & j = 1, \dots, n
\end{array} \tag{2.21}$$

2.2.3.3 Probability guarantees

To reduce the strict conservativeness of robust solutions, the modeler might allow for a certain degree of constraint violation. Probability guarantee are given on the feasibility of the model constraints for all the uncertain parameters taking values in the uncertainty set and beyond[18]. The use of probability indicator enables the modeler to measure the level of satisfaction of the constraints. In this case, the probability of constraint feasibility becomes akin to the chance - constrained model where one is interested in the guarantee level in which at least a constraint is violated. Bertsimas & Sim (2004) remarked that if nature is restricted in its behavior in that only a subset of the uncertain parameters change, a guarantee for the robust model is required to ensure that the robust solution will be feasible deterministically and with high probability. Using their proposed budget of uncertainty for the robust counterpart, they suggested that the solution of the robust model will remain feasible if up to $\lfloor \Gamma_i \rfloor$ of the uncertain coefficients change within their bound and one coefficient a_{it} changes by $(\Gamma_i - \lfloor \Gamma_i \rfloor)\hat{a}_{it}$. Further, Bertsimas & Sim (2004) further proved that the probability of constraint violation, $\Pr\left(\sum_j \tilde{a}_{ij} x_j^* > b\right)$ is bounded above by $\exp\left(-\frac{\Gamma^2}{2|J_i|}\right)$. Similarly, Ben-Tal & Nemirovski (2000) proved the bound $\exp\left(-\frac{\Omega^2}{2}\right)$ for the robust counterpart with interval-based ellipsoidal uncertainty set. Bertsimas et al. (2004) proved the bound $\frac{1}{1+\Delta^2}$ for the robust counterpart with norm-based uncertainty set[19]. Usually, if the probability distribution for the uncertainty exists, then it is desirable to determine the upper bound for constraint violation or lower bound for constraint satisfaction apriori or posterior. A stronger bound obtained indicates that the robust solution is feasible with high probability.

[18] It is important to note that, in reality the uncertainty defined does not cover the whole uncertain space containing all the possible realization of the uncertain parameters otherwise the probability guarantee for constraint feasibility will be equal to 1.

[19] Note that the dynamics of uncertainty assumed to have bounded and symmetric distributions.

2.2.4 Motivating example

Considering model (2.15), the decision variable is a positive variable (i.e. $x_j \in \mathbb{R}_+^n$) and so is the robust counterpart uncertain constraint (2.17). This is the main focus of Chapter 3. We compare the computational complexity of this approach as against the formulation of Bertsimas & Sim (2004) and others where the decision variable is a free variable. Bertsimas & Sim (2004) robust counterpart where the x_j is a free variable with the bounds $l_j \leq x_j \leq u_j$ is given as:

$$
\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + p_i \Gamma_i + \sum_{j=1}^{n} q_{ij} \leq b_i & i = 1, \dots, m \\
p_i + q_{ij} \geq \hat{a}_{ij} y_j & \forall i, \forall j \in J_i \\
l_j \leq x_j \leq u_j & j = 1, \dots, n \\
-y_j \leq x_j \leq y_j & \forall i, \forall j \in J_i \\
q_{ij} \geq 0 & \forall i, \forall j \in J_i \\
p_i \geq 0 & i = 1, .., m \\
y_j \geq 0 & \forall i, \forall j \in J_i
\end{array}
\qquad (2.22)
$$

We compare the advantage model (2.21) has on model (2.22) when x_j is defined for only positive variables with an example and the leave the rest of the discussion for the next chapter. Consider the following simple example given in Bazaraa, Jarvis, & Sherali (2010), page 231:

$$
\begin{array}{l}
\min z = -2x_1 - 4x_2 - x_3 \\
\text{s.t.} \\
\qquad 2x_1 + x_2 + x_3 \leq 10 \\
\qquad x_1 + x_2 - x_3 \leq 4 \\
\qquad 0 \leq x_1 \leq 4 \\
\qquad 0 \leq x_2 \leq 6 \\
\qquad 1 \leq x_3 \leq 4
\end{array}
$$

The nominal values are (2, 1, 1) and (1, 1, -1) which are the coefficients of the first and second constraint respectively. The problem is in its minimal form and by the simplex algorithm, the optimal solution is $(x_1^*, x_2^*, x_3^*) = (0.67, 6, 2.67)$ and the optimal objective function value is -28 which can be computed under 4 iterations. Assume that the uncertain coefficients are 10% accurate approximations of the "true" vector of coefficients. Let $J_1 = \{1,3\}$ and $J_2 = \{2\}$. The corresponding robust counterpart example based on model (2.21) is:

$$
\begin{aligned}
\text{RCE}(1) \quad & \min t = -2x_1 - 4x_2 - x_3 \\
& \text{s.t.} \\
& 2x_1 + x_2 + x_3 + p_1\Gamma + p_3\Gamma + q_{11} + q_{13} \le 10 \\
& x_1 + x_2 - x_3 + p_2\Gamma + q_{22} \le 4 \\
& p_1 + q_{11} \ge 0.1x_1 \\
& p_3 + q_{13} \ge 0.2x_3 \\
& p_2 + q_{22} \ge 0.4x_2 \\
& q_{11}, q_{13}, q_{22} \ge 0 \\
& p_1, p_2, p_3 \ge 0 \\
& 0 \le x_3 \le 4 \\
& 0 \le x_2 \le 4 \\
& 1 \le x_3 \le 6
\end{aligned}
$$

On the hand, the robust counterpart based on model (2.22) which is the robust formulation of Bertsimas & Sim (2004) is:

$$
\begin{aligned}
\text{RCE}(2) \quad & \min s = -2x_1 - 4x_2 - x_3 \\
& \text{s.t.} \\
& 2x_1 + x_2 + x_3 + p_1\Gamma + p_3\Gamma + q_{11} + q_{13} \le 10 \\
& x_1 + x_2 - x_3 + p_2\Gamma + q_{22} \le 4 \\
& p_1 + q_{11} \ge 0.1y_1 \\
& p_3 + q_{13} \ge 0.2y_3 \\
& p_2 + q_{22} \ge 0.4y_2 \\
& q_{11}, q_{13}, q_{22} \ge 0 \\
& p_1, p_2, p_3 \ge 0 \\
& -y_1 \le x_1 \le y_1 \\
& -y_2 \le x_2 \le y_2 \\
& -y_3 \le x_3 \le y_3 \\
& 0 \le x_3 \le 4 \\
& 0 \le x_2 \le 4 \\
& 1 \le x_3 \le 6 \\
& y_1, y_2, y_3 \ge 0
\end{aligned}
$$

Note that the example provided is defined for positive values of x_j only. This makes RCE(1) more appropriate than RCE(2). Table 2.4 summarizes the optimal solutions for different Γ values. Note that whiles optimal solution to RCE(1) and RCE(2) are equal, the latter entails more iteration and execution time to solve the problem.

Table 2.4. Robust counterpart solutions to motivating example

Robust parameter	$\Gamma = 0$	$\Gamma = 0.4$	$\Gamma = 0.6$	$\Gamma = 1$
t^*	−28.0	−27.73	−27.57	−25.82
$(x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*)$	(0.67,6.0,2.67)	(0.26,6.00,3.22)	(0.05,6.00,3.48)	(0.00,5.52,3.73)
s^*	−28.0	−27.73	−27.57	−25.82
(x_1^*, x_2^*, x_3^*)	(0.67,6.0,2.67)	(0.26,6.00,3.22)	(0.05,6.00,3.48)	(0.00,5.52,3.73)

2.3 Robust DEA

Robust DEA (henceforth RDEA) is the application of RO in DEA. The first application of the robust optimization to DEA began in 2008 with Sadjadi & Omrani when they investigated the performance of utility service providers where the underlying data was uncertain. The authors focused on providing a robust and reliable performance ranking of DMUs for management decision in the utility service. Furthermore, the work of Sadjadi et al (2011a), Wang & Wei (2010) and Shokouhi et al (2010) bolstered the need for robust efficiency measure via the RO. An efficiency score that is robust is expected to withstand disturbances in order to keep its ranking stable. In fact, the term 'robust' in DEA is generic since every approach that seeks to preserve the efficiency scores of DMUs is termed as a robust approach. Here, the RDEA is robustly referred because of the RO techniques infused in. It is specifically an uncertainty driven efficiency measure to acceptable robust efficiency scores. In this study, the RDEA approach to robustness is defined as:

> *a non-parametric frontier tool that utilizes robust optimization methods to immunize uncertain inputs and outputs data of DMUs and provide probability guarantee for reliable efficiency scores, robust discrimination and ranking of DMUs.*

The RDEA performance measure can be measured through the uncertainty set – based robust approach or the discrete scenario - based robust approach. As demonstrated in the previous sections, the contribution made in this study relate to the uncertainty set – based robust approaches to DEA. We first review the few studies made in the RDEA and provide a characterization of RO in the DEA to different uncertainty sets. The review allows us to highlight key areas that require attention in the methodological development. For completeness, some of the concerns which are addressed in the succeeding chapters will be restated in this section.

2.3.1 A literature review of RDEA

As aforementioned, the concept of RO in DEA began with the work of Sadjadi & Omrani (2008). The authors in their attempt to measure the efficiency of the electricity distribution companies in Iran but faced with output data uncertainty proposed two robust CCR models based on the robust approaches of Ben-Tal & Nemirovski (2000) and Bertsimas & Sim (2004). They compared their proposed models to the stochastic frontiers analysis technique in order to understand the effect of different data perturbations to the DEA efficiency. To further determine the input and output targets values of the electricity companies, Sadjadi et al (2011a) proposed an interactive RDEA which searches the envelopment frontier by combining DEA and multi-objective linear programming method such as the STEM. Wang & Wei (2010) use the robust approach of Ben-Tal & Nemirovski (2000) to similarly propose an input and output robust CCR models. Like the earlier researchers, their models avoid uncertainty measurement in the DEA normalization constraints and so where an input-oriented model is adopted, uncertainty is measured in the output data and vice versa.

Shokouhi et al (2010) proposed a general RDEA model in which inputs and outputs are constrained in an uncertainty set with data uncertainties covering the interval DEA approach. They used the robust approach of Bertsimas & Sim (2004) where they embraced Monte Carlo simulation to compute for the range of Gamma values for the conformity of the ranking of the DMUs. A similar modeling approach is made in a modified RDEA in Shokouhi, Shahriari, Agrell, & Hatami-Marbini (2014). Sadjadi & Omrani (2010) propose a combine bootstrapping and RDEA models that overcome the effect of perturbation and sampling error inherent in the input and output data. The proposed model is used to measure the efficiency of telecommunication companies in Iran. A robust super-efficiency model based on the ellipsoidal uncertainty set in envelopment CCR model is proposed in Sadjadi et al (2011b) to measure the efficiency of gas companies in Iran.

Hafezalkotob et al. (2015) consider the discrete set of scenarios used in the RO of Mulvey, Vanderbei, & Zenios (1995) to propose an RDEA for the electricity distribution companies in Iran. Different scenario set – based RDEA with specified probability for input and output data are also studied in Zahedi-seresht, Jahanshahloo, & Jablonsky (2017) in an attempt to derive the ranking of DMUs, being robust with respect to the changes of inputs and outputs in the different scenarios. Esfandiari, Hafezalkotob, Khalili-Damghani, & Amirkhan (2017) propose a robust two-stage DEA under the discrete set of scenarios. In the two-stage structure, the authors proposed two approaches: robust centralized (cooperative) and robust decentralized (non-cooperative) games models with application to the banking industry. Amirkhan, Didehkhani, Khalili-Damghani, & Hafezalkotob (2018) combined fuzzy set approaches and the discrete set scenario – based RO approach to deal with mixed fuzzy - robust uncertainty in the input/output data of DMUs. Ranking the robust efficiency of small and medium -sized enterprises with the fuzzy – robust DEA (FR-DEA), the authors showed under the CRS and VRS conditions that the FR-DEA maintain the advantages of both fuzzy and robust DEA and is capable of calculating the lower and upper bound efficiency of DMUs.

Lu (2015) is the first to develop RDEA models from the variable returns to scale technology using both the Ben-Tal & Nemirovski (2000) robust model and Bertsimas & Sim (2004) budget of uncertainty set to evaluate algorithm performance. The developed models, however, fall short of uncertainty in the normalization constraint. Arabmaldar, Jablonsky, & Saljooghi (2017) propose a new RDEA model by considering uncertainty in the CCR equality constraint. The proposed model is extended to a robust super – efficiency measure for a set of DMUs. Toloo & Mensah (2018) propose a similar reduced robust DEA (RRDEA) model with inequality constraint for the BCC model using the robust approach of Bertsimas & Sim (2004). They show that the proposed RRDEA reduces model complexity by half for nonnegative decision variables which were demonstrated with 250 banks in Europe. Omrani (2013) introduces an RDEA to find the common set of weights (CSW) in DEA with uncertain data.

Table 2.5. Advances in robust optimization applications in DEA

Authors	*DEA model*	*Robust approach**	*Uncertainty var*	*Application area*
Sadjadi & Omrani (2008)	CCR	BN & BS	Output	Energy
Shokouhi et al. (2010)	CCR	BS & Interval	Input & output	No application
Sadjadi & Omrani (2010)[b]	CCR	BS & Bootstrap	Output	Telecommunication
Wang & Wei (2010)	CCR	BN	Input or output	No application
Sadjadi et al. (2011)[a]	SE^{a}	BS	Input & output	Energy
Sadjadi et al. (2011)[b]	MORO^{b}	BS	Input & output	Energy
Omrani (2013)	CSW^{c}	BS	Input & output	Energy
Lu (2015)	BCC	BN & BS	Output	Algorithm performance
Mardani & Salarpour (2015)	CCR	BS & Interval	Input & Output	Agriculture
Hafezalkotob et al. (2015)	CCR	Mulvey	Input & output	Energy
Atıcı & Gülpınar (2016)	CCR	BS	Output	Agriculture
Wu et al. (2017)	CCR	Maxmin	Output	Hybrid poplar clones
Zahedi-seresht et al. (2017)	CCR	Mulvey	Input & output	Engineering company
Esfandiari et al. (2016)	Two-Stage	Mulvey	Input & output	Bank branches
Arabmaldar et al. (2017)	CCR/SE	BS	Input & output	Energy/forest district
Toloo & Mensah (2018)	BCC	BS	Input & output	Banking
Salahi et al. (2016)	CCR-CSW	Interval set	Input & output	Energy/forest district
Hladík (2019)	Fractional	Norm-based	Input & output	No application
Aghayi & Maleki (2016)	DDF^{d}	BS	Input & output	Banking
Bayati & Sadjadi (2017)	Network	BN & BS	Input & output	Energy

*BN = Ben-Tal & Nemirovski, BS = Bertsimas & Sims, $\mathbf{SE}^{a}$ = Super Efficiency $\mathrm{MOR}\mathbf{O}^{b}$ = Multi-Objective Model for Ratio Optimization, $\mathrm{CS}\mathbf{W}^{c}$ = Common Set of Weights, $\mathrm{DD}\mathbf{F}^{d}$ = Directional Distance Function

The common set of weight under interval uncertainties are computed from the optimistic viewpoint in Salahi et al. (2016). In a related study, Aghayi, Tavana, & Raayatpanah (2016) presented a robust DEA with a CSW to varying degrees of conservatism. Here the authors used the goal programming technique to compute the relative efficiencies of the DMUs by producing CSWs in one run and in addition ranking the DMUs using the level of conservatism of the decision maker.

Majority of the contributions made so far in the RDEA are application driven. As noted in Sadjadi & Omrani (2008) and Lu (2015), the applications often require developments in methodology and raise some practical questions about existing RDEA models, generating challenges as to how to advance the field. The applications made are notably on utility service with ranking seeking a trade-off between robustness and performance. See Sadjadi & Omrani (2008); Salahi et al. (2016) ; Bayati & Sadjadi (2017); Soltani, Tabriz, & Sanei (2017). Other applied disciplines include agriculture – efficiency of potato production (Mardani & Salarpour, 2015), the performance of olive – oil production (Atıcı & Gülpınar, 2016), banking (Esfandiari et al., 2017; Aghayi & Maleki, 2016) and others (Kaviani & Abbasi, 2015). Table 2.5 summarizes the major robust modeling approaches in the RDEA literature. The application areas are provided in the last column.

2.3.2 Characterization of DEA models with alternative uncertainty sets

In this section, we characterize the DEA models to some uncertainty sets given in Table 2.3. The models are formulated using the following symbols:

$\tilde{x}_{ij} = (\tilde{x}_{i1}, \tilde{x}_{i2}, \dots, \tilde{x}_{in})$: the uncertain input vector of $\mathrm{DMU}_j, j = 1, \dots, n$
$\tilde{x}_{io} = (\tilde{x}_{o1}, \tilde{x}_{o2}, \dots, \tilde{x}_{on})$: the uncertain input vector of DMU_o
$\tilde{y}_{rj} = (\tilde{y}_{r1}, \tilde{y}_{r2}, \dots, \tilde{y}_{rn})$: the uncertain output vector of $\mathrm{DMU}_j, j = 1, \dots, n$
$\tilde{y}_{ro} = (\tilde{y}_{o1}, \tilde{y}_{o2}, \dots, \tilde{y}_{on})$: the uncertain output vector of DMU_o
$x_{ij} = (x_{i1}, x_{i2}, \dots, x_{in})$: the nominal input vector of $\mathrm{DMU}_j, j = 1, \dots, n$
$x_{io} = (x_{o1}, x_{o2}, \dots, x_{on})$: the nominal input vector of DMU_o
$y_{rj} = (y_{r1}, y_{r2}, \dots, y_{rn})$: the nominal output vector of $\mathrm{DMU}_j, j = 1, \dots, n$
$y_{ro} = (y_{o1}, y_{o2}, \dots, y_{on})$: the nominal output vector of DMU_o
$\hat{x}_{ij} = (\hat{x}_{i1}, \hat{x}_{i2}, \dots, \hat{x}_{in})$: the deviation from a nominal input vector of DMU_j
$\hat{y}_{rj} = (\hat{y}_{r1}, \hat{y}_{r2}, \dots, \hat{y}_{rn})$: the deviation from nominal output vector of DMU_j
$I_j = (I_1, I_2, \dots, I_n)$: set of inputs of DMU_j that are subject to uncertainty
$R_j = (R_1, R_2, \dots, R_n)$: set of outputs of DMU_j that are subject to uncertainty

Definition 2.8. A $\mathrm{DMU}_k = (\boldsymbol{x}_k, \boldsymbol{y}_k)$ is uncertain if there exist $i \in I_k$ or $r \in R_k$. Moreover, for each input and output data, uncertainty index will be implemented in the model accordingly as below:

$$\mathrm{DMU}_j = \begin{cases} 1 & i \in I_k \\ 0 & i \notin I_k \end{cases} \quad \text{or} \quad \mathrm{DMU}_j = \begin{cases} 1 & r \in R_k \\ 0 & r \notin R_k \end{cases}$$

In what follows, the uncertain inputs and outputs are defined by $\tilde{x}_{ij} = x_{ij} + \eta_{ij}^x \hat{x}_{ij}$ and $\tilde{y}_{rj} = y_{rj} + \eta_{rj}^y \hat{y}_{rj}$ where η_{ij}^x and η_{rj}^y are the random variables of the uncertain input and output respectively. Observe that $\mathcal{U} = \left\{ (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathbb{R}^{|I_j|+|R_j|} \middle| , \eta_{ij}^x \in \mathbb{Z}^x(\Gamma_j^x),\ \eta_{rj}^y \in \mathbb{Z}^y\left(\Gamma_j^y\right) \text{ or } \eta_{ij}^x \in \mathbb{Z}^x(\Omega_j^x),\ \eta_{rj}^y \in \mathbb{Z}^y\left(\Omega_j^y\right) \right\}$

where

$$\mathbb{Z}^x(\Gamma_j^x)=\left\{(\eta_{ij}^x)\in\mathbb{R}^{|I_j|}\middle|\sum_{i\in I_j}\eta_{ij}^x\le\Gamma_j^x,|\eta_{ij}^x|\le 1,\forall i\in I_j\right\}$$

$$\mathbb{Z}^y(\Gamma_j^y)=\left\{(\eta_{rj}^y)\in\mathbb{R}^{|R_j|}\middle|\sum_{r\in R_j}\eta_{rj}^y\le\Gamma_j^y,|\eta_{rj}^y|\le 1,\forall r\in R_j\right\}$$

is in the case of budget of uncertainty of Bertsimas & Sim (2004) and

$$\mathbb{Z}^x(\Omega_j^x)=\left\{(\eta_{ij}^x)\in\mathbb{R}^{|I_j|}\middle| x_{ij}+\sum_{i\in I_j}\rho_{ij}^x\eta_{ij}^x,\|\boldsymbol{\eta}_j^x\|_2\le\Omega_j^x,\forall i\in I_j\right\}$$

$$\mathbb{Z}^y(\Omega_j^y)=\left\{(\eta_{rj}^y)\in\mathbb{R}^{|R_j|}\middle| y_{rj}+\sum_{r\in R_j}\rho_{rj}^y\eta_{rj}^y,\|\boldsymbol{\eta}_j^y\|_2\le\Omega_j^y,\forall r\in R_j\right\}$$

in the case of ellipsoidal uncertainty set (Ben-Tal & Nemirovski, 2000).

In matrix form, the uncertain inputs and outputs can be expressed as the following:

$$\begin{bmatrix}\tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n}\\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn}\end{bmatrix}=\begin{bmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ x_{m1} & x_{m2} & \cdots & x_{mn}\end{bmatrix}+\boldsymbol{\eta}_j^x\begin{bmatrix}\hat{x}_{11} & \hat{x}_{12} & \cdots & \hat{x}_{1n}\\ \hat{x}_{21} & \hat{x}_{22} & \cdots & \hat{x}_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \hat{x}_{m1} & \hat{x}_{m2} & \cdots & \hat{x}_{mn}\end{bmatrix}$$

and

$$\begin{bmatrix}\tilde{y}_{11} & \tilde{y}_{12} & \cdots & \tilde{y}_{1n}\\ \tilde{y}_{21} & \tilde{y}_{22} & \cdots & \tilde{y}_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \tilde{y}_{s1} & \tilde{y}_{s2} & \cdots & \tilde{y}_{sn}\end{bmatrix}=\begin{bmatrix}y_{11} & y_{12} & \cdots & y_{1n}\\ y_{21} & y_{22} & \cdots & y_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ y_{s1} & y_{s2} & \cdots & y_{sn}\end{bmatrix}+\boldsymbol{\eta}_j^y\begin{bmatrix}\hat{y}_{11} & \hat{y}_{12} & \cdots & \hat{y}_{1n}\\ \hat{y}_{21} & \hat{y}_{22} & \cdots & \hat{y}_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \hat{y}_{s1} & v_{s2} & \cdots & \hat{y}_{sn}\end{bmatrix}$$

where

$$\begin{bmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ x_{m1} & x_{m2} & \cdots & x_{mn}\end{bmatrix},\begin{bmatrix}y_{11} & y_{12} & \cdots & y_{1n}\\ y_{21} & y_{22} & \cdots & y_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ y_{s1} & y_{s2} & \cdots & y_{sn}\end{bmatrix} \text{and} \begin{bmatrix}\hat{x}_{11} & \hat{x}_{12} & \cdots & \hat{x}_{1n}\\ \hat{x}_{21} & \hat{x}_{22} & \cdots & \hat{x}_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \hat{x}_{m1} & \hat{x}_{m2} & \cdots & \hat{x}_{mn}\end{bmatrix},\begin{bmatrix}\hat{y}_{11} & \hat{y}_{12} & \cdots & \hat{y}_{1n}\\ \hat{y}_{21} & \hat{y}_{22} & \cdots & \hat{y}_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \hat{y}_{s1} & v_{s2} & \cdots & \hat{y}_{sn}\end{bmatrix}$$

are respectfully the nominal value and deviation matrix of uncertain inputs and outputs. It will be assumed that the unknown inputs and outputs data take their values from the symmetric interval. i.e. $\tilde{x}_{ij}\in[x_{ij}-\hat{x}_{ij},x_{ij}+\hat{x}_{ij}]$ and $\tilde{y}_{rj}\in[y_{rj}-\hat{y}_{rj},y_{rj}+\hat{y}_{rj}]$ and the random variables, η_{ij}^x and η_{rj}^y are independent and distributed symmetrically in the interval

[-1, 1]. To model uncertainty in the DEA model (in the case of the multiplier model), we will make use of Theorem 2.4. This is very important since by the dint of the normalization constraint $\sum_{i=1}^{m} v_i x_{io} = 1$ in model (2.8) input uncertainty analyses in the constraint for robust analysis could lead to a restriction on the constraint and probable model infeasibility. Therefore model (2.23) with inequality in the normalization constraint and uncertainty in the output objective function expressed as a constraint becomes very useful for the robust DEA modelling. The subject matter uncertainty in equality constraint of DEA is treated in Section 3.4.1.

Theorem 2.4. The CCR model (2.6) is equivalent to the following model:

$$
\begin{array}{ll}
\max \theta & \\
\text{s.t.} & \\
\theta - \sum_{r=1}^{s} u_r \tilde{y}_{ro} \leq 0 & \\
\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1 & \\
\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 & j = 1, \dots, n \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s
\end{array}
\tag{2.23}
$$

Proof. See (Toloo, 2014).

See also Chapter 3 for the proof using the BCC model and Chapter 4 for the similar theorem on the output oriented model.

2.3.2.1 A robust production possibility set

The PPS under uncertainty comprises the set of all feasible production plan or input-output combinations available to DMU_j in the uncertain space. It is defined as the set:

$$\psi_r = \{(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) \in \mathbb{R}^{m+s} \mid \widetilde{\boldsymbol{y}} \text{ is produced from } \widetilde{\boldsymbol{x}}\} \tag{2.24}$$

We suppose that $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are constrained in an arbitrary uncertainty set, $\mathcal{U} \subset \mathbb{R}^{n\times(m+s)}$. To define a robust PPS (PPS_r) and obtain an accurate mathematical definition, we consider the following axioms:

Assumption 2. Axioms for the set ψ_r:

(A1): The following set of uncertain activities corresponding to the observed activity DMU_j is a subset of PPS_r (for $j = 1, \dots, n$):

$$S_j = \left\{ (\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \middle| \bar{x}_i = \begin{cases} \widetilde{x}_{ij} & \text{if } i \in I_j \\ x_{ij} & \text{otherwise} \end{cases}; \quad \bar{y}_r = \begin{cases} \widetilde{y}_{rj} & \text{if } r \in R_j \\ y_{rj} & \text{otherwise} \end{cases} \right\}$$

Note that all observed activities are feasible, i.e. $\mathrm{DMU}_j = (\boldsymbol{x}_j, \boldsymbol{y}_j)$ belongs to PPS_r for $= 1, \dots, n$ if we let $\eta_{ij}^x = \eta_{rj}^y = 0; \forall i \in I_j$ and $\forall r \in R_j$. In addition, if DMU_j is a certain observation, then S_j is a singleton, namely $S_j = \{(\boldsymbol{x}_j, \boldsymbol{y}_j)\}$.

(A2): All dominated activities of a feasible activity belong to PPS_r, i.e. if $(\boldsymbol{x}, \boldsymbol{y}) \in \mathrm{PPS}_r$, then $\forall \overline{\boldsymbol{x}} \geq \boldsymbol{x}, \forall \overline{\boldsymbol{y}} \leq \boldsymbol{y}, (\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \in \mathrm{PPS}_r$.

(A3): If an activity $(\boldsymbol{x}, \boldsymbol{y})$ is feasible, then $\forall t > 0, (t\boldsymbol{x}, t\boldsymbol{y})$ is a feasible activity $\left((\boldsymbol{x}, \boldsymbol{y}) \in \mathrm{PPS}_r \Longrightarrow \forall t > 0, (t\boldsymbol{x}, t\boldsymbol{y}) \in \mathrm{PPS}_r\right)$.

(A4): The convex combination of each two feasible activities is a feasible activity, i.e. $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \mathrm{PPS}_r \Rightarrow \forall \lambda \in [0,1], \lambda(\boldsymbol{x}_1, \boldsymbol{y}_1) + (1 - \lambda)(\boldsymbol{x}_2, \boldsymbol{y}_2) \in \mathrm{PPS}_r$.

We can define the following robust production possibility set satisfying (A1) through (A4):

$$\mathrm{PPS}_r = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq \overline{\mathbf{X}}\boldsymbol{\lambda}, \boldsymbol{y} \leq \overline{\mathbf{Y}}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}_n\} \tag{2.25}$$

where

$$\overline{\mathbf{X}} = [\bar{x}_{ij}] \text{ and } \overline{\mathbf{Y}} = [\bar{y}_{rj}].$$

Accordingly, the following models measure the robust technical efficiency of DMU_o with input and output orientations, respectively:

$$\begin{aligned} &\min \theta \\ &\mathrm{s.t} \\ &(\theta \boldsymbol{x}_o,\ \boldsymbol{y}_o) \in \mathrm{PPS}_r \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\max \varphi \\ &\mathrm{s.t} \\ &(\boldsymbol{x}_o, \varphi \boldsymbol{y}_o) \in \mathrm{PPS}_r \end{aligned} \tag{2.27}$$

Two criteria are independently used in obtaining the robust efficiency: worst-case and best-case criteria. A worst-case robust efficiency is a conservative or pessimistic approach concerned with a guaranteed level of performance for all feasible realization of uncertain inputs and outputs in an uncertainty set while the best-case provides an optimistic view and consist of minimizing (maximizing) the input (output) over the set of optimistic feasible constraints. The duality analysis with the worst-case is discussed in Chapter 4. In what follows, we characterize DEA models to alternative uncertainty sets based on the PPS_r defined.

2.3.2.2 RDEA induced by norm uncertainty set

Following Bertsimas et al., (2004), the general norm – induced symmetric uncertainty set for the inputs and outputs data is formulated as the following:

$$\mathcal{U}_{(\Delta_j)} = \left\{(X, Y) \,\middle|\, \left\|M_x\left(vec(\tilde{X}) - vec(X)\right)\right\| \leq \Delta_j^x, \left\|M_y\left(vec(\tilde{Y}) - vec(Y)\right)\right\| \leq \Delta_j^y\right\}$$

where $\tilde{X}_j = \{\tilde{x}_{ij}\}_{j=1,\dots,n}$, $\tilde{Y}_j = \{\tilde{y}_{rj}\}_{j=1,\dots,n}$, $X_j = \{x_{ij}\}_{j=1,\dots,n}$, $Y_j = \{y_{rj}\}_{j=1,\dots,n}$ and M_x and M_y are the invertible matrix of the input and output vector X_j and Y_j respectively. The robust RDEA to this set lead to the following model

$$
\begin{aligned}
& \max \theta \\
& \text{s.t.} \\
& \theta - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \Delta_j^y \|(Y_o^T)^{-1} u_r\|^* \le 0 \\
& \textstyle\sum_{i=1}^{m} v_i x_{io} + \Delta_j^x \|(X_o^T)^{-1} v_i\|^* \le 1 \\
& \textstyle\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + \\
& \Delta_o^y \|(Y_o^T)^{-1} u_r\|^* + \Delta_o^x \|(X_o^T)^{-1} v_i\|^* \le 0 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \\
& \Delta_j^y \left\|\left(Y_j^T\right)^{-1} u_r\right\|^* + \Delta_i^x \left\|\left(X_j^T\right)^{-1} v_i\right\|^* \le 0 \quad j \ne o \\
& v_i \ge 0 \qquad \forall i \\
& u_r \ge 0 \qquad \forall r
\end{aligned}
\tag{2.28}
$$

where $\|\cdot\|^*$ is the dual normal of $\|\cdot\|$ defined as $\|\mathbf{s}\|^* \doteq \max_{\{\|\mathbf{p}\| \le 1\}} \mathbf{s}^{\mathrm{T}}\mathbf{p}$ for the vector $w_i = \{w_1, \dots, w_n\}$. The norm $\|\cdot\|$ is arbitrary. We use the $l_p -$ norm defined as $\|\mathbf{w}\|_p = (\sum_{i=1}^{n} |w_i|^p)^{\frac{1}{p}}$ with $p \ge 1$ which is commonly considered for $\|\cdot\|$ in literature. Note that $\|\mathbf{w}\|_p^*$ is the $l_q -$ norm $\|\mathbf{s}\|_p^*$ with $q = 1 + 1/(p-1)$. Model (2.28) has the following properties:

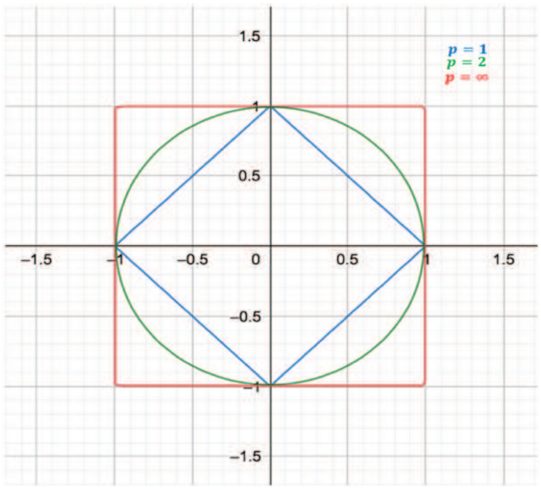


Figure 2.11: Norm in a unit circle for different values of p

Properties of the model:

- The terms $\Delta_j^y \|(Y_o^T)^{-1} u_r\|^*$ and $\Delta_j^x \|(X_o^T)^{-1} v_i\|^*$ provide the necessary protection that ensures the feasibility of the normalization and common constraints for each model run
- Let $\Delta_j = \Delta_j^x$ and Δ_j^y, the parameter Δ_j controls the tradeoff between robustness and performance.

- The model is convex and can be linear or nonlinear depending on the specific norm used.
- For the $l_p -$ norm, considering the random vector $\eta_{ij} = (\tilde{x}_{ij} - x_{ij})/\hat{x}_{ij}$ the following models emerge for special values of p.
  a. If $p = 1$ so that $\|\boldsymbol{\eta}\|_1 = \sum_{j=1}^{n}|\eta_{ij}|$, model (2.28) is reformulated as an LP with polyhedral uncertainty set.
  b. If $p = 2$ so that $\|\boldsymbol{\eta}\|_2 = \left(\sum_{j=1}^{n}|\eta_{ij}|^2\right)^{\frac{1}{2}}$, model (2.28) culminates into a second order cone problem with ellipsoidal uncertainty set.
  c. If $p = \infty$ so that $\|\boldsymbol{\eta}\|_\infty = \max_j|\eta_{ij}|$, model (2.28) is reformulated as an LP with box uncertainty set.

Figure 2.11 shows the norm uncertainty for different values of p.

2.3.2.3 RDEA induced by box uncertainty set

The first kind of uncertainty set proposed by Soyster (1973) is the interval set, a special case of the box uncertainty set when the parameter $\Phi_j = 1$. Considering the box uncertainty

$$\mathcal{U}_{(\Phi_j)} = \left\{(\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{y}}_r) \,\middle|\, \|(\tilde{x}_{ij} - x_{ij})/\hat{x}_{ij}\|_\infty \le \Phi_j^x, \|(\tilde{y}_{rj} - y_{rj})/\hat{y}_{rj}\|_\infty \le \Phi_j^y\right\}$$

for the uncertain inputs and outputs, the robust counterpart DEA has the following formulation:

$$\begin{aligned}
&\max \theta \\
&\text{s.t.} \\
&\theta - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \Phi_j^y \sum_{r\in R_j} u_r \hat{y}_{ro} \le 0 \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} + \Phi_j^x \sum_{i\in I_j} v_r \hat{x}_{io} \ \le 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + \\
&\Phi_o^y \textstyle\sum_{r\in R_o} u_r \hat{y}_{ro} + \Phi_o^x \sum_{i\in I_o} v_r \hat{x}_{io} \le 0 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \\
&\Phi_j^y \textstyle\sum_{r\in R_j} u_r \hat{y}_{rj} + \Phi_j^x \sum_{i\in I_j} v_r \hat{x}_{ij} \le 0 \qquad j \ne o \\
&v_i \ge 0 \qquad \forall i \\
&u_r \ge 0 \qquad \forall r
\end{aligned} \tag{2.29}$$

where Φ_i and Φ_r are the positive robust parameter for the model inputs and outputs. Model (2.29) has the following properties.

Properties of the model:

- The terms $\Phi_j^y \sum_{r\in R_j} u_r \hat{y}_{ro}$ and $\Phi_j^x \sum_{i\in I_j} v_r \hat{x}_{io}$ give the necessary protection that ensures the feasibility of the normalization and common constraints for each model run

- The model with the interval uncertainty $\left(\Phi_j^x = \Phi_j^y = 1\right)$ can be very conservative and with a bad performance.
- Let $\Phi_j = \Phi_j^x + \Phi_j^y$, the parameter Φ_j controls the tradeoff between robustness and performance. The model allows the DM a flexibility for robustness separately for the uncertain inputs and outputs.
- The model is an LP.

2.3.2.4 RDEA induced by polyhedral uncertainty set

Dimitris Bertsimas, Brown, & Caramanis (2011) considers a polyhedral uncertainty set in a case where the uncertainty affects the constraints in an affine manner. The polynomial uncertainty set can be viewed as a special case of the norm uncertainty set, i.e. when $p = 1$ in the $l_p -$ norm in model (2.28). To illustrate this, consider the uncertainty sets in terms of the uncertain inputs and outputs as below:

$$\mathcal{U}_{(\Gamma_j)} = \left\{(\widetilde{\boldsymbol{x}}_i, \widetilde{\boldsymbol{y}}_r) \middle| \left\|(\tilde{x}_{ij} - x_{ij})/\hat{x}_{ij}\right\|_1 \leq \Gamma_j^x, \left\|(\tilde{y}_{rj} - y_{rj})/\hat{y}_{rj}\right\|_1 \leq \Gamma_j^y\right\}$$

The corresponding robust model has the following formulation:

$$\begin{array}{ll}
\max \theta & \\
\text{s.t.} & \\
\theta - \sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y \leq 0 & \\
\sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x \leq 1 & \\
\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + p_o^y \Gamma_o^y + p_o^x \Gamma_o^x \leq 0 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + p_j^y \Gamma_j^y + p_j^x \Gamma_j^x \leq 0 & j \neq o \\
p_j^y \geq u_r \hat{y}_{rj} & \forall j, \forall r \in R_j \\
p_j^x \geq v_i \hat{x}_{ij} & \forall j, \forall i \in I_j \\
p_j^x, p_j^y \geq 0 & \forall j \\
v_i \geq 0 & \forall i \\
u_r \geq 0 & \forall r
\end{array} \tag{2.30}$$

where Γ_j^x and Γ_j^y are the positive robust parameter for the model inputs and outputs.

Properties of the model:

- The uncertainty affects the constraints in an affine manner.
- The model is a linear model.
- Let $\Gamma_j = \Gamma_j^x + \Gamma_j^y$, the model is flexible for the tradeoff between robustness and performance using parameter Γ_j.

2.3.2.5 RDEA induced by polyhedral and interval uncertainty set

The robust model induced by the polyhedral and interval uncertainty set is equivalent to the robust model based on the cardinality constrained uncertainty or the D −norm[20] proposed in Bertsimas & Sim (2004). The corresponding RDEA model is formulated as the below:

$$
\begin{array}{ll}
\max \theta & \\
\text{s.t.} & \\
\theta - \sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y + \sum_{r \in R_o} q_{ro} \leq 0 & \\
\sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x + \sum_{i \in I_o} w_{io} \leq 1 & \\
\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + p_o^y \Gamma_o^y + p_o^x \Gamma_o^x + \sum_{r \in R_o} q_{ro} + \sum_{i \in I_o} w_{io} \leq 0 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + p_j^y \Gamma_j^y + p_j^x \Gamma_j^x + \sum_{r \in R_j} q_{rj} + \sum_{i \in I_j} w_{ij} \leq 0 & j \neq o \\
p_j^y + q_{rj} \geq u_r \hat{y}_{rj} & \forall j, \forall r \in R_j \\
\mathrm{p}_j^x + w_{ij} \geq v_i \hat{x}_{ij} & \forall j, \forall i \in I_j \\
q_{rj}, w_{ij} \geq 0 & \forall j, \forall i \in I_j, \forall r \in R_j \\
p_j^x, p_j^y \geq 0 & \forall j \\
v_i \geq 0 & \forall i \\
u_r \geq 0 & \forall r
\end{array}
\tag{2.31}
$$

Model is (2.31) is proved and discussed in Chapter 3. Its duality analysis is studied in Chapter 4. The following are the properties of the model:

Properties of the model:

- The model is a linear model.
- The terms $p_o^y \Gamma_o^y + \sum_{r \in R_o} q_{ro}$ and $p_o^x \Gamma_o^x + \sum_{i \in I_o} w_{io}$ indicate the protection for uncertain data.
- A fixed number Γ_j of coefficients is allowed to deviate from their nominal values and constraint feasibility is guaranteed within the probability bound $\exp\left(-\frac{\Gamma^2}{2(|R_j|+|I_j|)}\right)$.
- Let $\Gamma_j = \Gamma_i + \Gamma_r$, the budget uncertainty parameter Γ_j ensures a flexible balance between robustness and performance of the model.

2.3.2.6 RDEA induced by ellipsoidal uncertainty set

The ellipsoidal uncertainty set induced RDEA model is equivalent to the RDEA model (2.28) induced by the l_2 − norm. We obtain model (2.32) using the uncertainty set below:

[20] The D −norm is defined as $\|(x,y)\|_{\Gamma_j}^{D} =$

$$
\max_{\left\{ \begin{array}{l} S \cup \{t_j\} \mid S_j \subseteq |R_j|, |S_j| = \lfloor \Gamma_j^y \rfloor, t_j \in \{|R_j|\} \setminus S_j \\ \bar{S}_j \cup \{\bar{t}_j\} \mid \bar{S}_j \subseteq |I_j|, |\bar{S}_j| = \lfloor \Gamma_j^x \rfloor, \bar{t}_j \in \{|I_j|\} \setminus \bar{S}_j \end{array} \right\}} \left\{ \begin{array}{c} \sum_{r \in S_j} \|y_{rj}\| + (\Gamma_j^y - \lfloor \Gamma_j^y \rfloor) \|y_{rt_j}\| + \\ \sum_{i \in \bar{S}_j} \|x_{ij}\| + (\Gamma_j^x - \lfloor \Gamma_j^x \rfloor) \|x_{i\bar{t}_j}\| \end{array} \right\}
$$

$$\mathcal{U}_{(\Omega_j)} = \left\{(\tilde{x}_{ij}, \tilde{y}_{rj}):\ x_{ij} + \sum_{i\in I_j} \eta_{ij}\rho_{ij} \mid \eta_{ij}^T\eta_{ij} \le \Omega_j^x;\ y_{rj} + \sum_{r\in R_j} \eta_{rj}\rho_{rj} \mid \eta_{rj}^T\eta_{rj} \le \Omega_j^y\right\}$$

$$\begin{aligned}
&\max\ \theta \\
&\text{s.t.} \\
&\theta - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \Omega_j^y \sqrt{\textstyle\sum_{r\in R_j} u_r^2 \hat{y}_{ro}^2} \le 0 \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} + \Omega_j^x \sqrt{\textstyle\sum_{i\in I_j} v_i^2 \hat{x}_{io}^2} \le 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + \\
&\Omega_o^y \sqrt{\textstyle\sum_{r\in R_o} u_r^2 \hat{y}_{ro}^2} + \Omega_o^x \sqrt{\textstyle\sum_{i\in I_o} v_i^2 \hat{x}_{io}^2} \le 0 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \\
&\Omega_j^y \sqrt{\textstyle\sum_{r\in R_j} u_r^2 \hat{y}_{rj}^2} + \Omega_j^x \sqrt{\textstyle\sum_{i\in I_j} v_i^2 \hat{x}_{ij}^2} \le 0 \qquad \forall j \\
&v_i \ge 0 \qquad \forall i \\
&u_r \ge 0 \qquad \forall r
\end{aligned} \tag{2.32}$$

where Ω_j^x and Ω_j^y are the positive robust parameter for the model inputs and outputs. This model is discussed for the ranking of banks efficiency in Chapter 5.

Properties of the model:

- The model is a nonlinear model and specifically a second order cone programming.
- Let $\Omega_j = \Omega_j^x + \Omega_j^y$, the model is flexible for the tradeoff between robustness and performance using the parameter Ω_j.

2.3.2.7 RDEA induced by ellipsoidal and interval uncertainty set

The model formulation follows Ben-Tal & Nemirovski (2000) when the uncertainty data are given by the random perturbation $\tilde{x}_{ij} = (1 + \epsilon\eta_{ij}^x)x_{ij}$ and $\tilde{y}_{rj} = (1 + \epsilon\eta_{rj}^y)y_{rj}$. Using the uncertainty set

$$\mathcal{U}_{(\Omega_j)} = \left\{ (\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{y}}_r) \left| \begin{array}{l} \left\|(\tilde{x}_{ij} - x_{ij})/\hat{x}_{ij}\right\|_\infty \le 1, \sum \frac{(\tilde{x}_{ij} - x_{ij})^2}{\hat{x}_{ij}^2} \le \Omega_j^{x2} \\ \left\|(\tilde{y}_{rj} - y_{rj})/\hat{y}_{rj}\right\|_\infty \le 1, \sum \frac{(\tilde{y}_{rj} - y_{rj})^2}{\hat{y}_{rj}^2} \le \Omega_j^{y2} \end{array} \right. \right\}$$

the correspond robust counterpart DEA is given as:

$$
\begin{aligned}
&\max \theta \\
&\text{s.t.} \\
&z - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \sum_{r\in R_o} \mu_r \hat{y}_{ro} + \Omega_j^y \sqrt{\sum_{r\in R_o} v_r^2 \hat{y}_{ro}^2} \le 0 \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} + \sum_{i\in I_o} \lambda_{io} \hat{x}_{io} + \Omega_j^x \sqrt{\sum_{i\in I_o} v_{io}^2 \hat{x}_{io}^2} \le 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + \sum_{r\in R_o} \mu_r \hat{y}_{ro} + \sum_{i\in I_o} \lambda_{io} \hat{x}_{io} + \\
&\Omega_o^y \sqrt{\textstyle\sum_{r\in R_o} v_{ro}^2 \hat{y}_{ro}^2} + \Omega_o^x \sqrt{\textstyle\sum_{i\in I_o} v_{io}^2 \hat{x}_{io}^2} \le 0 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{r\in R_j} \mu_r \hat{y}_{rj} + \sum_{i\in I_j} \lambda_{ij} \hat{x}_{ij} + \\
&\Omega_j^y \sqrt{\textstyle\sum_{r\in R_j} v_{rj}^2 \hat{y}_{rj}^2} + \Omega_j^x \sqrt{\textstyle\sum_{i\in I_j} v_{ij}^2 \hat{x}_{ij}^2} \le 0 && \forall j \ne o \\
&-\mu_{rj} \le u_r - v_{rj} \le \mu_{rj} && \forall r \in R_j \\
&-\lambda_{ij} \le v_i - v_{ij} \le \lambda_{ij} && \forall i \in I_j \\
&v_i \ge 0 && \forall i \\
&u_r \ge 0 && \forall r \\
&\lambda_{ij}, \mu_{rj} \ge 0 && \forall i \in I_j, \forall r \in R_j
\end{aligned}
\tag{2.33}
$$

We provide a discussion of this model and ranking of DMUs in Chapter 5 from the robust fractional programming point of view. The properties of this model are the same as the previous model.

2.4 Concluding remarks

Efficiency measurement with the DEA compares actual output from a given input with the maximally producible quantity of output. However, when the data for measurement is uncertain, the reference technology becomes distorted which results in unreliable efficiency scores and unstable performance ranking for management decisions. Beyond knowing the problems uncertain data cause, knowledge of how to overcome it is very necessary. The current chapter has thus attempted to offer basic concepts on dealing with uncertainty in DEA data and ensuring stable efficiency scores using the robust optimization. The insight into the modeling process is useful for the rest of the chapters.

Chapter 3: Robust optimization with nonnegative decision variables: A DEA approach

Summary

Robust optimization has become the state-of-the-art approach for solving linear optimization problems with uncertain data. Though relatively young, the robust approach has proven to be essential in many real-world applications. Under this approach, robust counterparts to prescribed uncertainty sets are constructed for general solutions to corresponding uncertain linear programming problems. It is remarkable that in most practical problems, the variables represent physical quantities and must be nonnegative. In this chapter, we propose alternative robust counterparts with nonnegative decision variables – a reduced robust approach which attempts to minimize model complexity. The new framework is extended to the robust Data Envelopment Analysis (DEA) with the aim of reducing the computational burden. In the DEA methodology, first we deal with the equality in the normalization constraint and then a robust DEA based on the reduced robust counterpart is proposed. The proposed model is examined with numerical data from 250 European banks operating across the globe. The results indicate that the proposed approach (i) reduces almost 50% of the computational burden required to solve DEA problems with nonnegative decision variables; (ii) retains only essential (non-redundant) constraints and decision variables without alerting the optimal value.

3.1 Introduction

The robust optimization has been proposed to handle uncertainties in the input data in classical mathematical programming problems. An alternative approach that immunizes uncertain parameters in some probability sense is that of the stochastic programming which dates back to Dantzig (1955). See Prékopa (1995), Birge & Louveaux (1997) for references.

Although the robust approach was introduced by Soyster (1973), it was until the late 1990s that it took a massive flurry of interest in the mathematical programming community. Since then, several robust models have been proposed (see Ben-Tal & Nemirovski, 1998; 1999; 2000; Bertsimas & Sim, 2004; Bertsimas, Pachamanova, & Sim, 2004; Ben-Tal, El Ghaoui, & Nemirovski, 2009) and the field continues to be explored due to its usefulness in application. The standard robust optimization adopts a conservative methodology that confines all uncertain parameters to a pre-defined uncertainty set so as to optimize the worst-case performance for all feasible realization of the uncertain parameters in the defined set. It is conceivable that, like the stochastic programming, the robust optimization approach leaves some theoretical and practical issues to be addressed. Key among these issues include (i) structure of the uncertainty set, (ii) tractability of the robust formulation, (iii) conservativeness and probability guarantees to the distribution of the uncertain parameters in the uncertainty set, (iv) complexity of the robust models and (v) quality of the robust solution (Bertsimas, Brown, & Caramanis, 2011; Gorissen, Yanıkoğlu, & den Hertog, 2015). In consequence, rather than finding a usual optimal solution, the concern has been to seek the best performance under most realizations of the uncertain parameters.

