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PhD in Computer Science and Computational Mathematics

Existence, non existence and uniqueness results for higher order elliptic systems

Candidate:
Delia Schiera
716179

Advisor:
Prof. Daniele Cassani

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Introduction

Consider the following

$$\begin{cases} L_1 u = \dfrac{\partial H(u,v)}{\partial v} & \text{in } \Omega \subseteq \mathbb{R}^N \\ L_2 v = \dfrac{\partial H(u,v)}{\partial u} & \\ B(u,v) = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where L_i are uniformly elliptic operators, $H \colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $B(u,v) = 0$ represents suitable boundary conditions in the case $\partial\Omega$ is not empty. This kind of systems in which two nonlinear PDEs are coupled in an hamiltonian fashion have attracted a lot of attention in the last two decades both from the Mathematical as well as Physical point of view as those models describe, among many others, nonlinear interaction between fields, see [10, 105]. Since this dissertation is of theoretical nature, here and in what follows we will only very briefly mention possible applications and give a few references.

When $L_1 = L_2 = -\Delta$, Ω is a smooth bounded domain,

$$H(u,v) = \frac{1}{q+1}\,|v|^{q+1} + \frac{1}{p+1}\,|u|^{p+1}$$

and $B(u,v) = 0$ represents Dirichlet boundary conditions, system (0.1) reads as

$$\begin{cases} -\Delta u = |v|^{q-1}\,v & \text{in } \Omega \subset \mathbb{R}^N,\ N > 2 \\ -\Delta v = |u|^{p-1}\,u & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

which corresponds to the coupling of two Lane–Emden equations, namely

$$\begin{cases} -\Delta u = |u|^{p-1}\,u & \text{in } \Omega \subset \mathbb{R}^N, N > 2 \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We refer the reader to the surveys [17, 86, 42] for a comprehensive discussion on results of existence/non-existence and various qualitative properties of solutions for systems of the form (0.2). We just recall here that there exists a critical curve [74], namely

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N},$$

such that there exists a nontrivial solution to (0.2) if (p, q) is below it, and no classical nontrivial solutions do exist on and above it (if Ω is a star-shaped domain). We also remark that the case of dimension $N = 2$ is deeply different, and one has exponential critical nonlinearities in place of power-like [86, 24, 25, 21].

A natural extension is given by systems in which in place of the Laplace operator one considers more general higher order operators, such as the polyharmonic operator, which is defined as

$$\Delta^s u = \Delta^{s-1} \Delta u,$$

where $s \in \mathbb{N}$, $s \geq 2$. Polyharmonic operators appear in the study of classical elasticity problems, such as the Kirchhoff plate equation. As related applications, we mention the analysis of stability of suspension bridges [47] and properties of Micro Electro-Mechanical Systems (MEMS) [22]. The prototype is the following

$$\begin{cases} (-\Delta)^{\alpha} u = |v|^{q-1} v & \\ & \text{in } \Omega \subseteq \mathbb{R}^N, N > \max\{2\alpha, 2\beta\} \\ (-\Delta)^{\beta} v = |u|^{p-1} u & \\ B(u, v) = 0 & \text{on } \partial\Omega \end{cases} \tag{0.3}$$

where $B(u, v) = 0$ represents suitable boundary conditions if any. We point out that (0.3) is non variational if $\alpha \neq \beta$, namely it is not clear how to define a functional whose critical points are weak solutions to (0.3). Moreover, we recall that the polyharmonic operator does not always satisfy a maximum principle. Indeed, if we take Dirichlet boundary conditions, i.e.

$$\frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, \quad \frac{\partial^t v}{\partial \nu^t} = 0,\ t = 0, \ldots, \beta - 1 \quad \text{on } \partial\Omega,$$

where ν is the outward pointing normal to the boundary, the maximum principle is known to hold only in the case Ω is a ball or a small deformation of a ball [47], whereas it fails for instance on ellipses with sufficiently big ratio of half axes. These features make the study of existence/non-existence of solutions to (0.3), as well as uniqueness, challenging and interesting problems. So far, only very partial Liouville–type non-existence results have been proved in the case $\Omega = \mathbb{R}^N$, see [76, 20, 69], whereas in [68] existence of nontrivial solutions is established if $\Omega = \mathbb{R}^N$, provided $\alpha = \beta$ and (p, q) is above the higher order critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2\alpha}{N}.$$

As for the case of a bounded domain, we refer to [93] for a priori bounds for systems with Navier boundary conditions, namely

$$\Delta^k u = 0,\ k = 0, \dots, \alpha - 1, \quad \Delta^j v = 0,\ j = 0, \dots, \beta - 1 \quad \text{on } \partial\Omega,$$

and to [28] for a critical point theorem which allows one to treat the variational case $\alpha = \beta$, both for Navier and Dirichlet boundary conditions. Unfortunately, the approach in [28] cannot be extended to the non-variational setting, nor to deal with more general nonlinearities than power-like. The aim of the present dissertation is to make advances in this direction, and furthermore to investigate the problem of uniqueness of solutions to (0.3) when Ω is a ball.

We first consider the variational case $\alpha = \beta$, for which we prove existence and non-existence results for (0.3) under Dirichlet boundary conditions, with Ω a sufficiently smooth bounded domain, and $N > 2\alpha$. We also consider more general nonlinearities than power-like. The existence result is achieved by means of a Linking Theorem on fractional Sobolev spaces, applied to the functional

$$I(u, v) = \int_\Omega A^s u A^{2\alpha - s} v \, dx - \int_\Omega H(u, v) \, dx,$$

where A^s represents a fractional power of $(-\Delta)^\alpha$ with Dirichlet boundary conditions, and H is the Hamiltonian associated to (0.3). This extends to the higher order case ideas in [43, 57], where the second order Lane–Emden system (0.2) is considered. We also prove, by exploiting a Pohozaev type identity [83], that no positive classical solutions to (0.3) exist on a ball, if (p, q) lies above the critical hyperbola.

However this approach fails in the non-variational case $\alpha \neq \beta$. Here, we restrict ourselves to the ball, and we exploit a different technique, adapting arguments of [8]. The proof relies on degree theory and a blow-up analysis to prove that if the only solution to (0.3) is the trivial one, then there exists a classical radial nontrivial solution to system (0.3) on $\Omega = \mathbb{R}^N$, provided the global maximum of solutions to (0.3) is attained at the center of the ball. The moving planes procedure together with Gidas–Ni–Nirenberg arguments [49] is used to establish that solutions to (0.3) are radially symmetric and strictly decreasing in the radial variable, see also [12]. Existence for $p, q > 1$ below the Serrin curves

$$2\beta q + N + 2\alpha pq - Npq = 0, \quad 2\alpha p + N + 2\beta pq - Npq = 0$$

is now ensured by contradiction, exploiting Liouville–type theorems for (0.3) on $\Omega = \mathbb{R}^N$ proved by Liu–Guo–Zhang [68] and Mitidieri–Pohozaev [76]. Furthermore, we prove a non-existence result in the spirit of [29], which allows us to extend the existence by assuming $pq > 1$, a more natural superlinearity condition than $p, q > 1$.

Moreover, we give some conditions under which existence for (0.3) on $\mathbb{R}^N$ holds, in the particular case $\alpha = 2$, $\beta = 1$. This is done by exploiting a result in [104], which proves that the existence of solutions to (0.3) on $\mathbb{R}^N$ is guaranteed by the non-existence of nontrivial solutions to the same system on a ball with Navier boundary conditions. Now, such a non-existence result is achieved extending ideas in [30], where a (s,t)-Laplace system is taken into account.

Concerning uniqueness, so far only the second order Lane–Emden system and the biharmonic equation have been studied [35, 33, 32, 41], or polyharmonic equations with sublinear nonlinearities, see [34]. We prove uniqueness of solutions to equations and systems up to order eight on a ball endowed with Dirichlet boundary conditions. The proof extends the classical argument by Gidas–Ni–Nirenberg [49]. Apparently, this approach works up to $\alpha \leq 4$, as technical difficulties arise when considering higher order operators, due to the fact that Dirichlet boundary conditions prescribe the behavior only of the first $\alpha - 1$ derivatives of the solution, and no information seems to be retained for higher order derivatives. We point out that these difficulties can be overcome by taking different boundary conditions which, from a physical point of view, correspond to vanishing higher order momenta along the boundary.

This dissertation is organized as follows.

Chapter 1 is a brief introduction on the topic. We first survey the main results regarding Lane–Emden type equations and systems, both in the second and higher order case. We recall definition and properties of the polyharmonic operators, and then we present our main results.

In *Chapter 2* we prove some existence and non existence results for the variational case $\alpha = \beta$, with Dirichlet boundary conditions. This is an extended transcription of [88].

Chapter 3 is devoted to the proof of existence of solutions in the non variational setting on a ball, see also [89]. We further extend our arguments to systems of an arbitrary number of equations and to the Navier case.

In *Chapter 4* we consider the problem of uniqueness for (0.3), following [89, 23]. We first take into account polyharmonic equations up to order eight, and then we show how to adapt the proof to treat polyharmonic systems.

Appendix A is devoted to the proof of an existence result for (0.3) on $\mathbb{R}^N$, in the particular case $\alpha = 2$, $\beta = 1$.

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Notations

$\mathbb{N}$	$= \{0, 1, 2, 3, \dots\}$
$\mathbb{R}^+$	$= \{x \in \mathbb{R}, x > 0\}$
$\lceil x \rceil$	the least integer greater than or equal to x
∂S	boundary of S
$\bar{S}$	closure of S
S^c	complement of S
$B(x, R)$	ball centered in x with radius R
B_R	$= B(0, R)$
ν	outward pointing normal to the boundary
$deg(f, \Omega, x)$	the Leray–Schauder degree of f in Ω over x
∂_{x_i}	$= \frac{\partial}{\partial x_i}$, the partial derivative in the direction x_i
D^α	$= \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$ (in the weak sense), with $\lvert\alpha\rvert = \sum_{i=1}^n \alpha_i$
∇	gradient operator
Δ	Laplace operator
$\Delta^s u$	$= \Delta(\Delta^{s-1} u)$, the polyharmonic operator of order $s \geq 1$
$G_\alpha(x, y)$	Green function for the polyharmonic operator of order α endowed with Dirichlet boundary conditions
$X \hookrightarrow Y$	X is embedded in Y, namely X is a subspace of Y and the identity operator defined on X into Y by $Ix = x$ is continuous
$X \oplus Y$	direct sum of X and Y
E'	dual space of E
$[X, Y]_s$	real interpolation space between X and Y of order s
a.e.	almost everywhere
$f \gneqq 0$	f is nonnegative and not identically zero
$\langle \cdot, \cdot \rangle$	Euclidean scalar product

$\lvert\cdot\rvert$	$= \sqrt{\langle\cdot,\cdot\rangle}$
$dist(x,\Omega)$	the Euclidean distance of x from the set Ω
$\lVert\cdot\rVert_E$	norm in the space E
$\lVert f\rVert_p$	$= \left(\int_\Omega \lvert f\rvert^p\right)^{1/p}$ for $p\in[1,\infty)$
$\lVert f\rVert_\infty$	$= \inf\{C\geq 0 : \lvert f(x)\rvert\leq C$ for a.e. $x\}$
$\lVert u\rVert_{W^{k,p}(\Omega)}$	$= \left(\sum_{\lvert\alpha\rvert\leq k}\int_\Omega \lvert D^\alpha u\rvert^p\,dx\right)^{1/p}$
$C^k(\Omega)$	space of k times continuously differentiable functions on Ω
$C^\infty(\Omega)$	$= \cap_{k\geq 0}C^k(\Omega)$
$C^k(\bar\Omega)$	functions in $C^k(\Omega)$ such that derivatives of order $\alpha\leq k$ admit a continuous extension to $\bar\Omega$
$C^{0,\gamma}(\bar\Omega)$	$= \{u\in C^0(\bar\Omega) : \sup_{x\neq y}\frac{\lvert u(x)-u(y)\rvert}{\lvert x-y\rvert^\gamma}<\infty\}$, with $0<\gamma<1$
$C^{k,\gamma}(\bar\Omega)$	$= \{u\in C^k(\bar\Omega) : D^j u\in C^{0,\gamma}(\bar\Omega)\quad \forall j,\lvert j\rvert\leq k\}$
$C_0^\infty(\Omega)$	functions in $C^\infty(\Omega)$ with compact support in Ω
$C_0^k(\bar\Omega)$	functions in $C^k(\bar\Omega)$ which vanish on $\partial\Omega$
$C^k(\mathbb{R}^N;\mathbb{R}^M)$	functions $f=(f_1,\ldots,f_M):\mathbb{R}^N\to\mathbb{R}^M$ such that $f_i\in C^k(\mathbb{R}^N)$
$L^p(\Omega)$	$= \{u:\Omega\to\mathbb{R}$ measurable such that $\lVert u\rVert_p<\infty\}$, with $1\leq p<\infty$
$L^\infty(\Omega)$	$= \{u:\Omega\to\mathbb{R}$ measurable such that $\lVert u\rVert_\infty<\infty\}$
$W^{k,p}(\Omega)$	$= \{\,u\in L^p(\Omega)$ s.t. $\forall\,\alpha,\lvert\alpha\rvert\leq k,\ D^\alpha u\in L^p(\Omega)\,\}$
$W_0^{k,p}(\Omega)$	closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$H^k(\Omega)$	$= W^{k,2}(\Omega)$
$H_0^k(\Omega)$	$= W_0^{k,2}(\Omega)$
$H_\theta^\alpha(\Omega)$	$= \{u\in H^\alpha(\Omega) :\ \Delta^j u=0$ on $\partial\Omega$ for $j<\alpha-1\}$
$E^s(\Omega)$	$= [L^2(\Omega),H^{2\alpha}(\Omega)\cap H_0^\alpha(\Omega)]_{s/(2\alpha)}$, for $0<s<2\alpha$.

Note: we omit the domain when it is clear from the context.

Chapter 1

Preliminaries and main results

The purpose of this dissertation is to investigate existence, non-existence and uniqueness of solutions for Hamiltonian elliptic systems of the following form [20, 69, 93]

$$\begin{cases} (-\Delta)^\alpha u = \dfrac{\partial H(u,v)}{\partial v} & \text{in } \Omega \subseteq \mathbb{R}^N, N > \max\{2\alpha, 2\beta\} \\ (-\Delta)^\beta v = \dfrac{\partial H(u,v)}{\partial u} & \\ B(u,v) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta^s u = \Delta(\Delta^{s-1}u), \quad \text{for } s = 2, 3, \ldots$$

is the polyharmonic operator [47], $\alpha, \beta \in \mathbb{N}$, $\alpha, \beta \geq 1$, $H \in C^1(\mathbb{R}^2; \mathbb{R})$, and $B(u,v) = 0$ represents suitable boundary conditions if any.
In the second order case $\alpha = \beta = 1$, if

$$H(u,v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1},$$

then problem (1.1) reduces to the so-called Lane–Emden system [17, 86, 42].
In this introductory chapter, we first review the main literature on second order Lane–Emden type equations and systems. Moreover, in Section 1.2 below we recall basic properties of the polyharmonic operator. Then, in Section 1.3 we introduce features and difficulties which arise in the higher order case, and we state our main results (see Section 1.4 and Section 1.5), which will be presented in full details in the next chapters. Finally, in Section 1.6 we discuss a few open questions as possible directions of future research.