Most remarkably, given the tractability of most robust linear programming, the robust technique has sparked interest in many applications in management science. Diverse areas of operations research applications including portfolio optimization, statistics, and learning, supply chain and inventory management, engineering etc. have been considered in the literature. For a comprehensive survey, we refer the reader to Bertsimas, Brown, & Caramanis (2011). However, despite the empirical success in these areas, the robust optimization has come under some practical concerns. As mentioned earlier, one of the critical practical issues of concern is the computational cost relating to the robust models. Although this issue is usually addressed with a flexible selection of uncertainty set, notwithstanding, it is important to note that the robust approaches considered usually provide general solutions of which some could be negative. In other words, the robust counterpart models provided in literature are generally defined with free-in-sign decision variables and a separate study looking at only nonnegative decision variables is not available. It is remarkable that in most practical problems, the variables represent physical quantities which are nonnegative (Bazaraa, Jarvis, & Sherali, 2010). Therefore, where decision variables are only nonnegative, a significant computational disadvantage is that the general robust counterpart models proposed in the literature present "unwanted parameters" that demand more computational resources. As a result, we believe that robust counterparts for nonnegative decision variables are needed. That is, there is the need for robust formulations, which by virtue of rendering some parameters redundant in the classical robust optimization models yield solutions faster than the former.

In this chapter, we consider different robust counterpart models and formulate alternative models when decision variables are nonnegative. We call them *reduced robust counterparts (*RRC). These reduced models are equivalent to the former models without any redundant variable or constraint. As a result, the problem size (that is the number of variables and constraints) of reduced model is significantly decreased which point out that the later models are more concise, more reliable, and more application-driven than the former models. In our pursuit of this cause, we use data envelopment analysis (DEA) method as an application-driven example to illustrate the practicality and computational usefulness of the RRC models. The DEA is one of the well-known linear programming (LP) applications that most often involves nonnegative variables.

Recent research in DEA has focused on the robust optimization application in DEA to ensure robustness in efficiency analysis. Although the field is quite new and developing, variants robust models encompassing different uncertainty sets and scenarios have been explored and introduced into the DEA (see Sadjadi & Omrani, 2008; Atıcı & Gülpınar, 2016). It is important to note that, equality constraints containing uncertain parameters restrict the feasible region and may lead to infeasibility issue (for more details see Ben-Tal et al, 2009, Chapter 2). The multiplier form of DEA models involves a normalization constraint which is in an equation form containing uncertain input data. Accordingly, most researchers find it difficult handling uncertainties in the inputs and outputs data *simultaneous*ly. Regarding the normalization constraint, an alternative formulation in which the constraint is feasible for *all* data uncertainties is adopted in this chapter for a feasible robust DEA. In other words, using a proposed formulation, we provide a coherent feasibility treatment to the normalization constraint as with all robust optimization compared to the treatment from other studies (c.f. Omrani, 2013 and Salahi et al., 2016) and therefore may be regarded a novelty approach for robust DEA to general uncertainty modeling. Moreover, unlike in previous studies, the robust DEA considered for computational studies is based on the aforementioned RRC. In summary, the main contributions of the chapter are the following:

1. We propose new robust counterpart optimization problems with nonnegative decision variables. This leads to an approach which is more applicable and computationally cost-effective for problems involving nonnegative decision variables.
2. The suggested robust approach is used to propose a new robust DEA model. Our robust DEA models (called reduced robust DEA) are compared to existing robust DEA models.
3. Prior to the reduced robust DEA formulation, we adjust the equality constraint in the normalization of the multiplier DEA models to inequality in order to allow for feasible and simultaneous consideration of uncertainties in the inputs and outputs data.
4. We consider a case study of 250 banks in Europe to validate our new approach. The obtained results point out that the proposed robust DEA model reduces 50% of the required computational burden.

Structure of the chapter. In Section 3.2, we review different robust counterparts models to the most commonly used uncertainty sets. Section 3.3 presents the robust counterparts models for nonnegative decision variables. In addition, the theoretical complexity of these models is analyzed which indicates less iteration required to solve the reduced models by any efficient algorithm. The DEA and robust DEA approaches are presented in Sections 3.4 which includes dealing with equality constraint in DEA. The proposed new reduced robust DEA is presented in Section 3.5. We provide a practical banking problem and test the complexity of the models in Section 3.6. The chapter ends with conclusion and remarks in the next section.

3.2 General robust counterpart formulations

In this section, we review different uncertainty sets and demonstrate their robust counterpart formulations. To this end, we first consider an uncertain linear programming model

$$\begin{aligned} & z_1 = \max \textstyle\sum_{j=1}^{n} c_j x_j \\ & \text{s.t.} \\ & \textstyle\sum_{j=1}^{m} \tilde{a}_{ij} x_j \le b_i \qquad i = 1, \dots, m \\ & l_j \le x_j \le u_j \qquad j = 1, \dots, n \end{aligned} \tag{3.1}$$

where $(c_1, \dots, c_n)$ is a cost coefficient vector, $\tilde{a}_{ij}$ represents the value of the technological coefficient that is subject to uncertainty, $(b_1, \dots, b_m)$ is the right-hand-side vector, $(x_1, \dots, x_n)$ is decision variable vector, and $l_j,\ u_j$ are the lower and upper bounds for decision variable x_j. Let X be the feasible region of model (3.1), i.e. $X = \{\boldsymbol{x}\colon \tilde{\boldsymbol{a}}_i \boldsymbol{x} \le b_i\ \forall i, \boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}\} \subset \mathbb{R}^n$ where $\tilde{\boldsymbol{a}}_i = (\tilde{a}_{1i}, \dots, \tilde{a}_{in})$, $\boldsymbol{l} = (l_1, \dots, l_n)$, and $\boldsymbol{u} = (u_1, \dots, u_n)$. By standard transformations, we can assume without loss of generality that l_j is finite for each $j = 1, \dots, n$. In addition, assume that only the technological coefficients are subjected to uncertainty and whiles their distribution may be unknown, they are known to be symmetric in an interval. Thus, let J_i represent the set of coefficients in row i that are subject to uncertainty, then the true value of each entry $a_{ij}, j \in J_i$ is modeled as a symmetric and bounded variable taking values in the interval $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$ (Bertsimas & Sim, 2004). The true value of the uncertain technological coefficient can be expressed as $\tilde{a}_{ij} = a_{ij} + \eta_{ij}\hat{a}_{ij}$ where a_{ij} is the *nominal value*, $\hat{a}_{ij}$ is the *maximum distance* that specifies how much the nominal value is likely to deviate from the true value, and η_{ij} denotes *random variable* that is symmetrically distributed in the interval $[-1, 1]$. Suppose an uncertainty set $\mathcal{U}$ (convex in structure) is constructed as an immunization region for the uncertain technological coefficients, the general robust counterpart (GRC) to a predefined uncertainty set for the classical uncertain LP model (3.1) can be formulated as:

$$
\begin{aligned}
& z_2 = \max \textstyle\sum_{j=1}^{n} c_j x_j \\
& \text{s.t.} \\
& \textstyle\sum_{j=1}^{n} a_{ij} x_j + \left[\max\limits_{\eta_{ij}\in\mathcal{U}} \left\{\sum_{j\in J_i} \eta_{ij} \hat{a}_{ij}\, x_j\right\}\right] \le b_i \quad && i = 1, \dots, m \\
& l_j \le x_j \le u_j && j = 1, \dots, n
\end{aligned}
\tag{3.2}
$$

Let X^{GRC} be the feasible region of model (3.2) involving the inner maximization problem $\max\limits_{\eta_{ij}\in\mathcal{U}}\left\{\sum_{j\in J_i} \eta_{ij}\hat{a}_{ij}\, x_j\right\}$, i.e. $X^{GRC} = \left\{\boldsymbol{x}: \boldsymbol{a}_i\boldsymbol{x} + \left[\max\limits_{\eta_{ij}\in\mathcal{U}}\left\{\sum_{j\in J_i}\eta_{ij}\hat{a}_{ij}\, x_j\right\}\right] \le b_i\ \forall i, \boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}\right\} \subset \mathbb{R}^n$. A vector $\boldsymbol{x}$ is a *robust feasible solution* if $\boldsymbol{x} \in X^{GRC}$. Note that $X^{GRC} \subseteq X$ which follows that the optimal objective value of the GRC model (3.2) is less than or equal to the optimal objective value of the uncertain LP model (3.1) . i.e., $z_2 \le z_1$. In other words, *taking uncertainty into consideration does not lead to improving the optimal objective value*. Rather, any optimal solution $\boldsymbol{x}^* \in X^{GRC}$ corresponds to a solution that maximizes the worst-case[21] objective function $\sum_{j=1}^{n} c_j x_j$ under all realizations of $\eta_{ij} \in \mathcal{U}$. Such worst-case solutions are obtained by first taking the dual of the inner maximization, which by the strong duality property (see Bazaraa, Jarvis, & Sherali, 2010) yield the same optimal objective values as its dual. Any solution to a specific robust counterpart to model (3.2) depends on the structure of the uncertainty set. The tractability[22] of the robust counterpart also depends on this set. For instance, suppose $\mathcal{U}$ is convex and the constraints are feasibly bounded in a convex region, then model (3.2) would lead to a computationally tractable solution (Ben-Tal, El Ghaoui, & Nemirovski, 2009). The general structure of the uncertainty set $\mathcal{U}$ is related to the distribution of uncertain parameters. For an unbounded distribution, the box, ellipsoidal, and polyhedral uncertainty sets can be used for the robust counterpart whereas interval constraint is necessary for bound distribution. The consideration of the latter case is necessary to avoid a more conservative solution. The box, interval+ellipsoidal, and interval+polyhedral uncertainty sets are reviewed in this chapter. As we mentioned earlier, the hint to deriving the robust solution to these uncertainty sets involves solving the subproblem (inner maximization problem) in model (3.2) using duality construction. For detailed information on these constructions, see Yuan, Li, & Huang (2016).

3.2.1 Robust counterpart to interval uncertainty set

In the robust optimization framework, the random variables, $\boldsymbol{\eta}$ are assumed to be independent in the interval [-1, 1] and the distribution over this interval is determined by the

[21] The modeling assumption of robust optimization is to seek an optimal solution that is best possible in the worst-case scenario. i.e. for a typical uncertain optimization problem $\max\limits_{x}\left\{\sum_{j=1}^{n} c_j x_j : \boldsymbol{A}(\boldsymbol{\zeta}) x_j \le b\right\}$ where $\boldsymbol{A} \in \mathbb{R}^{mxn}$ and $\boldsymbol{\zeta} \in \mathbb{R}^m$ denote the uncertain parameter belonging to the uncertainty set $\mathcal{U}$, we formulate a robust optimization that protects against the worst-case scenarios by minimizing over $\mathcal{U}$ the whole uncertain problem.

[22] By tractability, we mean the existence of an explicit polynomial-time algorithm to an equivalent formulation of the nominal optimization problem. Tractability issues are tantamount to solvability of the robust problem. See Ben-Tal, El Ghaoui, & Nemirovski, 2009) for more details.

nature of the uncertain parameters. For the LP model (3.1), consider that the perturbation bound for the uncertain coefficients is given by Φ. A box/interval uncertainty set created by the interaction of perturbation of the random variables can be described as follows:

$$\mathcal{U}_{Int}(\Phi) = \{\tilde{a}_{ij} = a_{ij} + \eta_{ij}\hat{a}_{ij} | \ \| \boldsymbol{\eta} \|_\infty \leq \Phi\} \tag{3.3}$$

The simplest case where knowledge about the distribution of $\boldsymbol{\eta}$ is known and the probability guarantees are given is when $\Phi \leq 1$ and $\mathrm{E}(\boldsymbol{\eta}) = 0$ (see Ben-Tal, El Ghaoui, & Nemirovski, 2009). The parameter Φ varies between 0 and 1 and the optimization model is robust when all the uncertain coefficients can be realized within the bound provided by the uncertainty set (3.3). The interval uncertainty set is shown in Figure 3.1-a for the distribution of random variables taking values within the bounds $\Phi = 1$. Note that $\| \boldsymbol{\eta} \|_\infty = 1$ coincides with the highly conservative robust formulation of Soyster (1973)[23]. Thus, taking $\mathcal{U} = \mathcal{U}_{Int}(1)$ in model (3.2), the subproblems $\max\limits_{\eta_{ij} \in \mathcal{U}_{Int}(1)\ \mathrm{U=U_{Int}(1)}} \{\sum_{j \in J_i} \eta_{ij} \hat{a}_{ij}\, x_j\}$ for $i = 1, \dots, m$ are linear optimization problems and hence substituting their related duals leads to the following robust counterpart:

$$\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\mathrm{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + \Phi \sum_{j \in J_i} \hat{a}_{ij}\, y_j \leq b_i & i = 1, \dots, m \\
-y_j \leq x_j \leq y_j & \forall i, \forall j \in J_i \\
l_j \leq x_j \leq u_j & j = 1, \dots, n \\
y_j \geq 0 & \forall i, \forall j \in J_i
\end{array} \tag{3.4}$$

Note that Soyster (1973) added a nonnegative variable y_j for each variable x_j; however, variable y_p is redundant if $\nexists i\colon p \in J_i$. As a result, we consider variable y_j if at least one of the coefficients of x_j in a constraint is uncertain. Note also that an accurate representation of the indices further reveals the actual number of variables and constraints in the model including those that are uncertain.

3.2.2 Robust counterpart to combined interval and ellipsoidal uncertainty set

Following the over-conservativeness of robust solutions using the interval uncertainty set, Ben-Tal & Nemirovski (1998, 1999) proposed using an ellipsoidal set which leads to solving the robust counterparts as a conic quadratic program. The proposed uncertainty set (Ben-Tal & Nemirovski, 2000) involves a combined interval and ellipsoidal uncertainty set to ensure a less conservative robust model. Specifically, the uncertainty set can be described as follows:

[23] Soyster (1973) robust formulation is one of the first models to immunize column-wise uncertainty in LP where uncertain parameters are confined to a convex set. Though the resulting robust model is linear, however, by taking the worst-case value of each uncertain parameter in the set, the approach becomes too conservative, sometimes producing results worse than the nominal problem.

$$\mathcal{U}_{Int+Ell}(\Phi,\Omega)=\{\tilde{a}_{ij}|\ \parallel \boldsymbol{\eta} \parallel_\infty \le \Phi, \parallel \boldsymbol{\eta} \parallel_2 \le \Omega\} \tag{3.5}$$

where Ω is a user-defined adjustment parameter that controls the trade-off between conservativeness and performance for the subsequent robust counterpart. We illustrate the distribution of $\boldsymbol{\eta}$ in the combined uncertainty set $\mathcal{U}_{Ell+Int}(1,1.14)$ in Figure 3.1-b. The robust counterpart to the use of this uncertainty set under the dual construction for model (3.2) is given as:

$$\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + \sum_{j\in J_i} \hat{a}_{ij}\, y_{ij} + \Omega\sqrt{\sum_{j\in J_i} \hat{a}_{ij}^2\, z_{ij}^2} \le b_i & i=1,\dots,m \\
-y_{ij} \le x_j - z_{ij} \le y_{ij} & \forall i, \forall j \in J_i \\
l_j \le x_j \le u_j & j=1,\dots,n \\
y_{ij} \ge 0 & \forall i, \forall j \in J_i
\end{array} \tag{3.6}$$

Note that the parameter Ω is defined as $\Omega \le (|J_i|)^{0.5}$ where $\Omega = (|J_i|)^{0.5}$ is the highest protection (i.e. the highest ellipsoid containing the box, $\{x_j: |x_j - z_j| \le y_j\}$) the decision maker can seek for the i^{th} constraint. The probability of violation of this i^{th} constraint is bounded above by $e^{-0.5\Omega^2}$ for any x_j^*. As mentioned before, $\mathcal{U}_{Ell+Int}(1,\Omega)$ is defined over the l_2 −norm. Therefore, the dual of the subproblem in model (3.2) involves quadratic functions $\left(\sum_{j\in J_i} \hat{a}_{ij}^2\, x_j^2\right)^{0.5}$ whose solution leads to second-order cone programming[24]. The robust counterpart of the optimization problem over the ellipsoidal uncertainty set or its intersection with the interval can lead to computationally tractable solutions or NP-hard problem. For large scale problems where interior points techniques can be harnessed, for instance, the ellipsoidal uncertainty set leads to practically tractable conic quadratic programming solutions (Ben-Tal & Nemirovski, 1998; 1999).

3.2.3 Robust counterpart to combined interval and polyhedral uncertainty set

Bertsimas & Sim (2004) relied on the family of a polyhedral set to propose a new robust formulation. The authors formulated an uncertainty set known as the budget of uncertainty set or the Bertsimas & Sim (2004) uncertainty set which is the commonly used uncertainty sets in practice because of its advantage in preserving the linearity of the nominal problem[25]. Ordinarily, the uncertainty set involves a combined interval and polyhedral set described as:

$$\mathcal{U}_{Int+Pol}(\Phi,\Gamma)=\{\tilde{a}_{ij}|\ \parallel \boldsymbol{\eta} \parallel_\infty \le \phi, \parallel \boldsymbol{\eta} \parallel_1 \le \Gamma\} \tag{3.7}$$

---

[24]. The main drawback of this formulation which makes it difficult to implement in practice is that it is computationally demanding since the robust counterpart is a nonlinear convex programming.

[25] This is because the polyhedron in $\mathbb{R}_+^2$ is simply an intersection of many finitely half spaces so the uncertainty set $\mathcal{U}_1$ affects the constraint in an affine manner. Computationally, the size of the robust counterpart polynomially grows in the dimension of $\mathcal{U}_1$ and the size of the nominal problem.

For simplicity, we consider $\Phi = 1$ and denote $\mathcal{U}_\Gamma$ as the budget uncertainty set where Γ is the parameter which the decision-maker can trade-off robustness and performance. Figure 3.1-c illustrates the distribution of $\boldsymbol{\eta}$ in the combined uncertainty set $\mathcal{U}_{Int+Pol}(1{,}1.5)$. The parameter, Γ_i of the i^{th} constraint takes values in $[0, |J_i|]$[26] and the robust solution is feasible if only less than Γ_i uncertain coefficients change. Besides, the uncertain parameters have maximum protection if at most Γ_i coefficient of the uncertain i^{th} constraints are allowed to deviate. Under this uncertainty set dynamics, the robust counterpart of the subproblem (3.2) to the uncertainty set $\mathcal{U}_\Gamma$ is the following:

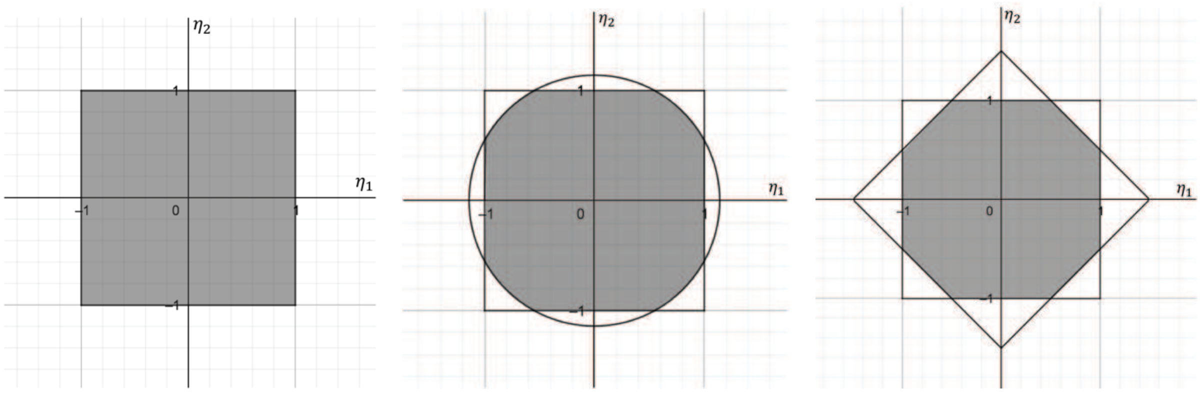
$$\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + \beta_i(\boldsymbol{y}, \Gamma_i) \leq b_i & i = 1, \dots, m \\
-y_j \leq x_j \leq y_j & \forall i, \forall j \in J_i \\
l_j \leq x_j \leq u_j & j = 1, \dots, n \\
y_j \geq 0 & \forall i, \forall j \in J_i
\end{array} \tag{3.8}$$

where $\beta_i(\boldsymbol{y}, \Gamma_i) = \max_{\{S_i \cup \{t_i\} |\, S_i \subseteq J_i, |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij}\, y_j + \left(\Gamma_i - \lfloor \Gamma_i \rfloor \hat{a}_{it_i} \right) y_{t_i} \right\}$ is *protection function* of the i^{th} constraint and $\boldsymbol{y} = (\dots, y_j, \dots) \in \mathbb{R}^{\sum_{i=1}^{m} |J_i|}$. Moreover, since model (3.8) is nonlinear, by strong duality to the subproblem, Bertsimas & Sim (2004) showed that an equivalent robust linear optimization has the formulation,

$$\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + p_i \Gamma_i + \sum_{j=1}^{n} q_{ij} \leq b_i & i = 1, \dots, m \\
p_i + q_{ij} \geq \hat{a}_{ij} y_j & \forall i, \forall j \in J_i \\
l_j \leq x_j \leq u_j & j = 1, \dots, n \\
-y_j \leq x_j \leq y_j & \forall i, \forall j \in J_i \\
q_{ij} \geq 0 & \forall i, j \in J_i \\
p_i \geq 0 & i = 1, .., m \\
y_j \geq 0 & \forall i, \forall j \in J_i
\end{array} \tag{3.9}$$

The robust solution parameter Γ_i regulates the number of $\tilde{a}_{ij}$ that may deviate from its nominal and obstruct the objection function. The higher value of a chosen Γ_i indicates a higher protection for the i^{th} constraint and vice versa. The probability for the violation of the i^{th} constraint is given by $e^{-\Gamma_i^2/2|J_i|}$.

[26] The budget of uncertainty parameter Γ_i may not assume integer value. However, where Γ_i takes an integer value, the last term in the protection function, $\beta_i(x^*, \Gamma_i)$ is excluded. Note that if $\Gamma_i = 0$, the problem attains a nominal solution and reduces to Soyster (1973) formulation if $\Gamma_i = |J_i|$.



$\mathcal{U}_{Int}(1)$ $\mathcal{U}_{Int+Ell}(1{,}1.14)$ $\mathcal{U}_{Int+Pol}(1{,}1.5)$

a) Box/interval uncertainty set b) Interval + ellipsoidal uncertainty set c) Interval + polyhedral uncertainty set

Figure 3.1. Illustration of uncertainty sets

3.3 Robust counterpart for nonnegative decision variables

In practice, a decision maker would prefer robust solutions for which the decision variable is positive. This condition is non-negotiable for many operations research problems such as robust efficiency scores via the data envelopment analysis, transportation problem and in some engineering and business applications. However, given that the unrestricted interval $l_j \leq x_j \leq u_j$ in the GRC could assume negative bounds and subsequently negative value for x_j, the goal of this section is to seek alternative reduced robust formulations that are restricted to the interval $0 \leq x_j \leq u_j$. We will compare robust optimization problems for the general LP model (3.1) with its reduced form when $l_j = 0$ and $u_j = \infty$ for $j = 1, \dots, n$ (or equivalently when $x_1, \dots x_n \geq 0$). We believe that taking the no-negativity constraints into consideration theoretically reduces the size of the corresponding robust counterpart optimization as well as practically decrease their required computational burden. To better understand the variations between the input sizes of the aforementioned robust counterparts, first, we look at the robust counterpart optimization confined to the positive half plane. As Aforementioned, we shall call it the *reduced robust counterpart* (RRC). The RRC considers robust counterpart optimization for the nonnegative decision variable,

$$\begin{aligned}
&\max \textstyle\sum_{j=1}^{n} c_j x_j \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} a_{ij} x_j + \left[\max\limits_{\eta_{ij}\in\mathcal{U}} \left\{\sum_{j\in J_i} \hat{a}_{ij}\, x_j\right\}\right] \leq b_i \quad && i = 1, \dots, m \\
&x_j \geq 0 && j = 1, \dots, n
\end{aligned} \tag{3.10}$$

Let X^{RRC} be the feasible region of model (3.10), i.e. $X^{RRC} = \left\{\boldsymbol{x}: \boldsymbol{a_i x} + \left[\max\limits_{\eta_{ij}\in\mathcal{U}} \left\{\sum_{j\in J_i} \eta_{ij} \hat{a}_{ij}\, x_j\right\}\right] \leq b_i \ \forall i, \boldsymbol{x} \geq 0\right\} \subseteq \mathbb{R}_+^n$, then $X^{RRC} \approx X^{GRC} \subseteq X$. The following theorem argues the redundancy of some constraints in the GRC model in the positive half plane which of course, reduces the computational burden for the reduced robust counterpart.

Theorem 3.1. *The tractable GRC constraint under nonnegativity condition, i.e.*

$$\begin{array}{ll} \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \\ \sum_{j=1}^{n} a_{ij} x_j + \sum_{j \in J_i} \hat{a}_{ij}\, y_j \leq b_i & i = 1, \dots, m \\ -y_j \leq x_j \leq y_j & \forall i, \forall j \in J_i \\ 0 \leq x_j \leq u_j & j = 1, \dots, n \\ y_j \geq 0 & \forall i, \forall j \in J_i \end{array} \tag{3.11}$$

is equivalent to the following model:

$$\begin{array}{ll} \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \\ \sum_{j=1}^{n} a_{ij} x_j + \sum_{j \in J_i} \hat{a}_{ij}\, x_j \leq b_i & i = 1, \dots, m \\ 0 \leq x_j \leq u_j & j = 1, \dots, n \end{array} \tag{3.12}$$

Proof. Let $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in \mathbb{R}^{n+|\cup_{i=1}^{m}\{J_i\}|}$ be an optimal solution of model (3.11). It is plain to verify that model (3.11) possesses alternative optimal solutions: consider the following set:

$$X^* = \{(\boldsymbol{x}^*, \boldsymbol{z}^*) | z_j^* \leq y_j^* \ \forall i, \forall j \in J_i\}$$

Clearly, $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ is a feasible solution for model (3.11) and its objective function value is equal to the objective function value of the optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. As a result, X^* is the set of all alternative optimal solutions of model (3.11). We arrive at model (3.12) when we let $z_j^* = x_j^* \ \forall j \in J_i$ (note that $x_j^* \leq y_j^*$) which completes the proof. ☐

Theorem 3.1 suggests fewer parameters in solving the robust counterpart for nonnegative decision variables. It is easy to see that, when the GRC involves positive decision variables, some variables and constraints become redundant which can be removed in order to reduce the extra computational effort without altering the optimal objective value. To see the implied usage of this suggested idea, we provide propositions which suggest that the robust counterparts discussed in Section 2 can be reformulated with fewer decision variables and constraints. The proof to these propositions can be inferred directly from the Theorem 3.1 and therefore omitted.

Proposition 3.1. *With the box uncertainty set $\mathcal{U}_\infty$ defined with nonnegative decision variable x, model (3.4) can be equivalently expressed as:*

$$\begin{array}{ll} \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \\ \sum_{j=1}^{n} a_{ij} x_j + \Phi \sum_{j \in J_i} \hat{a}_{ij}\, x_j \leq b_i & i = 1, \dots, m \\ x_j \geq 0 & j = 1, \dots, n \end{array} \tag{3.13}$$

The proof follows analogous reasoning from Theorem 3.1 since both $\hat{a}_{ij}$ and Φ are nonnegative parameters.

Proposition 3.2. *Given that the interval and polyhedral uncertainty set are intercepted in the positive half plane, the RRC with nonnegative decision variable x to model (3.9) can be* written as

$$
\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j + p_i \Gamma_i + \sum_{j \in J_i} q_{ij} \leq b_i & i = 1, \dots, m \\
p_i + q_{ij} \geq \hat{a}_{ij} x_j & \forall i, \forall j \in J_i \\
q_{ij} \geq 0 & \forall i, \forall j \in J_i \\
p_i \geq 0 & i = 1, \dots, m \\
x_j \geq 0 & j = 1, \dots, n
\end{array}
\tag{3.14}
$$

The proof follows an analogous reasoning from Theorem 3.1.

3.3.1 Complexity analysis

One of the main challenging questions in robust optimization relates to the structure and complexity of the robust counterpart towards different classes of uncertainty set, $\mathcal{U}$. For instance, for a given class of nominal problem and structured uncertainty set, what would be the complexity class of the corresponding robust problem? (Bertsimas et al, 2011). Computational complexity theories allow us to group optimization problems into different difficulty level based on the number of computational resources required to solve a problem. For a tractable robust counterpart, efficient running of an optimization algorithm depends on the model structure and the evaluation stages used in finding an optimal solution. Our concern here lies in the computational burden of the models and the iteration counts required in solving each of the robust models. Note that, the number of iterations or steps necessary to solve these problems depends on the input size of the problem and the algorithm used to solve the problem. Here we consider the simplex method which is the most popular and effective for solving linear programming problems. More formally, for a given optimization problem in the standard form

$$
\begin{array}{ll}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j = b_i & i = 1, \dots, m \\
x_j \geq 0 & j = 1, \dots, n
\end{array}
\tag{3.15}
$$

referencing Bazaraa et al. (2010), the average complexity of the simplex algorithm requires roughly m to $3m$ order of iterations to solve a problem. In most applications, the sparsity of the matrix $\mathbf{A} = \{a_{ij}\}$ may be exploited to obtain a more efficient solution by the algorithm. The analysis with the simplex algorithm excludes the robust counterpart to interval+ellipsoid uncertainty set since the problem requires a non-linear approach such as the interior point method. Let $k = \sum_{i=1}^{m} |J_i|$ represent the total number of uncertain data. The standard form of model (3.6) however, involves $m + 2(k + n)$ constraints and $(n + 2k) + (m + 2(k + n))$ nonnegative decision and slack variables. Table 3.1 summarizes the additions and multiplications involved in the iterations of the linear robust counterparts.

Table 3.1. Addition and multiplication in each iteration per operation

Robust Counterparts (RC)		RC – Box uncertainty set	RC – Polyhedral + interval uncertainty set
General RC	*Additions*	$(m + 2(k + n))(k + n + 1)$	$(m + 3k + 2n)(m + 2k + n + 1)$
	Multiplications	$(m + 2(k + n))(k + n) + (m + 3(k + n)) + 1$	$(m + 3k + 2n)(m + 2k + n) + (2m + 5k + 3n + 1)$
Reduced RC	*Additions*	$m(n + 1)$	$(m + k)(m + k + n + 1)$
	Multiplications	$mn + m + n + 1$	$(m + k)(m + k + n) + (2m + 2k + n + 1)$

The number of operation required in each iteration for the GRC in models (3.4) and (3.9) and the RRC models (3.13) - (3.14) using the simplex algorithm can be seen in Table 3.1. Clearly, the standard form of the GRC model (3.4) involves $m + 2(k + n)$ constraints and $(n + k) + m + 2(k + n)$ nonnegative decision and slack variables whiles, on the other hand, the associated standard form of the RRC model (3.13) requires m to $m + n$ iterations. Again, we note that the GRC model (3.9) contains $m + 3k + 2n$ constraints and $m + n + 2k$ decision variables and $m + 2n + 3k$ slack variables whereas the RRC (3.14) involves only $m + k$ constraints and $2(m + k) + n$ variables in all their standard forms. In summary, Table 3.1 indicates a significant reduction in the number of operations particularly for large-scale problems involving only nonnegative decision variables using models (3.13) and (3.14). From Theorem 3.1, if n is significantly larger than m, solving these robust counterparts would result in saving a considerable computer storage and time.

A further question however is, does the RRC models (3.13) and (3.14) generate equivalent optimal solution as its corresponding GRC models (3.4) and (3.9)?. Indeed the robust counterpart to a nonnegative optimization problem and its reduced form yield an equivalent solution as demonstrated by the following simple example in Bazaraa et al. (2010):

$$
\begin{aligned}
&\min z = -2x_1 - 4x_2 - x_3 \\
&\text{s.t.} \\
&\qquad 2x_1 + x_2 + x_3 \leq 10 \\
&\qquad x_1 + x_2 - x_3 \leq 4 \\
&\qquad 0 \leq x_1 \leq 4 \\
&\qquad 0 \leq x_2 \leq 6 \\
&\qquad 1 \leq x_3 \leq 4
\end{aligned}
\tag{3.16}
$$

The nominal values are (2, 1, 1) and (1, 1, -1) which are the coefficients of the first and second constraint respectively. The model is a minimization problem and its optimal solution and objective function value is $(x_1^*, x_2^*, x_3^*) = (0.67, 6, 2.67)$ and $z^* = -28$, respectively. Assume that the uncertain coefficients are 10% accurate approximations of the "true" vector of coefficients. Let $J_1 = \{1,3\}$ and $J_2 = \{2\}$. The corresponding robust counterpart based on the GRC model (3.4) is obtained as follows:

$$\begin{aligned}
&\min z = -2x_1 - 4x_2 - x_3 \\
&\text{s.t.} \\
&2x_1 + x_2 + x_3 + \Phi(0.2y_1 + 0.1y_3) \le 10 \\
&\; x_1 + x_2 - x_3 + \Phi 0.4 y_2 \le 4 \\
&\qquad -y_1 \le x_1 \le y_1 \\
&\qquad -y_2 \le x_2 \le y_2 \\
&\qquad -y_3 \le x_3 \le y_3 \\
&\qquad 0 \le x_1 \le 4 \\
&\qquad 0 \le x_2 \le 6 \\
&\qquad 1 \le x_3 \le 4 \\
&\qquad y_1, y_2, y_3 \ge 0
\end{aligned} \tag{3.17}$$

while the robust counterpart based on the RRC model (3.13) is clearly the following:

$$\begin{aligned}
&\min z = -2x_1 - 4x_2 - x_3 \\
&\text{s.t.} \\
&(2 + 0.2\Phi)x_1 + x_2 + (1 + 0.1\Phi)x_3 \le 10 \\
&x_1 + (1 + \Phi 0.4)x_2 - x_3 \le 4 \\
&\qquad 0 \le x_1 \le 4 \\
&\qquad 0 \le x_2 \le 6 \\
&\qquad 1 \le x_3 \le 4
\end{aligned} \tag{3.18}$$

Table 3.2. Robust counterpart optimal solutions and values to model (3.16)

Box parameter	GRC example (3.17)		RRC example (3.18)	
	z^*	$(x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*)$	z^*	(x_1^*, x_2^*, x_3^*)
$\Phi = 0$	-28.00	(0.67,6.00,2.67,0.67,6.00,2.67)	-28.00	(0.67,6.0,2.67)
$\Phi = 0.2$	-27.92	(0.48,6.00,2.96,0.48,6.00,2.96)	-27.92	(0.48,6.0,2.96)
$\Phi = 0.4$	-27.84	(0.29,6.00,3.25,0.29,6.00,3.25)	-27.84	(0.29,6.00,3.25)
$\Phi = 0.6$	-27.77	(0.11,6.00,3.55,0.11,6.00,3.55)	-27.77	(0.11,6.00,3.55)
$\Phi = 0.8$	-27.40	(0.00,5.90,3.79,0.00,5.90,3.79)	-27.40	(0.00,5.90,3.79)
$\Phi = 1.0$	-26.61	(0.00,5.67,3.94,0.00,5.67,3.94)	-26.61	(0.00,5.67,3.94)

It is plain to verify that the two robust counterparts lead to the same optimal solution under different box parameter Φ values. Table 3.2 summarizes the optimal solutions along with their optimal values for different Φ. As can be seen, the optimal values of both models are identical for various Φ which validates Theorem 1. We also emphasize the applicability of Proposition 3.2 by showing in Section 6, a real-world banking problem using the DEA approach

3.4 Data Envelopment Analysis approach

This section employs the robust DEA as a demonstration of the proposed models in Section 3. We focus on the BCC model with an input orientation as presented in model (2.12) of Chapter 2:

$$
\begin{array}{ll}
\vartheta^* = \max \sum_{r=1}^{s} u_r y_{ro} + u_o & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} = 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0 & j = 1, \dots, n \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s \\
u_o \text{ is free} &
\end{array}
\tag{3.19}
$$

Note that, this model (3.19) and other traditional DEA models use equality in the normalization constraint and can be formulated as inequality constraints (Toloo, 2014a and Gorissen et al., 2015). The next section focuses on uncertainty in the equality constraint for robust DEA analysis.

3.4.1 Equality constraint in uncertain DEA

In robust optimization, most often than not, the true values revolve in an unequal and symmetric interval. Equality constraints containing uncertain parameters are therefore required to be in the inequality form since the equality constraints can restrict the feasibility region or sometimes lead to the infeasibility of the robust model (Ben-Tal, El Ghaoui, & Nemirovski, 2009; Gorissen et al., 2015). However, uncertainty analysis carried out in DEA measures include modeling uncertainty in the normalization constraint which is equality constraints in model (2.12). In this case, though the DEA models are always feasible, they become infeasible for robust analysis. Unless the uncertain inputs and outputs are analyzed in the IECCR model (see Sadjadi et al, 2011 for example), these models become unsuitable for general robust efficiency measurement. Naturally, a solution which is feasible in the robust DEA sense requires inequality in the normalization constraint of the multiplier DEA models. Equality constraint containing uncertain parameters for general robust optimization problems have been analyzed in different ways in applications. See Gorissen et al. (2015) for a summary of alternative approaches. Salahi et al. (2016) dealt with this issue by converting equality in the normalization constraint of IMCCR model (2.8) to double inequality constraints. Omrani (2013) instead replaced the normalization constraint $\sum_{i=1}^{m} v_i x_{io} = 1$ with a superfluous constraint $\sum_{r=1}^{s} u_r - \sum_{i=1}^{m} v_i = 1$ for his common weight robust DEA model in order to avoid constraint infeasibility for the input uncertainty. As mostly considered, in DEA, alternative formulation converting the equality to inequality constraint is proposed (Toloo, 2014)a for the IMCCR model. Here, suppose the normalization constraint is fixed at any other positive parameter, model (3.19) can be restated as:

$$
\begin{array}{ll}
\max \sum_{r=1}^{s} u_r y_{ro} + u_o & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} = t & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \le 0 & j = 1, \dots, n \\
v_i \ge 0 & i = 1, \dots, m \\
u_r \ge 0 & r = 1, \dots, s \\
u_o \text{ free in sign} &
\end{array} \tag{3.20}
$$

where t is a positive parameter.

Remark 3.1. It can easily be verified that $(\boldsymbol{u}^*, \boldsymbol{v}^*, u_0^*)$ is an optimal solution for model (3.19) if and only if $(t\boldsymbol{u}^*, t\boldsymbol{v}^*, tu_0^*)$ is an optimal solution for model (3.20).

Theorem 3.2. *The following model is equivalent to the BCC model (3.19):*

$$
\begin{array}{ll}
\max \sum_{r=1}^{s} u_r y_{ro} + u_o & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} \le t & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \le 0 & j = 1, \dots, n \\
v_i \ge 0 & i = 1, \dots, m \\
u_r \ge 0 & r = 1, \dots, s \\
u_o \text{ free in sign} &
\end{array} \tag{3.21}
$$

where t is a positive parameter.

Proof. By primal-dual relationship, to show that the primal models (3.19) and (3.20) are equivalent, it is sufficient to compare the duals of these models as respectively below:

$$
\begin{array}{ll}
\min t\theta & \\
\text{s.t} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{i0} & i = 1, \dots, m \\
\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{r0} & r = 1, \dots, s \\
\sum_{j=1}^{n} \lambda_j = 1 & \\
\lambda_j \ge 0 & j = 1, \dots, n \\
\theta \text{ free is sign} &
\end{array} \tag{3.22}
$$

and

$$
\begin{aligned}
&\min t\theta \\
&\text{s.t} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{i0} \quad && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{r0} \quad && r = 1, \dots, s \\
&\textstyle\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge 0 && j = 1, \dots, n \\
&\theta \ \ge 0
\end{aligned}
\tag{3.23}
$$

Let θ^* be the optimal solution of the dual models. It is easy to see that $\theta^* > 0$ in model (3.22) since $\theta^* \le 0$ would mean $\boldsymbol{\lambda}^* = \mathbf{0}_n$ in the first constraint and subsequently violate the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ which is impossible. Therefore, models (3.22) and (3.23) are equivalent and their related primal models are also equivalent. □

Definition 3.1. Let the parameter t be fixed at 1, the appropriate DEA model for the robust optimization analysis is the following:

$$
\begin{aligned}
&\max z \\
&\text{s.t.} \\
&z - \textstyle\sum_{r=1}^{s} u_r y_{ro} - u_o \le 0 \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} \le 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \le 0 \quad && j = 1, \dots, n \\
&v_i \ge 0 && i = 1, \dots, m \\
&u_r \ge 0 && r = 1, \dots, s \\
&u_o \text{ free in sign}
\end{aligned}
\tag{3.24}
$$

Definition 3.1 provides a way to analyze uncertainties in both input and output data. Usually, for the majority of the DEA literature, considering the difficulty of uncertainty in the normalization constraint, when uncertainty appears in the input data, an input-oriented model is adopted whereas the output-oriented model is adopted with uncertain output data (Wang & Wei, 2010). Practically, as the choice of DEA orientation is not the prerogative of the decision analyst but mostly by the organization 's choice of production process model (3.24) is very useful for measuring the robust efficiency with uncertain inputs and outputs data in either the input or output orientation model.

3.4.2 Robust Data Envelopment Analysis

The robust optimization in DEA has been introduced by Sadjadi & Omrani (2008) based on the reviewed approaches in Section 2 of this chapter. Extension to other advanced DEA models and a varied application has since been made, particularly for energy efficiency measurement. We refer the reader to Sadjadi et al. (2011), Omrani (2013), Lu (2015), and Wu et al. (2016). The modeling of robust optimization in DEA follows three main approaches in literature: the robust approaches proposed by Mulvey et al. (1995), Ben-Tal & Nemirovski (2000) and Bertsimas & Sim (2004). The Bertsimas & Sim (2004) approach is employed in this study and as previously described in Section 2, the true values of the uncertain input and output data are expressed as $\tilde{x}_{ij} = x_{ij} + \eta^x_{ij}\hat{x}_{ij}$; $\tilde{y}_{rj} = y_{rj} + \eta^y_{rj}\hat{y}_{rj}$ where the independent random variables $\eta^x_{ij}, \eta^y_{rj}, \forall i, \forall r$ take values in the interval $[-1, 1]$ and the maximum deviations are defined as $\hat{x}_{ij} = e_i x_{ij}$ and $\hat{y}_{ij} = e_j y_{rj}$. Note that e is the percentage of perturbation specifying the amount of deviation from the uncertain inputs and outputs data from their true values. For each input data x_{ij}, $i \in I_j$ and output data y_{rj}, $r \in R_j$ the true values are modelled as variables $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ taking values in the symmetric interval $[x_{ij} - \hat{x}_{ij}, x_{ij} + \hat{x}_{ij}]$ and $[y_{rj} - \hat{y}_{rj}, y_{rj} + \hat{y}_{rj}]$ respectively, where I_j and R_j represent the set of inputs and outputs of DMUs that are subject to uncertainty. Note that $\mathrm{DMU}_k = (\boldsymbol{x}_k, \boldsymbol{y}_k)$ is uncertain if there exist $i \in I_k$ or $r \in R_k$. Using the uncertainty dynamics of Bertsimas & Sim (2004), note that the total (scaled) deviations $\eta^x_{ij} = (\tilde{x}_{ij} - x_{ij})/\hat{x}_{ij}$ and $\eta^y_{rj}, = (\tilde{y}_{rj} - y_{rj})/\hat{y}_{rj}$ which are symmetrically bounded in the interval $[-1, 1]$ and assume values between $-n$ and n are restricted to the budget of uncertainty parameter Γ_j. We assume that $\Gamma^x_j \in [0, |I_j|]$ and $\Gamma^y_j \in [0, |R_j|]$ are the budget of uncertainty parameter of the input and outputs vectors, respectively. We let $\Gamma_j = \Gamma^x_j + \Gamma^y_j \in [0, |I_j| + |R_j|]$ and $\sum_{i \in I_j} |\eta^x_{ij}| + \sum_{r \in R_j} |\eta^y_{rj}| \leq \Gamma_j$. In view of this, one can express the budget of uncertainty set as follows:

$$\mathcal{U}_{\Gamma_j} = \left\{ (\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{y}}_r) \left| \begin{array}{l} \tilde{x}_{ij} = x_{ij} + \eta^x_{ij}\hat{x}_{ij},\ \tilde{y}_{rj} = y_{rj} + \eta^y_{rj}\hat{y}_{rj},\ \forall i \in I_j, \forall r \in R_j \\ \sum_{i \in I_j} |\eta^x_{ij}| + \sum_{r \in R_j} |\eta^y_{rj}| \leq \Gamma_j, \forall i \in I_j, \forall r \in R_j \\ \eta^x_{ij}, \eta^y_{rj} \in [-1, 1],\ \forall i \in I_j, \forall r \in R_j \end{array} \right. \right\} \tag{3.25}$$

The level of the budget of uncertainty Γ_j allowed depicts the level of robustness allowed for each perturbation in the constraints of the uncertain model. From model (3.19) if the output in the objective function is subject to uncertainty, without loss of generality, the objective function can be expressed as max ω and the additional constraint $\omega - \sum_{r=1}^{s} u_r y_{ro} \leq 0$ added to the model. As a result, we extend the robust formulation of Sadjadi & Omrani (2008) to the following robust BCC model with both inputs and outputs uncertainty under the GRC approach:

$$
\begin{array}{ll}
\max \omega & \\
\text{s.t.} & \\
\omega - \sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y + \sum_{r \in R_o} q_{ro} + u_0 \leq 0 & \\
\sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x + \sum_{i \in I_o} w_{io} \leq 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o + p_j^y \Gamma_j^y + p_j^x \Gamma_j^x + \sum_{r \in R_j} q_{rj} + \sum_{i \in I_j} w_{ij} \leq 0 & j = 1, \dots, n \\
p_j^y + q_{rj} \geq e y_{rj} z_r^y & \forall j, \forall r \in R_j \\
p_j^x + w_{ij} \geq e x_{ij} z_i^x & \forall j, \forall i \in I_j \\
-z_r^y \leq u_r \leq z_r^y & r = 1, \dots, s \\
-z_i^x \leq v_i \leq z_i^x & i = 1, \dots, m \\
q_{rj}, w_{ij} \geq 0 & \forall j, \forall i \in I_j, \forall r \\
p_j^x, p_j^y \geq 0 & j = 1, \dots, n \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s \\
u_o \quad \text{free in sign} &
\end{array} \tag{3.26}
$$

where ω is the efficiency score of DMU_o; p_j^x, q_{rj} and p_j^y, w_{ij} are nonnegative variables of the set of uncertain inputs and outputs respectively; z_i^x and z_r^y are auxiliary variables of the absolute input and output weights; and Γ_o^x and Γ_o^y are the respective robust parameters of the uncertain inputs and outputs of the DMU under evaluation.

3.5 The new approach: Reduced Robust DEA (RRDEA)

Now, with a strong emphasis on the computational complexity pointed out in Section 3.1 and to formalize our argument in Theorem 3.1 with the DEA, a reduced robust DEA (RRDEA) model based on Proposition 3.2 is formulated for model (3.26). We suppose that all the uncertain inputs and outputs data are protected against with allowable deviation up to Γ_j^x and Γ_j^y and that an uncertain input and output data of DMU_j changes from their nominal values by $\left(\Gamma_j^x - \lfloor \Gamma_j^x \rfloor\right)\hat{x}_{i\bar{t}_j}$ and $\left(\Gamma_j^y - \lfloor \Gamma_j^y \rfloor\right)\hat{y}_{rt_j}$. Then from Section 2.3, the robust counterpart to the uncertain DEA (3.24) can be formulated as below:

$$
\begin{array}{ll}
\max \sum_{r=1}^{s} u_r y_{ro} + u_o + \beta(\boldsymbol{u}^*, \mathbf{0}_m, \Gamma_o^y) & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} + \beta(\mathbf{0}_s, \boldsymbol{v}^*, \Gamma_o^x) \leq 1 & \\
\sum_{r=1}^{s} u_r y_{rj} + u_o - \sum_{i=1}^{m} v_i x_{ij} + \beta(\boldsymbol{u}^*, \boldsymbol{v}^*, \Gamma_j^y + \Gamma_j^x) \leq 0 & j = 1, \dots, n \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s \\
u_o \quad \text{free in sign} &
\end{array} \tag{3.27}
$$

where $\beta(\boldsymbol{u}^*, \boldsymbol{v}^*, \Gamma_j^y + \Gamma_j^x)$ is the protection function corresponds to the uncertain data, and $\beta(\boldsymbol{u}^*, \mathbf{0}_m, \Gamma_j^y)$ and $\beta(\mathbf{0}_s, \boldsymbol{v}^*, \Gamma_j^x)$ are the protection functions corresponding to the uncertain outputs and inputs of DMU_j, respectively. Here

$$\beta(\boldsymbol{u}^*, \boldsymbol{v}^*, \Gamma_j^y + \Gamma_j^x) = \max_{\substack{\{S_j \cup \{t_j\} \mid S_j \subseteq |R_j|, |S_j| = \lfloor \Gamma_j^y \rfloor, t_j \in \{|R_j|\} \setminus S_j\} \\ \{\bar{S}_j \cup \{\bar{t}_j\} \mid \bar{S}_j \subseteq |I_j|, |\bar{S}_j| = \lfloor \Gamma_j^x \rfloor, \bar{t}_j \in \{|I_j|\} \setminus \bar{S}_j\}}} \left\{ \begin{array}{c} \sum_{r \in S_j} u_r \hat{y}_{rj} + \left(\Gamma_j^y - \lfloor \Gamma_j^y \rfloor \right) u_t \hat{y}_{rt_j} + \\ \sum_{i \in \bar{S}_j} v_i \hat{x}_{ij} + \left(\Gamma_j^x - \lfloor \Gamma_j^x \rfloor \right) v_t \hat{x}_{i\bar{t}_j} \end{array} \right\} \tag{3.28}$$

Therefore,

$$\beta(\boldsymbol{u}^*, \mathbf{0}_m, \Gamma_j^y) = \max_{\{S_j \cup \{t_j\} \mid S_j \subseteq R_j, |S_j| = \lfloor \Gamma_j^y \rfloor, t_j \in R_j \setminus S_j\}} \left\{ \sum_{r \in S_j} u_r \hat{y}_{rj} + \left(\Gamma_j^y - \lfloor \Gamma_j^y \rfloor \right) u_t \hat{y}_{rt_j} \right\} \tag{3.29}$$

An analogous definition can be made for $\beta(\mathbf{0}_s, \boldsymbol{v}^*, \Gamma_j^x)$. Note that $\Gamma_j^y = |R_j|$ and $\Gamma_j^x = |I_j|$ lead to the worst-case formulation meanwhile $\Gamma_j^y + \Gamma_j^x = 0$, $\beta(\boldsymbol{u}^*, \boldsymbol{v}^*, 0)$ for all j follow the nominal model (3.24). Therefore, the decision maker is able to make a trade-off between robustness and the level of conservatism of the solution by varying Γ_j^x in $[0, |I_j|]$ or Γ_j^y in $[0, |R_j|]$. If Γ_j^y and Γ_j^y are chosen as integer numbers, then we obtain

$$\beta(\boldsymbol{u}^*, \boldsymbol{v}^*, \Gamma_j^y + \Gamma_j^x) = \max_{\substack{\{S_j \mid S_j \subseteq |R_j|, |S_j| = \lfloor \Gamma_j^y \rfloor\} \\ \{\bar{S}_j \mid \bar{S}_j \subseteq |I_j|, |\bar{S}_j| = \lfloor \Gamma_j^x \rfloor\}}} \left\{ \sum_{r \in S_j} u_r \hat{y}_{rj} + \sum_{i \in \bar{S}_j} v_i \hat{x}_{ij} \right\} \tag{3.30}$$

Again, notice that the above model is nonlinear. We make use of Theorem 3.1 and follow Theorem 3.3 below.