1.1 Second order case

Let us first consider the so-called Lane–Emden equation

$$\begin{cases} -\Delta u = |u|^{p-1}\, u & \text{in } \Omega \subset \mathbb{R}^N, N > 2 \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p > 1$, and Ω is a smooth bounded domain. In astrophysics (1.2) plays an important role as it models several phenomena among which the thermal behavior of a spherical cloud of gas and the stellar structure, [27, 85]. Moreover, when $p = \frac{N+2}{N-2}$ it is related to the Yamabe problem, see for instance [99, Section 3.4]. Recall that $u \in H^1_0$ is a *weak solution* to (1.2) if for any $\varphi \in H^1_0$

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega |u|^p \, u\varphi \, dx,$$

where we denote with H^k the Sobolev space $W^{k,2}$, and with H^k_0 the closure of C^∞_0 in H^k. Problem (1.2) exhibits a variational structure, namely critical points of the energy functional $I : H^1_0 \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx$$

are weak solutions to (1.2). One can apply the Mountain Pass Theorem of Ambrosetti and Rabinowitz [6] to get the existence of a nontrivial critical point of I provided $p < \frac{N+2}{N-2}$, see for instance [39, Section 8.5.2]. Notice that $H^1_0 \hookrightarrow L^{p+1}$ if $p \le \frac{N+2}{N-2}$ and this embedding is compact if and only if the strict inequality holds [3, Chapter 4]. This solution turns out to be *classical*, in the sense that it belongs to $C^2(\Omega) \cap C(\bar{\Omega})$ and satisfies (1.2) pointwise, as follows by means of standard bootstrap arguments, combined with elliptic regularity theory [51, Theorem 8.9] and Sobolev embeddings. We also mention that one can prove existence of a ground state (namely, minimal energy) solution to (1.2) through minimization techniques on a Nehari manifold, see [78, 77] and [26, Theorem 2.4.2]. Further, non-existence for classical solutions in the case Ω is a star-shaped domain and $p \ge \frac{N+2}{N-2}$ is proved exploiting the Pohozaev identity

$$\int_\Omega \left(\frac{N}{p+1} |u|^{p+1} - \frac{N-2}{2} |\nabla u|^2 \right) dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle n, x \rangle \, dx,$$

see [81] and [39, Section 9.4.2]. We point out that the star-shapedness of Ω is crucial. Indeed, if $p = \frac{N+2}{N-2}$ and Ω is an annulus, then there exists a non-trivial solution to (1.2), [18, Section 1.3(3)].
Summarizing, $p = \frac{N+2}{N-2}$ plays the role of a critical threshold between existence and non-existence of classical solutions to (1.2), at least on star-shaped domains. It is worth recalling that in the case $\Omega = \mathbb{R}^N$, the situation is reversed,

and one proves existence of classical solutions above the critical exponent and non-existence below it [50].

Deep differences arise when taking into account the Lane–Emden system

$$\begin{cases} -\Delta u = |v|^{q-1}\, v & \text{in } \Omega \subset \mathbb{R}^N, N > 2 \\ -\Delta v = |u|^{p-1}\, u & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. Indeed, the energy functional related to (1.3) turns out to be

$$I(u,v) = \int_\Omega \nabla u \nabla v \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1}\, dx - \frac{1}{q+1} \int_\Omega |v|^{q+1}\, dx, \tag{1.4}$$

and in particular the quadratic part $\int_\Omega \nabla u \nabla v \, dx$ is strongly indefinite, in the sense that it is neither bounded from above nor from below on infinite dimensional spaces (just take pairs (u,u), $(u,-u)$). This prevents one from applying the Mountain Pass Theorem and the Nehari manifold minimization technique, and different tools have to be exploited in order to investigate existence for (1.3). One possible approach consists in a reduction by inversion originally due to Lions [67] combined with a critical point theorem due to Clément, Felmer and Mitidieri [28], which we recall below.

Theorem 1.1 (Theorem 2.2 in [28]). *Let $(E, \|\cdot\|)$ be a Banach space compactly embedded in $L^{p+1}(\Omega)$, and let $A : E \to L^{\frac{q+1}{q}}(\Omega)$ be an isomorphism. Assume $pq \neq 1$. Then*

$$J(u) = \frac{q}{q+1} \|Au\|_{\frac{q+1}{q}}^{\frac{q+1}{q}} - \frac{1}{p+1} \|u\|_{p+1}^{p+1}$$

has infinitely many critical points.

In particular, the functional

$$J(u) = \frac{q}{q+1} \int_\Omega |\Delta u|^{\frac{q+1}{q}}\, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1}\, dx \tag{1.5}$$

has a nontrivial critical point on the space $E = W^{2,\frac{q+1}{q}} \cap W_0^{1,\frac{q+1}{q}}$, provided (p,q) is below the following hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$$

and $pq \neq 1$. One can verify that, given a nontrivial critical point $u \in E$ of (1.5), then

$$(u, (-\Delta)^{-1}(|u|^{p-1}\, u))$$

is a nontrivial weak solution to (1.3), in other words, that (1.3) is equivalent to the following equation

$$\begin{cases} \Delta\left(|\Delta|^{\frac{1}{q}-1}\Delta u\right) = |u|^{p-1}u & \text{in } \Omega \\ u, \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

see [28, Theorem 2.3], see also [87] and [17, Section 4]. In [43, 57] a different strategy is presented, which requires some restrictions on the values for p, q, however it seems more suitable to treat different nonlinearities than power-like as in (1.3). The basic idea is to exploit fractional order Sobolev spaces and consider the energy functional for (1.3) such that derivatives are distributed asymmetrically in the quadratic part. Other techniques include Lyapunov–Schmidt type reduction [17, Section 5], finite dimensional approximation on Lorentz–Sobolev spaces [44], application of the dual variational principle [31], and a minimization on a generalized Nehari manifold, see [100]. We refer the reader to the extensive surveys on the topic [17, 86, 42] and to references therein. As for non-existence, one exploits a Pohozaev type identity as in [74, 83] to prove that no classical nontrivial solutions to (1.3) do exist on a star-shaped domain Ω, provided $p, q > 1$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}.$$

It turns out to be clear from what briefly recalled above the major role played by the critical curve

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \tag{1.6}$$

in particular as threshold between existence and non-existence of solutions to (1.3), as observed by Mitidieri in [74]. Notice that the energy functional (1.4) is well defined on $W^{1,s} \times W^{1,t}$ with $\frac{1}{s} + \frac{1}{t} = 1$, provided $\frac{1}{p+1} \geq \frac{1}{s} - \frac{1}{N}$ and $\frac{1}{q+1} \geq \frac{1}{t} - \frac{1}{N}$, due to the Sobolev embeddings. Therefore,

$$\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{N} = \frac{N-2}{N},$$

namely (p, q) lies below (1.6). Moreover, we observe that the intersection between the critical hyperbola (1.6) and the bisector $p = q$ is the point $p = \frac{N+2}{N-2}$, namely the critical exponent for (1.2). Finally, we mention that the problem of existence of positive solutions to (1.3) in the case $\Omega = \mathbb{R}^N$ is still not completely solved, except for the radial case (see [75, 90, 92]) and for dimensions $N = 3$ [82] and $N = 4$ [97]. Only partial results are known for higher dimensions, see [97, 98, 75, 91, 19].

1.2 Polyharmonic operators

The operator $\Delta^s : C^{2s} \to \mathbb{R}$, defined recursively as

$$\Delta^s u = \Delta(\Delta^{s-1} u), \quad \text{for } s = 2, 3, \dots,$$

is the *polyharmonic operator of order s* [47]. We recall that Δ^s is an elliptic operator, in the sense that the polynomial $L_s : \mathbb{R}^N \to \mathbb{R}$ such that $\Delta^s = L_s(\nabla)$ satisfies $L_s(\xi) > 0$ for any $\xi \neq 0$, see [4, p.625]. The polyharmonic operators appear in the study of classical elasticity problems, such as the Kirchhoff–Love plate equation [70, 103]. As related applications, we mention the analysis of stability of suspension bridges [58, 46] and Micro Electro–Mechanical Systems (MEMS) [22]. We refer to [47, Chapter 1] for these and other applications.
Higher order operators allow for plenty of possible boundary conditions and this is a main difference with the second order case. In this dissertation we will mainly deal with two boundary conditions, which are important in applications, for the problem

$$(-\Delta)^\alpha u = f \text{ in } \Omega \subset \mathbb{R}^N, N > 2\alpha.$$

On the one hand, we consider

$$\frac{\partial^k u}{\partial \nu^k} = 0, \quad k = 0, \dots, \alpha - 1, \text{ on } \partial\Omega \tag{D}$$

where ν is the outward pointing normal to $\partial\Omega$, which are the so-called *(polyharmonic) Dirichlet boundary conditions*, corresponding in the case of a deflecting plate to clamped conditions along the boundary. It is worth recalling that a function u is in H_0^α if and only if it satisfies (D), see [65, Proposition 1.2] and [3, Section 5.37]. On the other hand, we will refer to

$$\Delta^k u = 0 \quad k = 0, \dots, \alpha - 1, \text{ on } \partial\Omega \tag{N}$$

as the *Navier boundary conditions*, which correspond to the hinged plate model, by neglecting the contribution of the curvature of the boundary. Notice that in the second order case $\alpha = 1$ they both reduce to Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega.$$

Here and in what follows, given $g \in W^{m,p}(\Omega)$ and $j = 0, \dots, m$, we always interpret $\frac{\partial^j g}{\partial \nu^j} = 0$ on $\partial\Omega$ as $\gamma_j(g) = 0$, where

$$\gamma_j : W^{m,p}(\Omega) \to W^{m-j-1/p,p}(\partial\Omega)$$

is the trace operator which extends to $W^{m,p}(\Omega)$ the functional $g \mapsto \frac{\partial^j g}{\partial \nu^j}|_{\partial\Omega}$ defined on $C^m(\bar{\Omega})$ [65, Proposition 1.2]. In both cases, existence, uniqueness and regularity results for the linear theory can be proved given suitable assumptions on the function f. More precisely, one has the following results.

Theorem 1.2 (Theorem 15.2 in [4]). *Assume Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $N > 2\alpha$, and assume $f \in L^p$, where $p > 1$ is fixed. Then there exists a unique weak solution $u \in W^{2\alpha,p}$ to*

$$(-\Delta)^\alpha u = f \text{ in } \Omega$$

with Dirichlet boundary conditions (D) *or Navier boundary conditions* (N). *Moreover, there exists a constant $c > 0$ independent of u such that*

$$\|u\|_{W^{2\alpha,p}} \leq c \|f\|_p .$$

Theorem 1.3 (Theorem 9.3 in [4]). *Assume Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $N > 2\alpha$, and assume $f \in C^{0,\gamma}$, $\gamma \in (0, 1)$. Then there exists a unique classical solution $u \in C^{2\alpha,\gamma}$ to*

$$(-\Delta)^\alpha u = f \text{ in } \Omega$$

with Dirichlet boundary conditions (D) *or Navier boundary conditions* (N). *Moreover, there exists a constant $c > 0$ independent of u such that*

$$\|u\|_{C^{2\alpha,\gamma}} \leq c \|f\|_{C^{0,\gamma}} .$$

As for the validity of the maximum principle, in particular concerning positivity preserving property (namely, $(-\Delta)^\alpha u > 0$ implies $u > 0$), Navier and Dirichlet conditions exhibit quite different behaviors. Indeed, one can split the problem

$$\begin{cases} (-\Delta)^\alpha u = f & \text{in } \Omega \subset \mathbb{R}^N, N > 2\alpha \\ \Delta^k u = 0 \quad k = 0, \dots, \alpha - 1, & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

into α equations of order 2, as follows

$$\begin{cases} -\Delta u_k = u_{k+1} \quad k = 0, \dots, \alpha - 2 & \multirow{2}{*}{in Ω} \\ -\Delta u_{\alpha-1} = f & \\ u_k = 0 \quad k = 0, \dots, \alpha - 1, & \text{on } \partial\Omega, \end{cases}$$

since the α-polyharmonic operator with Navier boundary conditions coincides with α iterations of the Laplace operator, each endowed with second order Dirichlet boundary conditions. Consequently, (1.7) satisfies a maximum principle, namely $f \gneqq 0$ implies $u > 0$ on Ω, due to α applications of the classical maximum principle for the Laplace operator. These arguments fail for the Dirichlet problem

$$\begin{cases} (-\Delta)^\alpha u = f & \text{in } \Omega \subset \mathbb{R}^N, N > 2\alpha \\ \frac{\partial^k u}{\partial \nu^k} = 0, \quad k = 0, \dots, \alpha - 1, & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

However, Boggio [16, 15] proved that the Green function for (1.8) in the case Ω is the unit ball B_1 is explicitly given by

$$G_\alpha(x,y) = k_\alpha\, |x-y|^{2\alpha-N} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (\mu^2-1)^{\alpha-1}\mu^{1-N}\, d\mu,$$

and it turns out to be strictly positive, hence, if $f \gneqq 0$ is sufficiently regular and u is a classical solution to (1.8), then one can write

$$u(x) = \int_{B_1} G_\alpha(x,y) f(y)\, dy > 0.$$

Therefore, a maximum principle holds for the polyharmonic Dirichlet problem on a ball, see also [47, Proposition 3.6].

Theorem 1.4 ([16, 15])**.** *Let u be a classical solution to* (1.8) *and let $\Omega = B_1$. Assume that $f \in L^\infty$ satisfies $f \gneqq 0$. Then, $u > 0$ on Ω.*

This result has been extended to the case of bounded domains suitably "close" to the ball, see [47, Section 6.1.1] . As for other domains, an explicit representation of the Green function for (1.8) is in most cases out of reach, and further, numerous examples show that the Green function can also be sign changing (for instance, ellipses with sufficiently big ratio of half axes, bounded domains containing a corner, and the so-called *limaçons de Pascal*, see [47, Section 1.2] for a complete bibliography and a precise discussion on the topic). Consequently, the maximum principle fails on these domains. However, even if the corresponding Green function is sign changing, one expects that the negative part of it is comparatively small with respect to the positive part (first results in this direction can be found in [55, 52, 53]). Let us also mention that the positivity preserving property is related to the positivity of the first eigenfunction of the operator, see [47, Section 3.1.3].

We finally recall that a higher order analogue of the Hopf boundary lemma on a ball holds [54], [47, Theorem 5.7].

Theorem 1.5. *Let u be a solution to* (1.8) *with $\Omega = B_1$, and let $f \gneqq 0$. Then for every $x \in \partial B_1$*

$$\begin{cases} \Delta^{(\alpha/2)} u(x) > 0 & \text{if } \alpha \text{ even} \\ -\frac{\partial}{\partial\nu}\Delta^{(\alpha-1)/2} u(x) > 0 & \text{if } \alpha \text{ odd.} \end{cases}$$

1.3 Higher order Lane–Emden equations and systems

One may naturally wonder whether it is possible to extend, at least partially, the existence and non-existence results for the Lane–Emden equation and system recalled in Section 1.1 to similar problems involving the polyharmonic operators. Let us consider the following higher order Lane–Emden equation:

$$\begin{cases} (-\Delta)^\alpha u = |u|^{p-1}\, u & \text{in } \Omega \subset \mathbb{R}^N, N > 2\alpha \\ B(u,v) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $\alpha \in \mathbb{N}$, $\alpha \geq 1$, and $B(u,v) = 0$ represents either Dirichlet or Navier boundary conditions. As in the second order case $\alpha = 1$, (1.9) turns out to be variational, and the energy functional reads as

$$E(u) = \begin{cases} \frac{1}{2}\int_\Omega (\Delta^k u)^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} & \text{if } \alpha = 2k, \\ \\ \frac{1}{2}\int_\Omega \left|\nabla \Delta^k u\right|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} & \text{if } \alpha = 2k+1. \end{cases}$$

The functional E is defined on H_0^α if Dirichlet boundary conditions are considered, whereas if Navier boundary conditions are taken into account, then the natural space turns out to be H_θ^α, where

$$H_\theta^\alpha = \left\{ u \in H^\alpha : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } j < \alpha - 1 \right\}.$$

Compactness of the embedding $H_0^\alpha \hookrightarrow L^{p+1}$, and henceforth existence of solutions to (1.9), holds if $p < \frac{N+2\alpha}{N-2\alpha}$ [47, Section 7.2.2], and non-existence of classical solutions on a star-shaped domain can be proved if the exponent p satisfies $p \geq \frac{N+2\alpha}{N-2\alpha}$, see [83, Theorem 8] and [68, Corollary 2.7]. We refer to [14, 93, 96] for more general higher order equations.