Theorem 3.3. *The nonlinear model* (3.27) *is equivalent to the following reduced linear model:*

$$\begin{array}{ll}
\max \varpi & \\
\text{s.t.} & \\
\varpi - \sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y + \sum_{r \in R_o} q_{ro} + u_0 \le 0 & \\
\sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x + \sum_{i \in I_o} w_{io} \le 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o + p_j^y \Gamma_j^y + p_j^x \Gamma_j^x + \sum_{r \in R_j} q_{rj} + \sum_{i \in I_j} w_{ij} \le 0 & j = 1, \dots, n \\
p_j^y + q_{rj} \ge u_r \hat{y}_{rj} & \forall j, \forall r \in R_j \\
\mathrm{p}_j^x + w_{ij} \ge v_i \hat{x}_{ij} & \forall j, \forall i \in I_j \\
q_{rj}, w_{ij} \ge 0 & \forall j, \forall i \in I_j, \forall r \in R_j \\
p_j^x, p_j^y \ge 0 & j = 1, \dots, n \\
v_i \ge 0 & i = 1, \dots, m \\
u_r \ge 0 & r = 1, \dots, s \\
u_o \quad \text{free in sign} &
\end{array} \tag{3.31}$$

Proof. The protection function used in model (3.27) provides a simple way to generate a corresponding optimization problem for the input and output parameters. Given the optimal solution vector $(\boldsymbol{u}^*, \boldsymbol{v}^*)$, the protection function β $(\boldsymbol{u}^*, \boldsymbol{v}^*, \Gamma_j^y + \Gamma_j^x)$ can be formulated as the following linear optimization problem:

$$\begin{aligned}
&\max \textstyle\sum_{r\in R_j} |u_r|\hat{y}_{rj}\eta^y_{rj} + \sum_{i\in I_j} |v_i|\hat{x}_{ij}\eta^x_{ij} \\
&\text{s.t.} \\
&\textstyle\sum_{i\in I_j} \eta^x_{ij} \le \Gamma^x_j \\
&\textstyle\sum_{r\in R_j} \eta^y_{rj} \le \Gamma^y_j \\
&0 \le \eta^y_{rj} \le 1 && \forall r \in R_j \\
&0 \le \eta^x_{ij} \le 1 && \forall i \in I_j
\end{aligned} \tag{3.32}$$

As inspection makes clear, model (3.32) at optimality arrives at $\boldsymbol{\eta}^* = \left(\eta^x_{ij}, \eta^y_{rj}\right), \forall i \in I_j, \forall r \in R_j$ which is made up of Γ_j variables equal to 1 and another variable equal to $\Gamma^x_j - \lfloor \Gamma^x_j \rfloor$ and $\Gamma^y_j - \lfloor \Gamma^y_j \rfloor$. Equivalently, this implies selecting a subset of $\left\{S_j \cup \{t_j\} \middle| S_j \subseteq |R_j|, |S_j| = \lfloor \Gamma^y_j \rfloor, t_j \in \{|R_j|\} \backslash S_j;\ \bar{S}_j \cup \{\bar{t}_j\} \middle| \bar{S}_j \subseteq |I_j|, |\bar{S}_j| = \lfloor \Gamma^x_j \rfloor, \bar{t}_j \in \{|I_j|\} \backslash \bar{S}_j\right\}$ with the corresponding objective function $\sum_{r\in S_j} u_r\hat{y}_{rj} + \left(\Gamma^y_j - \lfloor \Gamma^y_j \rfloor\right) u_t\hat{y}_{rt_j} + \sum_{i\in S_j} v_i\hat{x}_{ij} + \left(\Gamma^x_j - \lfloor \Gamma^x_j \rfloor\right) v_t\hat{x}_{i\bar{t}_j}$, hence models (3.28) and (3.32) are rightfully equivalent. It should be noted that model (3.32) can be written as a linear minimization problem using strong duality theory (see Bazaraa et al., 2010). Let p^x_j, p^y_j, q_{rj} and w_{ij} be the dual variables of the first, second, third and fourth constraints respectively. Given $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ as optimal solution, at optimality, $z^y_r = |u_r|$ and $z^x_i = |v_i|$. The dual of the LP model (3.32) then follows:

$$\begin{aligned}
&\min \textstyle\sum_{r\in R_j} q_{rj} + \sum_{i\in I_j} w_{ij} + p^y_j\Gamma^y_j + p^x_j\Gamma^x_j \\
&\text{s.t} \\
&\mathrm{p}^y_j + q_{rj} \ge z^y_r\hat{y}_{rj} && \forall r \in R_j \\
&\mathrm{p}^x_j + w_{ij} \ge z^x_i\hat{x}_{ij} && \forall i \in I_j \\
&-z^y_r \le u_r \le z^y_r && \forall r \in R_j \\
&-z^x_i \le v_i \le z^x_i && \forall i \in I_j \\
&q_{rj}, w_{ij} \ge 0 && \forall i \in I_j, \forall r \in R_j \\
&p_j \ge 0
\end{aligned} \tag{3.33}$$

Model (3.32) is feasible and bounded for all Γ^x_j taking values in $[0, |\mathrm{I}_j|]$ and Γ^y_j taking values in $[0, |\mathrm{R}_j|]$. Therefore, by the strong duality theory, the dual model (3.33) is also feasible and bounded and their objective function values are equal. From Theorem 3.1, since u^*_r and v^*_i are nonnegative, model (3.33) can be simplified as model (3.34).

$$\begin{aligned}
&\min \textstyle\sum_{r\in R_j} q_{rj} + \sum_{i\in I_j} w_{ij} + p^y_j\Gamma^y_j + p^x_j\Gamma^x_j \\
&\text{s.t} \\
&\mathrm{p}^y_j + q_{rj} \ge u_r\hat{y}_{rj} && \forall r \in R_j \\
&\mathrm{p}^x_j + w_{ij} \ge v_i\hat{x}_{ij} && \forall i \in I_j \\
&q_{ij}, w_{ij} \ge 0 && \forall i \in I_j, \forall r \in R_j \\
&p_j \ge 0
\end{aligned} \tag{3.34}$$

Model (3.28) is equivalent to model (3.32) and by extension to model (3.34). Substituting model (3.34) into model (3.27) yields the resulting linear optimization model. ☐

Remark 3.2: Model (3.31) should be very relevant for large-scale problems since the number of variables and constraints can be reduced to aid efficient computation in less time.

3.6 Application to banking data

This section demonstrates the applicability of the RRDEA model using data from 250 banks in some 23 countries in the European Union. The section consists of two subsections. The performance of the RRDEA with respect to complexity is presented in subsection 7.1 and compared with the RDEA model (3.26). Subsection 7.2 provides a robust efficiency ranking of the banks using the RRDEA model. The banks analyzed comprise a conglomerate of European banks headquartered in the European Union with subsidiaries operating across the globe. They include, in terms of assets some large banks such as BNP Paribas, Deutsche Bank AG, HSBC Bank plc, Barclays Bank Plc, Société Générale SA whose operation extends beyond domestic European market. The data also comprises some less market share banks operating in the domestic market or solely in their single country. In Appendix B, we have provided descriptive summary and the raw data containing the financial statements indicators of the banks used. Next, we look at the analysis of efficiency with the bank data.

3.6.1 Selection of variables

We consider the selection of inputs and outputs which is crucial in the banking efficiency measurement. In the banking sector, similar to other sectors, a consensus is reached on the classification of some factors as inputs and outputs. However, the classification of others particularly deposit is unclear and controversial. The debate on bank deposit which in the DEA literature is termed as a flexible measure or dual role factors (see Toloo, 2012; Toloo, Allahyar, & Hančlová, 2018) is that, depending on the operational activities of the bank, in one hand, deposit could be regarded as an input (*intermediation approach*) and on the other hand as an output (*production approach*) or as a major component involved in the creation of added value (*value-added approach*). The first two are the main competing approaches as identified in Berger & Humphrey (1997) and so we explain them.

Intermediation approach: This approach involves examining how banks are organizationally efficient by using labor and capital in conjunction with financial liabilities; deposits to produce loans, mortgage, and other means of financing (e.g., investment). It therefore perceives banks as financial intermediaries between savers and investors and considers banks' liabilities as input. Efficiency of a bank by this approach signifies a strong indication of the strength of its lending ability which in turn is linked to the bank's ability to operate as a going concern (Paradi, Rouatt, & Zhu, 2011). One key important advantage of this approach is that it is better suited in capturing the management decisions to minimize the cost of financing mix, hence, it is seen more appropriate for evaluating the performance of financial institutions as a whole. According to a survey on banking efficiency by Fethi &

Pasiouras (2010), the intermediation approach has become a dominant approach used in the performance of whole banks since most banks are essentially financial intermediaries, whose main activities is to borrow funds from depositors and lend to others.

Production approach: This approach is of the view that banks are producers of services and product using capital, labour and other resources as inputs to produce loans and deposit account services including the number of transactions or document processing as outputs to customers. The production approach is a significant dimension of bank performance at the branch level. At the bank branch level, transactions are made face-face to customers and the branch is seen as a 'factory' of service rendering services in the form of transactions, loan processing or customer deposit account services. The approach is therefore recommended for bank branch performance, also, due to the fact that managers have limited control on making decision on financial mix (Berger & Humphrey, 1997)

It is important to note that different researchers select different measures using different approaches. There is no general consensus on the best approach to use in literature and the exact classification of deposit as input or outputs is even subject to controversy within a particular approach. For instance, although the intermediation approach is argued with deposit intake as input, it is too simplistic. Paradi, Rouatt, & Zhu (2011) argue that it is unfair to bank branches the classification of deposit as input since a significant amount of revenue is generated from deposits which further unfairly penalizes branches from taking in customers and their funds. Consequently, some studies consider deposit as an additional output in line with the value-added approach. In order to select the most appropriate bank features for this thesis, we follow Mostafa (2009) where 26 research papers done on the banking industry in different countries are surveyed. Reference is made to Toloo & Tichý (2015) on the percentage

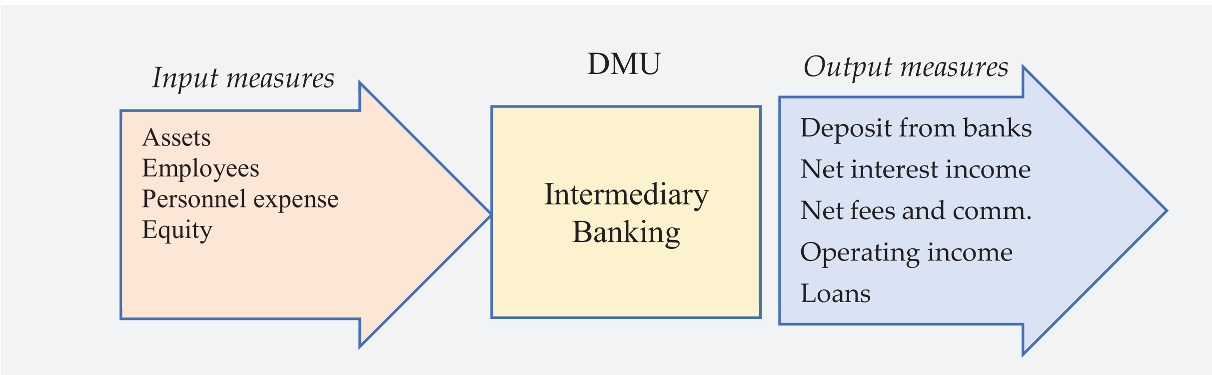


Figure 3.2. Input and output measures with banks as DMUs

of frequent selection of these banking measures presented in Mostafa (2009). Generally, employees are considered as an input variable and reasonably as fixed input. However, deposit is treated differently in banking studies; whiles 15.38% of research papers considered deposit as an input usually under the production approach, 26.92% of the surveyed papers measure it as output with the intermediate approach[27]. Figure 3.2 summarizes the approach adopted in this chapter. Assets, employees and equity are the most important measures considered as inputs and while loans, operating income and revenues (from interest and commissions) in addition to deposit are used here as output. Table 3.3 shows the descriptive statistics for the input and output measures. The specific sub regional descriptive details are also given in Appendix B. All the inputs and outputs variables are measured in millions of Euros. Employees - measured as the number of banking professionals and the non-banking staff is given in actual figures. As a result, the raw data are scaled for uniformity and to reduce round-off errors in the DEA models from excessively large values (Thanassoulis, 2001).

Table 3.3. Descriptive statistics for input and output measures

	Variables	Mean	SD	Min	Max
Inputs					
	Employees	14000	29277.42	217	193863
	Assets	140792.80	308512.70	10017.70	1994193
	Personnel expenses	1009.31	2270.90	14.2	16061
	Equity	8537.6	1046.2	226.3	100077
Outputs					
	Net interest revenue	1834.79	4001.07	52.3	33267
	Loans	66053.43	126218.9	305.6	758505
	Deposit from banks	18740.07	40327.94	58	263121
	Net fees & comm.	776.52	1714.09	6.6	12765
	Operating income	1220.27	2781.87	20.29	19805

In order to assess the performance and complexity of the RRDEA model compared to the RDEA model under the GRC, uncertainties compelling volatilities in banks specific variables were considered. Banking sector uncertainties may originate from forecast values of loans and deposit, missing values, and measurement errors, etc. A DMU_j is classified as uncertain if any of its inputs or outputs data is uncertain. Now we could consider the robust approach of Bertsimas & Sim (2004) to select the appropriate robust parameter Γ_j. For each DMU_j with $i \in I_j$ and $r \in R_j$, the percentage of perturbation, e_j of the nominal data is set to 0.01 and 0.05. For the choice of appropriate robustness level, it suffices to select Γ_j^x and Γ_j^y according to the number of uncertain input and output indices (see Sadjadi & Omrani, 2008, Omrani, 2013). Since the variable employee is given as fixed, there are three sources of uncertainties arising from the inputs and five sources of uncertainties for the output measures. To ensure full

[27] These performance measures are known as flexible measures (see Toloo, 2012; 2014) or dual-role factors (see Toloo & Barat, 2015; Toloo, Keshavarz, & Hatami-Marbini, 2018)

protection, Γ_j is set to 8; i.e. $\Gamma_j^x = 3$ and $\Gamma_j^y = 5$ which implies that the uncertain parameters are protected 100% taking their worst-case value in the uncertainty set.

3.6.2 Performance of the reduced robust DEA

As with Table 3.1, a computational comparison of the iterations counts of models under the GRC and RRC is conducted with the 250 DMUs. The goal of the comparison is to understand the numerical differences in the computational complexity of the RDEA and RRDEA as the size of the problem increases. To do this, we consider five independent groups including 50, 100, 150, 200, and 250 DMUs. The total number of iterations for each group is obtained by running the robust models (3.26) and (3.31) with CPLEX solver in GAMS. Table 3.4 shows the groups and average runtime result for each group. There are significant differences between the iteration used by the two models in solving the same problem

Table 3.4. Iteration counts

Groups	No. of DMUs	No. of iterations in RDEA	No. of iterations in RRDEA	Percent of reduction
1	50	333	232	30.3
2	100	591	404	31.6
3	150	819	469	42.7
4	200	1103	614	44.3
5	250	1306	704	46.1

For instance, suppose we run model (3.26) with 200 DMUs, the RDEA model requires 1103 iterations to generate the robust efficiency scores. On the other hand, the RRDEA model (3.31) requires only 614 iterations for the same number of DMUs, showing 44.3% reduction in the iterations count. It can be observed from column 3 that the robust model of under the GRC increases rapidly in the number of iterations as the number of DMU increases. This follows the exponential increase in the number of nonnegative variables and constraints that unwantedly increases the complexity of the model (De Klerk, 2008). Generally, from Theorem 3.1 it is noticeable that, particularly for large data set, the reduced robust model saves CPU time which by extension concludes a save in the computational cost of the reduced model to operations research problems with nonnegative decision variables.

3.6.3 Ranking of the Europeans banks

The proposed robust approach is applied to the ranking of banks in Europe to demonstrate the applicability of the RRDEA to efficiency measurement. The same observation used for the first group of DMUs is considered. The result from the classical DEA model and the robust

DEA models are reported in Table 3.5 which shows the efficiency scores and the rank of the DMUs (in bracket). A bank is efficient and operates on the efficient frontier under the VRS technology if its efficiency score is one. These efficient banks are shown in the 2nd column of Table 3.5 by the classical DEA model. There are several banks which are purely technical efficient as measured by the BCC model. Notice that the efficiency scores of the banks reduce in the case of the robust models.

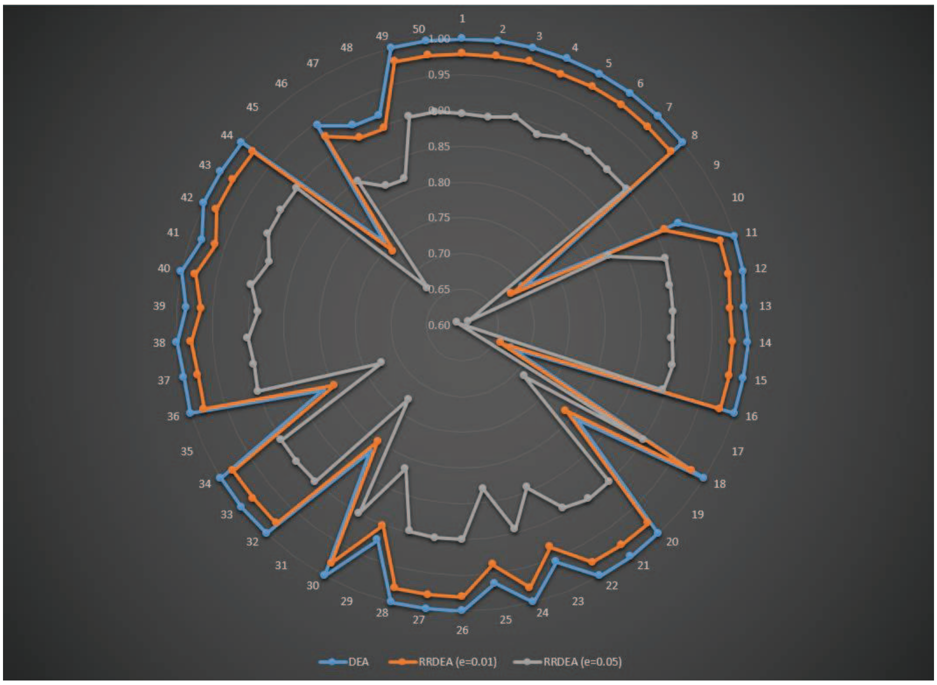


Figure 3.3. A plot of robust efficiency at different perturbation level

Under the robust technical efficiency assessment, considering the uncertainties in the data, we should know that the feasibility of the optimal solution as well as the execution time for the robust models, can be affected heavily by just a small perturbation of the data (see Ben-Tal & Nemirovski, 2000). The optimal solution decreases with each consideration of higher perturbation of the DEA input and output data. For this reason, the efficiency scores of the robust models are smaller than the efficiency of the classical DEA model (see Figure 3.3). In this chapter, the result of the robust efficiency is also reported for the full protection of uncertain inputs and outputs, i.e. $\Gamma_j = 8$ and $\Gamma_j^x = 3$ and $\Gamma_j^y = 5$. The result compares the efficiencies of the RRDEA and RDEA models. The last two pair of columns which are labeled "Robust DEA" and "Reduced robust DEA" practically validate our approach and illustrate that the robust models yield the same efficiency result at 1% and 5% perturbations of the uncertain data. The goal of reducing variables and constraints is therefore achieved without altering the optimal value and the information contemplated in decision making.

Notwithstanding the obtained result, the efficiency scores in both models decrease as the perturbation of the uncertain data increases (see Figure 3.3) Subsequently, the number of efficient banks in column 2 reduces when we trade-off optimality for performance. And so, only few banks are closed to efficient when perturbation of the uncertain data increases from 1% to 5%. The mean of the robust models at 1% and 5% perturbation is reported at 0.942, 0.861 respectively. For managers in the banking industry, this indicates a higher price to pay for robustness when uncertainty level increases (Bertsimas & Sim, 2004). Finally, per the output result not presented here, it is also observed that the execution time for solving each LP in the RDEA far exceeds the RRDEA.

Table 3.5. Ranking of banks based on the robust models

DMUs (Bank)	DEA	Robust DEA		Reduced robust DEA	
		e			
		0.01	0.05	0.01	0.05
1	1.000(1)	0.979	0.896	0.979(8)	0.896(15)
2	1.000(1)	0.979	0.893	0.979(11)	0.893(20)
3	1.000(1)	0.980	0.900	0.980(1)	0.900(6)
4	1.000(1)	0.977	0.886	0.977(13)	0.886(23)
5	1.000(1)	0.980	0.899	0.980(4)	0.899(9)
6	1.000(1)	0.980	0.900	0.980(1)	0.900(3)
7	1.000(1)	0.979	0.897	0.979(7)	0.897(12)
8	1.000(1)	0.980	0.898	0.980(5)	0.898(10)
9	0.700(16)	0.682	0.609	0.682(29)	0.611(36)
10	0.934(8)	0.912	0.826	0.912(21)	0.826(28)
11	1.000(1)	0.980	0.899	0.980(3)	0.899(8)
12	1.000(1)	0.979	0.896	0.979(8)	0.896(14)
13	0.995(2)	0.975	0.895	0.975(15)	0.895(17)
14	1.000(1)	0.979	0.894	0.979(10)	0.894(19)
15	1.000(1)	0.980	0.900	0.980(2)	0.900(7)
16	1.000(1)	0.979	0.895	0.979(9)	0.895(18)
17	0.676(17)	0.659	0.592	0.659(30)	0.592(37)
18	1.000(1)	0.980	0.900	0.980(1)	0.900(2)
19	0.807(14)	0.788	0.712	0.788(27)	0.713(34)
20	1.000(1)	0.980	0.900	0.980(1)	0.900(3)
21	1.000(1)	0.980	0.900	0.980(1)	0.900(5)
22	1.000(1)	0.978	0.892	0.978(12)	0.892(21)
23	0.956(6)	0.934	0.844	0.934(19)	0.844(26)
24	1.000(1)	0.979	0.895	0.979(8)	0.895(16)
25	0.964(5)	0.938	0.831	0.938(18)	0.831(27)

26	1.000(1)	0.980	0.900	0.980(1)	0.900(5)
27	1.000(1)	0.980	0.900	0.980(1)	0.900(5)
28	1.000(1)	0.980	0.898	0.980(6)	0.898(11)
29	0.923(9)	0.902	0.816	0.902(22)	0.816(31)
30	1.000(1)	0.980	0.900	0.980(1)	0.900(4)
31	0.819(12)	0.801	0.728	0.801(25)	0.728(32)
32	1.000(1)	0.980	0.900	0.980(1)	0.900(3)
33	1.000(1)	0.980	0.900	0.980(1)	0.900(3)
34	1.000(1)	0.980	0.900	0.980(1)	0.900(6)
35	0.816(13)	0.798	0.725	0.798(26)	0.725(33)
36	1.000(1)	0.980	0.900	0.980(1)	0.900(2)
37	0.996(2)	0.976	0.896	0.976(14)	0.896(13)
38	1.000(1)	0.980	0.900	0.980(1)	0.900(2)
39	0.987(3)	0.966	0.886	0.966(16)	0.888(22)
40	1.000(1)	0.980	0.900	0.980(1)	0.900(6)
41	0.983(4)	0.963	0.884	0.963(17)	0.884(24)
42	1.000(1)	0.980	0.900	0.980(1)	0.900(5)
43	1.000(1)	0.980	0.900	0.980(1)	0.900(6)
44	1.000(1)	0.980	0.900	0.980(1)	0.900(2)
45	0.759(15)	0.742	0.671	0.742(28)	0.671(35)
46	0.945(7)	0.926	0.848	0.926(20)	0.848(25)
47	0.918(10)	0.899	0.822	0.899(23)	0.822(29)
48	0.915(11)	0.896	0.820	0.896(24)	0.820(30)
49	1.000(1)	0.980	0.901	0.980(1)	0.901(1)
50	1.000(1)	0.980	0.900	0.980(1)	0.900(3)
Mean	0.962	0.942	0.861	0.942	0.861
SD	0.080	0.079	0.076	0.079	0.076
Min	0.676	0.659	0.591	0.659	0.591
Max	1.000	0.980	0.901	0.980	0.901

3.7 Concluding remarks

Robust counterpart optimization providing a general solution for decision variables has been the traditional way to study problems in operations research involving data uncertainty. However, in most practical problems where decision variables are nonnegative, the existing robust models present 'unwanted variables' that consume computational space particularly for large data set. The goal pursued in this chapter is to offer alternative robust counterparts for nonnegative decision variables. The chapter proposes reduced robust counterpart that attempts to minimize problem complexity without altering the optimality of the original solution.

In the DEA, whiles the decision variables are nonnegative, we find that the initial authors who proposed the robust DEA (Sadjadi & Omrani, 2008) and hence subsequent researchers consider the original formulation in robust optimization where the decision variables can be negative (free in sign). We have shown in this chapter that, such formulation involves many redundant constraints and decision variables which significantly increases the complexity of the robust DEA models and, of course, the required space and time for running the models. Addressing first the issue of infeasibility of simultaneous uncertainties in the DEA normalization constraint, we adjust the equality constraint in the normalization of the multiplier DEA models to inequality in order to allow for feasible and simultaneous consideration of uncertainties in the inputs and outputs data. Our complexity analysis using data from 250 European banks indicate that the proposed reduced robust DEA model renders some variables and constraints redundant in the RDEA models and reduces significantly the complexity in solving the same problem. In addition, almost 50% of the average reduction in iterations is found to save computational space. The proposed model while saving computational cost with problems with nonnegative decision variables also preserve the optimality of the original solution. By extension, the reduced robust counterpart for nonnegative decision variables can be used in many operations research applications such as portfolio analysis, supply chain management, and banking industry. Similar applications using the reduced robust DEA for the output orientation model and other uncertainty sets can be considered for further research in the future.

Chapter 4: Duality, classification and input – and output – orientations in robust DEA

Summary

The robust DEA has emerged as a fast-growing research area in operations research and has proven to be a useful tool in assessing managerial efficiency and productivity under uncertainty. While strong duality relations hold between the multiplier (primal) and envelopment (dual) in the DEA, the issue of rising importance is the relationship when uncertainty in data is introduced. In this chapter, the focus is to study through duality relations the link between the multiplier and envelopment models and among input- and output-orientation models in robust DEA. The groundwork is established with a robust fractional DEA which yields proper robust efficiency score for benchmarking of DMUs. For the proposed models, we provide a scheme for the classification of DMUs into fully robust efficient, partially robust efficient and robust inefficient. An application is made to study the performance of some banks in Germany.

4.1 Introduction

The underlying traditional DEA models estimate efficiency with precise data, which is given in either input – oriented multiplier form or its dual form - the envelopment model or the output-oriented multiplier model or its envelopment model. The assumption of precise data in the traditional DEA models presupposes that a fixed measure of efficiency can be obtained with the efficient frontier from the exact/precise amount of inputs and outputs (Park, 2010). Therefore, a DMU benchmarked as efficient is ranked based on its relative importance to the other DMUs with exact data. However, as aforementioned in previous chapters, benchmarking with precise data poses many challenges in the real-world setting since inputs and outputs of some/all DMUs are measured with noise and very uncertain. These uncertainties can displace the DEA efficiency frontier and affect the feasibility and optimality of the DEA model. In fact, the unique optimality of the primal-dual relationship among the DEA models can also be affected.

Duality is an important linear programming concept in DEA which offer flexibility in explaining the operational meaning of efficiency and of Pareto optimality in primal and dual spaces. The dual spaces in which efficiency is measured are those of *production* and of *value*. Therefore, by making it possible to switch from a production to a value-based context of efficiency assessment and vice versa, duality relations in DEA provides a natural link to and rationalization for the traditional productivity indexes (Thanassoulis, 2001). In the standard DEA context, the envelopment model (in the *production space*) is equivalent to the multiplier model (in the *value space*) which yields the DEA projection and the efficiency due to the linear programming duality[28]. However, under the DEA analysis with uncertain input and output data, such duality may not lead to a particular pair of robust multiplier and envelopment models, where frontier projections and divisional efficiency scores are generated in a single robust DEA model. This and indeed other advanced DEA models have their own primal-dual concepts and relationships. In some instances, the relationship is however not clear and contesting in literature. For instance, Chen et al, (2013) discussed network DEA pitfall and argued that a unique optimal solution does not necessarily exist between network DEA models built on multiplier and envelopment models. Lim and Zhu (2015) recently counter-argued this claim and demonstrated that duality in the standard DEA naturally migrates to the two-stage network DEA. Park (2010) studied the duality relations among multiplier and envelopment models of IDEA. He revealed that duality gap and hence efficiency gap exist between the dual pair models of DEA when imprecise data is incorporated. In this Chapter, a prime objective is to examine the duality relationship in the robust DEA case, herein from both the worst-case approach and the best-case approach.

Within the worst-case robust DEA approach, we also study the possibility of scheme that can interpret robust efficiency and classify DMUs into different robust classes. The robust efficiency of DMUs has been analyzed and interpreted differently in literature. Prior to the RDEA, research work on the deterministic analysis of uncertainty in DEA took the general discussion of Imprecise DEA (IDEA) (see also research work using fuzzy DEA from Hatami-Marbini et al., 2011). The IDEA proposed by Cooper et al (1999) deals with imprecise information in the input and output data such as data in bounded (interval), ratio or ordinal form. To provide a classification scheme for DMUs with imprecise data, Despotis & Smirlis (2002) suggested a variable transformation that treats interval DEA as a peculiar case of DEA with exact data. They provide lower and upper bound for efficiency score of DMUs and discriminate DMUs into fully efficient, efficient and inefficient units. Entani et al. (2002) provides a slightly different situation in which the efficiency of a mixture of interval and fuzzy

[28] Duality in linear programming is discussed thoroughly in Appendix A.1.

data are measured from both pessimistic and optimistic viewpoints. Park (2010) research into duality IDEA models makes two definitions: perfect efficiency and potential efficiency based on whether the optimal efficiency value of one is obtained for all the imprecise data values or some in each DMU. Wei & Wang (2017) classify DMUs with imprecise data into perfectly robust efficient, potentially robust efficient and robust inefficient units based on the lower and upper bounds formulated from the multiplier and envelopment models. These bounds according to the authors also represent the worst-case and best-case scenarios for the imprecise data.

In general, the robust classification of DMUs and its attendant meaning provides effective ranking strategies appealing to the risk preference level of the DM. Notwithstanding its significance, no attempt has been made on this issue in the RDEA. It is, therefore, the purpose of this chapter to extend the methodology in RDEA and examine duality relations and unit classification when the input and output data are uncertain. The main contributions of this chapter are in five folds:

1. we study duality relations and develop models that will establish a relationship between robust multiplier and envelopment models in the context of worst-case scenarios.
2. we prove that the normalization equation in the output-oriented model can be replaced by "≥". This is analogous to Theorem 3.2 for the BCC model in which it was showed that the traditional normalization constraint in the multiplier form of the input-oriented model which is in an equation form can be equivalently substituted by "≤".
3. More importantly, the chapter designs a classification scheme that utilizes the conservativeness of the decision maker and interprets robust efficiency solutions accordingly. Practically, in most cases, determining the robustness of the solution may depend on the risk preferences of the decision maker. Thus, unlike the classification scheme provided in Despotis & Smirlis (2002) and that of Park (2010) and Wei & Wang (2017) for the imprecise DEA, the proposed robust classification scheme is based on the conservativeness level of the DM and is further applicable and flexible to the risk preference of the DM.
4. The next contribution considers the characterization of the efficiency of solutions for all DMUs in both input and output orientations. The relationship between the robust input – oriented models and robust output – oriented models is thus studied.
5. Finally, a computational experiment is conducted with the proposed models using data on some banks in Germany. The result while validating the proposed models also provides an empirical estimation of the operational efficiency of banks in Germany under data uncertainty.

Structure of the chapter. In Section 2, we provide the motivation and background of the traditional DEA models and introduce a new output – oriented multiplier CCR model feasible for robust analysis, followed by introducing a robust fractional DEA programming. In Section 3, a robust classification scheme under the worst-case DEA model is proposed. The section also studies the multiplier and envelopment models in the worst-case and best-case scenario in the input – orientation. The same analysis is done with the output – orientation in Section 4. In Section 5, we present an application with the performance of some banks in Germany to demonstrate the efficacy of the proposed models. Finally, concluding remarks are made in Section 6.

4.2 The DEA models and motivation

Suppose there are n DMUs indexed as DMU_j $(j = 1, \dots, n)$ where each unit consumes m inputs $\boldsymbol{x}_j = (\dots, x_{ij}, \dots); i \in I = \{1, \dots, m\}$ to produce s outputs $\boldsymbol{y}_j = (\dots, y_{rj}, \dots); r \in R = \{1, \dots, s\}$. All inputs and outputs for all DMUs are non-negative and a DMU has at least one positive input and one positive output. For simplicity, we restate the CCR models with the given names below. The multiplier and envelopment (primal and dual) forms of the CCR model measuring the input-oriented technical efficiency of DMUs are expressed as the following, respectively:

IMCCR

$$
\begin{aligned}
&\theta^* = \max \textstyle\sum_{r=1}^{s} u_r y_{ro} \\
&\text{s.t.} \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} = 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad && j = 1, \dots, n \\
&v_i \ge 0 && i = 1, \dots, m \\
&u_r \ge 0 && r = 1, \dots, s
\end{aligned}
\tag{4.1}
$$

IECCR

$$
\begin{aligned}
&\min \theta \\
&\text{s.t} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} \quad && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r = 1, \dots, s \\
&\lambda_j \ge 0 && j = 1, \dots, n
\end{aligned}
\tag{4.2}
$$

Here, the v_i and u_r are the weights assigned to the i^{th} input and r^{th} output, respectively. DMU_o is CCR-efficient if $\theta^* = 1$, otherwise, it is CCR-inefficient. The technical efficiency measured by the output-oriented CCR models can be measured by the following:

OMCCR

$$
\begin{aligned}
&\varphi^* = \min \textstyle\sum_{i=1}^{m} v_i x_{io} \\
&\text{s.t.} \\
&\textstyle\sum_{r=1}^{s} u_r y_{ro} = 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad && j = 1, \dots, n \\
&v_i \ge 0 && i = 1, \dots, m \\
&u_r \ge 0 && r = 1, \dots, s
\end{aligned}
\tag{4.3}
$$

OECCR

$$
\begin{aligned}
&\max \varphi \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} \quad && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi y_{ro} && r = 1, \dots, s \\
&\lambda_j \ge 0 && j = 1, \dots, n
\end{aligned}
\tag{4.4}
$$

It is well established that the CCR input-oriented and output-oriented models are equivalent**.** To be more precise at the optimality we have $\varphi^* = 1/\theta^*$. In what follows, we are motivated by the relationships between models (4.1) - (4.4) when the input and output data are uncertain, using the robust optimization approach. The robust extension of the IMCCR model was given in Sadjadi & Omrani (2008) to assess the robust technical efficiency of utility companies. The IECCR model was extended to measure the robust super-efficiency by Sadjadi et al (2011b). Lu (2015) also studied the robust algorithm performance with the OMCCR model (4.3) under output uncertainty. However, Lu (2015) robust DEA model considers uncertainty in the normalization constraint which is to be avoided. The underlying modeling problem is that uncertainty analysis in equality constraint, in this case, the normalization constraint may lead to infeasibility issues since such constraint restricts the feasibility region (see Ben-Tal et al, 2009, Chapter 2). This is also generally the case for all DEA models which involve equality in a constraint or slack variables. Naturally, a solution which is feasible in the robust DEA sense requires inequality in the normalization constraint of models (4.1) and (4.3). Salahi et al. (2016) dealt with this issue by converting equality in the normalization constraint of model (4.1) to double inequality constraints. Omrani (2013) instead replaced the normalization constraint $\sum_{i=1}^{m} v_i x_{io} = 1$ with a superfluous constraint $\sum_{r=1}^{s} u_r - \sum_{i=1}^{m} v_i = 1$ for his common weight robust DEA model in order to avoid constraint infeasibility for the input uncertainty. To overcome such equality problem in model (4.1), in this chapter, we use the following model which is earlier proposed by Toloo (2014)a:

$$
\begin{array}{lr}
\max z & \\
\text{s.t.} & \\
z - \sum_{r=1}^{s} u_r y_{ro} \leq 0 & \\
\sum_{i=1}^{m} v_i x_{io} \leq 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad \forall j & \\
v_i \geq 0 \qquad\qquad\qquad\qquad\qquad\quad \forall i & \\
u_r \geq 0 \qquad\qquad\qquad\qquad\qquad\quad \forall r &
\end{array}
\tag{4.5}
$$

Toloo (2014)a proved that model (4.5) is the equivalent to the input-oriented model (4.1). Indeed, model (4.5) facilitate the efficiency measurement of both inputs and outputs data uncertainty and recently was adopted by Arabmaldar et al. (2017) to measure the robust super-efficiency of DMUs. To similarly deal with the output uncertainty issue in the normalization constraint of the OMCCR model (4.3), we consider the following theorem:

Theorem 4.1. *The following model is equivalent to the OMCCR model*:

$$
\begin{array}{ll}
\min \sum_{i=1}^{m} v_i x_{io} & \\
\text{s.t.} & \\
\sum_{r=1}^{s} u_r y_{ro} \geq 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & j = 1, \dots, n \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s
\end{array}
\tag{4.6}
$$

Proof. Consider the following dual of model (4.6):

$$
\begin{array}{ll}
\max \varphi & \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} & i = 1, \dots, m \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{ro} & r = 1, \dots, s \\
\lambda_j \geq 0 & j = 1, \dots, n \\
\varphi \geq 0 &
\end{array}
\tag{4.7}
$$

We show that the nonnegativity constraint $\varphi \geq 0$ is redundant. According to the strong duality property in linear programming (see Bazaraa, Jarvis, & Sherali, 2010), the primal and dual optimal objective values are equal. In other words, let $(\boldsymbol{v}^*, \boldsymbol{u}^*) \in \mathbb{R}^{m+s}$ and $(\varphi^*, \boldsymbol{\lambda}) \in \mathbb{R}^{1+n}$ be the optimal solution of models (4.3) and (4.6), respectively, then we can conclude that $\varphi^* = \sum_{i=1}^{m} v_i^* x_{io}$. In addition, from the first and second constrains set of model (4.6) we obtain that $1 \leq \sum_{r=1}^{s} u_r y_{rj} \leq \sum_{i=1}^{m} v_i x_{ij}$ which means for any feasible solution $(\boldsymbol{v}, \boldsymbol{u})$, including the optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$, we have $\sum_{i=1}^{m} v_i x_{ij} \geq 1$. As a result, we arrive at $(\sum_{i=1}^{m} v_i^* x_{io} =)\ \varphi^* \geq 1$ which proves that the last nonnegativity constraint in model (4.7) is non-geometrically[29] redundant. Hence, models (4.4) and (4.7) are equivalent and we can conclude that their duals (models (4.3) and (4.6)) are equivalent too which completes the proof. ☐

4.3 The Input-oriented robust DEA models

We devote this section to studying the robust efficiency of DMUs with input – orientation. The approach is based on the worst-case robust efficiency which is a conservative approach concerned with a guaranteed level of performance for all feasible realization of uncertain inputs and outputs in an uncertainty set. As a result, it reflects the worst-possible performance of a DMU. Later in the section, we shall provide a classification scheme for the robust efficiency under different conservatism level. In Section 3.2, we study the link between the robust multiplier and envelopment models.

In order to develop an explicit expression of the robust CCR model in the multiplier form, we consider model (4.5) and the variables $\tilde{x}_{ij}, \tilde{y}_{rj}$ taking values in the symmetric intervals $[x_{ij} - \hat{x}_{ij}, x_{ij} + \hat{x}_{ij}]$ and $[y_{rj} - \hat{y}_{rj}, y_{rj} + \hat{y}_{rj}]$. Then, using the fact that the random variations in the data are modeled such that $\sum_{i \in I} \eta_{ij}^x \leq \Gamma_j^x, |\eta_{ij}^x| \leq 1\ \forall i$, $\sum_{r \in R} \eta_{rj}^y \leq \Gamma_j^y, |\eta_{rj}^y| \leq 1, \forall r$ (see Bertsimas & Sim, 2004), it is sufficient to write the input-oriented multiplier robust CCR (IMRCCR) model according to the theorem below:

Theorem 4.2. *The IMRCCR model is expressed as the following:*

[29] A constraint is geometrically redundant if and only if the feasible region is identical with or without the constraint. A non-geometrically redundant constraint is a constraint which is redundant only at the optimality.

$$
\begin{aligned}
& I_m^* = \max z && \\
& \text{s.t.} && \\
& z - \textstyle\sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y + \sum_{r\in R_o} q_{ro} \leq 0 && \\
& \textstyle\sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x + \sum_{i\in I_o} w_{io} \leq 1 && \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + p_j^x \Gamma_j^x + p_j^y \Gamma_j^y + \sum_{r\in R_j} q_{rj} + \sum_{i\in I_j} w_{ij} \leq 0 && \forall j \\
& p_j^y + q_{rj} \geq u_r \hat{y}_{rj} && \forall j, \forall r \in R_j \qquad (4.8) \\
& p_j^x + w_{ij} \geq v_i \hat{x}_{ij} && \forall j, \forall i \in I_j \\
& q_{rj}, w_{ij} \geq 0 && \forall j, \forall i \in I_j, \forall r \in R_j \\
& p_j^x, p_j^y \geq 0 && \forall j \\
& v_i \geq 0 && \forall i \\
& u_r \geq 0 && \forall r
\end{aligned}
$$

where I_m^* is the IMR-efficiency score of DMU_o; p_j^x, q_{rj}, and p_j^y, w_{ij} are nonnegative variables correspond to the set of uncertain inputs and outputs respectively, and Γ_o^x and Γ_o^y are the respective robust parameters of the uncertain inputs and outputs of DMU_o.

Proof. Consider the following NLP model which is arrived at by substituting $\tilde{x}_{ij} = x_{ij} + \eta_{ij}^x \hat{x}_{ij}$ and $\tilde{y}_{rj} = y_{rj} + \eta_{rj}^y \hat{y}_{rj}$ into model (4.5):

$$
\begin{aligned}
& \max z && \\
& \text{s.t.} && \\
& z - \textstyle\sum_{r=1}^{s} u_r \left(y_{ro} + \eta_{ro}^y \hat{y}_{ro}\right) \leq 0 && \\
& \textstyle\sum_{i=1}^{m} v_i (x_{io} + \eta_{io}^x \hat{x}_{io}) \leq 1 && \\
& \textstyle\sum_{r=1}^{s} u_r \left(y_{rj} + \eta_{rj}^y \hat{y}_{rj}\right) - \sum_{i=1}^{m} v_i \left(x_{ij} + \eta_{ij}^x \hat{x}_{ij}\right) \leq 0 && \forall j \qquad (4.9) \\
& -1 \leq \eta_{ij}^x \leq 1 && \forall j, \forall i \in I_j \\
& -1 \leq \eta_{rj}^y \leq 1 && \forall j, \forall r \in R_j \\
& v_i \geq 0 && \forall i \\
& u_r \geq 0 && \forall r
\end{aligned}
$$

The model can be equivalently written as follows:

$$
\begin{aligned}
& \max z && \\
& \text{s.t.} && \\
& z - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \sum_{r\in R_j} u_r \eta_{ro}^y \hat{y}_{ro} \leq 0 && \\
& \textstyle\sum_{i=1}^{m} v_i x_{io} + \sum_{i\in I_j} v_i \eta_{io}^x \hat{x}_{io} \leq 1 && \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{r\in R_j} u_r \eta_{rj}^y \hat{y}_{rj} - \sum_{i\in I_j} v_i \eta_{ij}^x \hat{x}_{ij} \leq 0 && \forall j \qquad (4.10) \\
& -1 \leq \eta_{ij}^x \leq 1 && \forall j, \forall i \in I_j \\
& -1 \leq \eta_{rj}^y \leq 1 && \forall j, \forall r \in R_j \\
& v_i \geq 0 && \forall i \\
& u_r \geq 0 && \forall r
\end{aligned}
$$

The robust counterpart of model (4.10) is the following:

$$
\begin{aligned}
&\max z\\
&\text{s.t.}\\
&z-\textstyle\sum_{r=1}^{s}u_r y_{ro}+\max\limits_{\substack{\sum_{i\in I_o}\eta_{ro}^{y}\le\Gamma_j^{y}\\ 0\le\eta_{ro}^{y}\le 1}}\left\{\textstyle\sum_{r\in R_j}u_r\eta_{ro}^{y}\hat{y}_{ro}\right\}\le 0\\
&\textstyle\sum_{i=1}^{m}v_i x_{io}+\max\limits_{\substack{\sum_{i\in I_o}\eta_{io}^{x}\le\Gamma_j^{x}\\ 0\le\eta_{io}^{x}\le 1}}\left\{\textstyle\sum_{i\in I_j}v_i\eta_{io}^{x}\hat{x}_{io}\right\}\le 1\\
&\textstyle\sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}+\max\limits_{\substack{\sum_{r\in R_j}\eta_{rj}^{y}\le\Gamma_j^{y}\\ \sum_{i\in I_j}\eta_{ij}^{x}\le\Gamma_j^{x}\\ \eta_{rj}^{y},\eta_{ij}^{x}\in[0,1]}}\left\{\textstyle\sum_{r\in R_j}u_r\eta_{rj}^{y}\hat{y}_{rj}+\sum_{i\in I_j}v_i\eta_{ij}^{x}\hat{x}_{ij}\right\}\le 0 \qquad \forall j\\
&v_i\ge 0 \qquad \forall i\\
&u_r\ge 0 \qquad \forall r
\end{aligned}
\tag{4.11}
$$

Note that all the weights and input-output data are non-negative and hence we could consider η_{ij}^{x} and η_{rj}^{y} as positive variables in the inner-problems of model (4.11). Using the dual variables p_j^{y} and q_{rj} for the output and p_j^{x} and w_{ij} for the input in the inner–problem of the third constraints set, we arrive at the following equivalent linear model[30]:

$$
\begin{aligned}
&\min \textstyle\sum_{r\in R_j}q_{rj}+\sum_{i\in I_j}w_{ij}+p_j^{y}\Gamma_j^{y}+p_j^{x}\Gamma_j^{x}\\
&s.t\\
&\mathrm{p}_j^{y}+q_{rj}\ge u_r\hat{y}_{rj} \qquad \forall r\in R_j\\
&\mathrm{p}_j^{x}+w_{ij}\ge v_i\hat{x}_{ij} \qquad \forall i\in I_j\\
&q_{ij},w_{ij}\ge 0 \qquad \forall i\in I_j,\forall r\in R_j\\
&p_j\ge 0
\end{aligned}
\tag{4.12}
$$

A similar formulation for the first and second constraints sets and subsequent substitutions into model (4.11) completes the proof. ☐

It should be mentioned that the IMRCCR model (4.8) incorporates variables that protect the objective function and constraints in evaluating the robust efficiency of the DMUs. The variables (p_j^{x},p_j^{y}) and (q_{rj},w_{ij}) quantify the sensitivity of the inputs and outputs data to infinitesimal changes in the level of conservativeness. The quantities $p_j^{y}\Gamma_j^{y}+\sum_{r\in R_j}q_{rj}$ and $p_j^{x}\Gamma_j^{x}+\sum_{i\in I_j}w_{ij}$ represent the worst-case deviations of the uncertain outputs and inputs from their nominal values subject to the budget of uncertainty. Model (4.8) therefore involves $n+1+n\left(\sum_{j=1}^{n}|I_j|+\sum_{j=1}^{n}|R_j|\right)$ constraints, and $m+s$ weights and $2n+n\left(\sum_{j=1}^{n}|I_j|+\sum_{j=1}^{n}|R_j|\right)$ robust decision variables in each n instance of obtaining a solution. The robust model maximizes the weights (objective function) with respect to the worst-case perturbation of the

[30] Note that in the inner-problem it is assumed that $v_i,\forall i$ and $u_r,\forall r$ are constants that turns the problem to an LP model.

inputs and outputs data in the uncertainty set with a pessimistic view. The dual expression of the IMRCCR model (4.8) is given below:

$$
\begin{array}{ll}
I_d^* = \min \theta & \\
\text{s.t} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} - \sum_{j \in J_i^x} \gamma_{ij}^x \hat{x}_{ij} \leq \theta x_{io} & \forall i \in I \\
\sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{j \in J_r^y} \gamma_{rj}^y \hat{y}_{rj} \geq y_{ro} & \forall r \in R \\
\lambda_o \Gamma_o^x - \sum_{i \in I_o} \gamma_{io}^x + \theta \Gamma_o^x \geq 0 & \\
\lambda_j \Gamma_j^x - \sum_{i \in I_j} \gamma_{ij}^x \geq 0 & \forall j \neq o \\
\lambda_o \Gamma_o^y - \sum_{r \in R_o} \gamma_{ro}^y \geq -\Gamma_o^y & \\
\lambda_j \Gamma_j^y - \sum_{r \in R_j} \gamma_{rj}^y \geq 0 & \forall j \neq o \\
\lambda_o - \gamma_{io}^x + \theta \geq 0 & \forall i \in I_o \\
\lambda_j - \gamma_{ij}^x \geq 0 & \forall j \neq o, \forall i \in I_j \\
\lambda_o - \gamma_{ro}^y + 1 \geq 0 & \forall r \in R_o \\
\lambda_j - \gamma_{rj}^y \geq 0 & \forall j \neq o, \forall r \in R_j \\
\gamma_{ij}^x \geq 0 & \forall j, \forall i \in I_j \\
\gamma_{rj}^y \geq 0 & \forall j, \forall r \in R_j \\
\lambda_j \geq 0 & \forall j
\end{array}
\tag{4.13}
$$

where J_i^x and J_r^y are introduced indices for the uncertain inputs and outputs in the envelopment model (here and after); θ is the technical efficiency score of DMU_o; γ_{ij}^x and γ_{rj}^y are nonnegative variables corresponding to the set of uncertain inputs and outputs respectively. Note that ($I_m^* = I_d^*$) by the strong duality theorem. The following proposition states that the robust efficiency score obtained in models (4.8) and (4.13) for each DMU j is less than or equal to one.