The prototype system throughout this dissertation will be the following

$$\begin{cases} (-\Delta)^\alpha u = |v|^{q-1}\, v & \\ (-\Delta)^\beta v = |u|^{p-1}\, u & \text{in } \Omega \subseteq \mathbb{R}^N, N > \max\{2\alpha, 2\beta\} \\ B(u,v) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

where $\alpha, \beta \in \mathbb{N}$, $\alpha, \beta \geq 1$ and $B(u,v) = 0$ represents Dirichlet or Navier boundary conditions if Ω is bounded. If $\alpha = \beta$, (1.10) has a similar structure to that of the Lane–Emden system (1.3), and

$$I(u,v) = \begin{cases} \int_\Omega \Delta^{\frac{\alpha}{2}} u \Delta^{\frac{\alpha}{2}} v - \int_\Omega H(u,v) & \text{for even } \alpha \\ \\ \int_\Omega \nabla(\Delta^{\frac{\alpha-1}{2}} u) \nabla(\Delta^{\frac{\alpha-1}{2}} v) - \int_\Omega H(u,v) & \text{for odd } \alpha, \end{cases}$$

where $H(u,v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}$, plays the role of energy functional for it. Hence, if $\alpha = \beta$ then the natural candidate to be the critical threshold between existence and non-existence results is given by

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2\alpha}{N}, \tag{1.11}$$

as follows by Sobolev embedding theorems once we impose I to be well defined on the natural space related to the boundary conditions. The situation in which two different polyharmonic operators are taken into account in (1.10), namely if $\alpha \neq \beta$, seems to be the most challenging, since it does not exhibit a variational structure, and in particular, even the identification of the curve which could serve as critical threshold for (1.10) turns out to be a nontrivial problem. In the remainder of the present section, we recall some known facts concerning (1.10), in order to better contextualize our results, which will be presented in full details in the next chapters of this dissertation.

Let Ω be a bounded domain of $\mathbb{R}^N$. In [93] existence to (1.10) with Navier boundary conditions (N), also in the non-variational case, is obtained provided suitable a priori bounds on the solutions hold, see also [94] (actually there more general operators are taken into account). Theorem 3 in [57] states the existence of solutions to (1.10) with $\alpha = \beta$ in the subcritical case, namely below (1.11) with $p, q > 1$ and Navier boundary conditions.
In analogy to what happens in the second order case (1.3), also in the higher order variational case (1.10) where $\alpha = \beta$, one can exploit Theorem 1.1 to obtain a nontrivial critical point of the functional

$$J(u) = \frac{q}{q+1} \int_\Omega |\Delta^\alpha u|^{\frac{q+1}{q}}\, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1}\, dx \tag{1.12}$$

on the space $E = W^{2\alpha, \frac{q+1}{q}}(\Omega) \cap W_0^{\alpha, \frac{q+1}{q}}(\Omega)$, provided (p, q) is below the critical hyperbola (1.11) and $pq \neq 1$. One verifies that, given a nontrivial critical point u of (1.12), then

$$(u, ((-\Delta)^\alpha)^{-1}(|u|^{p-1}\, u))$$

is a nontrivial weak solution to (1.10), see [28, Theorem 2.3], see also [87]. The comments above convey into the following

Theorem 1.6 (Theorem 2.3 in [28]). *Let $\alpha = \beta$. Assume $pq \neq 1$ and let (p, q) below the critical hyperbola* (1.11). *Then there exists a nontrivial solution to* (1.10) *with boundary conditions* (N) *or* (D).

Finally, we recall that in [68] non existence is established above the hyperbola (1.11) for positive classical solutions to (1.10) with Navier boundary conditions, if $\alpha = \beta$ and Ω is a star-shaped domain.
We stress that the case (1.10) with Dirichlet conditions (D) and $\alpha \neq \beta$ has not been considered yet, nor the case (1.10), (D) with $\alpha = \beta$ and more general nonlinearities than power-like.

Let us now turn our attention to the case $\Omega = \mathbb{R}^N$. To the best of our knowledge, the only available result providing existence in this case is the following

Theorem 1.7 (Theorem 1.3 in [68]). *Let $\alpha = \beta$. Assume (p, q) lies above the corresponding critical hyperbola* (1.11). *Then there exist infinitely many* *radially symmetric positive classical solutions to*

$$\begin{cases} (-\Delta)^{\alpha} u = |v|^{q-1}\, v \\ (-\Delta)^{\beta} v = |u|^{p-1}\, u \end{cases} \quad in\ \mathbb{R}^N, N > \max\{2\alpha, 2\beta\}. \tag{1.13}$$

The first step in the proof consists in reducing, by means of a shooting technique, the problem of existence to (1.13) to the problem of non existence for the same system on a ball with Navier boundary conditions (N), see [68, 104], see also [62, 61] for similar problems studied through this method. In turn, the non existence result relies on a Pohozaev identity, which however is proved only in the particular case $\alpha = \beta$, and it cannot be extended straightforwardly to general α and β.

As for non-existence to (1.13), the situation turns out to be more involved, even in the variational case, except from the radial setting, for which the following result holds.

Theorem 1.8 (Theorem 5.1 in [20]). *Let $\alpha = \beta$. Let $p, q > 1$ and (p, q) below the critical hyperbola* (1.11). *Then the only positive classical radially symmetric solution to* (1.13) *is the trivial one.*

In the non radial case only a few partial results are known, which we recall below.

Theorem 1.9 (Theorem 1.2' in [69]). *If $1 < p, q < \min\{\frac{N+2\alpha}{N-2\beta}, \frac{N+2\beta}{N-2\alpha}\}$ then the only classical nonnegative solution to* (1.13) *is the trivial one.*

Theorem 1.10 (Theorem 1.1 in [106]). *Assume $\alpha = \beta$. If $p, q > 1$ and*

$$\frac{2\alpha(q+1)}{pq-1}, \frac{2\alpha(p+1)}{pq-1} \in \left[\frac{N-2}{2}, N-2\alpha\right),$$

then there is no positive classical solution to (1.13).

We also recall that non-existence for super solutions can be obtained by means of capacity estimates [76], see also [69, 20] for alternative proofs.

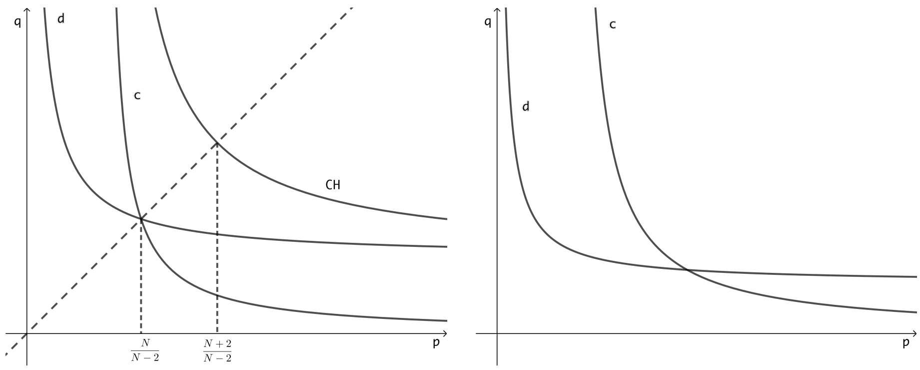
Theorem 1.11 (Theorem 19.1 in [76]). *Let us consider the following*

$$\begin{cases} (-\Delta)^{\alpha} u \geq |v|^{q} \\ (-\Delta)^{\beta} v \geq |u|^{p} \end{cases} \quad in\ \mathbb{R}^N, N > \max\{2\alpha, 2\beta\}. \tag{1.14}$$

If $p, q > 1$ and $2\beta q + N + 2\alpha pq - Npq \geq 0$ or $2\alpha p + N + 2\beta pq - Npq \geq 0$, then there exist no weak solutions to (1.14).