Proposition 4.1. $I_m^* \leq 1$

Proof. From the first constraint of model (4.8) $z \leq \sum_{r=1}^{s} u_r y_{ro} - p_o^y \Gamma_o^y - \sum_{r \in R_o} q_{ro}$. Since p_o^y and $q_{ro} \forall r \in R_o$ are nonnegative variables we have $\sum_{r=1}^{s} u_r y_{ro} - p_o^y \Gamma_o^y - \sum_{r \in R_o} q_{ro} \leq \sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y + \sum_{r \in R_o} q_{ro}$. Now, considering the second constraint and the third constraints set for $j = o$ we arrive at $\sum_{r=1}^{s} u_r y_{ro} + p_o^y \Gamma_o^y + \sum_{r \in R_o} q_{ro} \leq \sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x + \sum_{i \in I_o} w_{io} \leq 1$. Henceforth, $z \leq 1$ and the fact that the model is a maximization problem leads to $I_m^* \leq z$ which completes the proof. □

From Proposition 4.1, DMU_o is robust efficient or simply denoted R-efficient if $I_m^* = 1$. Otherwise, it is R- inefficient. The interpretation of the R-efficiency in the IMRCCR model (4.8) is given as the radial contraction rate of input levels of a DMU in order to reach the robust efficiency frontier (Wei & Wang, 2017). Similarly, the optimal solution of model (4.13) for the DMU_o apprises the minimum θ^* that decreases the input vector $\boldsymbol{x}_o$ radially to $\theta^* \boldsymbol{x}_o$ while restricting the deviation $(\hat{x}_{ij}, \hat{y}_{rj})$ of the uncertain inputs and outputs. Note that in each of the models, the efficiency score of the DMUs are preserved by the robust frontier since the PPS_r underlying the robust models are protected by the budget of uncertainty.

4.3.1 A classification scheme for DMUs

According to the IMRCCR model (4.8), an evaluated DMU adjust not only to the weights but also the conservativeness of the DM. A DMU can be partition into three main robust efficiency classes: a fully robust efficient, partially robust efficient, and robust inefficient units. The classification of the DMUs is determined based on the level of conservativeness of the DM that is controlled by the robust parameters, Γ_j^x, and Γ_j^y which are defined by the budget of uncertainty or the number of uncertain inputs and outputs. In the RDEA, we wish to control the level of conservativeness so that a reasonable trade-off between DMUs performance and robustness can be achieved. Let $\Gamma_j = \Gamma_j^x + \Gamma_j^y$, the values of Γ_j ranges from 0 to $|I_j| + |R_j|$ where $\Gamma_j = |I_j| + |R_j|$ admit the highest protection for the uncertain inputs and outputs because all the uncertain data are protected against. The first step in classifying DMUs under the IMRCCR model (4.8) is to determine the number of uncertain inputs and outputs for the budget of uncertainty.

$\Gamma_j = 0$ $\qquad$ $0 < \Gamma_j < |I_j| + |R_j|$ $\qquad$ $\Gamma_j = |I_j| + |R_j|$

Figure 4.1. Different degree of conservativeness

Figure 4.1 shows the different degree of conservativeness corresponding to the budget of uncertainty. Note that the optimal objective value z^* decreases as Γ_j increases, hence fewer units are expected to be robust efficient as Γ_j increases. The following definition provides a formal classification scheme for the DMUs in the robust DEA setting.

Definition 4.1. Let Γ_j takes values in the interval $[0, |I_j| + |R_j|]$ in model (4.8). Referencing to Proposition 4.1 the DMUs are classified into the following sets:

- (*Full robust efficiency*). DMU_o is fully R-efficient if and only if $I_m^* = 1$ when $\Gamma_j = |I_j| + |R_j|, \forall j$.
- (*Partial robust efficiency*). DMU_o is partially R- efficient if $I_m^* < 1$ when $\Gamma_j = |I_j| + |R_j|$ and there exist $\Gamma_j > 0$ such that $I_m^* = 1$

- (*Robust inefficiency*). A DMU$_o$ is said to be R- inefficient if $I_m^* < 1$ for all $\Gamma_j \in (0, |I_j| + |R_j|]$.

The above efficiency classification can be denoted as RE^{++} (full R-efficiency), RE^{+}(partial robust efficiency or PR-efficiency) and RE^{-} (R-inefficiency). The set RE^{++} consist of DMUs that are robust R-efficient in any combination of uncertain inputs and outputs at all robust level defined by the DM. This category of efficient DMUs is obtained under the most conservative evaluation of the uncertain data. So, logically, a DMU is robust efficient if and only if it is fully robust efficiency. The set RE^{+} consists of DMUs that are R-efficient at maximal sense but there are certain conservative levels for inputs and outputs combinations for which they cannot maintain R-efficiency. The PR-efficiency is therefore obtained in a less stringent manner than the full robust efficiency and as a result, the latter outperforms the former for all uncertain input and output data. Finally, the set RE^{-} consists of R-inefficient DMUs for all input and output combinations. It must also be noted that a unit can be efficient without being R-efficient or PR-efficient. The contrary is false.

4.3.2 Primal-Dual relationship: input orientation

The issue of rising importance in the RDEA is the relationship between the dual–pair of the DEA models, i.e. the multiplier and envelopment forms, when one incorporates uncertainty and builds a robust model either from the pessimistic or optimistic viewpoints. A prior knowledge generally exists in IDEA models (see Park, 2010) and Interval DEA models (see Entani et al., 2002) which presume that the presence of imprecise data invalidates the linear duality principle as well as create efficiency gap between the multiplier IDEA models and primal IDEA models. In the traditional DEA, it is known that IMCCR and IECCR are mutually dual. We will verify whether this property remains the same for the corresponding robust models. In order to do that, we introduce the following Input–oriented Envelopment robust CCR (IERCCR) model for the IECCR model (4.2) which, similar to the IMRCCR model (4.8), considers a worst-case formulation.

Theorem 4.3. *The IECCR model* (4.2) *with the worst-case criterion is equivalent to the following model*:

$$
\begin{array}{ll}
I_e^* = \min\theta & \\
\text{s.t.} & \\
\sum_{j=1}^n \lambda_j x_{ij} - \theta x_{io} + \sum_{j\in J_i^x}\delta_{ij}^x + \rho_i^x \Upsilon_i^x \le 0 & \forall i \in I \\
\sum_{j=1}^n \lambda_j y_{rj} - y_{ro} - \sum_{j\in J_r^y}\delta_{rj}^y - \rho_r^y \Upsilon_r^y \ge 0 & \forall r \in R \\
\rho_i^x + \gamma_{ij}^x \ge \lambda_j \hat{x}_{ij} & \forall i \in I,\ \forall j \in J_i^x (j \ne o) \\
\rho_i^x + \gamma_{io}^x \ge \alpha_o \hat{x}_{io} & o \in J_i^x, \forall i \in I \\
\rho_r^y + \gamma_{rj}^y \ge \lambda_j \hat{y}_{rj} & \forall r \in R, \forall j \in J_r^y (j \ne o) \\
\rho_r^y + \gamma_{ro}^y \ge \beta_o \hat{y}_{ro} & o \in J_r^y, \forall r \in R \\
-\alpha_o \le \lambda_o - \theta \le \alpha_o & \\
-\beta_o \le \lambda_o - 1 \le \beta_o & \\
\alpha_o, \gamma_{ij}^x, p_i^x \ge 0 & \forall i \in I, \forall j \in J_i^x \\
\beta_o, \gamma_{rj}^y, p_r^y \ge 0 & \forall r \in R, \forall j \in J_r^y \\
\lambda_j \ge 0 & \forall j
\end{array}
\tag{4.14}
$$

Proof. First, note that the IECCR model (4.2) equivalent to the formulation:

$$
\begin{array}{ll}
\min\theta & \\
\text{s.t.} & \\
\sum_{j=1(j\ne o)}^n \lambda_j x_{ij} + (\lambda_o - \theta) x_{io} \le 0 & \forall i \\
\sum_{j=1(j\ne o)}^n \lambda_j y_{rj} - (1 - \lambda_o) y_{ro} \ge 0 & \forall r \\
\lambda_j \ge 0 & \forall j
\end{array}
\tag{4.15}
$$

The pessimistic robust counterpart leads to the following model:

$$
\begin{array}{ll}
\min\theta & \\
\text{s.t.} & \\
\sum_{j=1}^n \lambda_j x_{ij} - \theta x_{io} + \max\limits_{\substack{\sum_{j\in J_i^x}\xi_{ij}^x \le \Upsilon_i^x \\ 0\le\xi_{ij}^x\le 1\ (\forall j\in J_i^x)}} \left\{ \sum_{\substack{j\in J_i^x \\ j\ne o}} \lambda_j \xi_{ij}^x \hat{x}_{ij} + |\lambda_o - \theta| \xi_{io}^x \hat{x}_{io} \right\} \le 0 & o \in J_i^x, \forall i \in I \\
\sum_{j=1}^n \lambda_j x_{ij} - \theta x_{io} + \max\limits_{\substack{\sum_{j\in J_i^x}\xi_{ij}^x \le \Upsilon_i^x \\ 0\le\xi_{ij}^x\le 1 (\forall j\in J_i^x)}} \left\{ \sum_{\substack{j\in J_i^x \\ j\ne o}} \lambda_j \xi_{ij}^x \hat{x}_{ij} \right\} \le 0 & o \notin J_i^x, \forall i \in I \\
\sum_{j=1}^n \lambda_j y_{rj} - y_{ro} - \max\limits_{\substack{\sum_{j\in J_r^y}\xi_{rj}^y \le \Upsilon_r^y \\ 0\le\xi_{rj}^y\le 1 (\forall j\in J_r^y)}} \left\{ \sum_{\substack{j\in J_i^y \\ j\ne o}} \lambda_j \xi_{rj}^y \hat{y}_{rj} + |\lambda_o - 1| \xi_{ro}^y \hat{y}_{ro} \right\} \ge 0 & o \in J_r^y, \forall r \in R \\
\sum_{j=1}^n \lambda_j y_{rj} - y_{ro} - \max\limits_{\substack{\sum_{j\in J_r^y}\xi_{rj}^y \le \Upsilon_r^y \\ 0\le\xi_{rj}^y\le 1 (\forall j\in J_r^y)}} \left\{ \sum_{\substack{j\in J_i^y \\ j\ne o}} \lambda_j \xi_{rj}^y \hat{y}_{rj} \right\} \ge 0 & o \notin J_r^y, \forall r \in R \\
\lambda_j \ge 0 & \forall j
\end{array}
\tag{4.16}
$$

Let the inner problems of the first and third constraints be denoted by the protection functions $\psi(\boldsymbol{\lambda}, \theta, \Upsilon_i^x)$ and $\psi(\boldsymbol{\lambda}, \theta\, \Upsilon_r^y)$ and let $(\rho_i^x, \boldsymbol{\delta}_i^x) \in I$, and $\left(\rho_r^y, \boldsymbol{\delta}_r^y\right) \in R$ be the dual variables, respectively, where $\boldsymbol{\delta}_i^x = \left(\dots, \delta_{ij}^x, \dots\right) \in \mathbb{R}^{|J_i|}$ and $\boldsymbol{\delta}_r^y = \left(\dots, \delta_{rj}^y, \dots\right) \in \mathbb{R}^{|J_r|}$. Using the strong duality theorem, we obtain the following equivalent models:

$$
\begin{array}{lll}
\psi(\boldsymbol{\lambda}, \theta, \Upsilon_i^x) = \min \sum_{j \in J_i^x} \delta_{ij}^x + \rho_i^x \Upsilon_i^x & & \\
\text{s.t} & & \\
\rho_i^x + \delta_{ij}^x \geq \lambda_j \hat{x}_{ij} & \forall j \in J_i^x (j \neq o) & \qquad (4.17) \\
\rho_i^x + \delta_{io}^x \geq |\lambda_o - \theta| \hat{x}_{io} & & \\
\delta_{ij}^x, p_i^x \geq 0 & \forall j \in J_i^x &
\end{array}
$$

$$
\begin{array}{lll}
\psi\left(\boldsymbol{\lambda}, \theta, \Upsilon_r^y\right) = \min \sum_{j \in J_r^y} \delta_{rj}^y + \rho_r^y \Upsilon_r^y & & \\
\text{s.t} & & \\
\rho_r^y + \delta_{rj}^y \geq \lambda_j \hat{y}_{rj} & \forall j \in J_r^y (j \neq o) & \qquad (4.18) \\
\rho_r^y + \delta_{ro}^y \geq |\lambda_o - 1| \hat{y}_{ro} & & \\
\delta_{rj}^y, p_r^y \geq 0 & \forall j \in J_r^y &
\end{array}
$$

Let $\alpha_o = |\lambda_o - \theta|$ and $\beta_o = |\lambda_o - 1|$ in models (4.17) and (4.18) respectively. We can write the following equivalent models:

$$
\begin{array}{lll}
\psi(\boldsymbol{\lambda}, \theta, \Upsilon_i^x) = \min \sum_{j \in J_i^x} \delta_{ij}^x + \rho_i^x \Upsilon_i^x & & \\
\text{s.t} & & \\
\rho_i^x + \delta_{ij}^x \geq \lambda_j \hat{x}_{ij} & \forall j \in J_i^x (j \neq o) & \qquad (4.19) \\
\rho_o^x + \delta_{io}^x \geq \alpha_o \hat{x}_{io} & & \\
-\alpha_o \leq \lambda_o - \theta \leq \alpha_o & & \\
\alpha_o, \delta_{io}^x, p_j^x \geq 0 & \forall j \in J_i^x &
\end{array}
$$

$$
\begin{array}{lll}
\psi\left(\boldsymbol{\lambda}, \theta, \Upsilon_r^y\right) = \min \sum_{j \in J_r^y} \delta_{rj}^y + \rho_r^y \Upsilon_r^y & & \\
\text{s.t} & & \\
\rho_r^y + \delta_{rj}^y \geq \lambda_j \hat{y}_{rj} & \forall j \in J_r^y (j \neq o) & \qquad (4.20) \\
\rho_o^y + \delta_{ro}^y \geq \beta_o \hat{y}_{ro} & & \\
-\beta_o \leq \lambda_o - 1 \leq \beta_o & & \\
\beta_o, \delta_{ro}^y, p_j^y \geq 0 & \forall j \in J_r^y &
\end{array}
$$

Analogously, we can develop a couple of models to measure the second and fourth inner problems and substitute all the built models into model (4.16). Then finally, considering that all the inputs and outputs data of the DMUs are uncertain we arrive at the following model:

$$
\begin{aligned}
& I_e^* = \min \theta \\
& \text{s.t.} \\
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} - \theta x_{io} + \sum_{j \in J_i^x} \delta_{ij}^x + \rho_i^x \Upsilon_i^x \leq 0 && \forall i \in I \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro} - \sum_{j \in J_r^y} \delta_{rj}^y - \rho_r^y \Upsilon_r^y \geq 0 && \forall r \in R \\
& \rho_i^x + \gamma_{ij}^x \geq \lambda_j \hat{x}_{ij} && \forall i \in I,\ \forall j \in J_i^x (j \neq o) \\
& \rho_i^x + \gamma_{io}^x \geq \alpha_o \hat{x}_{io} && o \in J_i^x, \forall i \in I \\
& \rho_r^y + \gamma_{rj}^y \geq \lambda_j \hat{y}_{rj} && \forall r \in R, \forall j \in J_r^y (j \neq o) \\
& \rho_r^y + \gamma_{ro}^y \geq \beta_o \hat{y}_{ro} && o \in J_r^y, \forall r \in R \\
& -\alpha_o \leq \lambda_o - \theta \leq \alpha_o \\
& -\beta_o \leq \ \lambda_o - 1 \leq \beta_o \\
& \alpha_o, \gamma_{ij}^x,\ p_i^x \geq 0 && \forall i \in I, \forall j \in J_i^x \\
& \beta_o, \gamma_{rj}^y, p_r^y \geq 0 && \forall r \in R, \forall j \in J_r^y \\
& \lambda_j \geq 0 && \forall j
\end{aligned}
\tag{4.21}
$$

which complete the proof. ☐

In the robust optimization framework, the dual of the robust counterpart results in an optimal solution is different from the robust counterpart of the dual regardless of the choice of uncertainty set (see Gabrel & Murat, 2010). The following theorems prove that the optimal objective value of the IERCCR model (4.14) is indeed greater than the optimal objective value of the IMRCCR (4.8).

Theorem 4.4. $I_e^* > I_m^*$

Proof. By definition, $I_m^* = I_d^*$. Suppose by contradiction that $I_e^* = I_m^*$. Let $(\boldsymbol{\lambda}^*, \boldsymbol{\zeta}^{\boldsymbol{x}*}, \boldsymbol{\zeta}^{\boldsymbol{y}*}) \in \mathbb{R}^{1+n+\sum_{j=1}^{n}(|I_j|+|R_j|)}$ be the optimal solution of model (4.13) where $\boldsymbol{\lambda}^* = \left(\dots, \lambda_j^*, \dots\right)_{j=1,..,n}$, $\boldsymbol{\zeta}^{\boldsymbol{x}*} = \left(\dots, \zeta_{ij}^{x*}, \dots\right)_{i \in I, j \in J_i^x}$ and $\boldsymbol{\zeta}^{\boldsymbol{y}*} = \left(\dots, \zeta_{rj}^{y*}, , \dots\right)_{r \in R, j \in J_r^y}$. The optimal objective value I_d^* is less than or equal to one according to Proposition 4.1 A feasible solution $(\boldsymbol{\lambda}^*, \boldsymbol{\delta}^{\boldsymbol{x}*}, \boldsymbol{\delta}^{\boldsymbol{y}*}) \in \mathbb{R}^{m+s+n+\sum_{j=1}^{n}(|I_j|+|R_j|)}$ of model (4.14) correspond with a feasible solution for model (4.13). However, the optimal objective function value, I_e^* from the feasible solution in model (4.14) is greater than the optimal objective functions values I_d^* and I_m^* which is a contradiction to the earlier claim. ☐

We have shown with Theorem 4.4 that the linear duality principle between the IECCR and IMCCR models (see appendix A1) breaks down when the underlying data is uncertain. This proof support that the similar result obtained for the imprecise DEA in see Park (2010). We therefore lay emphasis on the fact that the IECCR model differs from the IMRCCR model if the DMUs are benchmarked in the light of the worst-case scenario for uncertain inputs and outputs data. More importantly, the efficiency scores of the IERCCR model is higher than the

IMRCCR model. The differences in the optimal objective value of the models result from the ultraconservative strategy used in obtaining the two models. It is clear in the robust setting now that a DMU efficient in the envelopment model is not necessarily efficient in the multiplier model.

4.4 The output-oriented robust CCR model

We devote this section to study the robust efficiency of DMUs with output – orientation. In Sections 4.1 and 4.2, a classification of the robust efficiency under different conservatism level and duality relation between the multiplier and envelopment models are respectively studied. The latter section shall also establish an all-important relationship between the input – oriented and output-oriented robust models. Referencing models (4.8) and (4.14), the formulated input-oriented robust models minimize the input of DMUs while at least maintaining the given output level under the worst-case and best-case scenarios of the uncertain data. Here, the output-oriented robust models maximize the output level while using no more than the given amounts of inputs under a worst-case and best-case scenario of the uncertain data. The output-oriented multiplier robust CCR (OMRCCR) model is obtained similarly as follows:

$$
\begin{array}{ll}
O_m^* = \min \omega & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} + p_o^x \Gamma_o^x + \sum_{i \in I_o} w_{io} - \omega \leq 0 & \\
\sum_{r=1}^{s} u_r y_{ro} - p_o^y \Gamma_o^y - \sum_{r \in R_o} q_{ro} \geq 1 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + p_j^x \Gamma_j^x + p_j^y \Gamma_j^y + \sum_{r \in R_j} q_{rj} + \sum_{i \in I_j} w_{ij} \leq 0 & \forall j \\
p_j^y + q_{rj} \geq u_r \hat{y}_{rj} & \forall j, \forall r \in R_j \\
p_j^x + w_{ij} \geq v_i \hat{x}_{ij} & \forall j, \forall i \in I_j \\
q_{rj}, w_{ij} \geq 0 & \forall j, \forall i \in I_j, \forall r \in R_j \\
p_j^x, p_j^y \geq 0 & \forall j \\
v_i \geq 0 & \forall i \\
u_r \geq 0 & \forall r
\end{array}
\qquad (4.22)
$$

where the decision variable ω is the efficiency of DMU_o. Model (4.22) similarly entails $n + 1 + n\left(\sum_{j=1}^{n} |I_j| + \sum_{j=1}^{n} |R_j|\right)$ constraints and $m + s + 2n + n\left(\sum_{j=1}^{n} |I_j| + \sum_{j=1}^{n} |R_j|\right)$ decision variables. The dual model is given as follows:

$$
\begin{aligned}
& O_d^* = \max \phi && \\
& \text{s.t} && \\
& \textstyle\sum_{j=1}^{n} \mu_j x_{ij} - \sum_{j \in J_i^x} \zeta_{ij}^x \hat{x}_{ij} \le x_{io} && \forall i \in I \\
& \textstyle\sum_{j=1}^{n} \mu_j y_{rj} + \sum_{j \in J_r^y} \zeta_{rj}^y \hat{y}_{rj} \ge \phi y_{ro} && \forall r \in R \\
& \textstyle\sum_{i \in I_o} \zeta_{io}^x - \mu_o \Gamma_o^x \le \Gamma_o^x && \\
& \textstyle\sum_{i \in I_j} \zeta_{ij}^x - \mu_j \Gamma_j^x \le 0 && \forall j \ne o \\
& \textstyle\sum_{r \in R_o} \zeta_{ro}^y - \mu_o \Gamma_o^y - \phi \Gamma_o^y \le 0 && \\
& \textstyle\sum_{r \in R_j} \zeta_{rj}^x - \mu_j \Gamma_j^y \le 0 && \forall j \ne o \\
& \zeta_{io}^x - \mu_o \le 1 && \forall i \in I_o \\
& \zeta_{ij}^x - \mu_j \le 0 && \forall j \ne o, \forall i \in I_j \\
& \zeta_{ro}^y - \mu_o - \phi \le 0 && \forall r \in R_o \\
& \zeta_{rj}^y - \mu_j \le 0 && \forall j \ne o, \forall r \in R_j \\
& \zeta_{ij}^x \ge 0 && \forall j, \forall i \in I_j \\
& \zeta_{rj}^y \ge 0 && \forall j, \forall r \in R_j \\
& \mu_j \ge 0 && \forall j
\end{aligned}
\tag{4.23}
$$

It can be similarly verified that $O_d^* = O_m^*$ by the strong duality theorem (see Bazaraa, Jarvis, & Sherali, 2010). We know by now that the variables q_{rj}, w_{ij}, p_j and $\zeta_{ij}^x, \zeta_{rj}^y$ in models (4.22) and (4.23), respectively are the robust variables introduced to protect the inputs and outputs data from dislocating the efficient frontier. As a result, the optimal values ω^* and ϕ^* are preserved and as inspection makes clear, we have $1 \le \omega^* < \infty$ and $1 \le \phi^* < \infty$. The robust efficiency obtained by the OMRCCR model (4.22) is stated in the following proposition.

Proposition 4.2. $O_m^* \ge 1$.

Proof. Let $(\omega^*, \boldsymbol{v}^*, \boldsymbol{u}^*, \boldsymbol{p}^{x^*}, \boldsymbol{p}^{y^*}, \boldsymbol{q}^*, \boldsymbol{w}^*)$ be an optimal solution for model (4.22). From the first constraint, we obtain $\omega^* \ge \sum_{i=1}^{m} v_i^* x_{io} + p_o^{x^*} \Gamma_o^x + \sum_{i \in I_o} w_{io}^*$. Taking the second constraint along with along the third constraints set for $j = o$ into consideration leads to $\omega^* \ge \sum_{i=1}^{m} v_{i^*} x_{io} + p_o^{x^*} \Gamma_o^x + \sum_{i \in I_o} w_{io}^* \ge \sum_{r=1}^{s} u_r^* y_{ro} + p_o^{y^*} \Gamma_o^y + \sum_{r \in R_o} q_{ro}^*$. Since p_o^y and $q_{ro} \forall r \in R_o$ are nonnegative variables we can obviously obtain $\sum_{r=1}^{s} u_r^* y_{ro} + p_o^{y^*} \Gamma_o^y + \sum_{r \in R_o} q_{ro}^* \ge \sum_{r=1}^{s} u_r^* y_{ro} - p_o^{y^*} \Gamma_o^y - \sum_{r \in R_o} q_{ro}^* \ge 1$. Consequently, $(O_m^* =)\ \omega^* \ge 1$ which completes the proof. □

4.4.1 A classification scheme for the OMRCCR

The classification of robust efficiency in Section 3.1 indicates that a DMU has a better rank robustly if it has an optimal objective value of one or closer to one for higher values Γ_j. From Proposition 4.2, DMU_o is robust R-efficient if $\omega^* \ge 1$ and R-inefficient if $\omega^* = 0$. In fact, the evaluation of DMUs from model (4.22) is based not only to the weights but also the conservativeness of the DM. Likewise Definition 4.1, we can provide a classification scheme for the DMUs into RE^{++}, RE^{+}, and RE^{-}. The formal classification of the R-efficiency for DMUs with the output orientation in the robust setting follows the definition below:

Definition 4.2. Suppose Γ_j takes values in the interval $[0, |I_j| + |R_j|]$ for every $i \in I_j$ and $r \in R_j$ in model (4.22). Referencing Proposition 4.2, the DMUs are classified into the following sets:

- (*Full robust efficiency*). A DMU_o is fully robust efficient if and only if $O_m^* = 1$ when $\Gamma_j = |I_j| + |R_j|$
- (*Partial robust efficiency*). A DMU_o is partially robust efficient if $\omega^* = 1$ when $\Gamma_j \geq 0$ and $O_m^* > 1$ when $\Gamma_j = |I_j| + |R_j|$
- (*Robust inefficiency*). A DMU_o is said to be robust inefficient if $O_m^* > 1$ for all $\Gamma_j \in (0, |I_j| + |R_j|]$.

4.4.2 Primal-Dual relationship: output orientation

It is established with the input orientation in Section 3.2 that the IMRCCR and IERCCR models are not mutually dual. Indeed, the dual of the IERCCR model exceeds the IMRCCR. We carry out a similar study of the duality relationship in the output-oriented robust model. Consider the following equivalent model of the OECRR model (4.4):

$$
\begin{aligned}
& O_e^* = \max \phi \\
& \text{s.t.} \\
& \textstyle\sum_{j=1(j\neq o)}^{n} \mu_j x_{ij} + (\mu_o - 1)x_{io} \leq 0 \quad && \forall i \\
& \textstyle\sum_{j=1(j\neq o)}^{n} \mu_j y_{rj} - (\phi - \mu_o)y_{ro} \geq 0 \quad && \forall r \\
& \mu_j \geq 0 && \forall j
\end{aligned}
\tag{4.24}
$$

Using model (4.24) and considering similar reasoning apriori model (4.14) the Output–oriented Envelopment robust CCR (OERCCR) model under the worst-case scenario can be expressed as the following:

$$
\begin{array}{ll}
O_e^* = \max \phi & \\
\text{s.t.} & \\
\sum_{j=1}^{n} \mu_j x_{ij} - x_{io} + \sum_{j \in J_i^x} \delta_{ij}^x + \rho_i^x \Upsilon_i^x \leq 0 & \forall i \in I \\
\sum_{j=1}^{n} \mu_j y_{rj} - \phi y_{ro} - \sum_{j \in J_r^y} \delta_{ij}^y - \rho_r^y \Upsilon_r^y \geq 0 & \forall r \in R \\
\rho_i^x + \delta_{ij}^x \geq \mu_j \hat{x}_{ij} & \forall i \in I,\ \forall j \in J_i^x (j \neq o) \\
\rho_i^x + \delta_{io}^x \geq \alpha_o \hat{x}_{io} & o \in J_i^x, \forall i \in I \\
\rho_r^y + \delta_{rj}^y \geq \mu_j \hat{y}_{rj} & \forall r \in R, \forall j \in J_r^y (j \neq o) \\
\rho_r^y + \delta_{ro}^y \geq \beta_o \hat{y}_{ro} & o \in J_r^y, \forall r \in R \\
-\alpha_o \leq \lambda_o - 1 \leq \alpha_o & \\
-\beta_o \leq \ \phi - \lambda_o \leq \beta_o & \\
\alpha_o, \delta_{ij}^x,\ p_i^x \geq 0 & \forall i \in I, \forall j \in J_i^x \\
\beta_o, \delta_{rj}^y, p_r^y \geq 0 & \forall r \in R, \forall j \in J_r^y \\
\mu_j & \forall j
\end{array}
\tag{4.25}
$$

The formulation of the robust model (4.25) follows the same modeling procedure in Theorem 4.3. To see this, let $\sigma_o = |\mu_o - 1|$ and $\pi_o = |\phi - \mu_o|$ in models (4.17) and (4.18) and let $\psi(\boldsymbol{\mu}, \phi, \Upsilon_i^x)$ and $\psi(\boldsymbol{\mu}, \phi,\ \Upsilon_r^y)$ be the protection functions while $(\rho_i^x, \boldsymbol{\delta}_i^x) \in I$, and $\left(\rho_r^y, \boldsymbol{\delta}_r^y\right) \in R$ remain the dual variables, respectively. Then we obtain the following equivalent models:

$$
\begin{array}{ll}
\psi(\boldsymbol{\mu}, \phi, \Upsilon_i^x) = \min \sum_{j \in J_i^x} \delta_{ij}^x + \rho_i^x \Upsilon_i^x & \\
\text{s.t} & \\
\rho_i^x + \delta_{ij}^x \geq \mu_j \hat{x}_{ij} & \forall j \in J_i^x (j \neq o) \\
\rho_o^x + \delta_{io}^x \geq \sigma_o \hat{x}_{io} & \\
-\sigma_o \leq \mu_o - 1 \leq \sigma_o & \\
\sigma_o, \delta_{io}^x, p_j^x \geq 0 & \forall j \in J_i^x
\end{array}
\tag{4.26}
$$

$$
\begin{array}{ll}
\psi\left(\boldsymbol{\mu}, \phi, \Upsilon_r^y\right) = \min \sum_{j \in J_r^y} \delta_{rj}^y + \rho_r^y \Upsilon_r^y & \\
\text{s.t} & \\
\rho_r^y + \delta_{rj}^y \geq \mu_j \hat{y}_{rj} & \forall j \in J_r^y (j \neq o) \\
\rho_o^y + \delta_{ro}^y \geq \pi_o \hat{y}_{ro} & \\
-\pi_o \leq \phi - \mu_o \leq \pi_o & \\
\pi_o, \delta_{ro}^y, p_j^y \geq 0 & \forall j \in J_r^y
\end{array}
\tag{4.27}
$$

Moreover, putting (4.26) and (4.27) into the pessimistic robust counterpart for the output model:

$$
\begin{aligned}
&\min \phi \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \mu_j x_{ij} - x_{io} + \max_{\substack{\sum_{j\in J_i^x} \xi_{ij}^x \le \Upsilon_i^x \\ 0\le \xi_{ij}^x \le 1\ (\forall j\in J_i^x)}} \left\{ \sum_{\substack{j\in J_i^x \\ j\neq o}} \mu_j \xi_{ij}^x \hat{x}_{ij} + |\mu_o - 1|\ \xi_{io}^x \hat{x}_{io} \right\} \le 0 && o \in J_i^x, \forall i \in I \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} - x_{io} + \max_{\substack{\sum_{j\in J_i^x} \xi_{ij}^x \le \Upsilon_i^x \\ 0\le \xi_{ij}^x \le 1 (\forall j\in J_i^x)}} \left\{ \sum_{\substack{j\in J_i^x \\ j\neq o}} \mu_j \xi_{ij}^x \hat{x}_{ij} \right\} \le 0 && o \notin J_i^x, \forall i \in I \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - \phi y_{ro} - \max_{\substack{\sum_{j\in J_r^y} \xi_{rj}^y \le \Upsilon_r^y \\ 0\le \xi_{rj}^y \le 1 (\forall j\in J_r^y)}} \left\{ \sum_{\substack{j\in J_i^y \\ j\neq o}} \mu_j \xi_{rj}^y \hat{y}_{rj} + |\phi - \mu_o| \xi_{ro}^y \hat{y}_{ro} \right\} \ge 0 && o \in J_r^y, \forall r \in R \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - \phi y_{ro} - \max_{\substack{\sum_{j\in J_r^y} \xi_{rj}^y \le \Upsilon_r^y \\ 0\le \xi_{rj}^y \le 1 (\forall j\in J_r^y)}} \left\{ \sum_{\substack{j\in J_i^y \\ j\neq o}} \mu_j \xi_{rj}^y \hat{y}_{rj} \right\} \ge 0 && o \notin J_r^y, \forall r \in R \\
&\mu_j \ge 0 && \forall j
\end{aligned}
\tag{4.28}
$$

and assuming the constraints for uncertain input and output arrives at model (4.25).

Referencing Theorem 4.4 in which the optimal objective values of the robust multiplier model exceed the robust envelopment model, in contrast, we prove by the following Theorem 4.5 that the optimal objective value of the OERCCR model (4.25) is less than the optimal objective value of the OMRCCR model (4.22).

Theorem 4.5. $O_e^* < O_m^*$

Proof. By contradiction, let $O_e^* = O_m^*$ and for all DMUs recall that $O_d^* \le O_m^*$. It is easily verifiable that $(\boldsymbol{\mu}^*, \boldsymbol{\delta}^{\boldsymbol{x}*}, \boldsymbol{\delta}^{\boldsymbol{y}*}) \in \mathbb{R}^{1+n+\sum_{j=1}^{n}(|I_j|+|R_j|)}$ is a feasible solution to models (4.23) and (4.25). However, the optimal objective function value, O_e^* of this feasible solution is less than the optimal objective functions values O_d^* and O_m^* which is a contradiction to the earlier claim. □

From model models (4.22) and (4.8), similar property for the input – and output- oriented relationship in the CCR model holds in the RDEA setting. The relation between the IMRCCR model and the OMRCCR model can be shown via the following theorem.

Theorem 4.6. The input-oriented model (4.13) and the output-oriented model (4.23) are equivalents and their optimal objective functions are related by $\phi^* = 1/\theta^*$.

Proof. Let $(\phi, \boldsymbol{\mu}, \boldsymbol{\zeta}^x, \boldsymbol{\zeta}^y)$ be a feasible solution to model (4.23). We show that $(\theta, \boldsymbol{\lambda}, \boldsymbol{\gamma}^x, \boldsymbol{\gamma}^y) = (\frac{1}{\phi}, \frac{\mu}{\phi}, \frac{\zeta^x}{\phi}, \frac{\zeta^y}{\phi})$ is also a feasible solution to model (4.13). (4.23)To do this, consider the first two constraints of model (4.23). The first constraint $\sum_{j=1}^{n}\mu_j x_{ij} - \sum_{j\in J_i^x}\zeta_{ij}^x \hat{x}_{ij} \le x_{io}$ becomes $\sum_{j=1}^{n}\frac{\mu_j}{\phi}x_{ij} - \sum_{j\in J_i^x}\frac{\zeta_{ij}^x}{\phi}\hat{x}_{ij} \le \frac{x_{io}}{\phi}$ or $\sum_{j=1}^{n}\lambda_j x_{ij} - \sum_{j\in J_i^x}\gamma_{ij}^x \hat{x}_{ij} \le \theta x_{io}$ which is the first constraint of model (4.13). Similarly, the second constraint $\sum_{j=1}^{n}\mu_j y_{rj} + \sum_{j\in J_r^y}\zeta_{rj}^y \hat{y}_{rj} \ge \phi y_{ro}$ becomes $\sum_{j=1}^{n}\frac{\mu_j}{\phi}y_{rj} + \sum_{j\in J_r^y}\frac{\zeta_{rj}^y}{\phi}\hat{y}_{rj} \ge y_{ro}$ or $\sum_{j=1}^{n}\lambda_j y_{rj} + \sum_{j\in J_r^y}\gamma_{rj}^y \hat{y}_{rj} \ge y_{ro}$ which leads to the second constraint of model (4.13). All the other constraints can be similarly transformed and the fact that models (4.13) and (4.23) are minimization and maximization problems, respectively, validates the transformation. At optimality, $\boldsymbol{\mu}^* = \boldsymbol{\lambda}^*/\phi^*$, $\boldsymbol{\zeta}^{x^*} = \boldsymbol{\gamma}^{x^*}/\theta^*$, $\boldsymbol{\zeta}^{y^*} = \boldsymbol{\gamma}^{y^*}/\theta^*$ and $\phi^* = 1/\theta^*$ which completes the proof. □

4.5 Application to banking efficiency in Germany

To demonstrate the application of the proposed models, i.e. the robust multiplier input - and output-orientated models and their dual models, we analyze the performance of the major banks in Germany. First, we provide an overview of the banks and their operation in Germany.

4.5.1 Contextual setting: Uncertainties and banking efficiency in Germany

The German banking industry is one of the largest European banking markets with huge capital market for industries across Europe. The banks operate under a common set of rules by Deutsche Bundesbank which is the independent central bank of the Federal Republic of Germany. The central bank forms part of the Euro-system that shares responsibility with other national central banks and the European Central Bank in managing the European currency, the euro. Currently, the Bundesbank takes oversight of the roughly 2,000 credit institutions and 1,500 financial services institutions active throughout Germany. The banking system in Germany comprises of three pillars – private commercial banks, public banks, and co-operative banks, which primarily differ in terms of their legal form and ownership structure. The private banks such as the most popular Deutsche Bank and Commerzbank, Unicredit Bank AG (HypoVereinsbank), Deutsche Postbank AG represent the pillar with the largest asset, accounting for about 40% of the total asset in the banking industry. The public banks with government involvement such as the KfW (Kreditanstalt für Wiederaufbau) banks make up the large regional banks called the *Landesbanken* and the savings banks. Their asset share represents about 26% of total assets and their geographical business area is limited to the local government owners. The co-operative banks (e. g. Volksbanks and Raiffeisenbanks) are largely banks with mutual structure in which the shareholding is largely composed of depositors and borrowers. They represent about 17% of the total bank assets (European Banking Federation, 2018). While the private banks may be highly profit-oriented, the objective or the mandate in the case of co-operative banks and public banks is not to maximize

Table 4.1. Financial indicators of selected banks in Germany

DMU	Bank name	Input			Output		
		Employee	Assets	Equity	Deposits	Loans	Revenue
1	Deutsche Bank AG	101104.00	1629130.0	67624.00	44710.00	428521.00	15881.00
2	Commerzbank AG	51305.00	532641.00	30407.00	91633.00	203895.00	5779.00
3	DZ Bank AG	30029.00	408341.00	19729.00	97227.00	124829.00	2988.00
4	UniCredit Bank AG	16310.00	298745.00	20766.00	63079.00	113175.00	2797.00
5	Sparkassen-Finanzgruppe Hessen-Thuringen	26679.00	260313.00	20085.00	37399.00	152441.00	3269.00
6	Deutsche Postbank AG	14758.00	150597.00	7158.00	15444.00	97474.00	2403.00
7	Deutsche Bank Privat-und Geschaftskunden AG	12368.00	136927.30	2746.30	26741.20	68119.10	1940.50
8	Hamburger Sparkasse AG (HASPA)	5000.00	42638.50	3218.00	4618.70	30192.20	742.10
9	Santander Consumer Bank AG	3805.00	42124.20	3068.30	4991.30	30027.90	1177.40
10	State Street Bank GmbH	4102.00	37612.00	2211.90	849.20	691.80	82.70
11	Deutsche Apotheker- und Aerztebank eG	2139.00	36444.40	2195.60	7213.80	27892.90	675.10
12	KfW Ipex-Bank Gmbh	648.00	28369.70	3831.00	22761.50	24350.00	333.70
13	Sparkasse KölnBonn	4351.00	26511.00	1675.00	2443.50	19054.40	432.50
14	SEB AG	811.00	22398.70	2065.70	8387.00	12181.20	69.20
15	Frankfurter Sparkasse	1837.00	17985.60	903.30	1272.40	7335.90	288.50
16	Stadtsparkasse München	2669.00	17070.30	1546.10	437.10	12131.00	279.30
17	Sparkasse Hannover	2102.00	13491.60	1137.00	914.70	10456.90	283.20
18	Ostsächsische Sparkasse Dresden	1748.00	12170.30	741.60	1198.20	5069.70	224.10
19	Berliner Volksbank eG	2193.00	11677.30	975.90	367.00	7516.90	261.60
20	Nassauische Sparkasse	1754.00	11280.90	910.10	1291.60	8822.20	279.10
21	Die Sparkasse Bremen	1449.00	11058.70	736.60	2193.00	8762.80	233.50
22	Stadtsparkasse Düsseldorf	2080.00	10921.90	1294.30	496.00	7657.60	244.30
23	Sparkasse Pforzheim Calw	2005.00	10764.40	720.10	2251.30	7082.40	170.00
24	Kreissparkasse Muenchen Starnberg Ebersberg	1618.00	10722.10	790.80	1074.80	7339.00	227.20
25	Sparkasse Nürnberg	1890.00	10627.40	920.50	912.60	5563.90	210.40
26	Sparkasse Aachen	2047.00	10124.00	1280.30	912.10	7342.80	238.70

profit but rather to support the sustainable growth of businesses of their members and the social development of their local, regional or national economy (Behr & Schmidt, 2015). However, one thing is common; almost all German banks offer at least some services that one can classify as commercial and investment banking services. With the exception of specialist banks which for some historical reasons limit their activities to selected businesses, the rest of the banks operate a universal banking system. As a result, the banks face competition among each other and compete fiercely for market share both locally and internationally. The activities of these banks fall under the supervision and regulation of the Bundesbank and Federal Financial Supervisory Authority (BaFin). Given such homogeneity, an assessment of the efficiency of German banks seem very prudent.

The efficiency of banks in Germany particularly the private banks is a larger concern for managers and shareholders to determine best practice units, sustainability and increased market share. A more efficient bank is assumed to foster growth as it is able to select optimal projects that generate higher returns on assets and investment, and also play a crucial role in the allocation of financial resources at both the micro and macro level of the economy. Within the context of efficiency, a couple of studies, very few have been done on German banks. For example, Fiorentino et al. (2006) took a sample of 34, 192 observations on all universal German banks and assessed their efficiency between the period, 1993 – 2004. They concluded that efficiency rank stability is very high in the short run, however, the DEA efficiency is sensitive to measurement error and outliers as compared to the SFA. In explaining the efficiency differences among German and Austrian banks, Hauner (2005) attributed the cost-efficiency of the former banks to higher competition relatively to the latter. Ahn & Le (201) propose a DEA framework incorporating rationality concepts of decision making to derive appropriate performance of Germany saving banks. Their findings reveal stable scale efficiency pattern and suggest that savings banks are more efficient in fulfilling their public mandate than in generating profit.

Generally, the banking industry is vulnerable to global challenges (e.g. shocks in oil price, global financial crises, monetary policy, etc.) that leads to uncertainties in lending and borrowing activities and underpins new regulatory regime. For instance, the global financial crises in 2008 which put much stress on banks liquidity, decreased loans to borrowers by 47% and real investment such as working capital by 14%, low lending power and banks vulnerability through banking panic and in particular from banks co-syndication of credit line with the defunct crushed-down Lehman Brothers[31] prior to and during the crises affected many banks including German banks (Ivashina & Scharfstein, 2010). With exception of most cooperative banks whose activities were less of investment, the more investment engaging banks such Deutsche Bank and Commerzbank were largely affected due to their off-balance sheet activities and overly risky investment (Behr & Schmidt, 2015). Together with other banks, the crises meant various government interventions including bailout (with the exception of Deutsche bank) and new regulatory regime such as higher capital requirements and strict liquidity rules had to be instituted.

[31] Lehman Brothers Holdings Inc. was the fourth-largest investment bank in the United States which for the financial crises in 2008 was declared bankrupt. The financial activities of the firm included investment banking, fixed-income sales, investment management, private banking and equity.

In fact, uncertainties are common phenomena in the banking industry which affect their operations and efficiency. Uncertainties in the banking sector imply that planning on both inputs and outputs data becomes unreliable. Financial data tend to be undoubtedly random and imprecise. As a measure, Buch, Buchholz, Tonzer, & Buch (2014) compute uncertainty among banks as a cross-sectional dispersion of volatilities to bank's specific variables. Most of the uncertainties in bank data are idiosyncratic in nature, observable by bank insiders who are privy to detailed knowledge of the bank's portfolio, data, and its dynamics. Among many of the sources are listed below:

- *Prediction error*: Banks are occasionally involved in forecasting of financial data. For instance, projections on returns on investment, liabilities, loans, etc., are made towards preparation of balance sheet and financial report. These variables are subject to prediction errors.
- *Measurement error*: Banks record data assuming away random errors, many of which cannot be measured exactly. Measured data revolve around their mean value and have inherent errors.
- *Implementation error:* Some of the banking data used for analysis cannot be implemented exactly as computed. These data are subject to implementation error. e.g. the intensity of technological usage of banks equipment.

It is evident that these idiosyncratic errors observed in the data generating process result from differences in actual true data and observed or available data and produce uncertainties for which the measured efficiency scores become unstable and unreliable. The proposed models in this chapter rectify this problem of uncertainty and hence attempt to produce efficiency results which are robust to perturbations and uncertain conditions.

4.5.2 Data and variable selection

To analyze the efficiency of the banks in Germany with the DEA, we first look at the characterization of the financial indicators of banks which is based on the precise definition of banking activities. The definition of inputs and outputs result from the functions exerted by a bank (Berger & Humphrey 1997). The three broad approaches used in determining input and output measures in banking studies are explained in Chapter 3. To reminisce, a common difference for the two major approaches – the intermediary approach and the production approach refers to the treatment of deposits. According to Allen & Gale (1995), the German banking system are intermediaries predominant, thus the banks operate as intermediaries between investors and savers, mainly transmitting capital and labor (inputs) to loans and securities (outputs). The quality of this approach by a bank is measured by its efficiency in converting inputs into outputs while ensuring cost minimization or profit maximization. Within the context of this approach, we consider Kao & Liu (2014) in which demand deposit is treated as outputs in the intermediation approach and Mostafa (2009) in which a higher percentage of surveyed papers measure deposit as output with the intermediate approach.

Moreover, we consider data with the financial indicators from BankScope. The accounting year for the data is 2016. This data including other dataset on European banks is obtained from the published dataset in Alfiero et al. (2019). Table 4.1 shows the data employed and the variables considered. According to the analysis of banking financial measures above, we assume that banks demand as inputs fixed assets and employees which is used to produce outputs, such as loans to customers, deposit, and net interest revenue. As in chapter 3, the measurement of all these variables are taken in monetary terms except the number of employees which counted as the number of banking personnel employed in the said accounting year.

4.5.3 Results from model testing

In order to first make a comparison between the robust multiplier and envelopment models and second, the input- and output- oriented robust models, we use the data in Table 4.1 considering that the exact values of the data are unknown except their nominal values. Besides, as standard in robust DEA settings (Sadjadi & Omrani, 2008; Omrani, 2013; Lu, 2015; Arabmaldar et al., 2017), the perturbation of the input and input data from their nominal values is set to 5%. i.e. the true values of the uncertain data are expected to lie in the symmetric where (x_{ij}, y_{rj}) are the observed nominal values of the input and output data. Here, all interval, $\tilde{x}_{ij} \in [x_{ij} - 0.05 \times x_{ij},\ x_{ij} + 0.05 \times x_{ij}]$, and $\tilde{y}_{rj} \in [y_{rj} - 0.05 \times y_{rj},\ y_{rj} + 0.05 \times y_{rj}]$ variables are considered uncertain and so the robust parameter is selected for $\Gamma_j^x = 3$ and $\Gamma_j^y = 3$. i.e. $\Gamma_j = |I_j| + |R_j| = 6$. Table 4.2 shows the results obtained for the input - and output - oriented robust models including the CCR models. Based on the IMCCR and OMCCR models, six banks were efficient (efficiency score of 1) representing 23% of the banks under consideration here. The very least performing bank is State Street Bank GmbH. As visible from Table 1, the output of this bank is very low, e.g. with a relative low net interest revenue of 82.7. Another private bank, Deutsche Bank AG is the next least performing which is surprising given its assets and number of branches. We note that this efficiency of banks here is 'technical' without consideration to the ownership, size, age, no of branches etc of the firm. On the technical basis, referencing Grigorian & Manole (2002), conditions such as imperfect competition, prudential requirements and leverage concerns could be the driving force of the inefficiency of some banks. Also, in Germany, since the private sector banks are more subject to high market discipline, hence closer supervision and higher capital requirements than the public sector banks, the performance of these banks could be limited.

Table 4.2. The result of input and output models

DMU	IMCCR	IMRCCR	OMCCR	OMRCCR
1	0.575	0.471	1.739	2.124
2	0.607	0.497	1.647	2.012
3	0.692	0.566	1.446	1.767
4	0.617	0.500	1.622	2.002
5	0.752	0.612	1.365	1.668
6	0.951	0.779	1.052	1.285
7	1.000	0.819	1.000	1.222
8	0.884	0.724	1.131	1.382
9	1.000	0.819	1.000	1.222
10	0.104	0.086	9.575	12.000
11	1.000	0.819	1.000	1.222
12	1.000	0.819	1.000	1.222
13	0.924	0.756	1.082	1.322
14	0.746	0.610	1.341	1.638
15	0.728	0.596	1.374	1.679
16	0.875	0.717	1.142	1.396
17	0.973	0.796	1.028	1.256
18	0.741	0.607	1.349	1.648
19	0.857	0.701	1.168	1.426
20	1.000	0.819	1.000	1.222
21	1.000	0.819	1.000	1.222
22	0.900	0.736	1.112	1.358
23	0.851	0.696	1.176	1.436
24	0.885	0.725	1.130	1.380
25	0.723	0.592	1.384	1.691
26	0.938	0.768	1.066	1.303

A clearer picture of robust efficiency emerges when uncertainties are considered in the Banks' operations and data. The method developed here improve on efficiency measurement as decision-makers can trade-off efficiencies of the DMUs for robust performance under uncertain circumstances. Note that uncertainties are differently considered in some of the variables of the DMUs. Using the IMRCCR model (4.8), we observe from Table 4.3 that only two banks (Deutsche Bank Privat-und Geschaftskunden AG and KfW Ipex-Bank Gmbh) are robust efficient. The specific robust efficiency and hence the robust classification of banks, however, is obtained based on the conservative level of managers. The classification scheme designed Definition 4.1 considers different robust parameters ($\Gamma_j = 1$ for $\Gamma_j^x = 0$ and $\Gamma_j^y = 1$,

Table 4.3. Robust classification of banks

DMU	$\Gamma_j = 0$	$\Gamma_j = 1$	$\Gamma_j = 2$	$\Gamma_j = 3$	$\Gamma_j = 4$	$\Gamma_j = 5$	$\Gamma_j = 6$	Classification
1	0.575	0.534	0.502	0.497	0.479	0.479	0.479	RE^-
2	0.607	0.583	0.549	0.535	0.526	0.526	0.526	RE^-
3	0.692	0.657	0.628	0.628	0.625	0.625	0.625	RE^-
4	0.617	0.582	0.561	0.561	0.556	0.556	0.556	RE^-
5	0.752	0.667	0.653	0.653	0.651	0.651	0.651	RE^-
6	0.951	0.926	0.884	0.884	0.867	0.867	0.867	RE^-
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
8	0.884	0.842	0.833	0.832	0.832	0.832	0.832	RE^-
9	1.000	0.955	0.951	0.913	0.913	0.913	0.913	RE^-
10	0.104	0.100	0.094	0.094	0.092	0.092	0.092	RE^-
11	1.000	1.000	0.981	0.981	0.977	0.977	0.977	RE^+
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
13	0.924	0.888	0.866	0.866	0.866	0.866	0.866	RE^-
14	0.746	0.705	0.672	0.672	0.670	0.670	0.670	RE^-
15	0.728	0.658	0.624	0.624	0.624	0.624	0.624	RE^-
16	0.875	0.842	0.822	0.820	0.819	0.819	0.819	RE^-
17	0.973	0.935	0.917	0.914	0.913	0.913	0.913	RE^-
18	0.741	0.681	0.644	0.644	0.632	0.632	0.632	RE^-
19	0.857	0.829	0.800	0.797	0.797	0.797	0.797	RE^-
20	1.000	0.995	0.925	0.919	0.910	0.910	0.910	RE^-
21	1.000	0.960	0.956	0.921	0.921	0.921	0.921	RE^-
22	0.900	0.881	0.804	0.804	0.804	0.804	0.804	RE^-
23	0.851	0.824	0.797	0.797	0.797	0.797	0.797	RE^-
24	0.885	0.834	0.790	0.780	0.772	0.772	0.772	RE^-
25	0.723	0.687	0.649	0.624	0.624	0.624	0.624	RE^-
26	0.938	0.917	0.876	0.876	0.876	0.876	0.876	RE^-

$\Gamma_j = 2$ for $\Gamma_j^x = 1$ and $\Gamma_j^y = 1$, $\Gamma_j = 3$ for $\Gamma_j^x = 1$ and $\Gamma_j^y = 2$, $\Gamma_j = 4$ for $\Gamma_j^x = 2$ and $\Gamma_j^y = 2$, $\Gamma_j = 5$ for $\Gamma_j^x = 2$ and $\Gamma_j^y = 3$, and $\Gamma_j = 6$ for $\Gamma_j^x = 3$ and $\Gamma_j^y = 3$) which each result in an acceptable robust level efficiency. The last column of Table 4.3 indicates the robust classification of the DMUs; fully robust efficient banks, $RE^{++} = \{\mathrm{DMU}_7, \mathrm{DMU}_{12}\}$, partially robust efficient, $RE^+ = \{\mathrm{DMU}_{11}\}$ and the rest of the DMUs are robust inefficient DMUs, RE^-. Similar classification under Definition 4.2 holds for the output-oriented models.

Table 4.4. The result of duality relations

DMU	IMRCCR	IERCCR	OMRCCR	OERCCR
1	0.471	0.703	2.124	1.423
2	0.497	0.742	2.012	1.348
3	0.566	0.845	1.767	1.184
4	0.500	0.742	2.002	1.347
5	0.612	0.895	1.668	1.118
6	0.779	1.079	1.285	0.927
7	0.819	1.105	1.222	0.905
8	0.724	1.040	1.382	0.962
9	0.819	1.105	1.222	0.905
10	0.086	0.128	12.000	7.838
11	0.819	1.105	1.222	0.905
12	0.819	1.105	1.222	0.905
13	0.756	1.064	1.322	0.940
14	0.610	0.911	1.638	1.098
15	0.596	0.889	1.679	1.125
16	0.717	1.035	1.396	0.967
17	0.796	1.091	1.256	0.917
18	0.607	0.905	1.648	1.104
19	0.701	1.023	1.426	0.977
20	0.819	1.105	1.222	0.905
21	0.819	1.105	1.222	0.905
22	0.736	1.050	1.358	0.953
23	0.696	1.020	1.436	0.981
24	0.725	1.041	1.380	0.961
25	0.592	0.883	1.691	1.133
26	0.768	1.072	1.303	0.933

Returning to Table 4.2, the relationship between the input – and output – oriented robust models including the DEA efficiency under the IMCCR model (2.8) and OMCRR model (2.10) are observed. Since full uncertainty for all the variables and $\Gamma_j^x = 3$ and $\Gamma_j^y = 3$ is considered for all the DMUs, the results are more conservative. The efficiency under the IMCCR and OMCCR models are the equivalent with the robust models (4.8) and (4.22) respectively, when $\Gamma_j = 0$. More importantly, the optimal objective values of the IMRCCR models (4.8) and OMRCCR model (4.22) given in columns 3 and 5 of Table 4.2 confirm the equivalent relationship between the two models as established in Theorem 4.6. . Observe that, this relationship also works equivalently for the IERCCR model (4.14) and the OERCCR model (4.25). For each DMU, O_m^* expresses output enlargement while I_m^* describes input reduction rate in uncertain conditions. The relationship between the models supposes that like the traditional CCR models (see Cooper et al., 2006), the IMRCCR model will be efficient for any DMU if and only if it is also efficient when the OMRCCR model is used to evaluate its robust performance. On the other hand, the relationship between the robust CCR and their dual models expound an efficiency gap. The result in Table 4.4 shows that the input (output) -oriented multiplier robust CCR efficiency scores are different from the input (output) - oriented robust envelopment models. In fact, the pair columns show that the efficiency scores under the IERCCR model is greater than the efficiency scores under the IMRCCR and similarly, the efficiency scores from the OERCCR model is less than the efficiency scores from the OMRCCR. We conclude that the efficiency of robust envelopment models suffers from usual intuitive interpretation and must be used with caution.

4.6 Concluding remarks

We have introduced here a robust fractional DEA that is used to propose robust DEA models for efficiency measurement uncertainty. The major highlights from the models in the chapter are the following:

1. We proposed different robust models and studied their duality relations. We proved that the presence of uncertain data invalidates the linear duality principle as well as create efficiency gap between the multiplier robust DEA models and envelopment robust DEA models.
2. Complementary to Chapter 3 where it was showed that the traditional normalization constraint in the input-oriented BCC model which is in an equation form can be equivalently substituted by "≤" as similarly proved for the CCR model in (Toloo, 2014a), analogously, we proved that the normalization constraint in the output-oriented DEA model can be replaced by "≥".
3. Under the proposed models, a classification scheme that utilizes the conservativeness of the decision maker and interprets robust efficiency solutions accordingly were designed. According the scheme proposed, DMUs can be classified into fully robust efficient, partially robust efficient and robust inefficient.
4. A characterization of the efficiency solution for all DMUs in both input and output orientations is made and an equivalent relationship is established.
5. We showed the efficacy of the proposed models with an application on the performance of banks in Germany and demonstrated the applicable measurement of bank efficiency measurement under data uncertainty.

Chapter 5: Robust efficiency measurement under ellipsoidal uncertainty sets

Summary

The current chapter extends the conventional DEA models to a robust DEA framework by proposing new DEA models for evaluating the efficiency of a set of homogeneous decision-making units (DMUs) under ellipsoidal uncertainty set. There are four contributions that are made in this chapter:

i). We propose new robust CCR models based on two uncertainty sets: an ellipsoidal set that models uncertainty of epistemic type, and an interval-based ellipsoidal uncertainty set that models aleatory uncertainty. We study the relationship between robust DEA models of these two sets.

ii). We provide a robust classification scheme where DMUs can be classified into fully robust efficient, partially robust efficient and robust inefficient in the framework of ellipsoid protected based robust efficiency scores.

iii). The proposed models are extended to the additive DEA model, i.e. a robust additive DEA model is proposed, and its efficacy is analyzed with two imprecise additive DEA models in the literature. Numerical examples are provided.

iv). Finally, the practicability of the proposed models is demonstrated by studying the performance of the banking industry in Italy. Considering uncertainty in the banks' data, we showed that only a few banks are resilient in their performance and can be robustly classified as partially efficient or fully efficient compared to the DEA efficiency.

5.1 Introduction

In the past few years, different robust concepts such as the reliability of the robust solution (Ben-Tal & Nemirovski, 2000), the control of the price of robustness (Bertsimas & Sim, 2004), and the adjustable robustness (Ben-Tal, Goryashko, Guslitzer, & Nemirovski, 2004) among others have all been proposed. These concepts developed are largely based on two main uncertainty sets: ellipsoidal uncertainty set, and polyhedral uncertainty set or the so-called budget of uncertainty. Bertsimas & Sim (2004) budget of uncertainty is the more utilized for most of the RDEA modelling and is perhaps due to its advantage of preserving the linearity of the DEA model. Shokouhi et al (2010) proposed a general RDEA model in which inputs and outputs are constrained in an uncertainty set with data uncertainties covering the interval DEA approach. They used the robust approach of Bertsimas & Sim (2004) where they embraced Monte Carlo simulation to compute for the range of Gamma values for the conformity of the ranking of the DMUs. Omrani (2013) introduces an RDEA to find the common set of weights (CSW) in DEA with uncertain data using similar uncertainty set. In another paper, Arabmaldar et al. (2017) propose a robust super-efficiency DEA model by considering the uncertainty set of Bertsimas & Sim (2004).

This chapter focuses on the ellipsoidal uncertainty set introduced in Ben-Tal & Nemirovski (1999, 2000) to identify inefficiencies of DMU with the risk preference of the decision maker (DM). From the mathematical point of view, the ellipsoidal uncertainty set provides a convenient entity and offers the decision maker the ability to control the conservativeness of the efficiency solution to different data perturbations via the semi-axis of the ellipsoid. As aforementioned, almost all the applications of the robust optimization in uncertain DEA have dwelled on the budget of uncertainty of Bertsimas & Sim (2004). Models with ellipsoidal uncertainty, however, seem relatively unexplored. Sadjadi & Omrani (2008), Lu (2015) and Wu et al. (2017) are the few researchers who have made advances to RDEA considering uncertainty in an ellipsoid. However, all the proposed models have so far been limited to output data uncertainty due to the larger concern of considering input data uncertainty in the normalization constraint. Uncertainty in the equality constraint of the DEA models must be strictly satisfied to obtain a feasible solution for the RDEA counterpart. The issue is well addressed in Toloo (2014)a and Toloo & Mensah (2018) which subsequently enables this chapter to address uncertainties in both input and output data. Beyond our model formulation, this chapter also provides a classification scheme based on the proposed models. As such, we provide a scheme which allows DMUs to be classified into fully robust efficient, partially robust efficient and robust inefficient. Our robust approach is extended to the non-radial additive model. The newly proposed additive is compared with peer IDEA (Imprecise DEA) models proposed in Lee, Sam Park, & Kim (2002) and Matin, Jahanshahloo, & Vencheh (2007).

Structure of the chapter. In Section 5.2 we provide the background of the DEA and RO models. Two ellipsoidal uncertainty set – based RDEA models for both input and output uncertainty are developed and discussed in Section 5.3 with the extension to the additive model given in Section 5.4. A numerical example comparing the efficacy of the proposed with the IDEA models is given in this section. The penultimate section illustrates the applicability of the models with banking studies in Italy. Finally, we make conclusions and some further research direction in Section 5.6.

5.2 Robust optimization modeling

Consider the uncertain optimization problem

$$\begin{aligned}&\min_{x} f(\boldsymbol{x},\boldsymbol{\xi})\\&\text{s.t.}\\&F(\boldsymbol{x},\boldsymbol{\xi}) \in \mathrm{K} \subset \mathbb{R}^m\\&\boldsymbol{x} \geq \boldsymbol{0}_n\end{aligned} \tag{5.1}$$

where

- $\boldsymbol{x} \in \mathbb{R}^n$ is a vector of decision variables whose values are independent of the uncertain parameters.
- $\boldsymbol{\xi} \in \mathbb{R}^m$ is the data element of the optimization problem which is unknown at the time the values of $\boldsymbol{x}$ is being determined.
- $F(\boldsymbol{x},\boldsymbol{\xi}) \in \mathrm{K}$ is the constraints set.
- K is a convex cone.
- $\boldsymbol{0}_n$ is the origin in $\mathbb{R}^n$ space.

The RO methodology deals with the uncertain model (5.1) as a two-stage methodology: the first stage focuses on determining a deterministic uncertainty set for the uncertainty parameters whiles the second stage deals with solving a worst-case formulation known as the robust counterpart. The following is the robust counterpart for the model (5.1):

$$\min_{\boldsymbol{x}\in\mathbb{R}^n}\left\{\sup_{\xi\in\mathcal{U}} f(\boldsymbol{x},\boldsymbol{\xi}) : F(\boldsymbol{x},\boldsymbol{\xi}) \in \mathrm{K}\right\} \tag{5.2}$$

A vector $\boldsymbol{x}$ is called a robust feasible solution of model (5.1) if it satisfies all the possible realization of the constraints: $A(\boldsymbol{x},\boldsymbol{\xi}) \in \mathrm{K},\ \forall\boldsymbol{\xi} \in \mathcal{U}$. Typically, a robust solution of model (5.2) depends on the type of the uncertainty set, $\mathcal{U}$ used whose selection is motivated by the available information, user preference and the tractability of uncertainty set. Soyster (1973) considers the interval uncertainty set for the robust counterpart which leads to an aggressive conservatism of the robust solution. In this chapter, we consider the Ben-Tal & Nemirovski (1999) robust counterpart which is based on a $k-$ dimensional ellipsoidal uncertainty set:

$$\mathcal{U}_\Omega = \{\Phi(\boldsymbol{\xi})|\ \parallel \mathbf{P}\boldsymbol{\xi} \parallel_2 \leq \Omega\} \tag{5.3}$$

where

- $\boldsymbol{\xi} \rightarrow \Phi(\boldsymbol{\xi})$ is an affine embedding of $\mathbb{R}^m$ into $\mathbb{R}^k$,
- $\mathbf{P} \in \mathbb{R}^{n \times m}$ is a non-singular matrix of perturbations,
- $\|\cdot\|_2$ is the standard Euclidean norm[32], and
- Ω is the safety or robust parameter defined by the DM

The basic premise of the ellipsoidal uncertainty set adopted in Ben-Tal & Nemirovski (1998, 1999, 2000) is to control the risk tolerance of the DM by controlling the size of the ellipsoid via the parameter Ω. As the size of the ellipsoid increases, so does the risk aversion of the DM and vice versa. The approach overcomes the aggressive conservatism of the robust solution of Soyster (1973) and as shown in Ben-Tal & Nemirovski (1999), the robust counterpart with respect to this set although nonlinear has a tractable formulation in the form of second order quadratic programming. Specifically, the robust counterpart optimization over ellipsoidal uncertainty set or its intersection with the interval in most cases lead to computationally conic programs which has many solution algorithms (see Ben-Tal & Nemirovski, 2001; Grant et al, 2008).

5.3 Robust DEA models under ellipsoidal uncertainty sets

In this section, we will obtain different robust DEA models based on the ellipsoidal uncertainty sets proposed in Ben-Tal & Nemirovski (1999; 2000). We distinguish between two kinds of uncertainty: epistemic and aleatory. The following two ellipsoidal uncertainty sets will be used to derive a robust DEA model for the different types of uncertainties:

1. ellipsoidal uncertainty set and
2. box (interval) - based ellipsoidal uncertainty set.

The objective of our proposed robust models to these sets is to obtain efficiency solution to management decisions that can withstand data uncertainty while maintaining the performance of the DMUs. These sets are also practically useful for modelling correlation (if they exist) among the inputs (output) data which is relevant to prevent the effect of correlation on the efficiency mean (Farzipoor Sean, Memariani, & Lotfi, 2005). Specifically, the ellipsoidal uncertainty set is useful for the modelling of the epistemic type uncertainty and unbounded distribution of the uncertainty sets. On the other hand, the interval-based ellipsoidal set is practically useful for modelling aleatory uncertainty and bounded random distribution. Below, we discuss in detail the ellipsoidal uncertainty sets and their robust DEA formulation.

5.3.1 The usual ellipsoid case

Let $\mathbf{A} = \mathbf{A}^T$ be a positive definite matrix. The simplest case of an ellipsoid in (5.3) with $\Omega = 1$ represents the uncertainty for the matrix $\mathbf{A}$ such that:

[32] The standard Euclidean norm, $\|\cdot\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$

$$\mathcal{U}(\boldsymbol{\alpha}, \mathbf{A}) = \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x} - \boldsymbol{\alpha})^{\mathrm{T}} \mathbf{A}^{-1} (\boldsymbol{x} - \boldsymbol{\alpha}) \leq 1\} \tag{5.4}$$

where the vector $\boldsymbol{a} \in \mathbb{R}^n$ is the center of the ellipsoid and $\mathbf{A}$ is referred as the covariance matrix that defines the deviation of elements from the centre[33]. Let l_i be the axis-length of the ellipsoid, defined in the direction of π_i such that $\lambda_i = \left(\frac{1}{l_i}\right)^2$, where λ_i and π_i are respectively the eigenvalues and eigenvectors corresponding to the matrix $\mathbf{A}$. Figure 5.1 shows an ellipsoid in $\mathbb{R}^2$ (shaded) with centre $\boldsymbol{a}$ and axis – length l_1 and l_2 in the direction π_1 and π_2 . The ellipsoidal uncertainty sets form relatively a wide family. Here, we are concerned with the representation that can handle different cases of the ellipsoid including "ellipsoidal cylinders" and "flat"

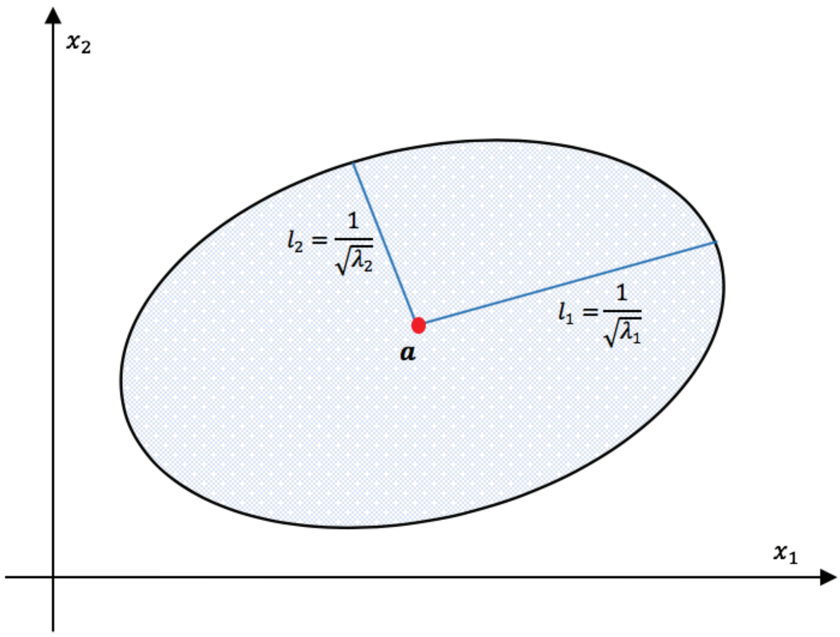


Figure 5.1. Geometry of an ellipsoid in $\mathbb{R}^2$.

ellipsoids such as points and intervals. This representation is equivalent to that used in Ben-Tal & Nemirovski (1999). Thus, throughout this section, our robust model will focus on the ellipsoid defined as:

$$\mathcal{U}_e = \{\boldsymbol{a} + \mathbf{P}\boldsymbol{u} | \ \|\boldsymbol{u}\|_2 \leq 1\} \tag{5.5}$$

where the Rank $(\mathbf{P}) = m \leq n$ is the *shape matrix* of the ellipsoid and $\boldsymbol{u} \in \mathbb{R}^n$. Note that if $\mathbf{P}$ is a symmetric positive definite matrix, the ellipsoid in (5.4) is identical to the expression in (5.5).

[33]Notice that $\mathbf{A}$ is symmetric positive definite, and we can obtain the real eigenvalues of the matrix using the Cholesky decomposition. The eigenvector π_i with eigenvalue λ_i of (5.4) represent the orientations of the principal axes of the ellipsoid. That is geometrically, π_i is the axis-vectors of the ellipsoid since it shows the direction of the ith axis of the ellipsoid.

Adapting the latter uncertainty set (5.5), we describe and model the dynamics of the uncertain input and output of DMU[34] in the following way.

First, let $(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)$ be the true values of the uncertain input and output data for DMU_j with maximum deviations, $(\hat{\boldsymbol{x}}_j, \hat{\boldsymbol{y}}_j)$ from their nominal values, $(\boldsymbol{x}_j, \boldsymbol{y}_j)$. Furthermore, let $\mathbf{P}^x = [\bar{\rho}_{ij}^x] \in \mathbb{R}^{m\times n}$, $\mathbf{P}^y = [\bar{\rho}_{rj}^y] \in \mathbb{R}^{s\times n}$, $\Xi^x = [\bar{\xi}_{ij}^x] \in \mathbb{R}^{m\times n}$, and $\Xi^x = [\bar{\xi}_{rj}^y] \in \mathbb{R}^{s\times n}$ where $\bar{\rho}_{ij}^x = \begin{cases}\rho_{ij}^x & \text{if } i \in I_j \\ 0 & \text{otherwise}\end{cases}$, $\bar{\rho}_{rj}^y = \begin{cases}\rho_{rj}^y & \text{if } r \in R_j \\ 0 & \text{otherwise}\end{cases}$, $\bar{\xi}_{ij}^x = \begin{cases}\xi_{ij}^x & \text{if } i \in I_j \\ 0 & \text{otherwise}\end{cases}$, $\bar{\xi}_{rj}^y = \begin{cases}\xi_{rj}^y & \text{if } r \in R_j \\ 0 & \text{otherwise}\end{cases}$. Then for all DMUs, the simple ellipsoid where $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{m+s}$ is described as:

$$\begin{aligned}\mathcal{U}_j^e &= \left\{ (\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{Y}}) \middle| \begin{matrix} \widetilde{\boldsymbol{X}} = (\mathbf{X} + \mathbf{P}^x \Xi^x), \left\|\boldsymbol{\xi}_j^x\right\|_2 \le 1 \\ \widetilde{\boldsymbol{Y}} = (\mathbf{Y} + \mathbf{P}^y \Xi^y), \left\|\boldsymbol{\xi}_j^y\right\|_2 \le 1 \end{matrix} ;\ j = 1, \dots, n \right\} \subset \mathbb{R}^{n\times(m+s)} \\ &= \left\{ (\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{Y}}) \middle| \begin{matrix} \tilde{x}_{ij} = x_{ij} + \sum_{i\in I_j} \rho_{ij}^x \xi_{ij}^x, & \left\|\boldsymbol{\xi}_j^x\right\|_2 \le 1 \\ \tilde{y}_{rj} = y_{rj} + \sum_{r\in R_j} \rho_{rj}^y \xi_{rj}^y, & \left\|\boldsymbol{\xi}_j^y\right\|_2 \le 1 \end{matrix} ;\ j = 1, \dots, n \right\}\end{aligned} \tag{5.6}$$

We refer to $\boldsymbol{\rho}_j^x$ and $\boldsymbol{\xi}_j^x$ ($\boldsymbol{\rho}_j^y$ and $\boldsymbol{\xi}_j^y$) as deviation vector defining the deviation of inputs (outputs) from their nominal values and the length of the inputs (outputs) vector, respectively, for DMU_j. Also, I_j and R_j represent the set of inputs and outputs of DMU_j that are subject to uncertainty and hence $I_j = \emptyset$ and $R_j = \emptyset$ present the case where there is no uncertainty in $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$. By definition, DMU_k with $(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is uncertain if there exist $i \in I_k$ or $r \in R_k$. Now following Wu et al., (2017), the weight vectors $\boldsymbol{u}$ or $\boldsymbol{v}$ is respectively mapped by the following relationships:

$$\boldsymbol{\xi}_j^x = \frac{(\boldsymbol{\rho}_j^x)^T \boldsymbol{v}}{\| (\boldsymbol{\rho}_j^x)^T \boldsymbol{v} \|_2}, \qquad \boldsymbol{\xi}_j^y = \frac{(\boldsymbol{\rho}_j^y)^T \boldsymbol{u}}{\| (\boldsymbol{\rho}_j^y)^T \boldsymbol{u} \|_2}$$

The RCCR model using the uncertainty set (5.6) considers the following theorem.

Theorem 5.1. *The robust CCR described under the ellipsoidal set (5.6) can be formulated by the following model:*

[34] The uncertainty description of the data here is epistemic uncertainty without random variations in the input and inputs data.

$$
\begin{aligned}
&\max\ w\\
&\text{s.t.}\\
&w - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \sqrt{\textstyle\sum_{r\in R_o} u_r^2 \hat{y}_{ro}^2} \le 0\\
&\textstyle\sum_{i=1}^{m} v_i x_{io} + \sqrt{\textstyle\sum_{i\in I_o} v_i^2 \hat{x}_{io}^2} \le 1\\
&\textstyle\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - \sqrt{\textstyle\sum_{r\in R_o} u_r^2 \hat{y}_{ro}^2} - \sqrt{\textstyle\sum_{i\in I_o} v_i^2 \hat{x}_{io}^2} \le 0 && (5.7)\\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sqrt{\textstyle\sum_{r\in R_j} u_r^2 \hat{y}_{rj}^2} + \sqrt{\textstyle\sum_{i\in I_j} v_i^2 \hat{x}_{ij}^2} \le 0 && \forall j \ne o\\
&v_i \ge 0 && i = 1, \dots, m\\
&u_r \ge 0 && r = 1, \dots, s
\end{aligned}
$$

Proof. Beginning with the fractional programming in Chapter 4, Section 2.2, the robust counterpart DEA with the ellipsoid in (5.6) (see details in Appendix C) is obtained by the following model:

$$
\begin{aligned}
&\max\ w\\
&\text{s.t.}\\
&w - \textstyle\sum_{r=1}^{s} u_r y_{ro} + \inf\limits_{\|\xi_o^y\|_2 \le 1} \left(\textstyle\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y \right)\\
&\textstyle\sum_{i=1}^{m} v_i x_{io} + \sup\limits_{\|\xi_o^x\|_2 \le 1} \left(\textstyle\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right) \le 1 && (5.8)\\
&\left(\textstyle\sum_{r=1}^{s} u_r y_{ro} + \inf\limits_{\|\xi_o^y\|_2 \le 1} \left\{\textstyle\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y \right\}\right) - \left(\textstyle\sum_{i=1}^{m} v_i x_{io} + \sup\limits_{\|\xi_o^x\|_2 \le 1} \left\{\textstyle\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right\}\right) \le 0\\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sup\limits_{\|\xi_j^y\|_2 \le 1} \sup\limits_{\|\xi_j^x\|_2 \le 1} \left\{\textstyle\sum_{r\in R_j} u_r \xi_{rj}^y \rho_{rj}^y + \sum_{i\in I_j} v_i \xi_{ij}^x \rho_{ij}^x\right\} \le 0 && \forall j \ne o\\
&v_i \ge 0 && \forall i\\
&u_r \ge 0 && \forall r
\end{aligned}
$$

Let us consider for each constraint a case and provide a simplified robust counterpart optimization.

Case 1: Robust counterpart for the first constraint:

$$
\begin{aligned}
\max_{\boldsymbol{u}\in\mathbb{R}^s} \inf_{\tilde{y}_o} \left\{\textstyle\sum_{r=1}^{s} u_r y_{ro} + \sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y \right\} &= \max_{\boldsymbol{u}\in\mathbb{R}^s} \left\{\textstyle\sum_{r=1}^{s} u_r y_{ro} + \inf\limits_{\|\xi_o^y\|_2 \le 1} (\boldsymbol{\xi}_o^y)^T (\boldsymbol{\rho}_o^y)^T \boldsymbol{u}\right\}\\
&= \max_{\boldsymbol{u}\in\mathbb{R}^s} \left\{\textstyle\sum_{r=1}^{s} u_r y_{ro} - \frac{\boldsymbol{u}^T \boldsymbol{\rho}_o^y (\boldsymbol{\rho}_o^y)^T \boldsymbol{u}}{\|(\boldsymbol{\rho}_o^y)^T \boldsymbol{u}\|_2}\right\}\\
&= \max_{\boldsymbol{u}\in\mathbb{R}^s} \left\{\textstyle\sum_{r=1}^{s} u_r y_{ro} - \frac{\|(\boldsymbol{\rho}_o^y)^T \boldsymbol{u}\|_2^2}{\|(\boldsymbol{\rho}_o^y)^T \boldsymbol{u}\|_2}\right\}\\
&= \max_{\boldsymbol{u}\in\mathbb{R}^s} \left\{\textstyle\sum_{r=1}^{s} u_r y_{ro} - \| (\boldsymbol{\rho}_o^y)^T \boldsymbol{u} \|_2\right\}
\end{aligned}
$$

Case 2: Robust counterpart for the second constraint:

$$
\begin{aligned}
\sup_{\tilde{x}_o} \left\{\textstyle\sum_{i=1}^{m} v_i x_{io} + \sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right\} &= \sup_{\|\xi_o^x\|_2 \le 1} \left\{\textstyle\sum_{i=1}^{m} v_i x_{io} + (\boldsymbol{\xi}_o^x)^T (\boldsymbol{\rho}_o^x)^T \boldsymbol{v}\right\}\\
&= \textstyle\sum_{i=1}^{m} v_i x_{io} + \| (\boldsymbol{\rho}_o^x)^T \boldsymbol{v} \|_2
\end{aligned}
$$

Case 3: Robust counterpart for the fourth constraint:

$$\sup_{\tilde{y}_j} \sup_{\tilde{x}_j} \left\{\sum_{r=1}^{s} u_r y_{rj} + \sum_{r\in R_j} u_r \xi_{rj}^{y} \rho_{rj}^{y} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{i\in I_j} v_i \xi_{ij}^{x} \rho_{ij}^{x}\right\}$$
$$= \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sup_{\left\|\xi_j^y\right\|_2 \le 1} \sup_{\left\|\xi_j^x\right\|_2 \le 1} \left\{(\boldsymbol{\xi}_j^y)^T (\boldsymbol{\rho}_j^y)^T \boldsymbol{u} + (\boldsymbol{\xi}_j^x)^T (\boldsymbol{\rho}_j^x)^T \boldsymbol{v}\right\}$$
$$= \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \| (\boldsymbol{\rho}_j^y)^T \boldsymbol{u} \|_2 + \| (\boldsymbol{\rho}_j^x)^T \boldsymbol{v} \|_2$$

The robust counterpart for the third constraint can be inferred from the fourth constraint. For each $r \in R_j$ we can write $\rho_{ij}^{y} = \hat{y}_{rj}$, with $\hat{y}_{rj} = \delta^y y_{rj}$ where δ^y is a given uncertainty level or percentage of perturbation. We write $\| \boldsymbol{\rho}_j^y \boldsymbol{u} \|_2 = \sqrt{\sum_{r\in R_j} u_r^2 \hat{y}_{rj}^2}$ for $r \in R_j$ and equivalently $\| \boldsymbol{\rho}_j^x \boldsymbol{v} \|_2 = \sqrt{\sum_{i\in I_j} v_i^2 \hat{x}_{ij}^2}$ for $i \in I_j$. Subsequently, we substitute cases 1 – 3 into model (5.8) which completes the proof. □

5.3.2 The combined interval and ellipsoid case

Suppose the input and output data of the DMUs are not only uncertain but are also random, i.e. they have a probability distribution. The type of uncertainty saturated by random data is known as aleatory uncertainty which can be modeled with probability theory (Ben-Tal & Nemirovski, 2000). Assume that the actual values of the input and output data are unknown, but the distribution of the random data is known to be symmetric in an interval. This entails the assumption that the true values, $(\tilde{x}_{ij}, \tilde{y}_{rj})$ of the uncertain input and output data are obtained from their nominal values through random perturbation:

$$\begin{aligned} \tilde{x}_{ij} &= (1 + \delta^x \zeta_{ij}^x) x_{ij} \\ \tilde{y}_{rj} &= (1 + \delta^y \zeta_{rj}^y) y_{rj} \end{aligned} \tag{5.9}$$

where $\{\zeta_{ij}^x\}_{i\in I_j}$ and $\{\zeta_{rj}^y\}_{r\in R_j}$ ($\zeta_{ij}^x = \zeta_{rj}^y = 0$ for $i \notin I_j, r \notin R_j$) are the independent random variables symmetrically distributed in the interval bound $[-1, 1]$ and where δ^x and δ^y are given uncertainty levels of the uncertain inputs and outputs. We use an interval-based ellipsoidal uncertainty set to describe and model the uncertainty dynamics in (5.9). See Ben-Tal & Nemirovski (2000). Considering the true definition of the uncertain data, $\tilde{x}_{ij}$ and $\tilde{y}_{ij}$, the uncertainty set is defined with both the $l_2 -$ norm and the infinity norm as follows:

$$\mathcal{U}_j^{i+e} = \left\{ (\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{Y}}) \middle| \begin{array}{ll} \tilde{x}_{ij} = (1 + \delta^x \zeta_{ij}^x) x_{ij}, & \left\|\boldsymbol{\zeta}_j^x\right\|_\infty \le 1, \left\|\boldsymbol{\zeta}_j^x\right\|_2 \le \Omega_j^x \\ \tilde{y}_{rj} = (1 + \delta^y \zeta_{rj}^y) y_{rj} & \left\|\boldsymbol{\zeta}_j^y\right\|_\infty \le 1, \left\|\boldsymbol{\zeta}_j^y\right\|_2 \le \Omega_j^y \end{array} ; j = 1, \ldots, n \right\} \tag{5.10}$$

where $\left\{\zeta_{ij}^x\right\}_{i\in I_j}$ and $\left\{\zeta_{rj}^y\right\}_{r\in R_j}$ are corresponding input and output data perturbations and Ω_j^x and Ω_j^y are the lengths of semi-axes of the ellipsoid for the uncertain input and output data

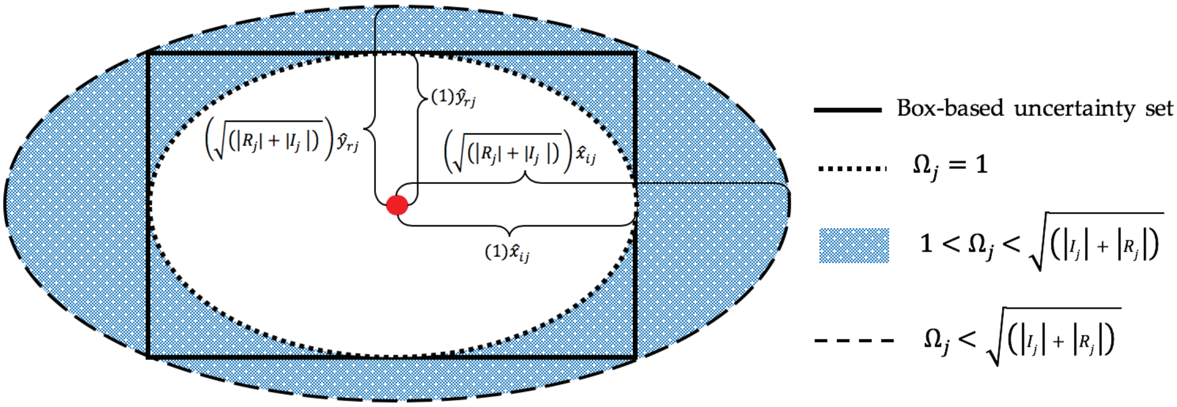


Figure 5.2. Illustration of feasible region for varying Ω_j values.

respectively. Let $\Omega_j = \Omega_j^x + \Omega_j^y$, where $\Omega_j \leq \left(|R_j| + |I_j|\right)^{0.5}$ depicts the allowable conservative preference of the DM and $|I_j|$ and $|R_j|$ are the cardinalities of the uncertain inputs and outputs respectively. According to Ben-Tal & Nemirovski (1999), with an allowable "uncertainty intervals" $\Delta_i = [x_{ij} - \hat{x}_{ij}, x_{ij} + \hat{x}_{ij}]$ and $\Delta_r = [y_{rj} - \hat{y}_{rj}, y_{rj} + \hat{y}_{rj}]$, the uncertainty set prescribed by Soyster (1973) is exactly the box $B = \left\{(\tilde{x}_{ij}, \tilde{y}_{rj}) \,\middle|\, |\tilde{x}_{ij} - x_{ij}| \leq \hat{x}_{ij};\ |\tilde{y}_{rj} - y_{rj}| \leq \hat{y}_{rj}\ \ i = 1, \dots, m;\ r = 1, \dots, s\right\}$. Therefore, the largest volume ellipsoid contained in the box occurs when $\Omega_j = 1$ and the smallest volume ellipsoid containing the box occurs when $\Omega_j = \left(|R_j| + |I_j|\right)^{0.5}$. Although it is possible to have $\Omega_j > \left(|R_j| + |I_j|\right)^{0.5}$, without loss of generality for the uncertain input and output data, we consider in this chapter that $0 \leq \Omega_j \leq \left(|R_j| + |I_j|\right)^{0.5}$. Figure 5.2 illustrates the different scenarios of the feasible region for the ellipsoid intersection with the box which is adapted similarly from Hanks, Weir, & Lunday (2017). To propose a robust DEA based on the uncertainty dynamics above, we consider the following lemma.

Lemma 5.1. *An uncertain problem with the constraint* $\boldsymbol{a}^T\boldsymbol{x} + \max_{\zeta \in \mathcal{U}_\infty} \boldsymbol{\zeta}^T\boldsymbol{x} \leq \boldsymbol{b}$ *under the interval type uncertainty set,* $\mathcal{U}_\infty = \{\boldsymbol{\zeta} |\ \| \boldsymbol{\zeta} \|_\infty \leq 1\}$ *has the following robust counterpart constraint:* $\sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{n} \hat{a}_j z_j \leq b\,, -z_j \leq x \leq z_j$.

Proof. See Yuan et al (2016)

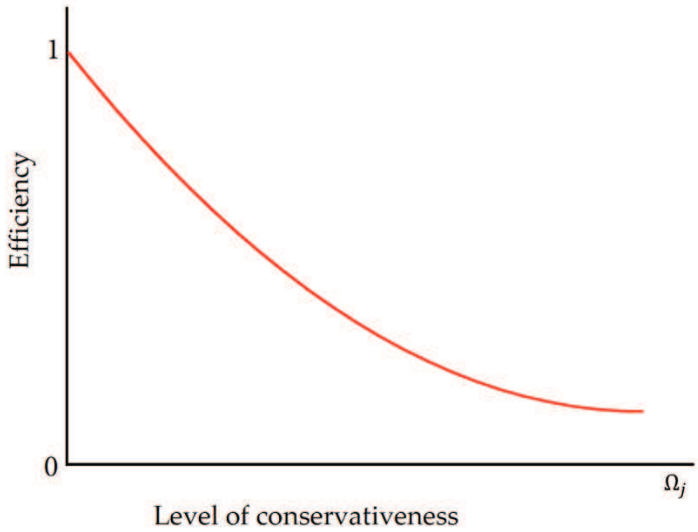


Figure 5.3. The trade-off between robustness and efficiency

Then using the uncertainty set (5.10) the proposed robust DEA is obtained according to the following theorem:

Theorem 5.2. *The robust CCR model described under the ellipsoidal set (5.11) is the following:*

$$
\begin{array}{ll}
\max z & \\
\text{s.t.} & \\
z - \sum_{r=1}^{s} u_r y_{ro} + \sum_{r \in R_o} \mu_r \hat{y}_{ro} + \Omega_o^y \sqrt{\sum_{r \in R_o} v_r^2 \hat{y}_{ro}^2} \leq 0 & \\
\sum_{i=1}^{m} v_i x_{io} + \sum_{i \in I_o} \lambda_{io} \hat{x}_{io} + \Omega_o^x \sqrt{\sum_{i \in I_o} v_{io}^2 \hat{x}_{io}^2} \leq 1 & \\
\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - \sum_{r \in R_o} \mu_r \hat{y}_{ro} - \sum_{i \in I_o} \lambda_{io} \hat{x}_{io} - & \\
\Omega_o^y \sqrt{\sum_{r \in R_o} v_{ro}^2 \hat{y}_{ro}^2} - \Omega_o^x \sqrt{\sum_{i \in I_o} v_{io}^2 \hat{x}_{io}^2} \leq 0 & \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{r \in R_j} \mu_r \hat{y}_{rj} + \sum_{i \in I_j} \lambda_{ij} \hat{x}_{ij} + & \\
\Omega_j^y \sqrt{\sum_{r \in R_j} v_{rj}^2 \hat{y}_{rj}^2} + \Omega_j^x \sqrt{\sum_{i \in I_j} v_{ij}^2 \hat{x}_{ij}^2} \leq 0 & j \neq o \\
-\mu_{rj} \leq u_r - v_{rj} \leq \mu_{rj} & \forall r \in R_j \\
-\lambda_{ij} \leq v_i - v_{ij} \leq \lambda_{ij} & \forall i \in I_j \\
v_i \geq 0 & i = 1, \dots, m \\
u_r \geq 0 & r = 1, \dots, s \\
\lambda_{ij}, \mu_{rj} \geq 0 & \forall i \in I_j, \forall r \in R_j
\end{array}
\quad (5.11)
$$

where μ_{rj} and λ_{ij} are auxiliary output and input variables; v_{rj} and v_{ij} are interval uncertainty parameters[35].

[35]Note that the RDEA to the ellipsoidal uncertainty set is practically tractable and convenient to handle with nonlinear solvers (such as Gurobi, MOSEK, BARON) and any efficient optimization software (see Sadjadi & Omrani 2008; Wu, Ding, Koubaa, Chaala, & Luo, 2017).

Proof. The proof follows similarly from Lemma 5.1 and Theorem 5.1

Models (5.7) and (5.11) provides certainty level, $\rho = \mathrm{P_r}(\boldsymbol{\zeta} \in \mathcal{U})$ that guarantees robust solutions for uncertain inputs and outputs. The robust model (5.11) has a feasible solution if all the constraints are satisfied with the probability guarantee $k = \exp(\Omega_j/2)$ (Ben-Tal & Nemirovski, 2000). Moreover, using model (5.11) the DM is at will to vary Ω_j according to his robust preference in the following:

- If $\Omega_j = 0$, the robust model shrinks to the nominal DEA problem.
- If $\Omega_j = 1$, the uncertainty denotes the largest volume of ellipsoid contained in the interval and
- If $\Omega_j = (|R_j| + |I_j|)^{0.5}$ it implies that the highest robust solution is sought for the uncertain inputs and outputs in the model. Here, since all the uncertain inputs and outputs are immunized or protected against.

Furthermore, the DM can seek different robust efficiency solution between $\Omega_j = 0$ and $\Omega_j = (|R_j| + |I_j|)^{0.5}$. It is important to note that the DM trade-off efficiency for robustness according to the tuning of Ω_j. Figure 5.3 depicts the relationship. For higher values of Ω_j, the robust efficiency scores decrease since increasing Ω_j implies that the uncertainty set is enlarged leading to high assurance for robustness. On the other hand, this implies worsening the performance of DMUs or paying the price for robustness (Bertsimas & Sim, 2004). The specific value of Ω_j to the model is therefore carefully chosen so as to avoid overly conservative solution.

Theorem 5.3: *The optimal objective values of model* (5.7) *is less than or equal to 1.*

Proof. Let $(w^*, \boldsymbol{v}^*, \boldsymbol{u}^*)$ be the optimal objective value of model (5.7). We have $w^* \leq \sum_{r=1}^{s} u_r^* y_{ro} - \sqrt{\sum_{r \in R_o} {u_r^*}^2 \hat{y}_{ro}^2}$ according to the first constraint of model (5.7). For every $(x_{ij}, y_{rj}), i \in I_o, r \in R_o$, we have $\sum_{r=1}^{s} u_r^* y_{ro} - \sqrt{\sum_{r \in R_o} {u_r^*}^2 \hat{y}_{ro}^2} \leq \sum_{r=1}^{s} u_r^* y_{ro} + \sqrt{\sum_{r \in R_o} {u_r^*}^2 \hat{y}_{ro}^2}$ and since $\sqrt{\sum_{r \in R_o} {v_r^*}^2 \hat{y}_{ro}^2}$ is nonnegative, taking the second and third sets of constraints arrive at $\sum_{r=1}^{s} u_r y_{ro} + \sqrt{\sum_{r \in R_o} v_r^{*2} \hat{y}_{ro}^2} \leq \sum_{i=1}^{m} v_i^* x_{io} + \sqrt{\sum_{i \in I_o} v_i^{*2} \hat{x}_{io}^2} \leq 1$ for each $j = o$. Consequently $w^* \leq 1$ which completes the proof. ☐

Theorem 5.4: *The optimal objective values of model* (5.11) *is less than or equal to 1.*

Proof: The proof is similar to Theorem 5.3. ☐

Remark 5.1. The relationship between models (5.7) and (5.11) are observed interestingly. When $\Omega_j = 1$, the ellipsoid is exactly inscribed by the box/interval as shown in Figure 5.2 and the optimal objective values of model (5.7) and model (5.11) are equal. Furthermore, if a DMU fully efficient in the later model is also efficient in the former model. The reverse, however, is not true.

Definition 3. A DMU_o is *R-efficient*, if and only if it satisfies the following two conditions:

(i) it is CCR-efficient and
(ii) $w^* = 1$ or $z^* = 1$.

Alternatively, a DMU that is CCR – inefficient *is* obviously robust – inefficient. Generally, the efficiency of model (5.11) is obtained through the parameter $\Omega_j \leq \left(|R_j| + |I_j|\right)^{0.5}$. Let ϑ_j^r be the optimal objective function value of model (5.11). The following definitions hold.

Definition 4. DMU_j is *R-efficient* if $\vartheta_j^r = 1$ for all values of Ω_j; otherwise, it is *R-inefficient.*

Definition 5. The performance of DMU_j is better than DMU_k if $\vartheta_j^r > \vartheta_k^r, \ \forall j, \forall k.$

Remark 5.2. If a DMU is characterized as inefficient by the robust model (5.11) then it is also characterized as inefficient by the robust model (5.7).

We provide some suggestions for the classification of DMUs under the RDEA by the following definition.

Definition 6. The robust efficiency for DMUs under the robust model (5.11) can be classified into three mutually exclusive subsets as follows.

- (*Full robust efficiency*). DMU_o is fully R-efficient if and only if $\vartheta_j^r = 1$ when $\Omega_j = \left(|R_j| + |I_j|\right)^{0.5}, \ \forall j.$
- (*Partial robust efficiency*). DMU_o is partially R- efficient if $\vartheta_j^r < 1$ when $\Omega_j = \left(|R_j| + |I_j|\right)^{0.5}$ and there exist $\Omega_j > 0$ such that $\vartheta_j^r = 1$
- (*Robust inefficiency*). A DMU_o is said to be R-inefficient if $\vartheta_j^r < 1$ for all $\Omega_j \in \left(0, \left(|R_j| + |I_j|\right)^{0.5}\right].$

Note that the classification of DMUs under the RDEA depends on the DM conservativeness level and risk preference. In other words, a robust ranking of DMUs using model (5.11) is based on the values of Ω_j which gives a further interpretation to robustness. For the classification scheme $(i) - (iii)$, denote by $RE^{++} \sim$ full R-efficiency, $RE^{+} \sim$ partial robust efficiency or PR-efficiency and $RE^{-} \sim$ R-inefficiency. The set RE^{++} consist of DMUs that are robust CCR efficient in any combination of uncertain inputs and outputs at all conservative level defined by the DM. The set RE^{+} consist of DMUs that R- efficient at maximal sense but there are certain conservative levels for inputs and outputs combinations for which they cannot maintain CCR- efficiency. Finally, the set RE^{-} consist of robust inefficient DMUs for all input and output combinations. The classification scheme above is analogous to the classification of interval DEA efficiency provided in Despotis & Smirlis (2002).