The curves

$$2\beta q + N + 2\alpha pq - Npq = 0, \quad 2\alpha p + N + 2\beta pq - Npq = 0 \tag{1.15}$$



(a) *Case* $\alpha = \beta = 1$, $N = 3$.   (b) *Case* $\alpha = 2$, $\beta = 1$, $N = 5$.

Figure 1.1: *In the two figures, c and d represent the two Serrin curves* (1.15). *The curve CH is the critical hyperbola* (1.6).

are the higher order extension of the so-called Serrin curves

$$2q + N + 2pq - Npq = 0, \quad 2p + N + 2pq - Npq = 0 \tag{1.16}$$

which constitute the threshold between existence and non-existence of super solutions to (1.3), see [75, 76], in the sense that if (p, q) is below one of the two curves (1.16) then the only (weak) super solution to (1.3) is the trivial one, whereas if (p, q) is above both the curves, then

$$u(x) = \frac{\varepsilon}{(1 + |x|^2)^{\frac{q+1}{pq-1}}}, \quad v(x) = \frac{\varepsilon}{(1 + |x|^2)^{\frac{p+1}{pq-1}}}$$

yield a super solution to (1.3), provided ε is sufficiently small. Unfortunately, it does not seem easy to extend such counterexamples to the higher order case, not even in the variational setting, and therefore to prove that (1.15) are actually sharp. Notice further that the intersection between the Serrin curves (1.16) and the bisector $p = q$ is $p = \frac{N}{N-2}$, namely the critical exponent for the existence of super solutions to (1.2), see also Figure 1.1.

1.4 Main results: existence and non-existence to system (1.10)

1.4.1 Variational case $\alpha = \beta$

All the results in the present subsection appeared in [88]. In Chapter 2 we give full details of the proofs, which adapt and extend considerations in [43] to the higher order case, see also [57]. The main idea is to exploit fractional order Sobolev spaces and consider a energy functional such that derivatives are distributed asymmetrically in the quadratic part. This approach, differently from the reduction by inversion in [28], allows one to consider more general nonlinearities than power-like, at the price of requiring some extra assumptions on the exponents p, q. In what follows, Ω is a fairly general smooth bounded domain ($C^{2\alpha,\gamma}$ regularity is enough).

Theorem 1.12. *Let $N > 2\alpha$ and let p, q be such that $pq > 1$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2\alpha}{N}, \quad \max\{(N-4\alpha)p, (N-4\alpha)q\} < N + 4\alpha. \tag{1.17}$$

Let $H\colon \mathbb{R}^2 \to \mathbb{R}$ be of class C^1 such that

(H1) $H(u, v) \geq 0$ for all $(u, v) \in \mathbb{R}^2$

(H2) There exists $R > 0$ such that, if $(u, v) \in \mathbb{R}^2$ satisfies $|(u, v)| \geq R$, then

$$\frac{1}{p+1}\partial_u H(u, v) \cdot u + \frac{1}{q+1}\partial_v H(u, v) \cdot v \geq H(u, v) > 0$$

(H3) There exist $r > 0$ and $a > 0$ such that, if $(u, v) \in \mathbb{R}^2$ satisfies $|(u, v)| \leq r$, then

$$H(u, v) \leq a(|u|^{p+1} + |v|^{q+1})$$

(H4) There exists $b > 0$ such that

$$|\partial_u H(u, v)| \leq b(|u|^p + |v|^{p(q+1)/(p+1)} + 1)$$
$$|\partial_v H(u, v)| \leq b(|v|^q + |u|^{q(p+1)/(q+1)} + 1).$$

Then, there exists a nontrivial solution

$$(u, v) \in W^{2\alpha,\frac{q+1}{q}}(\Omega) \cap W_0^{\alpha,\frac{q+1}{q}}(\Omega) \times W^{2\alpha,\frac{p+1}{p}}(\Omega) \cap W_0^{\alpha,\frac{p+1}{p}}(\Omega)$$

to problem

$$\begin{cases} (-\Delta)^\alpha u = \partial_v H(u, v) & \text{in } \Omega \subset \mathbb{R}^N \\ (-\Delta)^\alpha v = \partial_u H(u, v) & \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

Remark 1.1. If $p, q > 1$, then condition (1.17) reduces to

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2\alpha}{N}.$$

As a corollary, we point out that one can recover the existence result for the Lane–Emden system with Dirichlet boundary conditions given by Theorem 1.6.

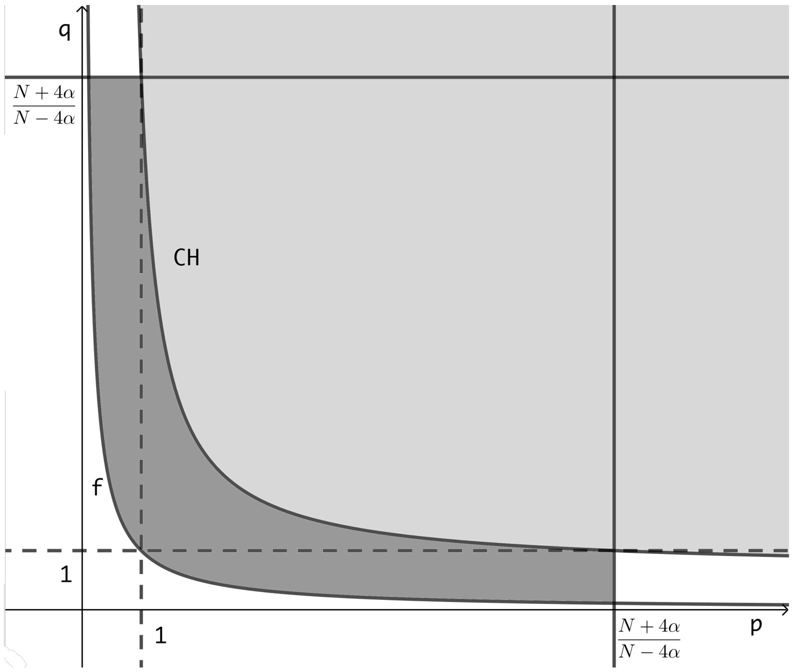


Figure 1.2: *Case $\alpha = 2$, $N = 10$. The curve f represents $pq = 1$, whereas CH is the critical hyperbola* (1.11)*. The dark grey area is the domain of existence for* (1.19) *given by Corollary 1.1, the light grey area is the non-existence domain, see Corollary 1.2.*

Corollary 1.1. *Let $N > 2\alpha$ and let p, q be such that $pq > 1$ and* (1.17) *holds. Then there exists a non trivial solution to*

$$\begin{cases} (-\Delta)^{\alpha} u = |v|^{q-1}\, v & \\ & \text{in } \Omega \subset \mathbb{R}^N \\ (-\Delta)^{\alpha} v = |u|^{p-1}\, u & \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, & \text{on } \partial\Omega \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, & \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

see Figure 1.2.

Moreover, we prove the following non-existence result as a corollary of the Pucci–Serrin identity in [83].

Theorem 1.13. *Let Ω be a ball in $\mathbb{R}^N$ and $N > 2\alpha$. Assume that there exists $a \in \mathbb{R}$ such that*

$$NH(u, v) - au\partial_u H(u, v) - (N - 2\alpha - a)v\partial_v H(u, v)) \leq 0.$$

Then, no classical positive solutions to (1.18) *exist.*

As by-product of Theorem 1.13 we have the following

Corollary 1.2. *Let Ω be a ball in $\mathbb{R}^N$ and $N > 2\alpha$. Assume $p, q > 0$ to be such that*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2\alpha}{N}.$$

Then, no positive classical solutions to (1.19) *exist, see Figure 1.2.*

Proof of Corollary 1.2. It is enough to choose $a = \frac{N}{p+1}$ in Theorem 1.13. Thus, condition

$$NH(u,v) - au\partial_u H(u,v) - (N - 2\alpha - a)v\partial_v H(u,v)) \leq 0$$

reads as follows

$$\left(\frac{N}{q+1} - N + 2\alpha + \frac{N}{p+1}\right) \int_\Omega |v|^{q+1} \leq 0,$$

which is equivalent to

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2\alpha}{N}. \qquad \square$$

We point out that both in Theorem 1.13 and Corollary 1.2 we deal only with balls. This is necessary as we exploit Theorem 1.5, which is not available in a general (even star-shaped) domain.

1.4.2 Nonvariational case $\alpha \neq \beta$

In this subsection we present some results obtained in [89], concerning existence for the nonvariational case. The first step is to prove that non existence to (1.10) on $\Omega = B_1$ with Dirichlet boundary conditions implies existence of the same system on $\mathbb{R}^N$ by adapting and extending considerations in [8]. The main tools are a continuation argument and a blow up analysis, together with the moving planes procedure [13, 12].

Theorem 1.14. *If the only solution to*

$$\begin{cases} (-\Delta)^\alpha u = |v|^q & \\ & \text{in } B_1 \subset \mathbb{R}^N, N > \max\{2\alpha, 2\beta\} \\ (-\Delta)^\beta v = |u|^p & \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial B_1 \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \dots, \beta - 1, & \text{on } \partial B_1. \end{cases} \tag{1.20}$$

is the trivial one, and if $pq > 1$, then there exists a classical nontrivial radially symmetric and nonnegative solution to (1.13).

Therefore, as we did in [89], one can apply Theorem 1.9 and Theorem 1.11 above to obtain existence to (1.20).

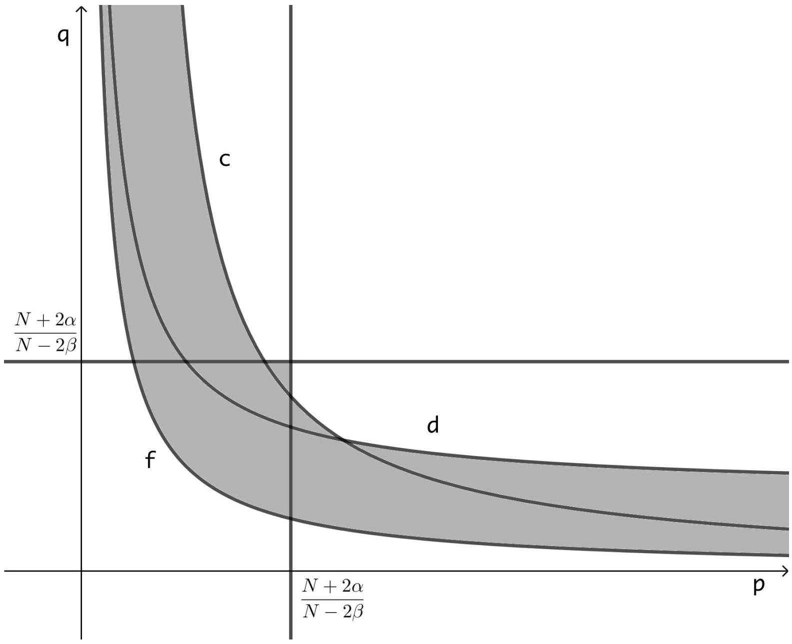


Figure 1.3: *Case* $\alpha = 3$*,* $\beta = 1$*,* $N = 10$*. The curves c and d represent the Serrin curves* (1.15)*, whereas f is* $pq = 1$*. The grey area is the domain of existence for* (1.20) *provided by Corollary 1.4.*

Corollary 1.3. *Assume* $p, q > 1$*. Assume further that one of the following is satisfied:*

(i) $2\beta q + N + 2\alpha pq - Npq \geq 0$ *or* $2\alpha p + N + 2\beta pq - Npq \geq 0$*;*

(ii) $p, q < \min\{\frac{N+2\alpha}{N-2\beta}, \frac{N+2\beta}{N-2\alpha}\}$.

Then there exists a positive classical solution to (1.20).

Here, by exploiting [96, Proposition 2], and proving a suitable non existence result in the spirit of [29], we extend Corollary 1.3 to the more natural superlinearity condition $pq > 1$.

Corollary 1.4. *Assume* $pq > 1$*. Assume further that one of the following is satisfied:*

(i) $2\beta q + N + 2\alpha pq - Npq \geq 0$ *or* $2\alpha p + N + 2\beta pq - Npq \geq 0$*;*

(ii) $p, q < \min\{\frac{N+2\alpha}{N-2\beta}, \frac{N+2\beta}{N-2\alpha}\}$.

Then there exists a positive classical solution to (1.20)*, see Figure 1.3.*

To the best of our knowledge, these are the first results concerning non-variational Lane–Emden systems on a bounded domain with Dirichlet boundary conditions. In Chapter 3 below we prove Theorem 1.14 and Corollary 1.4, and we extend these results to systems with an arbitrary number of equations. Also, we point out how to adapt the proof in order to treat the Navier case.

Remark 1.2. All the results above hold on a generic ball $B(x, R)$.

1.4.3 Complementary results: the system on $\mathbb{R}^N$

In Appendix A we study the problem of existence to higher order Lane–Emden systems on the entire space $\mathbb{R}^N$ in the particular case $\alpha = 2$, $\beta = 1$. The first step, as in [68], is to prove that non existence to (1.13) implies existence to the same system on a ball with Navier boundary conditions. This is obtained by means of a shooting technique [104], see also [62, 61] for similar problems studied through this method.

Theorem 1.15 (Theorem 1 in [104])**.** *If for any $R > 0$, $N > \max\{2\alpha, 2\beta\}$, $p, q > 1$ the following system*

$$\begin{cases} (-\Delta)^\alpha u = |v|^{q-1}\, v & \text{in } B_R \\ (-\Delta)^\beta v = |u|^{p-1}\, u & \text{in } B_R \\ \Delta^s u = 0, \quad s \leq \alpha - 1 & \text{on } \partial B_R \\ \Delta^t v = 0, \quad t \leq \beta - 1 & \text{on } \partial B_R \end{cases} \tag{1.21}$$

has no classical radial positive solutions, then (1.13) *has infinitely many classical, radially symmetric and positive solutions.*

In [68] a Pohozaev type identity, which holds only provided $\alpha = \beta$, is exploited to prove non-existence for (1.21). This is not trivially extended to the nonvariational case $\alpha \neq \beta$. Here, we prove the following result in the spirit of [30], in the particular case $\alpha = 2$, $\beta = 1$.