5.3.3 Numerical example with uncertain data

We study the proposed models with numerical data from Hatami-Marbini & Toloo (2017). Table 5.1 shows the input and output data which is assumed to be uncertain. In order to compare the robust models (5.7) and (5.11), we assume 5% perturbation of the input and output data from their nominal values. The results for the CCR model and robust CCR models are shown in Table 5.2. We considered for omega values in model (5.11), $\Omega_j = 0$, 0.5, 1, 2 and 2.8 which were chosen arbitrary reckoning that $\Omega_j = 0$ is equivalent to the CCR efficiency in column two of Table 5.2 and $\Omega_j = \left(|R_j| + |I_j|\right)^{0.5} \cong 2.8$ is the highest conservatism the DM can tolerate for the uncertain data. It can be observed that, except DMU_9, all other DMUs are CCR efficient. However, as uncertainty is considered in the data and Ω_j increases in the robust model (5.11), the efficiency scores decrease, a fact already indicated in Figure 5.3. shows the efficiency of DMUs as Ω_j increases from $\Omega_j = 0$ to $\Omega_j = 2.8$ at an interval of 0.4. Note that the robust efficiency of DMU_1, DMU_3, DMU_4, DMU_6 and DMU_{10} remain the same at 1 for all values of Ω_j. These DMUs are called R – efficient. Considering columns three and five of Table 5.2, it is evident the equivalency of the robust models (5.7) and (5.11) when the ellipsoid is inscribed by the box/interval. Thus, the robust efficiency scores of model (5.11) include that of model (5.7) at $\Omega_j = 1.0$, Here, DMUs which are R –inefficient in the later model are also R – inefficient in the former model. However, as mentioned earlier, it is possible that the maximum realization of the uncertain data may occur at the corners of the interval (see Figure 5.2) which means that model (5.11) can be more conservative and with higher complexity than

Table 5.1. Data for uncertainty analysis adapted from Hatami-Marbini & Toloo (2017)

DMU	Input				Output			
	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
1	32	50	82	46	47	93	54	65
2	61	56	68	37	88	56	92	80
3	42	58	45	34	94	65	80	80
4	73	39	88	81	50	53	93	97
5	45	38	68	41	47	42	70	52
6	86	62	44	32	86	45	100	47
7	38	74	71	74	83	91	62	74
8	61	54	70	62	79	60	72	98
9	84	52	38	47	85	68	51	41
10	87	47	31	52	78	95	70	92

Table 5.2. Results of CCR and RCCR models

DMU	θ^*	w^*	z^*				Classification
			$(\Omega_j = 0.5)$	$(\Omega_j = 1)$	$(\Omega_j = 2)$	$(\Omega_j = 2.8)$	
1	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
2	1.000	0.944	1.000	0.944	0.899	0.899	RE^{+}
3	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
4	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
5	1.000	0.884	0.959	0.884	0.842	0.853	RE^{-}
6	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
7	1.000	0.956	1.000	0.956	0.931	0.931	RE^{+}
8	1.000	0.875	0.947	0.875	0.839	0.839	RE^{-}
9	0.994	0.833	0.905	0.833	0.812	0.798	RE^{-}
10	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}

model (5.7) at full protection of the uncertain data. Although, it can be observed from Table 5.2 that the robust efficient DMUs in model (5.7) remain the same for model (5.11), the efficiency of the inefficient DMUs decrease in the latter model. The full classification of the DMUs under model (5.11) is shown in the last column of Table 5.2. It is observed that the R-efficient DMUs are $RE^{++} = \{\mathrm{DMU}_1, \mathrm{DMU}_3, \mathrm{DMU}_4, \mathrm{DMU}_6, \mathrm{DMU}_{10}\}$, the PR-efficient DMUs are $RE^{+} = \{\mathrm{DMU}_2, \mathrm{DMU}_7\}$ while finally, the R-inefficient DMUs are $RE^{-} = \{\mathrm{DMU}_5, \mathrm{DMU}_8, \mathrm{DMU}_9\}$. Note that the classification of the DMUs, right in own sense corresponds to the uncertainty considered in the input and output data.

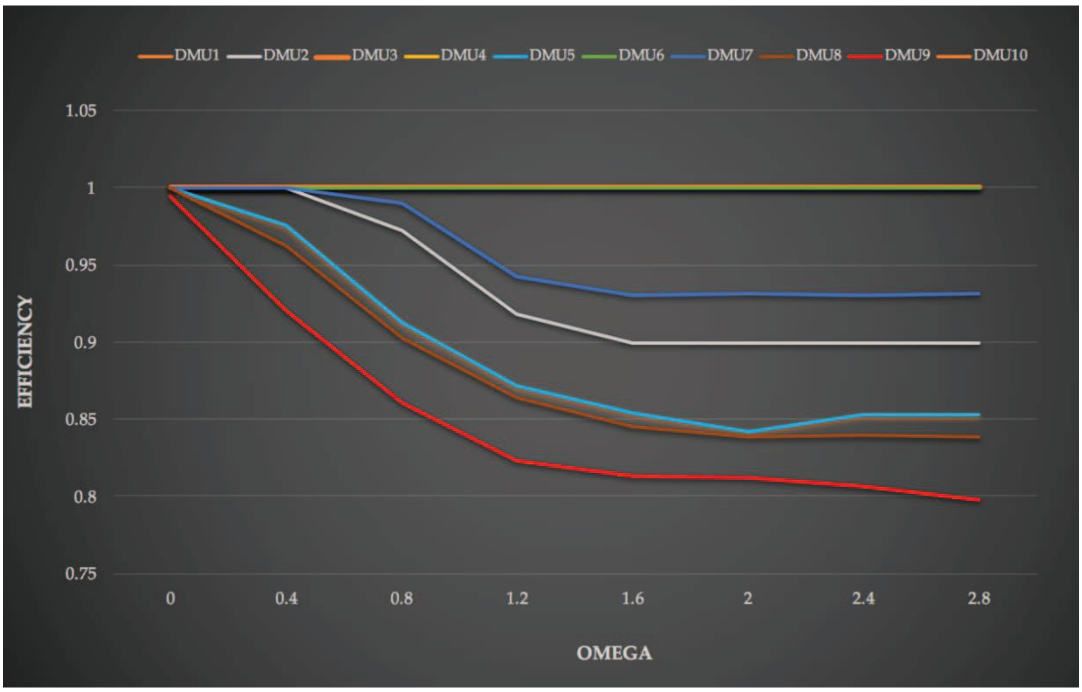


Figure 5.4. Efficiency scores of DMUs under different values of omega

5.4 Extension to the additive DEA model and imprecise data

In this section, we extend the robust approach to the additive (ADD) model with imprecise data and compare inefficiencies of the proposed model with imprecise additive models in literature.

5.4.1 The robust additive model

Consider the additive (ADD) model proposed in Charnes, Cooper, Golany, Seiford, & Stutz (1985) to evaluate the efficiency of DMUs:

$$\begin{aligned}
& s_o^* = \max \textstyle\sum_{j=1}^{m} s_i^- + \sum_{j=1}^{s} s_r^+ \\
& \text{s.t} \\
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{i0} && i = 1, \dots, m \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{r0} && r = 1, \dots, s \\
& \lambda_j \geq 0 && j = 1, \dots, n \\
& s_i^- \geq 0 && i = 1, \dots, m \\
& s_r^+ \geq 0 && r = 1, \dots, s
\end{aligned} \tag{5.12}$$

where s_i^- and s_r^+ are the slacks for the input and output respectively. In order to extend the RO to the additive model above, we first take the dual formulation in order to avoid possible infeasibility as a result of uncertainty analysis in the equality constraints of model (5.12). The dual of model (5.12) is expressed as the following:

$$\begin{aligned}
& \min \omega \\
& \text{s.t.} \\
& \omega - \textstyle\sum_{i=1}^{m} v_i x_{io} + \sum_{r=1}^{s} u_r y_{ro} \leq 0 \\
& \textstyle\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \geq 0 && j = 1, \dots, n \\
& v_i \geq 0 && i = 1, \dots, m \\
& u_r \geq 0 && r = 1, \dots, s
\end{aligned} \tag{5.13}$$

where ω is the efficiency of DMU_o. It is easily verifiable that $\omega^* \geq 0$; thus, an efficient point (x_{ij}, y_{rj}) will lie on the facet-defining hyperplane with equation $\sum_{i=1}^{m} v_i^* x_{io} - \sum_{r=1}^{s} u_r^* y_{ro} = 0$. Then, a DMU j is efficient if $\omega^* = 0$ and inefficient if $\omega^* > 0$ or alternatively, $\omega^* > 0$ and $(\boldsymbol{v}^*, \boldsymbol{u}^*) \geq \mathbf{1}_{m+s}$ measures the inefficiencies of the DMUs. In particular, to obtain an efficiency preserving unit to data perturbation, we consider the ellipsoidal-interval uncertainty defined in (5.10) and similarly to model (5.11) we propose the following robust additive model (RADD):

$$
\begin{array}{ll}
\min \varpi & \\
\text{s.t.} & \\
\varpi \ - \sum_{i=1}^{m} v_i x_{io} + \sum_{r=1}^{s} u_r y_{ro} + \sum_{i \in I_o} \lambda_{io} \hat{x}_{io} + & \\
\Omega_o^x \sqrt{\sum_{i \in I_o} v_{io}^2 \hat{x}_{io}^2} + \sum_{r \in R_o} \mu_{ro} \hat{y}_{ro} + \Omega_o^y \sqrt{\sum_{r \in R_o} v_{ro}^2 \hat{y}_{ro}^2} \ \leq 0 & \\
\sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s} u_r y_{ro} + \sum_{r \in R_o} \mu_{ro} \hat{y}_{ro} + \sum_{i \in I_o} \lambda_{io} \hat{x}_{io} & \\
\Omega_o^y \sqrt{\sum_{r \in R_o} v_{ro}^2 \hat{y}_{ro}^2} + \Omega_o^x \sqrt{\sum_{i \in I_o} v_{io}^2 \hat{x}_{io}^2} \geq 0 & \\
\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + \sum_{r \in R_j} \mu_{rj} \hat{y}_{rj} + \sum_{i \in I_j} \lambda_{ij} \hat{x}_{ij} & \\
\Omega_j^y \sqrt{\sum_{r \in R_j} v_{rj}^2 \hat{y}_{rj}^2} + \Omega_j^x \sqrt{\sum_{i \in I_j} v_{ij}^2 \hat{x}_{ij}^2} \geq 0 & j \neq o \\
-\mu_{rj} \leq u_r - v_{rj} \leq \mu_{rj} & \forall r \in R_j \\
-\lambda_{ij} \leq v_i - v_{ij} \leq \lambda_{ij} & \forall i \in I_j \\
v_i \geq 1 & i = 1, \dots, m \\
u_r \geq 1 & r = 1, \dots, s \\
\lambda_{ij}, \mu_{rj} \geq 0 & \forall i \in I_j, \forall r \in R_j
\end{array}
\tag{5.14}
$$

where ϖ is the robust additive efficiency of DMU_o.

5.4.2 Numerical example with exact and imprecise data

We intend to compare the RADD model proposed in this section to the imprecise additive models developed in Lee et al. (2002) and Matin et al. (2007) and so we use the numerical example given in Cooper et al. (1999) and presented in Table 5.3. The column headings indicate the data to be dealt with in ordinal and bounded forms as well as in the customary exact forms represented by the conditions $y_r \in D_r^+, x_i \in D_i^-$ where D_r^+ and D_i^-. DEA models described by these data are nonlinear and usually converted to linear standard DEA with exact data by using the transformation approach suggested in Zhu (2003). It is to be noted

Table 5.3. Exact and imprecise data adapted from Cooper et al. (1999)

	Inputs		Outputs	
	Exact	Bound	Exact	Ordinal
DMU	Cost	Judgment	Revenue	Satisfaction
j	x_{1j}	x_{2j}^a	y_{1j}	y_{2j}^b
1	100	[0.6, 0.7]	2000	4
2	150	[0.8, 0.9]	1000	2
3	150	1	1200	5
4	200	[0.7, 0.8]	900	1
5	200	1	600	3

[a]Ordinal ranking such that 5 = highest rank, …, 1= lowest rank (i.e. $y_{23} \geq \cdots \geq y_{24}$)
[b]Ratio bound based on the reference DMUs 3 or 5 (e.g., $0.6 \leq x_{21} \leq 0.7$ with $x_{23} = 1$)

Table 5.4. Retrieved exact data adapted from Lee et al. (2002)[c]

DMUs	x_{1j}	x_{2j}	y_{1j}	y_{2j}
1	100	0.7	2000	50
2	150	0.8	1000	20
3	150	1	1200	100
4	200	0.8	900	10
5	200	1	600	20

^{c}Exact data when DMU_j ($j = 2, 3, 4, 5$) is under-evaluation. Note: $\text{DMU}_{o\equiv 1}$ for $x_{21} = 0.6$

that, the robust model is not able to deal with ordinal and bounded data. The approach adopted in this chapter follows the transformation of bound and ordinal data in Table 5.3 to exact data by Lee et al. (2002). The result of the retrieved exact data is given in Table 5.4. Using this retrieved data, we compare the result from the RADD model to the two-stage imprecise additive model of Lee et al. (2002) and the one – stage imprecise additive model of Matin et al. (2007). Table 5.5 presents the inefficiency of DMUs proposed by the different methods. The efficiency of DMUs provided by the proposed robust model ($\Omega_j = 0$) indicated in Table 5.5 is the same as the former two methods where the RADD model yields larger scores for the inefficient DMUs and with higher discriminating power. The performance of DMUs on the three models are indifferent and their efficiency score according to Table 5.5 is ranked as follows:

$$\text{DMU}_1 \sim \text{DMU}_3 \succ \text{DMU}_2 \succ \text{DMU}_4 \succ \text{DMU}_5$$

where the symbol ''∼'' denotes ''indifferent to'' and the symbol ''≻'' denotes ''superior to''. It should be noted that the RADD model lightens the computational burden compared to the imprecise DEA models and provides the flexibility for controlling the conservativeness of solution to data perturbations. Thus, for some imprecise data, the proposed model in this chapter is more computationally effective and flexible in robustly ranking the efficiency of DMUs.

Table 5.5. Computed inefficiency with different additive models

DMUs	Lee at al. (2002)^d	Martin et al. (2007)^e	RADD model
1	0	0	0
2	1321.429	1050.1	1358.57
3	0	0	0
4	1200	1200	1518.57
5	2314.286	1500.3	2365.71

^{d}Result here are taken from Lee et al. (2002) two-stage approach.

^{e}Result here are taken from Matin et al. (2007) one-stage approach.

5.5 Application to banking efficiency in Italy

In order to demonstrate the real world application of the proposed robust CCR models under the ellipsoidal and interval-based ellipsoidal uncertainty, this section analyses the performance of banks operating in Italy with the proposed models.

5.5.1 Banking efficiency in Italy

The Italian banking market is of particular interest to measure robust efficiency, particularly for the analysis of uncertainties that have characterized the banking industry for some time now. The market is emerging from a prolonged period of distress following the global financial crises in 2008 and the slowdown of the Italian economy[36]. Notwithstanding the lengthy recessions, the banking system in Italy has shown enough resilience and recovering although, some of the largest mutual banks (the so-called "banche popolari") are still facing challenges with nonperforming loans (Giordano, Mbriani, & Lopes, 2013). While there exist enormous optimism among shareholders and investors on returns, it is important to note that Italian banks face competition in a global uncertain environment particularly in Europe which has meant that they have had to compare themselves to other commercial and universal banks in Europe and beyond. This requires that banks are operating efficiently locally and robustly under the changing environment.

Exploring the efficiency of banks in Italy, this study considers banks as decision-making units which consume inputs, for instance, a number of assets and working staff required to generate a certain amount of output level; interest on loans or overall revenue, etc. To examine the performance of these banks, data comprising 29 main banks in Italy for the accounting year 2015 were collected from the Bureau van Dick – Bankscope database (Bank scope 2015). The selected banks operate under a common set of rules and regulations set up by the Bank of Italy and by extension the Central Bank of Europe which implies that they have a common current denominator for which comparison of performance can be smoothly made. The selection of variables as input and output is done stem from the consideration of either the intermediation approach, production approach or the value-added approach of banking

[36] The banking crises that engulfed Italy and ongoing mildly can be attributed to two main sources. First is the financial market crises in 2008 that were caused by mortgage crises and largely the failure of the Lehman Brothers. The second one stem from the sovereign debt crises that affected Greece and some peripheral countries of the European monetary union: Italy, Spain, Portugal, Ireland. The Italian government through the bank of Italy in its supervisory capacity instituted measures such as the provision of liquidity, strengthening and supporting of banks, recapitalization of distress banks and including the so-called "Tremonti bond". The measures were to revitalize the banking industry, protect depositors and also finance the economy.

Table 5.6. Data for 29 major banks in Italy in 2015

Bank	Inputs			Outputs		
	Employees	Assets	Equity	Deposits	Loans	Revenue
B01	125510.00	860433.40	53485.50	163050.90	445293.90	12764.70
B02	90807.00	676496.00	48593.00	79743.00	350010.00	10259.00
B03	25731.00	169012.00	9622.70	28068.10	106680.40	2905.50
B04	16972.00	120509.60	8546.80	24078.00	71902.80	1843.70
B05	17718.00	117200.80	10517.80	16626.80	83815.70	1724.50
B06	13371.00	77494.50	5649.00	22222.70	60523.20	1750.30
B07	11447.00	61261.20	5651.80	7385.60	43702.60	1362.30
B08	8197.00	51373.20	5138.10	4290.60	36462.50	1017.50
B09	7743.00	50203.30	4647.40	9000.70	33953.90	870.50
B10	2651.00	44710.20	2070.10	8803.90	7430.60	322.20
B11	5273.00	39783.40	2552.10	9973.50	25068.30	584.70
B12	6019.00	37455.30	2479.70	6190.70	22649.40	509.30
B13	3195.00	35537.60	2649.40	3029.20	23290.40	547.10
B14	6263.00	33349.30	2153.40	6913.00	22012.20	556.70
B15	5034.00	30298.90	2489.10	3352.70	20395.20	383.80
B16	5868.00	27916.70	1687.20	449.90	18004.80	697.50
B17	4123.00	26901.70	2187.70	4194.10	18263.50	544.90
B18	3927.00	24186.20	1663.40	5811.20	19070.70	543.90
B19	3588.00	21861.10	2323.70	2323.10	18736.10	426.20
B20	3064.00	14968.20	1297.20	1661.60	13121.70	352.90
B21	3194.00	14809.50	1084.70	2680.60	9414.80	266.30
B22	916.00	13852.60	226.30	3731.30	4942.40	141.70
B23	2208.00	13545.30	1387.60	1467.40	12295.50	261.00
B24	2570.00	13205.90	1258.50	2137.90	7945.80	277.20
B25	1863.00	12276.90	1006.20	2375.20	6795.10	175.50
B26	2371.00	12248.10	922.10	905.20	9386.30	285.70
B27	1207.00	11769.20	750.10	1612.40	6394.80	175.40
B28	2443.00	11615.50	724.70	471.10	9328.80	292.30
B29	2989.00	10765.90	771.20	381.50	6734.00	184.50
Mean	13319.38	90863.50	6328.84	14583.86	52193.98	1449.20
SD	27280.19	192987.03	12690.11	32430.64	99685.54	2879.44
Max	916.00	10765.90	226.30	381.50	4942.40	141.70
Min	125510.00	860433.40	53485.50	163050.90	445293.90	12764.70

activities as discussed in Chapter 3 and 4. In Italy, Casu & Girardone (2002) examined the cost efficiency of banks conglomerates by assessing the cost characteristics of bank parent companies and their subsidiaries. Favoring the intermediation approach, they considered as inputs labor cost, deposits, and physical capital whiles total loans and other earning assets were used as outputs. Aiello & Bonanno (2016) considered the role of banks in Italy as an intermediary and used deposits, capital and labor as input factors whiles they used loans, securities and commission income as output factors. Within this context and following the survey of Mostafa (2009) in which deposit is mostly used as outputs, we select as input factors; employees, assets and equity and as output factors; deposits from banks, loans, and revenue. Table 5.6 shows the input and output factors and statistics for the Italian banks used for this study (see Table B. in Appendix B for details of the banks). All the inputs and outputs are expressed in monetary values. It is assumed that the actual values of some of the input and output factors are uncertain. A bank has uncertainty characterization if any of its input or output data for the performance measurement is uncertain. Here, we perceive uncertainty in banking data to be the result of errors from measurement and statistical computations and other errors such as from forecast values of loans, non-performing loans, deposit, etc. Following this development, we then apply the proposed models (5.7) and (5.11) to assess the robust performance of the banks.

5.5.2 Robust efficiency results

In the proposed robust models, we seek to obtain an acceptable performance level of banks by optimizing the worst-case values of the uncertain inputs and outputs values in the ellipsoids. For each bank, uncertainty is considered in some or all the inputs and outputs where the realization of their values are restricted to the uncertainty sets. We suppose that the inputs and outputs deviate from their nominal values by a percentage of perturbation, $\delta = 0.05$. The result of the model implementation is reported in Table 5.7. The third column shows the efficiency ranking by the IMCCR model (2.8) and the fifth and last column show the efficiency ranking by the robust models (5.7) and (5.11). Here, the result obtained in model (5.11) for $\Omega_j = \left(|R_j| + |I_j|\right)^{0.5} \cong 2.5$ indicates the highest conservativeness of decision makers which occurs at the full protection of the inputs and outputs against all uncertainties. Column 2 of Table 5.7 shows 8 banks with efficiency score equal to 1 which are efficient under the CCR model and two banks (B13 and B22) which are $\mathrm{R}-$efficient in the two robust models. As evidenced from Figure 5.3, the robust efficiency decreases relatively to the DEA efficiency which indicates the worst-case and reliable performance of the banks in uncertain conditions. The DEA result indicates an average overall technical efficiency (0.898) for the banks under study. Although the banks are performing averagely well, the number of banks which are efficient with or without uncertainty analysis is quite small. The lowest performing bank includes UniCredit SpA (B01) with an efficiency score of 0.738.

Table 5.7. Efficiency scores and ranking – CCR and RCCR models

Banks	IMCCR (2.8)	Rank	RCCR (5.7)	Rank	RCCR (5.11) [a]	Rank
B01	0.738	29	0.649	27	0.602	29
B02	0.769	26	0.648	28	0.629	26
B03	0.897	16	0.767	16	0.733	13
B04	0.847	21	0.729	21	0.692	20
B05	0.876	18	0.748	17	0.717	16
B06	1.000	1	0.963	4	0.947	3
B07	0.932	13	0.792	13	0.728	15
B08	0.899	15	0.780	15	0.713	18
B09	0.860	20	0.737	20	0.702	19
B10	0.815	22	0.676	24	0.667	21
B11	0.975	9	0.840	10	0.791	9
B12	0.785	25	0.667	25	0.642	25
B13	1.000	1	1.000	1	1.000	1
B14	0.874	19	0.748	18	0.714	17
B15	0.803	23	0.683	22	0.656	22
B16	1.000	1	0.859	8	0.833	8
B17	0.938	12	0.793	12	0.745	12
B18	1.000	1	0.921	5	0.850	6
B19	0.972	10	0.842	9	0.791	9
B20	1.000	1	0.882	7	0.840	7
B21	0.792	24	0.680	23	0.648	23
B22	1.000	1	1.000	1	1.000	1
B23	1.000	1	0.912	6	0.868	5
B24	0.887	17	0.738	19	0.643	24
B25	0.759	27	0.653	26	0.617	27
B26	0.963	11	0.829	11	0.765	11
B27	0.922	14	0.784	14	0.731	14
B28	1.000	1	0.969	3	0.900	4
B29	0.753	28	0.629	29	0.614	28

[a]Note that this result is obtained for $\Omega_j = 2.5$

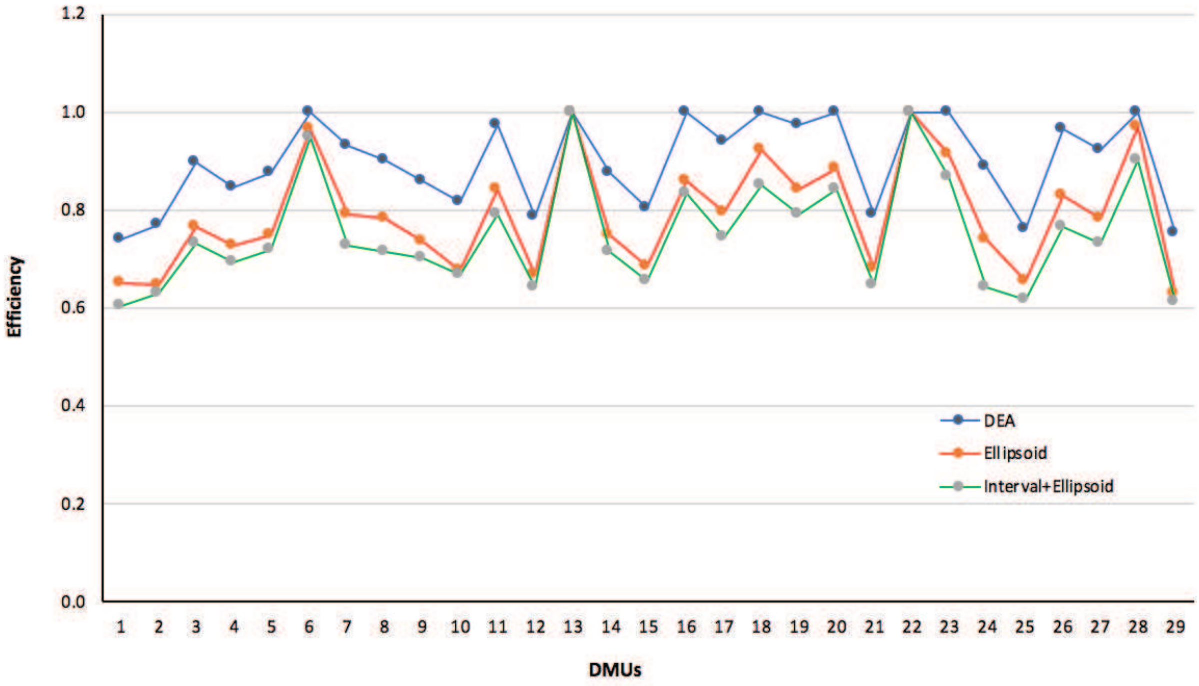


Figure 5.5. The result from ellipsoid and interval-based ellipsoid ($\Omega_j = 2.5$) sets

For the robust classification of banks, the robust parameter Ω_j is set to a range from 0 when no uncertainty in data is anticipated to $\Omega_j = 2.5$ when full protection for uncertainty is anticipated. The choice of appropriate Ω_j within this range is selected arbitrarily. Table 5.8 shows the result of the robust classification of the banks. In exchange for higher guaranteed robustness, higher values of Ω_j is selected. The efficiency of banks decrease as Ω_j increases and the DM can express preferences with different values of Ω_j and robust efficiency which is similar to the approach proposed in Ben-Tal & Nemirovski (2000) and Sadjadi & Omrani (2008). At full protection of the inputs and outputs, only B13 and B22 has full $\mathrm{R}-$ efficiency. Banks B06, B16, B18, B20, B23, and B28 are $\mathrm{PR}-$ efficient at different conservativeness level. The rest of the DMUs are $\mathrm{R}-$inefficient. The last column of Table 5.8 shows the classification of the banks as given in Definition 6. Considering the fact that many banks were inefficient in the traditional DEA evaluation, it is unsurprising the number efficient banks which are partially or fully robust efficient. As observed, 2 banks and 6 banks are fully or partially robust efficient at different levels from the 8 DEA efficient banks.

Table 5.8. Classification of banks based on the robust model (5.11)

Banks	$\delta = 0.05$							Classification
	$\Omega_j = 0$	$\Omega_j = 0.1$	$\Omega_j = 0.5$	$\Omega_j = 1.0$	$\Omega_j = 1.5$	$\Omega_j = 2.0$	$\Omega_j = 2.5$	
B01	0.738	0.728	0.690	0.649	0.611	0.603	0.602	RE^-
B02	0.769	0.755	0.704	0.648	0.629	0.629	0.629	RE^-
B03	0.897	0.882	0.827	0.767	0.741	0.733	0.733	RE^-
B04	0.847	0.835	0.786	0.729	0.693	0.693	0.692	RE^-
B05	0.876	0.863	0.810	0.748	0.717	0.717	0.717	RE^-
B06	1.000	1.000	1.000	0.963	0.951	0.947	0.947	RE^+
B07	0.932	0.916	0.856	0.792	0.749	0.735	0.728	RE^-
B08	0.899	0.886	0.837	0.780	0.732	0.713	0.713	RE^-
B09	0.860	0.847	0.796	0.737	0.703	0.703	0.702	RE^-
B10	0.815	0.799	0.739	0.676	0.667	0.667	0.667	RE^-
B11	0.975	0.960	0.905	0.840	0.798	0.794	0.791	RE^-
B12	0.785	0.772	0.723	0.667	0.649	0.642	0.642	RE^-
B13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
B14	0.874	0.859	0.807	0.748	0.715	0.715	0.714	RE^-
B15	0.803	0.790	0.740	0.683	0.664	0.657	0.656	RE^-
B16	1.000	1.000	0.931	0.859	0.836	0.834	0.833	RE^+
B17	0.938	0.922	0.859	0.793	0.766	0.759	0.745	RE^-
B18	1.000	1.000	0.985	0.921	0.873	0.858	0.850	RE^+
B19	0.972	0.957	0.904	0.842	0.791	0.791	0.791	RE^-
B20	1.000	1.000	0.960	0.882	0.854	0.848	0.840	RE^+
B21	0.792	0.780	0.733	0.680	0.651	0.648	0.648	RE^-
B22	1.000	1.000	1.000	1.000	1.000	1.000	1.000	RE^{++}
B23	1.000	1.000	0.988	0.912	0.885	0.875	0.868	RE^+
B24	0.887	0.870	0.806	0.738	0.698	0.668	0.643	RE^-
B25	0.759	0.748	0.704	0.653	0.621	0.617	0.617	RE^-
B26	0.963	0.948	0.893	0.829	0.784	0.767	0.765	RE^-
B27	0.922	0.906	0.847	0.784	0.758	0.748	0.731	RE^-
B28	1.000	1.000	1.000	0.969	0.916	0.900	0.900	RE^+
B29	0.753	0.739	0.688	0.629	0.614	0.614	0.614	RE^-

5.6 Concluding remarks

Robust solutions which is one of the most significant components of managerial efficiency decisions has become an integral research focus for many organizations faced with data imprecision and uncertainty. Therefore, achieving a robust solution with feasibility for both inputs and outputs uncertainty is very essential for productivity and efficiency analysis. In this chapter, we proposed new robust DEA models based on the ellipsoidal uncertainty and interval-based ellipsoidal uncertainty sets designed in Ben-Tal & Nemirovski (1999, 2000). This has been done in a manner that immunizes arbitrary uncertainties partly or in all inputs and outputs data simultaneously. By constraining the uncertain data in an ellipsoidal uncertainty sets, the models developed in this chapter become less pessimistic and in contrast offer advantage over the interval DEA models which mostly evaluate the performance of DMUs based on their extreme lower and upper bounds of the efficiency. A very important advantage derived from the proposed models is the ability to control the conservativeness of efficiency scores to different data perturbations via the radius of the ellipsoid. The models offer the DM the flexibility of controlling the level of robustness. Numerical examples illustrating the proposed models are given especially with a robust additive model which is compared with some IDEA models to show the efficacy, potential, and applicability of the robust additive model. Furthermore, the proposed robust models are applied for the evaluation and classification of banks in Italy. Using the proposed model, bank managers can now classify banks into fully robust efficient, partially robust efficient and robust inefficient units. It is notable that these models are input-oriented but can be extended to output-oriented and BCC models. In addition, incorporating uncertainties into the envelopment side of the additive model and extension to the slacks-based model can be considered topics for future research.

Chapter 6: Robust multi-objective transportation problem with network efficiency

Summary

Transportation problem (TP) deals with shipping products from several sources to several destinations which either minimizes the total transportation cost (min-type) or maximizes the total transportation profit (max-type) under the intrinsic assumption of certain data. A network efficiency measurement of the TP arises when shipment arcs involve multiple min-type (inputs) and multiple max-type (outputs) factors. DEA method is an optimization approach which can measure the network efficiency by assigning weight to each min-type and max-type factors and then maximizes the ratio of the weighted sum of max-type factors over the weighted sum of min-type factors. Given that different conflicting objectives under unknown conditions exist concurrently in practice, this chapter analyses the TP with network efficiency focus under the multiple objective linear programming (MOLP) framework. The DEA and MOLP are integrated to minimize arc inefficiencies and other min-type factors while maximizing max-type factors. We provide a linear programming robust model through goal programming (GP) approach in the presence of uncertain demands and supplies. Finally, we provide a numerical example to illustrate the applicability of the proposed model.

6.1 Introduction

The transportation problem (TP) is one of the intriguing yet nontrivial linear programming (LP) problems in operations research. The conventional single objective TP suggested by Hitchcock (1941) is a network-type structure that deals with shipping products (goods) from several sources to several destinations which either minimizes the total transportation cost or maximizes the total transportation profit. The TP modelled as multi-objective optimization problem can be solved by standard algorithms to enumerate all the non-dominated solutions (Isermann, 1979) or by interactive algorithms and compromising solution procedures (Ringuest & Rinks, 1987; Bit, Biswal, & Alam, 1992; El-Wahed & Lee, 2006).

Along a compendium of literature studies on the TP in recent times and indeed in real world situations, the following practical needs of the decision maker (DM) arises, which we seek to address in this chapter:

- Each possible shipment from the sources to the destinations entails several conflicting objectives in which the DM seeks a compromise solution from the multiple objectives (Li & Lai, 2000; Das, Goswami, & Alam, 1999; Gupta & Kumar 2012; Narayanamoorthy & Anukokila, 2014).
- Each product shipment may involve multiple inputs (min-type) and multiple outputs (max-type) factors where assessment of the performance of each shipment might be required (Chen & Lu, 2007; Amirteimoori, 2011).
- In practice, the parameters of the TP (i.e., cost, demand, supply, etc.) are precisely unknown in terms of delivery and quantity (including defected product). As a result, a robust approach is needed to immunize the uncertain parameters (Das et al., 1999; Gabrel, Lacroix, Murat, & Remli, 2014; Narayanamoorthy & Anukokila, 2014).

Several researchers have carried out investigations into the multiple objective transportation problem (MOTP) to deal separately with these thematic areas; in fact, the decision-making process has mainly focused on trade-off among conflicting objectives and desired compromising solutions rather than a complete set of non-dominated solutions. One of the effective strategies to generate a satisfactory compromise is to use the GP technique. Charnes, Cooper, & Ferguson (1955) proposed the GP concept in which they sought to study executive compensation plan by minimizing total deviations between realized goals and expected goals. The linear GP provides an analytical framework by which the DM can optimize multiple, conflicting objectives. The decision aspect in the design of goals makes the DM at least as important as the modeler in resolving conflict of accomplishment between specific transportation supply or demand goals and the possible maximization of profit (Abdelaziz 2007). However, rarely the transportation parameters are fixed in practice to satisfy requirements and generate the exact DM goals.

Uncertainties and imprecision in the parameters of multiple objective problems makes the mathematical expression difficult to solve with traditional methods since any feasible solution would have to consider the feasibility of the uncertain parameters. There are several approaches that have been proposed to deal with imprecision and uncertainties in MOLP, namely the robust optimization approach, stochastic programming, interval and fuzzy programming techniques. Kwak & Schniederjans (1985) were among the first authors to propose a generalized goal programming to overcome variations in the supply and demand requirement of the TP. Das et al. (1999) presented a MOTP where the interval cost, source, and destination parameters are solved with fuzzy programming technique. Fuzzy membership description of ambiguous transport parameters, e.g. fuzzy demand, fuzzy product and fuzzy cost have been extensively discussed in literature (see, Bit, Biswal, & Alam, 1992; Li & Lai 2000; Gupta & Kumar, 2012). A new treatment combines the fuzzy and goal programming techniques. For example, Zangiabadi & Maleki (2007) and Narayanamoorthy & Anukokila (2014) proposed fuzzy goal programming methods to deal with fuzzy goals and interval cost respectively. El-Wahed & Lee (2006) also combine the goal programming, fuzzy programming and interactive programming to provide a more realistic preferred compromise solution.

Robust optimization on the other hand is a set – based deterministic approach used to protect DM decisions against uncertainties and provide guarantees for stable and quality solutions. The approach uses uncertainty sets through which the uncertain parameters are immunized. The robust optimization literature is faced with three well known formulations that share similar minimal assumption of the uncertainty set, nonetheless with different ways of representation. The robust formulation of Soyster (1973) considers box uncertainty set to linear robust counterpart optimization, the Bertsimas & Sim (2004) polyhedral uncertainty set also lead to linear robust counterpart optimization while the Ben-Tal & Nemirovski (2000) use of ellipsoidal uncertainty set transform the LP to set of conic and quadratic programming. The primary objective of the uncertainty set design is to provide a guarantee for robust solution whenever the uncertain constraints are feasible in the set. It is imperative to know that no robust optimization approach has so far been considered for the MOTP. Gabrel, Lacroix, Murat, & Remli (2014) proposed a two-stage robust formulation for the location transportation problem with uncertain demand. The robust approach utilizes the Bertsimas & Sim (2004) cardinality-constrained and interval-based uncertainty set with Kelley's algorithm to iteratively search for an optimal solution. However, the extreme solution points generated by their recourse problem entails just a solution to a single objective TP. Kuchta (2004) developed a robust goal programming for the general multi-criteria programming using Bertsimas & Sim (2004) concept of robustness. Hanks et al. (2017) recently extended Kuchta's model to further leverage for the constraint uncertainties via the norm-based and ellipsoidal uncertainty set. The robust goal programming is applied to the multi-objective portfolio selection problem in Ghahtarani & Najafi (2013). Herein, we propose a robust goal model under tight dual formulation of the MOTP.

Moreover, we consider the efficiency of shipment arcs with incommensurate multiple inputs and multiple outputs, which in earlier research is suggested for the classical extended TP, the extended assignment problem and the extended shortest path problem (see Chen & Lu 2007; Amirteimoori, 2011; Amirteimoori, 2012). In these extended LP problems, DEA is employed for the aggregated performance measurement of the shipments plan from the source to the destinations. Keshavarz & Toloo (2014) and Keshavarz & Toloo (2015) studied the efficiency status of the feasible solutions of these extended problems in the MOLP and the DEA framework.

In our proposed approach, we adapt the initial origin- and destination-orientations efficiency measurement concept of Amirteimoori (2011) and provide a comprehensive approach to solve the MOTP in which network efficiencies, compromising goal target and robust methods are combined to generate realistic solution preferred by the DM. Rather than focusing only on the maximum efficiencies of outlets, we seek in addition to optimize the min- and max-type objective functions presented by each shipment arc. From the viewpoint of multi-objective optimization, we consider p min-type and q max-type objective functions including minimizing the inefficiencies of each arc. The chapter uses the technique of robust goal programming to generate a compromise solution of an equivalent LP for the uncertain MOTP.

Structure of the chapter. In Section 6.2, we provide the background of the study including introduction to the DEA. Section 6.3 describes our model formulation whiles Section 6.4 illustrate the robust goal programming as the solution method of this study. We conclude the chapter in Section 6.5.

6.2 Background

This section provides a brief background to the TP with network efficiency. It introduces the DEA approach as a multicriteria tool which further helps us to measure the efficiency on each transportation arc and develop a new MOTP in Section 3.

6.2.1 DEA for multi-criteria decision analysis

DEA is a multi-criteria optimization tool that is widely used for evaluating the relative efficiency of a set of decision-making units (DMUs) with multiple max- and min-type factors. At the heart of DEA is an LP that measures the relative efficiency of DMUs as the ratio of weighted sum of outputs to the weighted sum of inputs. The first two DEAs model, known as the CCR and BCC models are due to Charnes, Cooper, & Rhodes (1978) and Banker et al. (1984), respectively, under constant and variable returns-to-scale assumptions. Suppose there are N DMUs; DMU_J, $(J = 1, \dots, N)$ that use I inputs $(x_{iJ}, = 1, \dots, I)$, to produce R outputs $(y_{rJ}, r = 1, \dots, R)$. The IMCCR model for the relative efficiency of DMUs is given apriori follows:

$$
\begin{array}{ll}
\max \theta = \sum_{r=1}^{s} u_r y_{ro} & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io} = 1 & \\
\sum_{r=1}^{s} u_r y_{rJ} - \sum_{i=1}^{m} v_i x_{iJ} \leq 0 & J = 1, \dots, N \\
v_i \geq \varepsilon & i = 1, \dots, I \\
u_r \geq \varepsilon & r = 1, \dots, R
\end{array}
\tag{6.1}
$$

where the outputs weights u_r and input weights v_i are required to be greater than a small positive number ε (known as non-Archimedean infinitesimal) to forestall weights from being zero (see Toloo, 2014d). The DEA efficiency benchmarking in the CCR model is characterized by the identification of an efficient frontier determined by the non-dominated DMUs. In other words, DEA efficiency is a dominance-based concept which follows similar Pareto optimality conditions as the MOLP. A DMU is efficient (Pareto optimal) if and only if it is not possible to improve the performance of any input or output without worsening at least one other input or output. See Keshavarz & Toloo (2015) on further established efficiency status of the MOLP solutions as pertain to the DEA. The set of all efficient units is obtained by solving at least one optimization problem for each DMU in (6.1); whence $\theta = 1$ indicates a Pareto efficient unit. Figure 6.1 shows a unitized space for one unitized input and two output case where P_1P_2 indicates the piecewise DMU has efficient solution

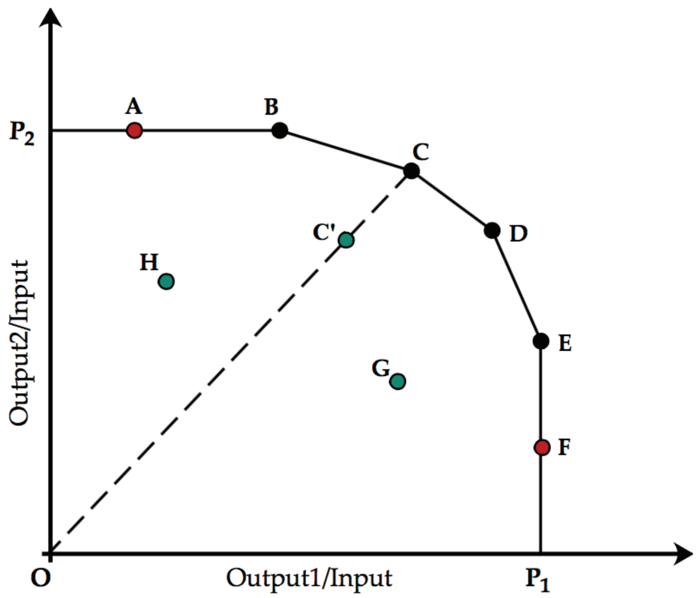


Figure 6.1. Efficiency of DMUs in unitized space.

linear efficient frontier. DMUs A and F are *weakly* efficient, usually obtained if the lower bound ε is removed and $\theta = 1$. Other DMUs on the frontier are Pareto efficient or *strongly* efficient. Notice that the distance measure CC' indicate an efficient projection of the dominated $\mathrm{DMU}_{\mathrm{C}'}$ to the frontier.

6.2.2 The transportation problem with arc efficiency

First, consider a transportation network made of m origions O_i $(i = 1, \dots, m)$ and n distinations D_j $(j = 1, \dots, n)$. For each transportation made, a supply quantity s_i from source i is dispatched which is received as demand d_j at destination j. It is clear that the supply and demand indicate physical quantities and must be nonnegative, i.e. $s_i \geq 0\ \forall i,\ d_j \geq 0\ \forall j.$ The process is represented as a network with m source nodes, n sink nodes, and a set of $m \times n$ directed arcs (links). Let $\mathbf{X} = [x_{ij}]_{m\times n}$ denote the matrix of decision variables, in other words, the quantity of goods to be transported from all the sources to all the destinations. The conventional TP is the problem of minimizing the total transportation cost of the whole distribution and is mathematically expressed as:

$$
\begin{array}{ll}
z(\mathbf{X}) = \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
\text{s.t.} & \\
\sum_{j=1}^{n} x_{ij} \le s_i & i = 1, \dots, m \\
\sum_{i=1}^{m} x_{ij} \ge d_j & j = 1, \dots, n \\
x_{ij} \ge 0 & i = 1, \dots, m,\ j = 1, \dots, n
\end{array}
\tag{6.2}
$$

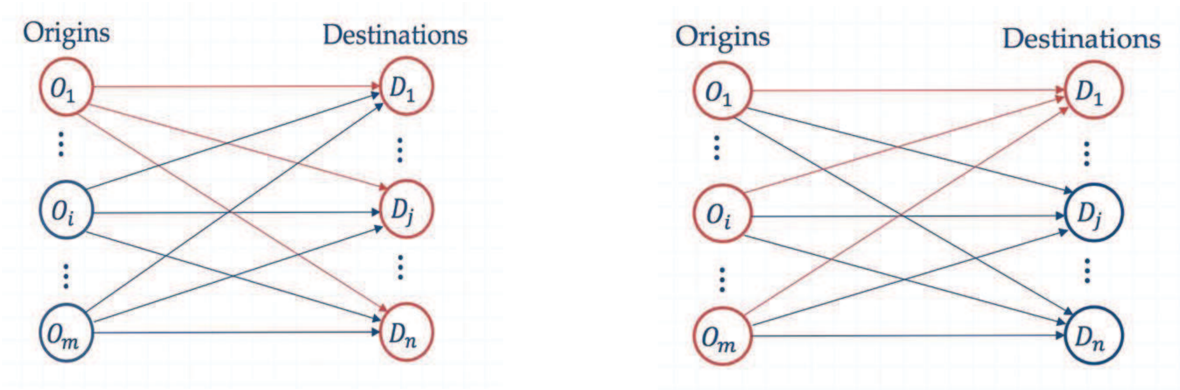
where c_{ij} represent the cost of transporting the products from source i to destination j. The condition $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$ is imposed for feasibility of the problem. The transportation problem considering arc efficiency developed under the generic name 'extended transportation problem' (ETP) is studied in Amirteimoori (2011). The ETP adopts DEA to measure efficiency of incommensurate multiple max-type and multiple min-type factors of transportation arcs. The idea was earlier introduced in Chen & Lu (2007) for the assignment problem[37]. The ETP considers a network system with single product made of multiple inputs and multiple outputs. Each arc (i, j), as a DMU, has T min-type factors (inputs) $\left(c_{ij}^{(1)}, c_{ij}^{(2)}, \dots, c_{ij}^{(T)}\right)$ and Q max-type factors (output) $\left(p_{ij}^{(1)}, p_{ij}^{(2)}, \dots, p_{ij}^{(Q)}\right)$ where $\left(c_{ij}^{(1)}, c_{ij}^{(2)}, \dots, c_{ij}^{(T)}\right) \ge \mathbf{0}_T$ and $\left(p_{ij}^{(1)}, p_{ij}^{(2)}, \dots, p_{ij}^{(Q)}\right) \ge \mathbf{0}_Q$. Two scenarios are used to measure the efficiency of each link; the origin – oriented scenario and destination – oriented scenario. The DEA model (6.3) measures the efficiency at the origin

$$
\begin{array}{ll}
\theta_{ij}^{(1)} = \max\ \sum_{q=1}^{Q} u_q p_{ij}^{(q)} & \\
\text{s.t.} & \\
\sum_{t=1}^{T} v_t\, c_{ij}^{(t)} = 1 & \\
\sum_{q=1}^{Q} u_q p_{il}^{(q)} - \sum_{t=1}^{T} v_t\, c_{il}^{(t)} \le 0 & l = 1, \dots, n \\
u_q \ge \varepsilon & q = 1, \dots, Q \\
v_t \ge \varepsilon & t = 1, \dots, T
\end{array}
\tag{6.3}
$$

[37] An extended assignment problem addresses each assignment of n jobs to n individuals with the maximum efficiency over profit or cost.

$$
\begin{aligned}
&\theta_{ij}^{(2)} = \max \ \textstyle\sum_{q=1}^{Q} u_q p_{ij}^{(q)} && \\
&\text{s.t.} && \\
&\textstyle\sum_{t=1}^{T} v_t \, c_{ij}^{(t)} = 1 && \\
&\textstyle\sum_{q=1}^{Q} u_q p_{kj}^{(q)} - \sum_{t=1}^{T} v_t \, c_{kj}^{(t)} \le 0 && k = 1, \dots, m \\
&u_q \ge \varepsilon && q = 1, \dots, Q \\
&v_t \ge \varepsilon && t = 1, \dots, T
\end{aligned} \tag{6.4}
$$

While the DEA model (6.4) measures the efficiency at the destination. Here the vector u_q and v_t are the outputs weights and input weights respectively and the origin- and destination-efficiency are given by $\theta_{ij}^{(1)}$and $\theta_{ij}^{(2)}$. Figure 6.2 shows the transportation arcs of the two-scenario efficiency measurement.



(a) Origin – oriented scenario (b) Destination – oriented scenario

Figure 6.2. Transportation problem with network efficiency

6.2.2.1 Amirteimoori's Approach

Amirteimoori (2011) considers the maximum efficiency of the transportation arcs over profit and cost. However, unlike the extended assignment problem which suggests the use of composite efficiency index as a performance measure of each assignment (Chen & Lu, 2007), Amirteimoori (2011) considers the averages of the relative efficiency of the two scenarios of the transportation arc and compute the efficiency of the network with the following model:

$$
\begin{aligned}
&\min \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} (1 - e_{ij}) x_{ij} && \\
&\text{s.t.} && \\
&\textstyle\sum_{j=1}^{n} x_{ij} = s_i && i = 1, \dots, m \\
&\textstyle\sum_{i=1}^{m} x_{ij} = d_j && j = 1, \dots, n \\
&x_{ij} \ge 0 && i = 1, \dots, m, \ j = 1, \dots, n
\end{aligned} \tag{6.5}
$$

where $1 - e_{ij}$ is the inefficiency score of arc (i, j) and $e_{ij} = 1/2\left(e_{ij}^{(1)} + e_{ij}^{(2)}\right)$. As a matter of fact, model (6.5) is a traditional transportation problem where the unit cost of shipping from factory i to warehouse j is $1 - e_{ij}$ and provides a single objective of maximum efficiency.

6.2.3 The drawback of the extended approaches

While the two scenarios of efficiency measurement are fascinating for operations research problems such as the transportation and assignment problems, the approaches of Chen & Lu (2007) – a composite efficiency defined as the product of $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$ and Amirteimoori (2011) – average efficiency e_{ij}, induce a non-multiple criteria optimization framework characterizing the non-dominated solution of the problems. The main drawback of these extended approaches are pointed out in Keshavarz & Toloo (2015) and Shirdel & Mortezaee (2015). Shirdel & Mortezaee (2015) provides a counterexample to Amirteimoori (2011) ETP to show that solutions generated in the later are not necessarily a non-dominated solution, thus given that the transportation problem is usually a multiple choice or multiple objective problem requiring Pareto optimal to alternative feasible solutions (Roy, Maity, Weber, & Gök, 2017). On the other hand, Keshavarz & Toloo (2015) provides practical example to show that the efficient solution in the extended assignment problem of Chen & Lu (2007) is dominated. Keshavarz & Toloo (2014) further provide a multi-criteria framework to classify all the efficient solutions of the assignment problem.

6.3 Proposed multi-objective transformation problem

In this study, we seek to consider the efficiency of the networks whilst optimizing the max-type and min-type objective functions of the transportation problem. Let $h_{ij}^{(k)} \geq 0$ represent k^{th} unit cost for shipping along link (i, j) for $k = 1, \dots, K, i = 1, \dots, m, j = 1, \dots, n$. Under the crisp environment with precise values, the MOTP is the problem of minimizing K objective functions, $z^k(\mathbf{X}); k = 1, \dots, K$, below;

$$\begin{array}{ll}
\min z^k(\mathbf{X}) = \sum_{i=1}^{m} \sum_{j=1}^{n} h_{ij}^{(k)} x_{ij} & k = 1, \dots, K \\
\text{s.t.} & \\
\sum_{j=1}^{n} x_{ij} \leq s_i & i = 1, \dots, m \\
\sum_{i=1}^{m} x_{ij} \geq d_j & j = 1, \dots, n \\
x_{ij} \geq 0 & i = 1, \dots, m, \ j = 1, \dots, n
\end{array} \tag{6.6}$$

where $\mathbf{X} \in [x_{ij}] \in \mathbb{R}^{m \times n}$ is the matrix of decision variables, $z(\mathbf{X}) = (z^1(\mathbf{X}), z^2(\mathbf{X}), \dots, z^K(\mathbf{X}))$ is the vector of K objective functions with $z(\mathbf{X}) \geq \mathbf{0}_K$. We consider the function $z(\mathbf{X})$ comprising T min-type functions $\boldsymbol{c}^t(\mathbf{X});\ t = 1, \dots, T$ and Q max-type functions $\boldsymbol{p}^q(\mathbf{X});\ q = 1, \dots, Q$. Since the DM is interested in obtaining a transportation plan with maximum efficiency, we also consider $\boldsymbol{\theta}^l(\mathbf{X}) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_{ij}^{(1)}x_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n}\theta_{ij}^{(2)}x_{ij}\right) \in \mathbb{R}^2_{(0,1]}$, $l = 1,2$ comprising two max-type (efficiency) functions as shown in Figure 6.2 (Amirteimoori, 2011). Note that the function $\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_{ij}^{(1)}x_{ij}$ shows the total origin-efficiency score corresponding to the feasible solution X. We propose an MOTP with the following max-type and min-type functions:

$$
\begin{array}{ll}
\max\ \boldsymbol{p}^q(\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n}p_{ij}^{(q)}x_{ij} & q = 1, \dots, Q \\
\min\ \boldsymbol{c}^t(\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}^{(t)}x_{ij} & t = 1, \dots, T \\
\max\ \boldsymbol{\theta}^l(\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\theta_{ij}^{(l)}x_{ij} & l = 1, 2 \\
\text{s.t.} & \\
\sum_{j=1}^{n}x_{ij} \le s_i & i = 1, \dots, m \\
\sum_{i=1}^{m}x_{ij} \ge d_j & j = 1, \dots, n \\
x_{ij} \ge 0 & i = 1, \dots, m,\ j = 1, \dots, n
\end{array}
\tag{6.7}
$$

Note that problem (6.7) maintains the same feasibility structure as problem (6.6) and contains $Q + 2$ max-type and T min-type factors. The problem solution treating the max-type and min-type functions as multiple objectives in multiple dimensions has no unique solution. However, there are equally good mathematical solutions in the following: non-dominated solution, efficient solution, compromise solution, preferred solution, ideal solutions.

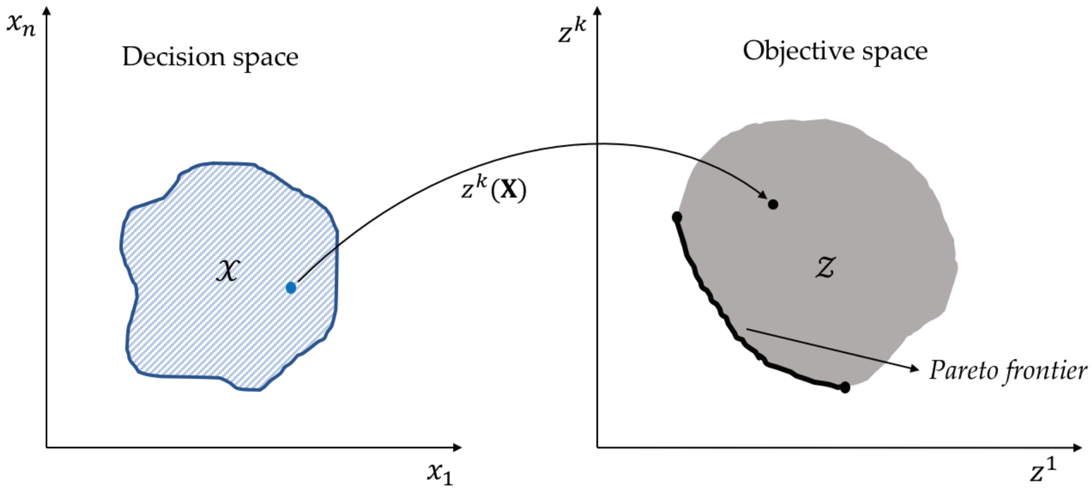


Figure 6.3. Mapping of decision space into objective space

Let $\mathcal{X} = \left\{\mathbf{X} \middle| \sum_{j=1}^{n}x_{ij} \le s_i, \forall i; \sum_{i=1}^{m}x_{ij} \ge d_j, \forall j\right\} \subset \mathbb{R}^{m\times n}$ denotes the feasible region in the decision space and subsequently $= \{(\boldsymbol{p}^1(\mathbf{X}), \dots, \boldsymbol{p}^Q(\mathbf{X}), \boldsymbol{c}^1(\mathbf{X}), \dots, \boldsymbol{c}^T(\mathbf{X}), \boldsymbol{\theta}^1(\mathbf{X}), \boldsymbol{\theta}^2(\mathbf{X})) : \mathbf{X} \in \mathcal{X}\} \subset \mathbb{R}^{Q+T+2}$ represents the feasible region in the objective space. See Figure 6.3. We define the following solutions;

Definition 6.1 (*Efficient/non-dominated solution*) A feasible matrix $\boldsymbol{X}^* \in \mathcal{X}$ is said to be non-dominated, efficient or Pareto optimal solution of (6.7) if there is no other feasible solution $\boldsymbol{X} \in \boldsymbol{\mathcal{X}}$ such that $\left(-\boldsymbol{p}^q(\boldsymbol{X}^*),\ \boldsymbol{c}^t(\boldsymbol{X}^*), -\boldsymbol{\theta}^l(\boldsymbol{X}^*)\right) \leq \left(-\boldsymbol{p}^q(\boldsymbol{X}),\ \boldsymbol{c}^t(\boldsymbol{X}),\ -\boldsymbol{\theta}^l(\boldsymbol{X})\right)$ $t = 1, \dots, T;\ , q = 1, \dots, Q;\ \ l = 1{,}2$ and $\left(\boldsymbol{p}^q(\boldsymbol{X}),\ \boldsymbol{c}^t(\boldsymbol{X}),, \boldsymbol{\theta}^l(\boldsymbol{X})\right) \neq \left(\boldsymbol{p}^q(\boldsymbol{X}^*),\ \boldsymbol{c}^t(\boldsymbol{X}^*), \boldsymbol{\theta}^l(\boldsymbol{X}^*)\right)$ for some $t, q\ and\ l$. Otherwise, the point $\boldsymbol{X}^*$ is not efficient.

The set of all efficient solution denoted by $\mathcal{X}_E$ is generally called a complete solution and the image of $\mathcal{X}_E$ in $\mathcal{Z}$ forms the non-dominated frontier $\mathcal{Z}_N$.

Definition 6.2 (*Ideal solution*) The ideal solution to (6.7) is a point $\boldsymbol{X}_i \in \mathcal{X}$ such that the objective values $(\boldsymbol{p}^1(\boldsymbol{X}_i), \dots, \boldsymbol{p}^Q(\boldsymbol{X}_i);\ \boldsymbol{c}^1(\boldsymbol{X}_i), \dots, \boldsymbol{c}^T(\boldsymbol{X}_i);\ \boldsymbol{\theta}^1(\boldsymbol{X}_i), \boldsymbol{\theta}^2(\boldsymbol{X}_i))$ are optimal for each sub problem max $\boldsymbol{p}^q(\boldsymbol{X})$, min $\boldsymbol{c}^t(\boldsymbol{X})$ and max $\boldsymbol{\theta}^l(\boldsymbol{X})$ subject to the constraints in (6.7).

Definition 6.3 (*Compromise solution*) A feasible vector $\boldsymbol{X}^0 \in \mathcal{X}$ is called a compromise solution of (6.7) if and only if $\boldsymbol{X}^0 \in \mathcal{X}_E$ and $\boldsymbol{c}^t(\boldsymbol{X}^0) \leq \wedge_{\boldsymbol{X} \in \mathcal{X}} \boldsymbol{c}^t(X)$; $\boldsymbol{p}^q(\boldsymbol{X}^0) \geq \vee_{\boldsymbol{X} \in \mathcal{X}} \boldsymbol{p}^q(\boldsymbol{X})$ and $\boldsymbol{\theta}^l(\boldsymbol{X}^0) \geq \vee_{\boldsymbol{X} \in \mathcal{X}} \boldsymbol{\theta}^l(\boldsymbol{X})$ where $\wedge$ and $\vee$ stands for “minimum” and “maximum”.

If the compromise solution meets the DMs maximum preferences (i.e. taking into consideration the various objective values), then it is called optimal compromise solution. On the one hand, it is practically impossible to enumerate all non-dominated solutions for most MOTPs. Therefore, it is only important to we concern ourselves with an optimal compromise solution which is the closest solution to the ideal point.

6.4 Solution methods

The common solution methods used by many researchers for the MOTP are fuzzy programming (Bit, Biswal, & Alam, 1992; Das et al., 1999; Li & Lai, 2000), GP and iterative approaches (El-Wahed & Lee, 2006), and sometimes a combination of them (Zangiabadi & Maleki, 2007; Narayanamoorthy & Anukokila, 2014). Generally, most popular a posteriori method for solving multi-objective optimization problems are the scalarization, ϵ −constraint or the goal programming method. However, for the MOTP, the goal programming is one of the main ideas portrayed in several alternative approaches to obtain a compromise solution. In this chapter, an efficient solution using the goal programming technique is explored. Later in this work, a robust goal programming is formulated to obtain realistic decision based on different uncertain scenarios of the uncertain parameters in the MOTP.

6.4.1 Algorithm for the MOTP

The general steps to solve the proposed MOTP model with GP and robust optimization approaches are described below.

> ***Step 1***. Measure the efficiency scores θ_{ij}^1 and θ_{ij}^2 for each arc (i, j) with both origin- and destination-oriented models (6.3) and (6.4), respectively.
> ***Step 2***. Build an MOTP with the benchmark efficiencies in *Step 1*.
> ***Step 3***. Solve the built model in *Step 2* as an LP using GP.
> ***Step 4.*** Determine the aspiration level of the DM from the ideal points of the MOTP model in *Step* 2.
> ***Step 5.*** If the uncertain parameters are found at the right hand of the LP, then find the dual of the goal programming model in *Step 3*; otherwise go to *Step 6*.
> ***Step 6***. Define the protection function for the uncertain parameters and solve the model using the linear robust optimization technique of Bertsimas and Sim (2004)[38].

6.4.2 Goal programming approach

The GP provides a special analytical approach to properly address the transportation problem with incommensurate max-type and min-type factors. It guarantees a compromise multi-criteria framework by which DM can optimize multiple, conflicting objectives concurrently to achieve satifiable solutions (Larbani & Aouni, 2011; El-Wahed & Lee, 2006). The GP has been applied to several real-world problems such as in finance, health, economics etc. where a multiple objectives decision aid tool was needed. The basic idea is to assign a specific goal (or aspirations level) g_t for each objective function $z^k(\mathbf{X})$, and then minimize the total deviations, $|z^k(\mathbf{X}) - g_k|$ from their target values.

For the MOTP with max-type and min-type factors, there are three different set of goals corresponding to each objective function in (6.7), set as below,

$$\text{DM Goals} = \begin{cases} \boldsymbol{p}^q(\mathbf{X}) \geq g_q & q = 1, \dots, Q \\ \boldsymbol{c}^t(\mathbf{X}) \leq \dot{g}_c & t = 1, \dots, T \\ \boldsymbol{\theta}^l(\mathbf{X}) \geq \ddot{g}_l & l = 1, 2 \end{cases}$$

where $g_q, \dot{g}_c$ *represents the aspiration levels of the* Q *max-type,* T *min-type functions while* $\ddot{g}_l$ *represent the goal for network efficiency.*

[38] The robust GP via the norm-based uncertainty set or the ellipsoidal uncertainty can be used at this stage (see Hanks, Weir, & Lunday, 2017).

The MOTP (6.7) can be written as the following goal model:

$$\begin{array}{ll}
\text{Min } \sum_{q=1}^{Q} w_q d_q^- + \sum_{t=1}^{T} \dot{w}_t \dot{d}_t^+ + \sum_{l=1}^{2} \ddot{w}_l \ddot{d}_l^- & \\
\text{s.t.} & \\
\boldsymbol{p}^q(\mathbf{X}) + d_q^- - d_q^+ = g_p & q = 1, \dots, Q \\
\boldsymbol{c}^t(\mathbf{X}) + \dot{d}_t^- - \dot{d}_t^+ = \dot{g}_t & t = 1, \dots, T \\
\boldsymbol{\theta}^l(\mathbf{X}) + \ddot{d}_l^- - \ddot{d}_l^+ = \ddot{g}_l & l = 1,2 \\
\sum_{j=1}^{n} x_{ij} \le s_i & i = 1, \dots, m \\
\sum_{i=1}^{m} x_{ij} \ge d_j & j = 1, \dots, n \\
d_q^-,\ d_q^+, \dot{d}_t^-,\ \dot{d}_t^+,\ \ddot{d}_l^-,\ \ddot{d}_l^+ \ge 0 & \forall q,\ \forall t,\ \forall l
\end{array} \tag{6.8}$$

where the variables $d_q^-,\ d_q^+, \dot{d}_t^-,\ \dot{d}_t^+,\ \ddot{d}_l^-,\ \ddot{d}_l^+$ indicate respectively the positive and negative deviations between the achievement levels $\boldsymbol{p}^q(\mathbf{X})$, $\boldsymbol{c}^t(\mathbf{X}), \boldsymbol{\theta}^l(\mathbf{X})$ as well the aspiration levels $g_q, \dot{g}_c, \ddot{g}_l$; and $w_q, \dot{w}_t,\ \ddot{w}_l$ are the Euclidean normalised weights respectively. Note that the normalized weight, for instance $w_q = a_q/\eta_q$ where a_q is the DM chosen weights and η_q is the Euclidean norm of the former is necessary for the commensurability of the goals. The DM aspirations can be determined by considering ideal solution points of model (6.7). We therefore require for the DM, an efficient solution that is close as possible to the ideal points of the defined objective functions. Assuming that the feasible region $\mathcal{X}$ is non-empty, compact and convex, and the objective functions $\boldsymbol{p}^q(\mathbf{X}), \forall q$; $\boldsymbol{c}^t(\mathbf{X}), \forall t$; $\boldsymbol{\theta}^1(\mathbf{X}), \boldsymbol{\theta}^2(\mathbf{X})$ are continuous in $\mathcal{Z}$; more formally, the efficient solution status to model (6.8) can be described by the following proposition which is a direct result of Larbani & Aouni's (2011) theorem.

Proposition 6.1. *Assume that $\mathcal{X}$ is compact and the functions $\boldsymbol{p}^q(\boldsymbol{X}), \boldsymbol{c}^t(\boldsymbol{X}), \boldsymbol{\theta}^l(\boldsymbol{X}), \forall q, \forall t, \forall l$ are continuous. Let $\overline{\mathbf{X}}$ be a solution vector to the GP formulation (6.8). Then any optimal solution of the following program is an efficient solution*

$$\max_{\mathbf{X} \in \bar{\mathcal{X}}} \ \sum_{q=1}^{Q} \boldsymbol{p}^q(\mathbf{X}) - \sum_{t=1}^{T} \boldsymbol{c}^t(\mathbf{X}) + \sum_{l=1}^{2} \boldsymbol{\theta}^l(\mathbf{X}) \tag{6.9}$$

where $\quad \bar{\mathcal{X}} = \left\{ \mathbf{X} \in \mathcal{X} \,\middle|\, \left(\boldsymbol{p}^q(\overline{\mathbf{X}}),\ -\boldsymbol{c}^t(\overline{\mathbf{X}}),\ \boldsymbol{\theta}^l(\overline{\mathbf{X}})\right) \le \left(\boldsymbol{p}^q(\mathbf{X}),\ -\boldsymbol{c}^t(\mathbf{X}),\ \boldsymbol{\theta}^l(\mathbf{X})\right), \forall q, \forall t, \forall l \right\}.$
Furthermore, if $\overline{\mathbf{X}}$ is an optimal solution to problem (6.9)*, then it is efficient; otherwise it is not efficient and it is dominated by all the optimal solutions of problem* (6.9).

6.4.3 Robust goal programming approach

Until now, the MOTP has been analyzed under crisp environment in which feasible solutions and Pareto optimal are quite often guaranteed. In practice, the MOTP involves uncertainties which may arise from ambiguity of the transport data or imprecise knowledge of demand or supply due to defected products, weather conditions, etc. and any feasible solution will also require feasibility of the uncertain constraint or objective (Ehrgott, Ide, & Schöbel, 2014). In this chapter, uncertainties are considered in some demands and supplies. We define the DM uncertainty as interval values of the demand and supply. Formally, let I and J indicate the set of uncertain demands and supply, respectively. Each true value $\tilde{s}_i \in I$ and $\tilde{d}_j \in J$ are symmetric and bounded random variables that take values in $[s_i - \hat{s}_i, s_i + \hat{s}_i]$ and $[d_j - \hat{d}_j, d_j + \hat{d}_j]$ respectively where $\hat{s}_i$ and $\hat{d}_j$ represent the maximum deviation from the nominal values s_i and d_j. We describe a robust solution that is feasible for all *possible realizations* of these uncertain parameters. Denoting $\mathcal{U}(\Gamma_1, \Gamma_2)$ as the *budget of uncertainty set*, we follow closely the robust model of Bertsimas & Sim (2004) where Γ_1, Γ_2 called the *budget of uncertainties parameters* are the maximum range of values allowed for the uncertain demand and supply, respectively, to simultaneously deviate from their nominal values. Define

$$\mathcal{U}(\Gamma_1, \Gamma_2) = \left\{ (\tilde{s}_i,\ \tilde{d}_j) \in \mathbb{R}^{|I \cup J|} \middle| \begin{array}{l} \tilde{s}_i = s_i\ + \xi_i \hat{s}_i, \forall i \in I,\ \xi_i \in \mathbb{Z}(\Gamma_1) \\ \tilde{d}_j = d_j + \xi_j \hat{d}_j, \forall j \in J, \xi_j \in \mathbb{Z}(\Gamma_2) \end{array} \right\} \tag{6.10}$$

where

$$\mathbb{Z}(\Gamma_1) = \{(\ \xi_i) \in \mathbb{R}^{|I|} |\ \ \textstyle\sum_{i \in I} \xi_i \leq \Gamma_1, |\ \xi_i| \leq 1, \forall i\} \tag{6.11}$$

$$\mathbb{Z}(\Gamma_2) = \{(\ \xi_j) \in \mathbb{R}^{|J|} |\ \ \textstyle\sum_{j \in J} \xi_j \leq \Gamma_2, |\ \xi_j| \leq 1, \forall j\} \tag{6.12}$$

Under the assumption that all the uncertain parameters will not take their worst-case values, a certain level of deviation is allowed (Bertsimas & Sim, 2004). Accordingly, the robust indicators Γ_1 and Γ_2 are restricted to intervals $\Gamma_1 \in [0, |I|]$ and $\Gamma_2 \in [0, |J|]$. For $\Gamma_1 = \Gamma_2 = 0$, the DM's value for the demand and supply is equal to their nominal values while $\Gamma_1 = |I|$ and $\Gamma_2 = |J|$ (or equivalently $\Gamma_1 + \Gamma_2 = |I \cup J|$) implies that the worst-case values of demand and supply are considered. Notice that the uncertain parameters of model (6.8) are on the right sides (i.e. demand and supply), we transfer the uncertainties from the right-hand sides to the objective function coefficient using the dual program (Gabrel & Murat, 2010). The dual model (6.8) is given below.

$$\begin{array}{ll}
\max \sum_{q=1}^{Q} g_q y_q - \sum_{t=1}^{T} \dot{g}_t \dot{y}_t + \sum_{l=1}^{2} \ddot{g}_l \ddot{y}_l + \sum_{i=1}^{m} s_i \lambda_i - \sum_{j=1}^{n} d_j \gamma_j & \\
\text{s.t.} & \\
\sum_{q=1}^{Q} p_{ij}^{(q)} y_q - \sum_{t=1}^{T} c_{ij}^{(t)} \dot{y}_t + \sum_{l=1}^{2} \theta^{(l)} \ddot{y}_l + \lambda_i - \gamma_j \leq 0 & \forall i, \forall j \\
y_q \leq w_q & \forall q \\
\dot{y}_t \geq -\dot{w}_t & \forall t \\
\ddot{y}_l \leq \ddot{w}_l & \forall l \\
\text{all variables are nonnegative} &
\end{array} \tag{6.13}$$

Considering the uncertainty set in (6.10) – (6.12) and using Bertsimas & Sim (2004) approach, the robust goal programming is formulated as follow

$$
\begin{aligned}
&\max z \\
&\text{s.t.} \\
&z - \textstyle\sum_{q=1}^{Q} g_q y_q + \sum_{t=1}^{T} \dot{g}_t \dot{y}_t - \sum_{l=1}^{2} \ddot{g}_l \ddot{y}_l - \sum_{i=1}^{m} s_i \lambda_i \\
&+ \textstyle\sum_{j=1}^{n} d_j \gamma_j + \beta(\boldsymbol{\lambda}, \boldsymbol{\gamma}, \Gamma_1, \Gamma_2) \le 0 \\
&\textstyle\sum_{q=1}^{Q} p_{ij}^{(q)} y_q - \sum_{p=1}^{P} c_{ij}^{(t)} \dot{y}_t + \sum_{l=1}^{2} \theta^{(l)} \ddot{y}_l + \lambda_i - \gamma_j \le 0 && \forall i, \forall j \\
&y_q \le w_q && \forall q \\
&\dot{y}_t \ge -\dot{w}_t && \forall t \\
&\ddot{y}_l \le \ddot{w}_l && \forall l \\
&\text{all variables are nonnegative}
\end{aligned}
\tag{6.14}
$$

where

$$
\beta(\boldsymbol{\lambda}, \boldsymbol{\gamma}, \Gamma_1, \Gamma_2) = \max_{\substack{\{S_1 \cup \{t_1\} |\ S_1 \subseteq I, |S_1| = \lfloor \Gamma_1 \rfloor, t_1 \in I \setminus S_1\} \\ \{S_2 \cup \{t_2\} |\ J_1 \subseteq J, |S_2| = \lfloor \Gamma_2 \rfloor, t_2 \in J \setminus S_2\}}} \Big\{ \textstyle\sum_{i \in S_1} \hat{s}_i \lambda_i + (\Gamma_1 - \lfloor \Gamma_1 \rfloor) \hat{s}_{it_1} + \sum_{j \in S_2} \hat{d}_j \gamma_j + (\Gamma_2 - \lfloor \Gamma_2 \rfloor) \hat{d}_{jt_2} \Big\}
\tag{6.15}
$$

If Γ_1 and Γ_2 are chosen as integer values, then we have

$$
\beta(\boldsymbol{\lambda}, \boldsymbol{\gamma}, \Gamma_1, \Gamma_2) = \max_{\substack{\{S_1 \cup \{t_1\} |\ S_1 \subseteq I, |S_1| = \lfloor \Gamma_1 \rfloor\} \\ \{S_2 \cup \{t_2\} |\ J_1 \subseteq J, |S_2| = \lfloor \Gamma_2 \rfloor\}}} \Big\{ \textstyle\sum_{i \in S_1} \hat{s}_i \lambda_i + \sum_{j \in S_2} \hat{d}_j \gamma_j \Big\}
\tag{6.16}
$$

Notice that model (6.14) is nonlinear. A linear model formulation is obtained in the following way (Bertsimas & Sim, 2004; Toloo & Mensah, 2018).

The function in (6.15) is equal to the following the linear optimization problem,

$$
\begin{aligned}
&\beta(\lambda_i^*, \gamma_j^*, \Gamma_1, \Gamma_2) = \max\ \textstyle\sum_{i \in I} \hat{s}_i |\lambda_i^*| \xi_i + \sum_{j \in J_2} \hat{d}_j |\gamma_j^*| \xi_j \\
&\text{s.t.} \\
&\textstyle\sum_{i \in I} \xi_i \le \Gamma_1 \\
&\textstyle\sum_{j \in J} \xi_j \le \Gamma_2 \\
&0 \le \xi_i \le 1 && \forall i \in I \\
&0 \le \xi_j \le 1 && \forall j \in J
\end{aligned}
\tag{6.17}
$$

The optimal value of model (6.17) is made up of $(\Gamma_1 + \Gamma_2)$ variables equal to 1 and two variables at $\Gamma_1 - \lfloor \Gamma_1 \rfloor$ and $\Gamma_2 - \lfloor \Gamma_2 \rfloor$ which are equivalent to the selection of the subset with objective functions maximized in (6.15). The dual of model (6.17) is the following:

$$
\begin{aligned}
&\min \textstyle\sum_{i \in I} P_i + z_1 \Gamma_1 + \sum_{j \in J} W_j + z_2 \Gamma_2 \\
&\text{s.t.} \\
&P_i + z_1 \ge \hat{s}_i |\lambda_i^*| && \forall\, i \in I \\
&W_j + z_2 \ge \hat{d}_j |\gamma_j^*| && \forall\, j \in J \\
&P_i \ge 0 && \forall i \in I \\
&W_j \ge 0 && \forall\, j \in J \\
&z_1, z_2 \ge 0
\end{aligned}
\tag{6.18}
$$

Now given that model (6.15) is feasible and bounded for all $\Gamma_1 \in [0, |I|\,]$ and $\Gamma_2 \in [0, |J|\,]$ respectively, by strong duality, the dual model (6.18) is also feasible and bounded. Therefore model (6.13) can be reformulated in the following linear form:

$$
\begin{array}{ll}
\max z & \\
\text{s.t.} & \\
z - \sum_{q=1}^{Q} g_q y_q + \sum_{t=1}^{T} \dot{g}_t \dot{y}_t - \sum_{l=1}^{2} \ddot{g}_l \ddot{y}_l - \sum_{i=1}^{m} s_i \lambda_i & \\
+ \sum_{j=1}^{n} d_j \gamma_j + \sum_{i \in I} P_i + z_1 \Gamma_1 + \sum_{j \in J} W_j + z_2 \Gamma_2 \leq 0 & \\
\sum_{q=1}^{Q} p_{ij}^{(q)} y_q - \sum_{p=1}^{P} c_{ij}^{(t)} \dot{y}_t + \sum_{l=1}^{2} \theta^{(l)} \ddot{y}_l + \lambda_i - \gamma_j \leq 0 & \forall i, \forall j \\
P_i + z_1 \geq \hat{s}_i u_i & \forall\, i \in I \\
W_j + z_2 \geq \hat{d}_j v_j & \forall\, j \in J \\
y_q \leq w_q & \forall q \\
\dot{y}_t \geq -\dot{w}_t & \forall t \\
\ddot{y}_l \leq \ddot{w}_l & \forall l \\
P_i \geq 0 & \forall\, i \in I \\
W_j \geq 0 & \forall\, j \in I \\
-u_i \leq \lambda_i \leq u_i & \\
-v_j \leq \gamma_j \leq v_j & \\
z_1 \geq 0,\ z_2 \geq 0 &
\end{array}
\tag{6.19}
$$

6.5 Concluding remarks

The ETP is a network-type structure with shipment arcs involving multiple inputs and multiple outputs. In this chapter, we analyze through the multi-objective framework the transportation problem with multiple objectives and multiple inputs and multiple output data. The concept of DEA and MOLP were utilized to provide unified linear programming model while also ensuring a robust solution to uncertain parameters of the problem. The Chapter further describes the sequential algorithm/solution procedure for the uncertain MOTP, mianly via the robust goal programming.

Chapter 7: Summary, conclusions and future research

7.1 Summary and conclusions

While various methods exist to provide solution to inexactness in DEA data (e.g. fuzzy DEA models, Imprecise DEA, Interval DEA, stochastic DEA models), the robust DEA (set-based or scenario-based) set its own unique path in characterizing uncertainty and ensuring probability guarantee for reliable efficiency scores, robust discrimination and ranking of DMUs. At the center of the robust DEA is the robust optimization technique which enables us to model uncertainty in the input and output data of DMUs. For the robust DEA to have impact in theory and application, we feel that methodologies that meet the requirements of computational tractability, guarantee for feasibility of the robust DEA solution in terms of uncertainty in both input and output data and feasibility in probability sense if the uncertainty dynamics obey some natural probability distributions are needed.

In this thesis, we focused on the set-based model for uncertainty within the context of robust optimization to advance the modeling of the robust DEA. We propose models which satisfy the robust optimization modeling technique and set the basis for robust DEA modeling and applications. Specifically, we contributed to the following:

1. **On robust counterpart to positive decision variables**: The framework of the robust optimization involves robust counterpart to general free-in-sign decision variables. The DEA like many other operations research problems present variables which are physical quantities and must be nonnegative. Within the context of robust optimization, in Chapter 3, we derive alternative robust counterpart optimization which shows that our approach significantly reduces the computational burden but preserve the optimality of the original solution for (big data) problems with nonnegative decision variables.

2. **On equality constraint and feasibility in the robust DEA**: Feasibility in RO requires the avoidance of uncertainty analysis in the equality constraint. In the robust DEA, this requires that the normalization constraint which is equality constraint is remodeled as inequality constraint. Omrani (2013) and Salahi et al. (2016) make some suggestions concerning how uncertainty can be considered in the normalization constraint. This topic is treated in detail in Chapter 3 particularly for the input-oriented model and for the output - oriented model in Chapter 4. Theorems suggesting the use of inequality sign are proved.

3. **On the robust frontier characterization**: We provide a characterization for robust efficiency beginning with a definition for robust PPS. In Chapter 2, the robust PPS is proposed with its axioms which is used as a basis for many of the proposed robust DEA models.

4. **On the classification of robust efficiency of DMUs**: While a classification scheme exists for DMUs with imprecise and interval data, a striking observation is its non-existence in the RDEA setting. This gap in the literature is resolved in Chapters 4. A classification scheme for DMUs considering the conservativeness of the decision maker and classifying DMUs into fully robust efficient, partially robust efficient and robust inefficient is proposed. A similar scheme is provided for the classification of DMUs with the robust DEA model under ellipsoidal uncertainty sets in Chapter 5.

5. **On duality relations in robust DEA**: The duality relations in the robust DEA setting is studied. We prove that the presence of uncertain data invalidates the linear duality principle as well as create efficiency gap between the multiplier robust DEA models and envelopment robust DEA models. Our result provides further interpretation to practitioners on the use of robust DEA models in terms of multiplier and envelopment models. Particularly, the worst-case robust envelopment models produce result which fail interpretation and must be used with caution.

6. **On input – output relationship in RDEA**: The relationship between the input and output – oriented models are known to be reciprocally equivalent with the CRS assumption. In the robust DEA setting, this relationship is studied, and the equivalency of the relation proved in Chapter 4.

7. **On multi-objective application with RDEA**: In Chapter 7, we solve a multi-objective transportation problem that focuses on minimizing arc inefficiencies in the presence of demand and supply uncertainty. The chapter entails an extension of the RDEA to multi-objectives decision making using the goal programming technique.

8. **On the characterization of DEA to different uncertainty set**: In Chapter 2, we provide different several robust DEA models to different uncertainty sets, i.e. norm, polyhedral, interval-based polyhedral, ellipsoidal, interval-based ellipsoidal uncertainty sets. Some of these models are well structured for robust DEA application in the other chapters and useful for extension is future studies.

9. **On various applications of RDEA**: While the theoretical and modelling expansion of the robust DEA has been the focus of this thesis, on the practical level, some profound contributions are made. Extensive analysis is done into these application areas rather than mere validation of the models proposed. First, we assessed the robust efficiency of the largest (in terms of assets) 250 banks in Europe in Chapter 3. The performance of some major 26 banks with the largest assets are also assessed with and without uncertainty in data. In Chapter 5, the performance of the major banks in Italy are evaluated with uncertainty restricted to the ellipsoid. Coupled with several numerical examples made in this thesis, the case study of these application areas further proves the applicability of the robust DEA models developed.

10. **On comparison with other robust approaches**: We observe from a comparison of a proposed robust additive DEA model with developed IDEA models of Lee et al. (2002) and Matin et al. (2007) using imprecise data that, the efficiency classification of DMUs with all the models are the same. However, this is after ordinal and bounded data are transformed to exact. Thus, generically, the robust DEA model is not able to directly deal with ratio data or ordinal data and somehow fuzzy data. A research in this direction for instance pay recourse to the modeling approach of Shokouhi et al (2010) and Amirkhan et al (2018) in which interval data and fuzzy data are modelled with the robust optimization approach.

7.2 Future research

Some further research in the area of robust DEA include the followings:

1. **Extension of robust technique to advanced DEA models**: A broader perspective and discussion on reliable and stable performance of DMUs can be found in extending the robust optimization technique to advanced DEA models. The models developed in this thesis are based on basic DEA models including the robust additive model. In Sadjadi et al. (2011)[a], Omrani (2013) , Salahi et al. (2016) , Esfandiari et al. (2016), Arabmaldar et al. (2017) and Salahi, et al (2018) are extended the super-efficiency, common set of weights, two stage DEA and Russel measure. It is desirable extending

the robust DEA to other models such as slack-based model, network DEA, Malmquist productivity index, etc.

2. **Inequality constraints and feasibility in robust DEA models:** Most of the envelopment form of DEA models entail slacks variables and hence equality constraint which lack feasibility for robust optimization modeling. In Chapters 3 and 4 we derive alternative models that involve inequality for the normalization constraints of the input - and output - oriented multiplier models. It is desirable to derive inequality constraint(s) for max-slack models and most of the DEA models involving equality constraints.

3. **Probability bounds for the distribution of the uncertain inputs and output data**: So far, the robust DEA relies on the probability bounds on feasibility of the robust DEA counterpart based on the specific uncertainty set used. Since the uncertainty in DMUs can be quite unique and complex, it will be interesting to derive probability bounds for more general distributions especially for advanced DEA models with higher model complexity.

4. **Computational and comparative studies**: There is a lack of comparative studies among robust DEA under different uncertainties, here by extension we also mean a comparison with the discrete-scenario based robust DEA. Furthermore, a computational comparison of robust DEA models, IDEA models, fuzzy DEA models and stochastic DEA approaches including the chance-constrained DEA would be a humble task to strike a uniform approach to performance evaluation under uncertainty.

Appendix A **Chapter 2**

A.1 Exposition on duality in DEA

This appendix overview linear programming and its duality relations. Such duality relation is very useful in DEA since the multiplier and envelopment models are concepts of duality in linear programming. The concepts here are largely based on Cooper, Seiford, & Tone (2006) and Bazaraa et al. (2010).

Suppose the primal (P) linear program (considered in the canonical form) is given by the following model:

$$\begin{array}{ll} (P) & \min \sum_{j=1}^{n} c_j x_j \\ & \text{s.t.} \\ & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \dots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \dots, n \end{array} \tag{A.1}$$

Then the dual (D) program is defined by:

$$\begin{array}{ll} (D) & \max \sum_{i=1}^{m} b_i y_i \\ & \text{s.t.} \\ & \sum_{j=1}^{n} a_{ij} y_i \leq c_j \quad j = 1, \dots, n \\ & y_i \geq 0 \qquad\qquad i = 1, \dots, m \end{array} \tag{A.2}$$

Note that the terms "primal" and "dual" are relative to the frame of reference and is chosen arbitrary. Basically, the dual of the "dual" is the "primal" itself. Let $x_j^0, j = 1, \dots, n$ and $y_i^0, i = 1, \dots, m$ be the feasible solution to the primal and dual program respectively. The following theorems verifiable from textbooks on linear programming are restated here.

Theorem A.1 (Weak duality theorem) *For each primal feasible solution x_j^0 and each dual feasible solution y_i^0,*

$$\sum_{j=1}^{n} c_j x_j^0 \geq \sum_{i=1}^{m} b_i y_i^o$$

That is, the objective function value of the dual maximizing problem never exceeds that of the primal minimizing problem.

Proof. The weak duality theorem follows from the respective feasibility conditions of the two solutions. In problem (P), feasibility of x_j^0 implies that

$$\sum_{j=1}^{n} a_{ij} x_j^0 \geq b_i, \;\; x_j^0 \geq 0$$

whereas feasibility of y_i^* in problem (D) implies

$$\sum_{i=1}^{m} a_{ij} y_i^0 \leq c_j, \quad y_i^0 \geq 0$$

Multiplying the ith constraint in problem (P) by y_i^0 gives

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j^0 y_i^0 \geq \sum_{i=1}^{m} b_i y_i^0$$

whereas multiplying the jth constraint in problem (D) by x_j^0

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} y_i^0 x_j^0 \leq \sum_{j=1}^{n} c_j x_j^0$$

Then since the LHS of the two inequalities are equal, we have

$$\sum_{j=1}^{n} c_j x_j^0 \geq \sum_{i=1}^{m} b_i y_i^0$$

□

The weak duality provides a bound on the optimal value of the objective function of either the primal or the dual. In other words, the value of the objective function for any feasible solution to the primal minimization problem is bounded from below by the value of the objective function for any feasible solution to its dual. Similarly, the value of the objective function for its dual is bounded from above by the value of the objective function of the primal. The equality of the primal-dual at optimality follows from the strong duality theorem.

Theorem A.2 (Strong duality theorem) *If there exist feasible solutions to both the primal and the dual, then there exists an optimal solution x_j^* to the primal and an optimal solution y_i^* to the dual such that*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof. See Bazaraa et al. (2010)

The strong duality theorem simply states that if both the primal and dual problems are feasible then they have the same optimal value. Below, we summarize duality relations including when the solution is infeasible or unbounded.

i) In a primal-dual pair of linear programs, if either the primal or the dual problem has an optimal solution, then the other does also, and the two optimal objective values are equal.

ii) If either the primal or the dual problem has an unbounded solution, then the other has no feasible solution.

iii) If either problem has no solution, then the other problem either has no solution or its solution is unbounded.

A more precise relationship for the optimal solution of the primal and dual models is given by defining the concept of complementary slackness.

Theorem A.3 (Complementary Slackness Theorem) *Assume problem* (P) *has a solution* x_j^* *and problem* (D) *has a solution* y_i^*.

i) *If* $x_j^* > 0$, *then the* j*th constraint in* (D) *is binding, i.e.* $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$.

ii) *If the* j*th constraint in* (D) *is not binding, i.e.* $\sum_{i=1}^{m} a_{ij} y_i^* < c_j$, *then* $x_j^* = 0$.

iii) *If* $y_i^* > 0$, *then the* i*th constraint in* (P) *is binding, i.e.* $\sum_{j=1}^{n} a_{ij} x_j^* > b_i$.

iv) *If the* i*th constraint in* (P) *is not binding, i.e.* $\sum_{j=1}^{n} a_{ij} x_j^* > b_i$, *then* $y_i^* = 0$.

Proof. It follows complementary from the strong duality theorem. See Bazaraa et al. (2010)

The complementary slackness theorem identifies a relationship between variables in one problem and associated constraints in the other problem. The statement of the theorem is indeed about "complementary slackness" in that there cannot be slack in both a constraint and the associated dual variable and so if in an optimal solution of a linear program, the value of the primal variable associated with a constraint is nonzero, then that constraint in the dual must be satisfied with equality. Moreover, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

Duality relation in DEA

We show the equivalency of the optimal solution of the dual CCR models with input orientation. Let's consider the primal CCR model given in Section 2.1.2 with row vector $\boldsymbol{v}$ for input multipliers and row vector $\boldsymbol{u}$ as output multipliers. The model (multiplier form) is given as:

$$
\begin{aligned}
(M) \qquad & \max_{\boldsymbol{u},\boldsymbol{v}} \sum_{r=1}^{s} u_r y_{ro} \\
& \text{s.t.} \\
& \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad && j = 1, \dots, n \\
& v_i \ge 0 && i = 1, \dots, m \\
& u_r \ge 0 && r = 1, \dots, s
\end{aligned}
\tag{A.3}
$$

The dual of problem (M) is expressed with the dual variable θ and a nonnegative intensity vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)^T$ for the constraints. Then using the slacks $\boldsymbol{s}^- = \{s_i^-\}^T \in \mathcal{R}^m$ and $\boldsymbol{s}^+ = \{s_r^+\}^T \in \mathcal{R}^s$ for the input and output constraints, the following equivalent dual model (E) is obtained:

$$
\begin{aligned}
(E) \qquad & \min_{\theta,\boldsymbol{\lambda},\boldsymbol{s}^-,\boldsymbol{s}^+} \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
& \text{s.t} \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad && i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} && r = 1, \dots, s \\
& \lambda_j \ge 0 && j = 1, \dots, n \\
& s_i^- \ge 0 && i = 1, \dots, m \\
& s_r^+ \ge 0 && r = 1, \dots, s
\end{aligned}
\tag{A.4}
$$

Consider the following optimality conditions which defines efficiency for the dual pair models.

Claim A.1

Let $(\theta^*, \boldsymbol{v}^*, \boldsymbol{u}^*)$ be the optimal solution of problem (M). DMU_o is said to be efficient if $\theta^* = 1$ and with at least one $\boldsymbol{v}^* > 0$ and $\boldsymbol{u}^* > 0$ or else DMU_o is inefficient.

Claim A.2

Let $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ be the optimal solution of problem (E). DMU_o is called efficient if $\theta^* = 1$ and $\boldsymbol{s}^{-*} = \boldsymbol{0}_m,\ \boldsymbol{s}^{+*} = \boldsymbol{0}_s$ or else DMU_o is inefficient.

Theorem A.4. The optimal solution of problems (M) and (E) are equivalent and the efficiency definition in Claim A.2 implies that of Claim A.1.

Proof. From the complementary slack theorem, it holds that for $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ of problem (M) and $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ of problem (E),

$$\boldsymbol{v}^* \boldsymbol{s}^{-*} = 0 \text{ and } \boldsymbol{u}^* \boldsymbol{s}^{+*} = 0$$

Claims 1 and 2 both imply the efficiency with $\theta^* = 1$. We explore alternate possibilities

a) Suppose $\theta^* = 1$ in Claim 2 and $s^{-*} \neq 0_m,\ s^{+*} \neq 0_s$, (an inefficiency condition by Claim 2) then by the complementary conditions, the elements of v^* and u^* corresponding to the positive slacks must be zero which result in inefficiency in Claim 1
b) If $\theta^* < 1$ in problem (M), then DMU_o is inefficient by Claim 1 and by the strong duality theorem, problems (E) has $\theta^* < 1$ which implies DMU_o is inefficient according to Claim 2.
c) If $\theta^* = 1$ and $s^{-*} = 0_m,\ s^{+*} = 0_s$, then, by the strong theorem of complementarity, problem (M) is assured of a positive optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ and hence DMU_o is efficient by Claim 1. ☐

Interpretation of Primal – Dual relation in DEA.

As already noted, duality relationship holds practical implication in DEA. Suppose an input orientation, a farm's objective of minimizing input in the envelopment model is equivalent to maximizing output in the multiplier model. Therefore, per the linear duality principle, any of these equivalent models can be solved. The two are held in the dual *production* and *value* based spaces (Thanassoulis, 2001). The *production space* is characterized with the envelopment model since it is directly derived from the PPS which is used as the framework within which productive efficiency is measured. In this space, the envelopment model looks for a combination of DMUs which may dominate the DMU being evaluated. Here, DMUs corresponding to positive decision variable λ at the optimality serve as the reference units to other DMUs.

The dual multiplier model on the other hand is characterized in the *value space* since it gives a value-based measure of efficiency of DMUs. The multipliers or weights in the model, u_r can be seen as an imputed marginal value or shadow price of output r. Similarly, v_i can be seen as the imputed marginal value or shadow price of output i. Note that the imputed marginal values of inputs and outputs are DMU-specific. The efficiency of DMUs corresponds to the maximum value of the ratio of the imputed marginal value of outputs levels to imputed marginal value of input levels. Ostensibly, the total imputed input value is normalized to some arbitrary level, usually 1, i.e. $\sum_{i=1}^{m} v_i^* x_{io}$ $(= 1)$ as in the case of problem (M) so that we can see the relative importance of each unit by reference to $\sum_{i=1}^{m} v_i^* x_{io}$.

A.2 Numerical construction of basic uncertainty sets

Consider a typical description of the uncertainty dynamics where the nominal value is (4,2), the deviation from the nominal is $(5, 3)$ is given as

$$\mathcal{U} = \{(4 + 5\eta_1, 2 + 3\eta_2)|-1 \le \eta_1, \eta_2 \le 1\}$$

A.2.1 Construction of the box uncertainty set.

For the box uncertainty set, the box uncertainty is given by $\mathcal{U}_i(\Phi) = \left\{a_{ij} + \eta_{ij}\hat{a}_{ij} |\ \| \boldsymbol{\eta} \|_\infty \le \Phi\right\}$ or equivalently $\mathcal{U}_i(\Phi) = \{(a_1 + \eta_1\hat{a}_1, a_2 + \eta_2\hat{a}_2)|\max\{|\eta_1|, |\eta_2|\} \le \Phi\}$. From the above dynamics, box uncertainty set would be given by the following:

$$\mathcal{U}_i(\Phi) = \{(4 + 5\eta_1, 2 + 3\eta_2)|-1 \le \eta_1, \eta_2 \le 1, \max\{|\eta_1|, |\eta_2|\} \le \Phi\}$$

with

$$\max\{|\eta_1|, |\eta_2|\} \le \Phi \rightarrow |\eta_1| \le \Phi \ \& \ |\eta_2| \le \Phi \ \rightarrow -\Phi \le \eta_1 \le \Phi, -\Phi \le \eta_2 \le \Phi,$$

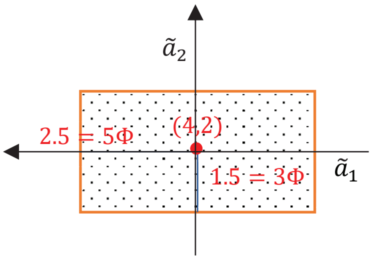
which further implies

$$\rightarrow -5\Phi \le 5\eta_1 \le 5\Phi, -3\Phi \le 3\eta_2 \le 3\Phi \rightarrow \begin{cases} -5\Phi + 4 \le 5\eta_1 + 4 \le 5\Phi + 4 \\ -3\Phi + 2 \le 3\eta_2 + 2 \le 3\Phi + 2' \end{cases}$$

The following properties of the uncertainty set are observed

i) $\mathcal{U}_i(\Phi) \subset \mathcal{U}$ for all $\Phi < 1$,
ii) $\mathcal{U}_i(1) = \mathcal{U}$,
iii) if $0 \le \Phi_1 < \Phi_2$, then $\mathcal{U}_i(\Phi_1) \subset \mathcal{U}_i(\Phi_2)$,
iv) $\mathcal{U}_i(0) = (a_1, a_2)$

Moreover, for $\Phi = 0.5$ and $\Phi = 1$, we obtain the following plots for the explicit box uncertainty sets.



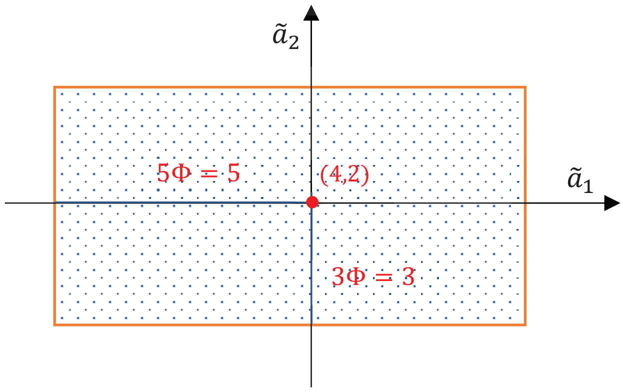


Figure A.2.1: Box uncertainty sets (*Left,* when $\Gamma = 0.5$ and *Right* when $\Gamma = 1.0$)

A.2.2 Construction of the ellipsoidal uncertainty set.

Consider the ellipsoidal uncertainty set given by $\mathcal{U}_e(\Omega) = \left\{a_{ij} + \eta_{ij}\hat{a}_{ij} \middle| \parallel \boldsymbol{\eta} \parallel_2 \leq \Omega\right\}$ or equivalently $\mathcal{U}_e(\Omega) = \{(a_1 + \eta_1\hat{a}_1, a_2 + \eta_2\hat{a}_2) \big|\ \eta_1^2 + \eta_2^2\} \leq \Omega^2\}$. and similarly, for the uncertainty dynamics described above, the ellipsoid can be obtained as follows:

$$
\begin{aligned}
\mathcal{U}_e(\Omega) &= \{(4 + 5\eta_1, 2 + 3\eta_2) \big| -1 \leq \eta_1, \eta_2 \leq 1, \eta_1^2 + \eta_2^2 \leq \Omega^2\} \\
&= \{(u_1, u_2) \big| u_1 = 4 + 5\eta_1, u_2 = 2 + 3\eta_2, -1 \leq \eta_1, \eta_2 \leq 1, \eta_1^2 + \eta_2^2 \leq \Omega^2\} \\
&= \left\{(u_1, u_2) \middle| \eta_1 = \frac{u_1 - 4}{5}, \eta_1 = \frac{u_2 - 2}{3} - 1 \leq \eta_1, \eta_2 \leq 1, \eta_1^2 + \eta_2^2 \leq \Omega^2\right\} \\
&= \left\{(u_1, u_2) \middle| \left(\frac{u_1 - 4}{5}\right)^2 + \left(\frac{u_2 - 2}{3}\right)^2 \leq \Omega^2 \right\} \\
&= \left\{(u_1, u_2) \middle| \left(\frac{u_1 - 4}{5\Omega}\right)^2 + \left(\frac{u_2 - 2}{3\Omega}\right)^2 \leq 1 \right\}
\end{aligned}
$$

Here, the properties of $\mathcal{U}_e(\Omega)$ in relation to $\mathcal{U}$ is as before. Similarly, for $\Omega = 0.5$ and $\Omega = 1$, the following uncertainty sets plots are eminent.

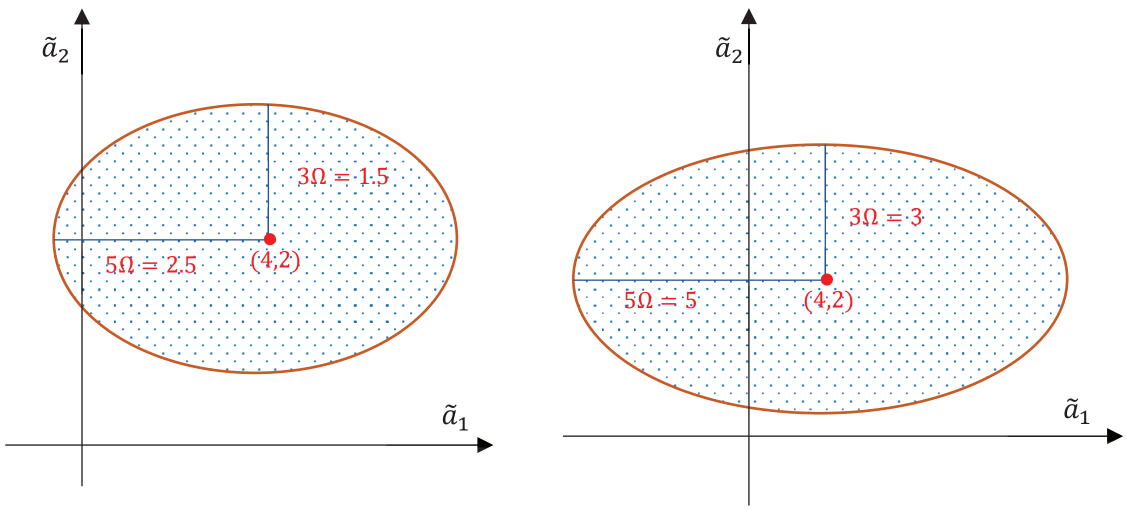


Figure A.2.2: Ellipsoidal uncertainty sets (Left, when $\Gamma = 0.5$ and Right when $\Gamma = 1.0$)

A.2.3. Construction of the polyhedral uncertainty set.

We consider the polyhedral set given by $\mathcal{U}_p(\Gamma) = \left\{a_{ij} + \eta_{ij}\hat{a}_{ij} \middle| \ \| \boldsymbol{\eta} \|_1 \leq \Gamma\right\}$ or equivalently $\mathcal{U}_p(\Gamma) = \{(a_1 + \eta_1\hat{a}_1, a_2 + \eta_2\hat{a}_2) | \ |\eta_1| + |\eta_2|\} \leq \Gamma\}$ and for the dynamics of the uncertainty described above, we obtain the following:

$$
\begin{aligned}
\mathcal{U}_p(\Gamma) &= \{(4 + 5\eta_1, 2 + 3\eta_2) | -1 \leq \eta_1, \eta_2 \leq 1, |\eta_1| + |\eta_2| \leq \Gamma\} \\
&= \{(u_1, u_2) | u_1 = 4 + 5\eta_1, u_2 = 2 + 3\eta_2, \ -1 \leq \eta_1, \eta_2 \leq 1, |\eta_1| + |\eta_2| \leq \Gamma\} \\
&= \left\{(u_1, u_2) \middle| \eta_1 = \frac{u_1 - 4}{5}, \eta_1 = \frac{u_2 - 2}{3}, -1 \leq \eta_1, \eta_2 \leq 1, |\eta_1| + |\eta_2| \leq \Gamma\right\} \\
&= \left\{(u_1, u_2) \middle| \left|\frac{u_1 - 4}{5}\right| + \left|\frac{u_2 - 2}{3}\right| \leq \Gamma \right\}
\end{aligned}
$$

Again, we envisage similar properties in relation with the set $\mathcal{U}$. Now we obtain the following plots of polyhedral uncertainty sets for different values of Γ.

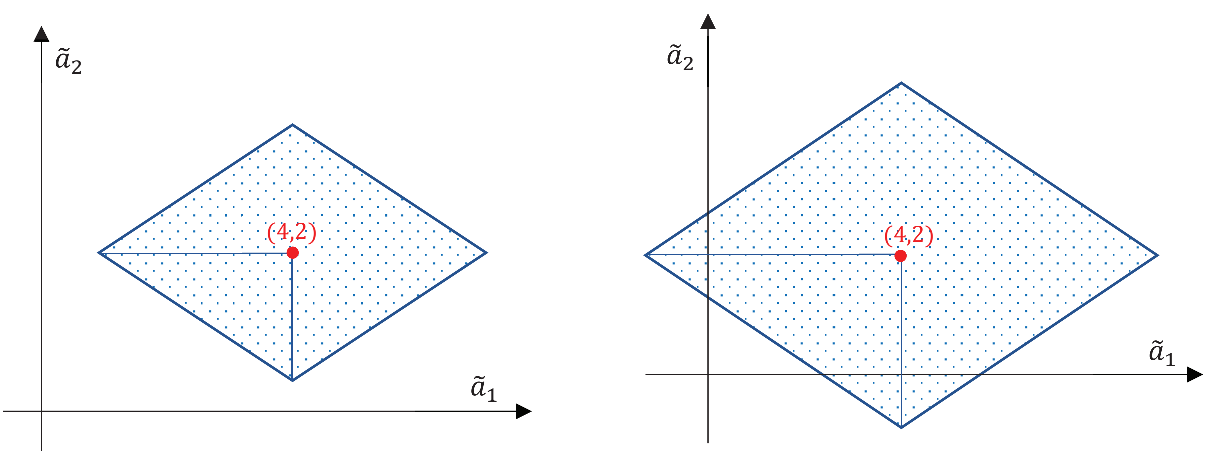


Figure A.2.3: Polyhedral uncertainty sets (*Left*, when $\Gamma = 0.5$ and *Right* when $\Gamma = 1.0$)

Appendix B Chapter 3

A dataset of European banks in performance evaluation under uncertainty[39]

Summary

This appendix explains the dataset containing financial indicators from the financial statements of 250 banks operating in Europe which were collated for the 2015 accounting year for the analysis in Chapter 3. First, the dataset is split into input and outputs measures. Then the preferred number of inputs and outputs in relation to the total number of data is selected according to the rule of thumb in data envelopment analysis (DEA).

B.1. Specifications Table

Subject area	Operations research and management science
More specific subject area	Data envelopment analysis
Type of data	Table, figure
How data was acquired	Obtainable from financial statements of banks from Bureau van Dick – Bankscope database
Data format	Raw, analyzed with descriptive and statistical data
Experimental factors	The sample consists of raw financial data of banks for the accounting year 2015.
Experimental features	Indicators of interest were systematically selected and collated.
Data source location	Global data
Data accessibility	Data is within this article. Also, largely accessible from the database of the current database host, Orbis Bank Focus: https://banks.bvdinfo.com/version-2018810/home.serv?product=OrbisBanks

[39] This appendix is published as a paper in Data in Brief Journal: https://doi.org/10.1016/j.dib.2018.11.048

B.2. Value of the data

- The raw data contains key financial statements indicators of 250 banks in Europe which were taken from the individual bank's financial statement in 2015.
- The data are arranged in order of the largest bank to the smallest bank in terms of assets
- The data is useful for measuring the performance of banks in Europe and for comparative analysis of sub-regional performances and beyond.
- The data can be used by researchers to evaluate a wide range of efficiency measures for the countries under consideration.

B.3. Data

The data comprises financial indicators in the financial statements of 250 public and private banks operating in Europe. Table B.1. shows the distribution of these banks according to the sub-region. Including data on indicators such as assets, employees, personnel expenses, equity, loans, net interest income, deposit from banks, operating income and net fees and commission, the detailed financial statements of the banks were obtained from the Bureau van Dick – Bankscope database for the 2015 accounting year. The summary of descriptive statistics of these indicators for each subregion is provided in Table B.2. All the financial indicators were measured in millions of Euros with the exception of employees which is measured in actual figures. The total number of employees is defined as the number of banking professionals and the non-banking staff is given employed in the accounting year.

Table B.1. Classification according to region

Region	Number of banks	Percentage
Western Europe	129	51.6
Eastern Europe	22	8.8
Northern Europe	33	13.2
Southern Europe	66	26.4
Total	**250**	**100**

Table B.2. Descriptive statistics for Eastern Europe

Financial indicators	**Mean**	**SD**	**Min**	**Max**
Inputs				
Employees	8471.59	7367.41	2952.00	38203.00
Assets	23006.93	13292.19	10517.04	62604.63
Equity	2611.49	1672.28	996.40	7097.94
Personnel Expenses	229.99	155.97	92.15	648.82

Outputs				
Deposits Banks	1602.50	1216.85	58.00	4484.65
Loans	14302.75	8981.52	4132.18	43617.70
Net Income Revenue	619.62	429.78	230.77	1752.56
Operating Income	330.48	239.11	82.44	919.25
Net Fees Commission	239.79	177.35	76.64	676.85

Table B.3. Descriptive statistics for Northern Europe

Financial indicators	**Mean**	**SD**	**Min**	**Max**
Inputs				
Employees	21295.39	31900.42	1374.00	129400.00
Assets	277515.12	382760.47	10231.98	1526980.04
Equity	16442.98	21190.44	1056.87	89950.27
Personnel Expenses	1801.82	2941.45	113.09	13570.41
Outputs				
Deposits Banks	27492.99	41157.08	208.60	168902.51
Loans	135471.03	163827.56	5097.33	620171.67
Net Income Revenue	3374.36	4595.32	121.26	18149.74
Operating Income	2061.49	3509.67	22.89	17641.53
Net Fees Commission	1197.31	2042.12	29.70	10785.48

Table B.4. Descriptive statistics for Southern Europe

Financial indicators	Mean	SD	Min	Max
Inputs				
Employees	15823.02	33227.16	217.00	193863.00
Assets	110047.41	224459.34	10267.48	1340260.00
Equity	8165.56	16031.01	226.30	98753.00
Personnel Expenses	966.23	1944.74	14.20	11107.00
Outputs				
Deposits Banks	17311.01	34733.78	81.40	185459.00
Loans	63093.07	123057.73	768.60	758505.00
Net Income Revenue	1957.18	4806.37	52.30	33267.00
Operating Income	1176.55	2271.45	20.29	12628.00
Net Fees Commission	844.13	1796.64	15.40	10033.00

Table B.5. Descriptive statistics for Western Europe

Financial indicators	Mean	SD	Min	Max
Inputs				
Employees	12144.32	28624.09	586.00	189000.00
Assets	141635.11	339445.33	10017.70	1994193.00
Equity	7716.21	16315.15	296.30	100077.00
Personnel Expenses	961.52	2382.19	64.40	16061.00
Outputs				
Deposits Banks	20154.79	45321.98	331.23	263121.00
Loans	58635.75	121719.18	305.60	735784.00
Net Income Revenue	1585.57	3628.48	69.20	23133.00
Operating Income	1179.20	3004.06	34.10	19805.00
Net Fees Commission	725.82	1711.31	6.60	12765.00

B.4. Experimental Design, Materials and Methods

Financial statements of banks were first downloaded from the Bankscope database Orbis Bank Focus (2016). Then data on the financial indicators mentioned above were compiled from 250 banks financial statements individually and collated. These banks are arranged in descending order of their assets size. Subsequently, for the performance analysis of the banks using the data envelopment analysis (DEA) tool, the financial indicators are split into two samples. The first is the input measures and the second is the output measures.

The separation of the financial indicators into inputs and outputs measures was done based on the selective measures described in Mostafa (2009) and Toloo & Tichý (2015). The approach adopted for selecting inputs and outputs is the intermediary approach of banking studies, which is shown in Table B.2. With the exception of deposit and loans refereed mostly in literature as dual role factors, it is unarguable the selection of the measures as input and inputs. In this appendix, the selection of deposit specifically as output corresponds to its treatment in Toloo & Tichý (2015) and Toloo & Mensah (2018). The number of DMUs in correspondence to the input and outputs measures is selected according to the rule of thumb in DEA as follows (for more details see Toloo et al. (2015) and Toloo & Allahyar (2018):

$$n \geq \max\{m \times s, 3(m + s)\}$$

where

n = total number of DMUs (observations)

m = number of inputs

s = number of outputs

All the raw data are scaled for uniformity and to reduce round-off errors from excessively large values prior to analysis.

Appendix C Chapter 5

C.1 Fractional robust DEA with ellipsoidal set

Recall that the robust fractional DEA is formulated as

$$\begin{aligned}
&\max_{(\tilde{x}_o,\tilde{y}_o)\in\mathcal{U}_o^e} \inf\left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}}\right\} \\
&\text{s.t.} \\
&\inf_{(\tilde{x}_o,\tilde{y}_o)\in\mathcal{U}_o^e} \left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}}\right\} \le 1 \\
&\sup_{(\tilde{x}_j,\tilde{y}_j)\in\mathcal{U}_j^e} \left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}}\right\} \le 1 \qquad \forall j \ne o \\
&v_i \ge 0 \qquad \forall i \\
&u_r \ge 0 \qquad \forall r
\end{aligned} \tag{C.1.1}$$

Then considering the uncertainty dynamics and using the ellipsoidal uncertainty set so defined; $\mathcal{U}_j^e = \left\{(\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}}) \middle| \begin{array}{ll} \tilde{x}_{ij} = x_{ij} + \sum_{i\in I_j} \rho_{ij}^x \xi_{ij}^x, & \left\|\xi_j^x\right\|_2 \le 1 \\ \tilde{y}_{rj} = y_{rj} + \sum_{r\in R_j} \rho_{rj}^y \xi_{rj}^y, & \left\|\xi_j^y\right\|_2 \le 1 \end{array};\quad j = 1,\dots,n\right\}$, we obtain the following formulation:

$$\begin{aligned}
&\max_{\boldsymbol{u}\in\mathbb{R}^s} \left\{\frac{\sum_{r=1}^{s} u_r y_{ro} + \inf_{\left\|\xi_o^y\right\|_2\le 1}\left(\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right)}{\sum_{i=1}^{m} v_i x_{io} + \sup_{\left\|\xi_o^x\right\|_2\le 1}\left(\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right)}\right\} \\
&\text{s.t.} \\
&\frac{\sum_{r=1}^{s} u_r y_{ro} + \inf_{\left\|\xi_o^y\right\|_2\le 1}\left\{\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right\}}{\sum_{i=1}^{m} v_i x_{io} + \sup_{\left\|\xi_o^x\right\|_2\le 1}\left\{\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right\}} \le 1 \\
&\frac{\sum_{r=1}^{s} u_r y_{rj} + \sup_{\left\|\xi_j^y\right\|_2\le 1}\left\{\sum_{r\in R_j} u_r \xi_{rj}^y \rho_{rj}^y\right\}}{\sum_{i=1}^{m} v_i x_{ij} + \inf_{\left\|\xi_j^x\right\|_2\le 1}\left\{\sum_{i\in I_j} v_i \xi_{ij}^x \rho_{ij}^x\right\}} \le 1 \qquad \forall j \ne o \\
&v_i \ge 0 \qquad \forall i \\
&u_r \ge 0 \qquad \forall r
\end{aligned} \tag{C.1.2}$$

which is equivalent to the following model:

$$
\max_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{R}^{m+s}} \left\{ \frac{\sum_{r=1}^{s} u_r y_{ro} + \inf_{\|\xi_o^y\|_2\le 1} \left(\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right)}{\sum_{i=1}^{m} v_i x_{io} + \sup_{\|\xi_o^x\|_2\le 1} \left(\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right)} \right\}
$$

s.t.

$$
\left(\sum_{r=1}^{s} u_r y_{ro} + \inf_{\|\xi_o^y\|_2\le 1} \left\{\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right\}\right) - \left(\sum_{i=1}^{m} v_i x_{io} + \sup_{\|\xi_o^x\|_2\le 1} \left\{\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right\}\right) \le 0 \qquad \text{(C.1.3)}
$$

$$
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sup_{\|\xi_j^y\|_2\le 1} \sup_{\|\xi_j^x\|_2\le 1} \left\{\sum_{r\in R_j} u_r \xi_{rj}^y \rho_{rj}^y + \sum_{i\in I_j} v_i \xi_{ij}^x \rho_{ij}^x\right\} \le 0 \qquad \forall j \ne o
$$

$$
v_i \ge 0 \qquad \forall i
$$

$$
u_r \ge 0 \qquad \forall r
$$

We suppose the following change of variable in the spirit of Cooper and Charnes (1962). Specifically, let $t = \frac{1}{\sum_{i=1}^{m} v_i x_{io} + \sup_{\|\xi_o^x\|_2\le 1} \left(\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right)} > 0$ and $u_r = tu_r, v_i = tv_i$. Model C.1.3 is then equivalent to the following model:

$$
\max_{\boldsymbol{u}\in\mathbb{R}^{s}} \sum_{r=1}^{s} u_r y_{ro} + \inf_{\|\xi_o^y\|_2\le 1} \left(\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right)
$$

s.t.

$$
\sum_{i=1}^{m} v_i x_{io} + \sup_{\|\xi_o^x\|_2\le 1} \left(\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right) = 1
$$

$$
\left(\sum_{r=1}^{s} u_r y_{ro} + \inf_{\|\xi_o^y\|_2\le 1} \left\{\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right\}\right) - \left(\sum_{i=1}^{m} v_i x_{io} + \sup_{\|\xi_o^x\|_2\le 1} \left\{\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right\}\right) \le 0 \qquad \text{(C.1.4)}
$$

$$
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sup_{\|\xi_j^y\|_2\le 1} \sup_{\|\xi_j^x\|_2\le 1} \left\{\sum_{r\in R_j} u_r \xi_{rj}^y \rho_{rj}^y + \sum_{i\in I_j} v_i \xi_{ij}^x \rho_{ij}^x\right\} \le 0 \qquad \forall j \ne o
$$

$$
v_i \ge 0 \qquad \forall i
$$

$$
u_r \ge 0 \qquad \forall r
$$

The normalization constraint is always binding at optimality even when the constraint is in an inequality form $\le$ as shown in Toloo (2014)a, which implies the model above can be written as:

$$
\max_{\boldsymbol{u}\in\mathbb{R}^{s}} + \inf_{\|\xi_o^y\|_2\le 1} \left(\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right)
$$

s.t.

$$
\sum_{i=1}^{m} v_i x_{io} + \inf_{\|\xi_o^x\|_2\le 1} \left(\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right) \le 1
$$

$$
\left(\sum_{r=1}^{s} u_r y_{ro} + \inf_{\|\xi_o^y\|_2\le 1} \left\{\sum_{r\in R_o} u_r \xi_{ro}^y \rho_{ro}^y\right\}\right) - \left(\sum_{i=1}^{m} v_i x_{io} + \sup_{\|\xi_o^x\|_2\le 1} \left\{\sum_{i\in I_o} v_i \xi_{io}^x \rho_{io}^x\right\}\right) \le 0 \qquad \text{(C.1.5)}
$$

$$
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sup_{\|\xi_j^y\|_2\le 1} \sup_{\|\xi_j^x\|_2\le 1} \left\{\sum_{r\in R_j} u_r \xi_{rj}^y \rho_{rj}^y + \sum_{i\in I_j} v_i \xi_{ij}^x \rho_{ij}^x\right\} \le 0 \qquad \forall j \ne o
$$

$$
v_i \ge 0 \qquad \forall i
$$

$$
u_r \ge 0 \qquad \forall r
$$

C.2 Italian banks used for the analysis

Table C.2. Major banks in Italy used for the analysis

Banks	Bank name
B01	UniCredit SpA
B02	Intesa Sanpaolo
B03	Banca Monte dei Paschi di Siena SpA-Gruppo Monte dei Paschi di Siena
B04	Banco Popolare - Società Cooperativa-Banco Popolare
B05	Unione di Banche Italiane Scpa-UBI Banca
B06	Banca Nazionale del Lavoro SpA
B07	Banca popolare dell'Emilia Romagna
B08	Cassa di Risparmio di Parma e Piacenza SpA
B09	Banca Popolare di Milano SCaRL
B10	Banca Mediolanum SpA
B11	Banca Popolare di Vicenza Societa cooperativa per azioni
B12	Credito Emiliano SpA-CREDEM
B13	Banca Popolare di Sondrio Societa Cooperativa per Azioni
B14	Veneto Banca scpa
B15	Banca Carige SpA
B16	Banco di Napoli SpA
B17	Banca Piccolo Credito Valtellinese-Credito Valtellinese Soc Coop
B18	Deutsche Bank SpA
B19	Banca Popolare di Bergamo SpA
B20	Cassa di Risparmio del Veneto SpA
B21	Banca Popolare di Bari Soc. Coop.P.A
B22	CheBanca SpA
B23	Banco di Brescia San Paolo Cab SpA
B24	Banco di Sardegna SpA
B25	Cassa di risparmio di Asti SpA
B26	Banco di Desio e della Brianza SpA-Banco Desio
B27	Banca di Credito Cooperativo di Roma
B28	Unipol Banca Spa
B29	Banca Sella SpA

Bibliography

Abd El-Wahed, W. F., & Lee, S. M. (2006). Interactive fuzzy goal programming for multi-objective transportation problems. *Omega*, *34*(2), 158–166.

Abdelaziz, F. Ben. (2007). Multiple objective programming and goal programming: New trends and applications. *European Journal of Operational Research*, *177*(3), 1520–1522.

Aghayi, N., & Maleki, B. (2016). Efficiency measurement of DMUs with undesirable outputs under uncertainty based on the directional distance function: Application on bank industry. *Energy*, *112*, 376–387.

Aghayi, N., Tavana, M., & Raayatpanah, M. A. (2016). Robust efficiency measurement with common set of weights under varying degrees of conservatism and data uncertainty. *European J. of Industrial Engineering*, *10*(3), 385.

Ahn, H., & Le, M. H. (2015). DEA efficiency of German savings banks: evidence from a goal-oriented perspective. *Journal of Business Economics*, *85*(9), 953–975.

Ahn, T., Charnes, A., & Cooper, W. W. (1988). Efficiency characterizations in different DEA models. *Socio-Economic Planning Sciences*, *22*(6), 253–257.

Aiello, F., & Bonanno, G. (2016). Bank efficiency and local market conditions: Evidence from Italy. *Journal of Economics and Business*, *83*, 70–90.

Alexander, W. R. J., Haug, A. A., & Jaforullah, M. (2010). A two-stage double-bootstrap data envelopment analysis of efficiency differences of New Zealand secondary schools. *Journal of Productivity Analysis*, *34*(2), 99–110.

Alfiero, S., Esposito, A., Mensah, E. K., & Toloo, M. (2019). A dataset of European banks in performance evaluation under uncertainty. *Data in Brief*, *22*, 214–217.

Allen, F., & Gale, D. (1995). A welfare comparison of intermediaries and financial markets in Germany and the US. *European Economic Review*, *39*(2), 179–209.

Amirkhan, M., Didehkhani, H., Khalili-Damghani, K., & Hafezalkotob, A. (2018). Mixed uncertainties in data envelopment analysis: A fuzzy-robust approach. *Expert Systems with Applications*, *103*, 218–237.

Amirteimoori, A. (2011). An extended transportation problem: A DEA-based approach. *Central European Journal of Operations Research*, *19*(4), 513–521.

Amirteimoori, A. (2012). An extended shortest path problem: A data envelopment analysis approach. *Applied Mathematics Letters*, *25*(11), 1839–1843.

Arabmaldar, A., Jablonsky, J., & Saljooghi, F. H. (2017). A new robust DEA model and super-efficiency measure. *Optimization*, *66*(5), 723–736.

Atıcı, K. B., & Gülpınar, N. (2016b). Robust DEA Approaches to Performance Evaluation of Olive Oil Production Under Uncertainty. In M. Doumpos, C. Zopounidis, & E. Grigoroudis (Eds.), *Robustness Analysis in Decision Aiding, Optimization, and Analytics* (p. 299–318.). Springer International Publishing.

Azadi, M., & Saen, R. F. (2012). Developing a new chance-constrained DEA model for suppliers selection in the presence of undesirable outputs. *International Journal of Operational Research*, *13*(1), 44–66.

Bandi, C., & Bertsimas, D. (2012). Tractable stochastic analysis in high dimensions via robust optimization. *Mathematical Programming*, 134(1), 23-70.

Banker, R. D. (1993). Maximum Likelihood, Consistency and Data Envelopment Analysis: A Statistical Foundation. *Management Science*, *39*(10), 1265–1273.

Banker, R. D. (1996). Hypothesis Tests Using Data Envelopment Analysis. *Journal of*

Productivity Analysis, *159*, 139–159.
Banker, R. D., Chang, H. H., & Cooper, W. W. (1996). Equivalence and implementation of alternative methods for determining returns to scale in data envelopment analysis. *European Journal of Operational Research*, *89*(3), 473–481.
Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science*, *30*(9), 1078–1092.
Banker, R. D., & Natarajan, R. (2008). Evaluating contextual using data variables envelopment affecting analysis. *Operations Research*, *56*(1), 48–58.
Banker, R. D., & Thrall, R. M. (1992). Estimation of returns to scale using data envelopment analysis. *European Journal of Operational Research*, *62*(1), 74–84.
Bayati, M. F., & Sadjadi, S. J. (2017). Robust network data envelopment analysis approach to evaluate the efficiency of regional electricity power networks under uncertainty, *Plos One,* 12(9) 1–20.
Bazaraa, M. S., Jarvis, J. J., & Sherali, H. D. (2010). *Linear programming and network flows* (4th ed.). John Wiley & Sons.
Behr, P., & Schmidt, R. H. (2015). The German Banking System: Characteristics and Challenges. *The Palgrave Handbook of European Banking*.
Ben-Tal, A., El Ghaoui, L., Nemirovskiĭ, A. S. *Robust optimization*. Princeton University Press.
Ben-Tal, A., Goryashko, A., Guslitzer, E., & Nemirovski, A. (2004). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, *99*(2), 351–376.
Ben-tal, A., & Nemirovski, A. (1998). Robust Convex Optimization. *Mathematics of Operations Research*, *23*(4), 769–805.
Ben-tal, A., & Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research Letters*, *25*(1), 1–13.
Ben-Tal, R., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, *88*(3), 411–424.
Berger, A. N., Humphrey, D. B., & Berger, A.N. and Humphrey, D. B. (1997). Efficiency of financial institutions: International survey and directions for future research. *European Journal of Operational Research*, *98*(2), 175–212.
Bertsimas, D., & Brown, D. B. (2009). Constructing Uncertainty Sets for Robust Linear Optimization. *Operations Research*, *57*(6), 1483–1495.
Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and Applications of Robust Optimization. *SIAM Review*, *53*(3), 464–501.
Bertsimas, D., Gupta, V., & Kallus, N. (2018). Data-driven robust optimization. *Mathematical Programming*, 167(2). 235-292.
Bertsimas, D., Pachamanova, D., & Sim, M. (2004). Robust linear optimization under general norms. *Operations Research Letters*, *32*(6), 510–516.
Bertsimas, D., & Sim, M. (2004). The Price of robustness. *Operations Research*, *52*(1), 35–53.
Birge, J., & Louveaux, F. (1997). *Introduction to Stochastic Programming*. Springer Series in Operations Research.
Bit, A. K., Biswal, M. P., & Alam, S. S. (1992). Fuzzy programming approach to multicriteria decision making transportation problem. *Fuzzy Sets and Systems*, *50*(2), 135–141.
Bogetoft, P., & Otto, L. (2011). *Benchmarking with DEA, SFA, and R* (Vol. 157). Springer Science and Business Media
Boles, J. N. (1971). *The 1130 Farrell efficiency system-multiple products, multiple factors*. Iannini Foundation of Agricultural Economics.
Buch, C. M., Buchholz, M., Tonzer, L., & Buch, C. (2014). Uncertainty and International

Banking. In conference paper series *"International Banking: Microfoundations and Macroeconomic Implications"* organized by the International Monetary Fund (IMF) and the Dutch National Bank (DNB).

Casu, B., & Girardone, C. (2002). A comparative study of the cost efficiency of Italian bank conglomerates. *Managerial Finance*, *28*(6), 34–45.

Cazals, C., Florens, J. P., & Simar, L. (2002). Nonparametric frontier estimation: A robust approach. *Journal of Econometrics*, *106*(1), 1–25.

Charnes, A., Cooper, W. ., Golany, B., Seiford, L., & Stutz, J. (1985). Foundations of DEA fo Pareto-Koopmans Efficient Empirical Production Functions. *Journal of Econometrics*, *30*, 91–107.

Charnes, A., Cooper, W. W., & Ferguson, R. O. (1955). Optimal estimation of executive compensation by linear programming. *Management Science*, *1*(2), 138–151.

Charnes, A., Cooper, W. W., Lewin, A. Y., Morey, R. C., & Rousseau, J. (1984). Sensitivity and stability analysis in dea. *Annals of Operations Research*, *2*(1), 139–156.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, *2*(6), 429–444.

Charnes, A., Roussea, J. J., & Semple, J. H. (1996). Sensitivity and stability of efficiency classifications in data envelopment analysis. *Journal of Productivity Analysis*, *7*(1), 5–18.

Chen, L. H., & Lu, H. W. (2007). An extended assignment problem considering multiple inputs and outputs. *Applied Mathematical Modelling*, *31*(10), 2239–2248.

Chen, T. Y. (2002). A comparison of chance-constrained dea and stochastic frontier analysis: Bank efficiency in taiwan. *Journal of the Operational Research Society*, *53*(5), 492–500.

Cooper, W., Huang, Z., & Li, S. (2004). Chance Constrained DEA. In *Handbook on Data Envelopment Analysis* (pp. 229–264).

Cooper, W. W., Huang, Z., Lelas, V., Li, S. X., & Olesen, O. B. (1998). Chance Constrained Programming Formulations for Stochastic Characterizations of Efficiency and Dominance in DEA. *Journal of Productivity Analysis*, *9*(1), 53–79.

Cooper, W. W., Park, K. S., & Yu, G. (1999). IDEA and AR-IDEA: Models for Dealing with Imprecise Data in DEA. *Management Science*, *45*(4), 597–607.

Cooper, W. W., Park, K. S., & Yu, G. (1999). IDEA and AR-IDEA: Models for Dealing with Imprecise Data in DEA. *Management Science*, *45*(4), 597–607.

Cooper, W. W., Seiford, L. M., & Tone, K. (2006). *Introduction to data envelopment analysis and its uses: With DEA-solver software and references*. Springer Science & Business Media

Dantzig, G. B. (1955). Linear programming under uncertainity. *Management Science*, *1*(3–4), 197–206.

Daraio, C., & Simar, L. (2006). A robust nonparametric approach to evaluate and explain the performance of mutual funds. *European Journal of Operational Research*, *175*(1), 516–542.

Daraio, C., & Simar, L. (2007). *Advanced Robust And Nonparametric Methods In Efficiency Analysis. Methodology And Applications*. Springer Science & Business Media.

Das, S. K., Goswami, A., & Alam, S. S. (1999). Multiobjective transportation problem with interval cost, source and destination parameters. *European Journal of Operational Research*, *117*(1), 100–112.

De Klerk, E. (2008). The complexity of optimizing over a simplex, hypercube or sphere: A short survey. *Central European Journal of Operations Research*, *16*(2), 111–125.

Debreu, G. (1951). The Coefficient of Resource Utilization. *Econometrica*, *19*(3), 273.

Deprins, D., Simar, L., & Tulkens, H. (1984). Measuring labor efficiency in post office. In *The Performance of Public Enterprises: Concepts and Measurement* (pp. 243–268).

Despotis, D. K., & Smirlis, Y. G. (2002). Data envelopment analysis with imprecise data, *European Journal of Operational Research*, *140*, 24–36.

Dorfman, R., Samuelson, P. A., & Solow, R. M. (1987). *Linear Programming and Economic Analysis*. Courier Corporation.

Ehrgott, M., Holder, A., & Nohadani, O. (2018). Uncertain Data Envelopment Analysis. *European Journal of Operational Research*, *268*(1), 231–242.

Ehrgott, M., Ide, J., & Schöbel, A. (2014). Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, *239*(1), 17–31.

Emrouznejad, A., & Yang, G. liang. (2018). A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Socio-Economic Planning Sciences*, *61*, 4–8.

Entani, T., Maeda, Y., & Tanaka, H. (2002). Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research*, *136*(1), 32–45.

Esfandiari, M., Hafezalkotob, A., Khalili-Damghani, K., & Amirkhan, M. (2017). Robust two-stage DEA models under discrete uncertain data. *International Journal of Management Science and Engineering Management*, *12*(3), 216–224.

European Banking Federation. (2018). Banking in Europe: EBF Facts & Figures 2018. *EBF Economic Outlook*, *Nr 43*.

Fare, R., & Grosskopf, S. (1985). A Nonparametric Cost Approach to Scale Efficiency. *The Scandinavian Journal of Economics*, *87*(4), 594–604.

Farrell, M. J., & Fieldhouse, M. (1962). Estimating Efficient Production Functions under Increasing Returns to Scale. *Journal of the Royal Statistical Society. Series A (General)*, *125*(2), 252.

Farzipoor Sean, R., Memariani, A., & Lotfi, F. H. (2005). The effect of correlation coefficient among multiple input vectors on the efficiency mean in data envelopment analysis. *Applied Mathematics and Computation*, *162*(2), 503–521.

Fethi, M. D., & Pasiouras, F. (2010). Assessing bank efficiency and performance with operational research and artificial intelligence techniques: A survey. *European Journal of Operational Research*, *204*(2), 189–198.

Fiorentino, E., Karmann, A., & Koetter, M. (2006). *The Cost Efficiency of German Banks: A Comparison of Sfa and DEA*. *SSRN*. https://doi.org/10.2139/ssrn.947340

French, S. (1995). Uncertainty and Imprecision: Modelling and Analysis. *The Journal of the Operational Research Society*, *46*(1), 70–79.

Gabrel, V., Lacroix, M., Murat, C., & Remli, N. (2014). Robust location transportation problems under uncertain demands. *Discrete Applied Mathematics*, *164*(PART 1), 100–111.

Gabrel, V., & Murat, C. (2010). Robustness and duality in linear programming. *Journal of the Operational Research Society*, *61*(8), 1288–1296.

Ghahtarani, A., & Najafi, A. A. (2013). Robust goal programming for multi-objective portfolio selection problem. *Economic Modelling*, *33*, 588–592.

Giordano, L., Mbriani, C., & Lopes, A. (2013). Analysis of the Italian banking system efficiency: A stochastic frontier approach. In *Multicriteria and Multiagent Decision Making with Applications to Economics and Social Sciences* (pp. 195–217). Springer, Berlin, Heidelberg.

Gorissen, B. L., Yanıkoğlu, İ., & den Hertog, D. (2015). A practical guide to robust optimization. *Omega*, *53*, 124–137.

Gregory, C., Darby-Dowman, K., & Mitra, G. (2011). Robust optimization and portfolio selection: The cost of robustness. *European Journal of Operational Research*, *212*(2), 417–428.

Grigorian, D. a., & Manole, V. (2002). Determinants of commercial bank performance in

transition: an application of data envelopment analysis. *IMF Working Paper Series*.
Guo, P., & Tanaka, H. (2001). Fuzzy DEA: A perceptual evaluation method. *Fuzzy Sets and Systems*, *119*(1), 149–160.
Gupta, A., & Kumar, A. (2012). A new method for solving linear multi-objective transportation problems with fuzzy parameters. *Applied Mathematical Modelling*, *36*(4), 1421–1430.
Hafezalkotob, A., Haji-Sami, E., & Omrani, H. (2015). Robust DEA under discrete uncertain data: a case study of Iranian electricity distribution companies. *Journal of Industrial Engineering International*, *11*(2), 199–208.
Hanks, R. W., Weir, J. D., & Lunday, B. J. (2017). Robust goal programming using different robustness echelons via norm-based and ellipsoidal uncertainty sets. *European Journal of Operational Research*, *262*(2), 636–646.
Hassan Shirdel, G., & Mortezaee, A. (2015). A DEA-based approach for the multi-criteria assignment problem. *Croatian Operational Research Review*, *6*(1), 145–154.
Hatami-Marbini, A., Emrouznejad, A., & Tavana, M. (2011). A taxonomy and review of the fuzzy data envelopment analysis literature: Two decades in the making. *European Journal of Operational Research*, *214*(3), 457–472.
Hatami-Marbini, A., & Toloo, M. (2017). An extended multiple criteria data envelopment analysis model. *Expert Systems with Applications.* 73, 201-219.
Hauner, D. (2005). Explaining efficiency differences among large German and Austrian banks. *Applied Economics*, *37*(9), 969–980.
Hitchcock, F. (1941). The distribution of a product from several sources to numerous localities. *Studies in Applied Mathematics*, *20*(1–4), 224–230.
Hladík, M. (2019). Universal efficiency scores in data envelopment analysis based on a robust approach. *Expert Systems with Applications*, *122*, 242-252.
Isermann, H. (1979). The enumeration of all efficient solutions for a linear multiple-objective transportation problem: *Naval Research Logistics Quarterly*, *26*(1), 123–139.
Ivashina, V., & Scharfstein, D. (2010). Bank lending during the financial crisis of 2008. *Journal of Financial Economics*, *97*(3), 319–338.
Kao, C., & Liu, S.-T. (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems*, *113*(3), 427–437.
Kao, C., & Liu, S. T. (2014). Measuring performance improvement of Taiwanese commercial banks under uncertainty. *European Journal of Operational Research*, *235*(3), 755–764.
Kaviani, M. A., & Abbasi, M. (2015). A Robust Data Envelopment Analysis Method with Bounded Data for Ranking Operations Strategies. *International Journal of Strategic Decision Sciences*, *6*(3), 49–64.
Keshavarz, E., & Toloo, M. (2014). Finding efficient assignments: An innovative DEA approach. *Measurement: Journal of the International Measurement Confederation*, *58*, 448–458.
Keshavarz, E., & Toloo, M. (2015). Efficiency status of a feasible solution in the Multi-Objective Integer Linear Programming problems: A DEA methodology. *Applied Mathematical Modelling*, *39*(12), 3236–3247.
Koopmans, T. C. (1951). An Analysis of Production as an Efficient Combination of Activities in Koopmans T.C. (Ed.). In *Activity Analysis of Production and Allocation* (pp. 225–87). Cowles Commission for Research in Economics, Monograph No. 13, Wiley, New York.
Kouvelis, P., & Yu, G. (1997). *Robust Discrete Optimization and Its Applications*. *Springer* (Vol. 14). K1uwer Academic Publishers.
Kuchta, D. (2004). Robust goal programming. *Control and Cybernetics*, *33*(3).
Kwak, N. K., & Schniederjans, M. J. (1985). Goal programming solutions to transportation

problems with variable supply and demand requirements. *Socio-Economic Planning Sciences*, *19*(2), 95–100.

Land, K. C., Lovell, C. A. K., Thore, S., Land, K. C., & Lovell, C. A. K. (1993). Chance-constrained Data Envelopment Analysis, *14*(6), 541–554.

Larbani, M., & Aouni, B. (2011). A new approach for generating efficient solutions within the goal programming model. *Journal of the Operational Research Society*, *62*(1), 175–182.

Lee, Y. K., Sam Park, K., & Kim, S. H. (2002). Identification of inefficiencies in an additive model based IDEA (imprecise data envelopment analysis). *Computers and Operations Research*, *29*(12), 1661–1676.

Lertworasirikul, S., Fang, S. C., Joines, J. A., & Nuttle, H. L. W. (2003). Fuzzy data envelopment analysis (DEA): A possibility approach. *Fuzzy Sets and Systems*, *139*(2), 379–394.

Li, L., & Lai, K. K. (2000). A Fuzzy Approach To The Multiobjective Transportation Problem. *Computers & Operations Research*, *27*(1), 43–57.

Lio, W., & Liu, B. (2017). Uncertain data envelopment analysis with imprecisely observed inputs and outputs. *Fuzzy Optimization and Decision Making*.

Lovell, C. K. (1993). "Production frontiers and productive efficiency", in Fried, H., Lovell, C., and Schmidt, S. (eds.). *The Measurement of Productive Efficiency: Techniques and Applications*.

Lu, C.-C. (2015). Robust data envelopment analysis approaches for evaluating algorithmic performance. *Computers & Industrial Engineering*, *81*, 78–89.

Mardani, M., & Salarpour, M. (2015). Measuring technical efficiency of potato production in Iran using robust data envelopment analysis. *Information Processing in Agriculture*, *2*(1), 6–14.

Matin, R. K., Jahanshahloo, G. R., & Vencheh, A. H. (2007). Inefficiency evaluation with an additive DEA model under imprecise data, an application on IAUK departments. *Journal of the Operations Research Society of Japan*, *50*(3), 163–177.

McDonald, J. (2009). Using least squares and tobit in second stage DEA efficiency analyses. *European Journal of Operational Research*, *197*(2), 792–798.

Mostafa, M. M. (2009). Modeling the efficiency of top Arab banks: A DEA-neural network approach. *Expert Systems with Applications*, *36*(1), 309–320.

Mulvey, J. M., Vanderbei, R. J., & Zenios, S. a. (1995). Robust optimization of large-scale systems. *Operations Research*, *43*(2), 264–281.

Mulvey, J. M., Vanderbei, R. J., Zenios, S. A,. (1995). Robust optimization of large-scale systems. *Operations Research*, *43*(2), 264–281.

Narayanamoorthy, S., & Anukokila, P. (2014). Goal programming approach for solving transportation problem with interval cost, *Journal of Intelligeent and Fuzzy Systems,* 26(3), 1143–1154.

Olesen, O. B., & Petersen, N. C. (1995). Chance Constrained Efficiency Evaluation. *Management Science*, *41*(3), 442–457.

Olesen, O. B., & Petersen, N. C. (2016). Stochastic data envelopment analysis - A review. *European Journal of Operational Research*, *251*(1), 2–21.

Omrani, H. (2013). Common weights data envelopment analysis with uncertain data: A robust optimization approach. *Computers and Industrial Engineering*, *66*(4), 1163–1170.

Orbis Bank Focus. (2016). Bankscope database.

Paradi, J. C., Sherman, H. D., & Tam, F. K. (2017). *Data Envelopment Analysis in the Financial Services Industry: A Guide for Practitioners and Analysts Working in Operations Research Using DEA* (Vol. 266). Springer.

Paradi, J. C., Rouatt, S., & Zhu, H. (2011). Two-stage evaluation of bank branch efficiency using data envelopment analysis. *Omega*, *39*(1).

Park, K. S. (2007). Efficiency bounds and efficiency classifications in DEA with imprecise data. *Journal of the Operational Research Society*, *58*(4), 533–540.

Park, K. S. (2010). Duality, efficiency computations and interpretations in imprecise DEA. *European Journal of Operational Research*, *200*(1), 289–296.

Pastor, J. T., & Ruiz, J. L. (2007). Variables with negative values in DEA. In *Modeling Data Irregularities and Structural Complexities in Data Envelopment Analysis* (pp. 63–84).

Prékopa, A. (1995). *Stochastic Programming*. Springer Netherlands.

Ramalho, E. A., Ramalho, J. J. S., & Henriques, P. D. (2010). Fractional regression models for second stage DEA efficiency analyses. *Journal of Productivity Analysis*, *34*(3), 239–255.

Ringuest, J. L., & Rinks, D. B. (1987). Interactive solutions for the linear multiobjective transportation problem. *European Journal of Operational Research*, *32*(1), 96–106.

Roy, S. K., Maity, G., Weber, G. W., & Gök, S. Z. A. (2017). Conic scalarization approach to solve multi-choice multi-objective transportation problem with interval goal. *Annals of Operations Research*, *253*(1), 599–620.

Sadjadi, S. J., & Omrani, H. (2008). Data envelopment analysis with uncertain data: An application for Iranian electricity distribution companies. *Energy Policy*, *36*(11), 4247–4254.

Sadjadi, S. J., & Omrani, H. (2010). A bootstrapped robust data envelopment analysis model for efficiency estimating of telecommunication companies in Iran. *Telecommunications Policy*, *34*(4), 221–232.

Sadjadi, S. J., Omrani, H., Abdollahzadeh, S., Alinaghian, M., & Mohammadi, H. (2011a). A robust super-efficiency data envelopment analysis model for ranking of provincial gas companies in Iran. *Expert Systems with Applications*, *38*(9), 10875–10881.

Sadjadi, S. J., Omrani, H., Abdollahzadeh, S., Alinaghian, M., & Mohammadi, H. (2011b). A robust super-efficiency data envelopment analysis model for ranking of provincial gas companies in Iran. *Expert Systems with Applications*, *38*(9), 10875–10881.

Sadjadi, S. J., Omrani, H., Makui, A., & Shahanaghi, K. (2011). An interactive robust data envelopment analysis model for determining alternative targets in Iranian electricity distribution companies. *Expert Systems with Applications*, *38*(8), 9830–9839.

Salahi, M., Toloo, M., & Hesabirad, Z. (2018). Robust Russell and enhanced Russell measures in DEA. *Journal of the Operational Research Society*, 1-9.

Salahi, M., Torabi, N., & Amiri, A. (2016). An optimistic robust optimization approach to common set of weights in DEA. *Measurement: Journal of the International Measurement Confederation*, *93*, 67–73.

Seiford, M., & Thrall, M. (1990). Recent Developments in DEA: The Mathematical Programming Approach to Frontier Analysis. *Journal of Econometrics*, *46*, 7–38.

Sengupta, J. K. (1992). A fuzzy systems approach in data envelopment analysis. *Computers & Mathematics with Applications*, *24*(8–9), 259–266.

Shokouhi, A. H., Hatami-Marbini, A., Tavana, M., & Saati, S. (2010). A robust optimization approach for imprecise data envelopment analysis. *Computers & Industrial Engineering*, *59*(3), 387–397.

Shokouhi, A. H., Shahriari, H., Agrell, P. J., & Hatami-Marbini, A. (2014). Consistent and robust ranking in imprecise data envelopment analysis under perturbations of random subsets of data. *OR Spectrum*, *36*(1), 133–160.

Simar, L. L., & Wilson, P. W. (2000). A general methodology for bootstrapping in non-

parametric frontier models. *Journal of Applied Statistics*, *27*(6), 779–802.
Simar, L., & Wilson, P. W. (1998). Sensitivity Analysis of Efficiency Scores: How to Bootstrap in Nonparametric Frontier Models. *Management Science*, *44*(1), 49–61.
Simar, L., & Wilson, P. W. (2007). Estimation and inference in two-stage, semi-parametric models of production processes. *Journal of Econometrics*, *136*(1), 31–64.
Soltani, A., Tabriz, A. A., & Sanei, M. (2017). Evaluation of the suggestions system performance using robust DEA model, the case of Esfahan's Steel Company. *International Journal of Productivity and Quality Management*, 15(1), 20-35.
Soyster, A. L. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, *21*(3), 1154–1157.
Talluri, S., Narasimhan, R., & Nair, A. (2006). Vendor performance with supply risk: A chance-constrained DEA approach. *International Journal of Production Economics*, 100(2), 212-222.
Tavana, M., Shiraz, R. K., Hatami-Marbini, A., Agrell, P. J., & Paryab, K. (2013). Chance-constrained DEA models with random fuzzy inputs and outputs. *Knowledge-Based Systems*, *52*, 32–52.
Thanassoulis, E. (1997). Duality in data envelopment analysis under constant returns to scale. *IMA Journal of Management Mathematics*, *8*(3), 253–266.
Thanassoulis, E. (2001). *Introduction To the Theory and Application of Data* (1st ed.). Dordrecht: Kluwer Academic Publishers.
Toloo, M. (2012). Alternative solutions for classifying inputs and outputs in data envelopment analysis. *Computers and Mathematics with Applications*, *63*(6), 1104–1110.
Toloo, M. (2014a). An epsilon-free approach for finding the most efficient unit in DEA. *Applied Mathematical Modelling*, *38*(13), 3182–3192.
Toloo, M. (2014b). *Data Envelopment Analysis with selected models and applications* (1st ed.). SAEL, Vol. 30. Ostrava: VSB-TU Ostrava.
Toloo, M. (2014c). Notes on classifying inputs and outputs in data envelopment analysis: a comment. *European Journal of Operational Research*, *235*, 810–812.
Toloo, M. (2014d). The role of non-Archimedean epsilon in finding the most efficient unit: With an application of professional tennis players. *Applied Mathematical Modelling*, *38*(21–22), 5334–5346.
Toloo, M., & Allahyar, M. (2018). A simplification generalized returns to scale approach for selecting performance measures in data envelopment analysis. *Measurement: Journal of the International Measurement Confederation*, *121*, 327–334.
Toloo, M., Allahyar, M., & Hančlová, J. (2018). A non-radial directional distance method on classifying inputs and outputs in DEA: Application to banking industry. *Expert Systems with Applications*, *92*, 495–506.
Toloo, M., & Barat, M. (2015). On considering dual-role factor in supplier selection problem. *Mathematical Methods of Operations Research*, *82*(1), 107–122.
Toloo, M., Barat, M., & Masoumzadeh, A. (2015). Selective measures in data envelopment analysis. *Annals of Operations Research*, *226*(1), 623–642.
Toloo, M., Keshavarz, E., & Hatami-Marbini, A. (2018). Dual-role factors for imprecise data envelopment analysis. *Omega*, *77*, 15–31.
Toloo, M., & Mensah, E. K. (2018). Robust optimization with nonnegative decision variables : A DEA approach. *Computers and Industrial Engineering*. https://doi.org/10.1016/j.cie.2018.10.006
Toloo, M., & Nalchigar, S. (2011). A new DEA method for supplier selection in presence of both cardinal and ordinal data. *Expert Systems with Applications*, *38*(12), 14726–14731.

Toloo, M., & Tichý, T. (2015). Two alternative approaches for selecting performance measures in data envelopment analysis. *Measurement*, *65*, 29–40.
Wang, K., & Wei, F. (2010). Robust data envelopment analysis based MCDM with the consideration of uncertain data. *Journal of Systems Engineering and Electronics*, *21*(6), 981–989.
Wang, Y. M., Greatbanks, R., & Yang, J. B. (2005). Interval efficiency assessment using data envelopment analysis. *Fuzzy Sets and Systems*, *153*(3), 347–370.
Wanke, P., & Barros, C. (2014). Two-stage DEA: An application to major Brazilian banks. *Expert Systems with Applications*, *41*(5), 2337–2344.
Wei, G., & Wang, J. (2017). A comparative study of robust efficiency analysis and Data Envelopment Analysis with imprecise data. *Expert Systems with Applications*, *81*, 28–38.
Wen, M. (2015). *Uncertain Data Envelopment Analysis*. Springer.
Wu, D., Ding, W. D., Koubaa, A., Chaala, A., & Luo, C. C. (2017). Robust DEA to assess the reliability of methyl methacrylate-hardened hybrid poplar wood. *Annals of Operations Research*, *248*(1–2), 515–529.
Yu, G. (1997). Robust economic order quantity models. *European Journal of Operational Research*, *100*(3), 482–493.
Yuan, Y., Li, Z., & Huang, B. (2016). Robust optimization under correlated uncertainty: Formulations and computational study. *Computers & Chemical Engineering*, *85*, 58–71.
Zadeh, L. A. (1965). Fuzzy_set_Zadeh-1965.pdf. *Information and Control*, *8*(8), 338–353.
Zahedi-seresht, M., Jahanshahloo, G., & Jablonsky, J. (2017). A robust data envelopment analysis model with different scenarios, *Applied mathematical modelling*, *52*, 306–319.
Zangiabadi, M., & Maleki, H. R. (2007). Fuzzy goal programming for multiobjective transportation problems. *Journal of Applied Mathematics and Computing*, *24*(1–2), 449–460.
Zhu, J. (2003). Imprecise data envelopment analysis (IDEA): A review and improvement with an application. *European Journal of Operational Research*, *144*(3), 513–529.
Zhu, J., & Cook, W. D. (2007). *Modeling Data Irregularities and Structural Complexities in Data Envelopment Analysis*. Springer Science & Business Media