Theorem 1.16. *Let $N > 4$. Assume $\alpha = 2, \beta = 1$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-4}{2N}. \tag{1.22}$$

Then, no positive classical radially symmetric solutions to (1.21) *do exist.*

By combining Theorem 1.16 and Theorem 1.15 one has

Corollary 1.5. *Let $N > 4$, $\alpha = 2$ and $\beta = 1$. There exist infinitely many nontrivial solutions to* (1.13)*, provided $p, q > 1$ and* (1.22) *is satisfied.*

Remark 1.3. The curve in (1.22) and the Serrin curves (1.15) relate consistently to each other. Indeed, (1.22) is equivalent to

$$(N-4)pq - (N+4)(p+q) - 4 - 3N > 0$$

and one has

$$0 < (N-4)pq - (N+4)(p+q) - 4 - 3N < -2\beta q - N - 2\alpha pq + Npq$$

and

$$0 < (N-4)pq - (N+4)(p+q) - 4 - 3N < -2\alpha p - N - 2\beta pq + Npq$$

for $\alpha + \beta = 3$. Also, if $p, q < \min\{\frac{N+2\alpha}{N-2\beta}, \frac{N+2\beta}{N-2\alpha}\} = \frac{N+4}{N-2}$, then

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N+1} > \frac{N-4}{2N}.$$

1.5 Uniqueness results

In what follows, we restrict ourselves to the problem of uniqueness for Lane–Emden type equations and systems on a ball. For a completely general bounded domain, uniqueness fails, see [9, 36]. It is conjectured that only one solution to (1.2) exists on convex domains, we refer to [79] for a survey on possible approaches and partial results in this context and an extensive bibliography.

We first recall the following seminal result by Gidas–Ni–Nirenberg and its proof.

Theorem 1.17 ([49]). *There exists at most one nontrivial positive solution to* (1.2) *with* $\Omega = B_1$ *and* $p > 1$*, which is radially symmetric and strictly decreasing in the radial variable.*

Proof. By means of the moving planes technique [13], one proves that any positive solution to (1.2) is radially symmetric and strictly decreasing in the radial variable [49, Theorem 1]. Now, assume u and w are two positive solutions to (1.2). Then

$$\tilde{w}(r) = \lambda^{\frac{2}{p-1}} w(\lambda r)$$

is a solution to (1.2) on the ball of radius $1/\lambda$, and $\tilde{w}(0) = u(0)$ once we choose

$$\lambda^{\frac{2}{p-1}} = \frac{u(0)}{w(0)}.$$

Therefore, u and $\tilde{w}$ satisfy the same second order ODE with the same initial conditions (notice that $u'(0) = \tilde{w}'(0) = 0$). The assumption $p > 1$ implies that the nonlinearity is Lipschitz continuous, hence $u = \tilde{w}$ on $[0, \min\{1, 1/\lambda\}]$ by classical ODE theory. In particular, $0 = u(1) = \tilde{w}(1) > 0$ if $\lambda < 1$, and $0 < u(1/\lambda) = \tilde{w}(1/\lambda) = 0$ if $\lambda > 1$, which implies $\lambda = 1$ and $u = w$. □

This proof has been extended to the biharmonic Dirichlet case

$$\begin{cases} \Delta^2 u = |u|^p & \text{in } B_1 \subset \mathbb{R}^N, N > 4 \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1 \end{cases}$$

by Dalmasso [35], see also [41]. We refer to [34] for uniqueness of solutions to

$$\begin{cases} (-\Delta)^\alpha u = |u|^p & \text{in } B_1 \subset \mathbb{R}^N, N > 2\alpha \\ \frac{\partial^k u}{\partial \nu^k} = 0, \, k \leq \alpha - 1 & \text{on } \partial B_1 \end{cases}$$

with a sublinear nonlinearity, namely $p < 1$.
Moreover, uniqueness holds for the Lane–Emden system (1.3) on a ball with $p, q > 1$, see [33]. We also recall that in [32] uniqueness is proved for

$$\begin{cases} -\Delta u = |v|^q & \\ -\Delta v = |w|^p & \text{in } B_1 \subset \mathbb{R}^N, N > 2 \\ -\Delta w = |u|^r & \\ u = v = w = 0 & \text{on } \partial B_1 \end{cases}$$

where $p, q, r > 1$.

In Chapter 4 below we recall results in [89, 23], where we adapt and extend the Gidas–Ni–Nirenberg proof [49] to the case of polyharmonic equations and systems up to order eight.

Theorem 1.18. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^\alpha u = |u|^p \, , & \text{in } B_1 \subset \mathbb{R}^N, N > 2\alpha \\ \frac{\partial^k u}{\partial \nu^k} = 0, & \text{on } \partial B_1 \, , \, k \leq \alpha - 1 \end{cases}$$

with $p > 1$ *and* $1 \leq \alpha \leq 4$.

Remark 1.4. Notice that Theorem 1.18 yields uniqueness of solutions in the sharp range of existence $1 < p < (N + 2\alpha)/(N - 2\alpha)$ as a consequence of [83, Theorem 8] and [47, Theorems 7.17–7.18].

Theorem 1.19. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j} \, , \, j = 1, \dots, m-1 & \text{in } B_1, \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m} & \text{in } B_1, \\ \frac{\partial^k u_j}{\partial \nu^k} = 0, \, k = 0, \dots, \alpha_j - 1, \, j = 1, \dots, m, & \text{on } \partial B_1, \end{cases} \tag{1.23}$$

with $p_j \geq 1$ *for any* j, $\prod_{j=1}^m p_j > 1$, $N > 2 \max\{\alpha_j\}_j$ *and* $1 \leq \alpha_j \leq 4$ *for any* $j = 1, \dots, m$, *where* $m \geq 1$.

As we are going to see, when trying to extend the proof of Theorem 1.18 to the case $\alpha \geq 5$, one has to face technical difficulties due to the fact that Dirichlet boundary conditions prescribe the behavior only of the first $\alpha - 1$ derivatives of the solution, and no information apparently can be retained for higher order derivatives. However, in this context new boundary conditions show up in a natural fashion for which we have the following

Corollary 1.6. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^{\alpha} u = |u|^{p}, & \text{in } B_1, \\ \Delta^{2k} u = 0,\ 2k \leq \alpha - 1, & \text{on } \partial B_1 \\ \frac{\partial}{\partial \nu} \Delta^{2k} u = 0,\ 2k+1 \leq \alpha - 1, & \text{on } \partial B_1 \end{cases} \tag{1.24}$$

with $N > 2\alpha$, $p > 1$ *and* $\alpha \in \mathbb{N}$, $\alpha \geq 1$.

Boundary conditions considered in (1.24), on one side from the mathematical point of view enable us to split the equation into a system of equations subject to Dirichlet boundary conditions, on the other side, the Physical constraint makes vanishing higher order momenta along the boundary.

Remark 1.5. Let us point out that the boundary conditions in (1.24) satisfy the complementing condition [4], which here reads as follows.

Definition 1.1. We say that the complementing condition holds for

$$\begin{cases} (-\Delta)^{\alpha} u = |u|^{p}, & \text{in } B_1 \\ B_j(x, D) u = h_j, \text{ for } j = 1, \ldots, \alpha, & \text{on } \partial B_1 \end{cases}$$

if, for any nontrivial tangent vector $\tau(x)$, the polynomials in t $B_j'(x; \tau + t\nu)$ are linearly independent modulo the polynomial $(t - i\,|\tau|)^{\alpha}$, where B_j' represents the highest order part of B_j.

Consider the particular case $\alpha = 4$ and let $|\tau| = 1$. One has $B_1(x, D)u = u$, $B_2(x, D)u = \frac{\partial u}{\partial \nu}$, $B_3(x, D)u = \Delta^2 u$, and $B_4(x, D)u = \frac{\partial}{\partial \nu}\Delta^2 u$. Therefore it follows that $B_1(x, s) = 1$, $B_2(x, s) = s \cdot \nu$, $B_3(x, s) = (s \cdot s)(s \cdot s)$, and $B_4(x, s) = (s \cdot \nu)(s \cdot s)(s \cdot s)$, whence by direct calculations $B_1'(x, \tau + t\nu) = 1$, $B_2'(x, \tau + t\nu) = t$, $B_3'(x, \tau + t\nu) = t^4 + 1$ and $B_4'(x, \tau + t\nu) = t^5 + t$. Dividing these polynomials by $(t - i)^4$, we get 1, t, $4it^3 + 6t^2 - 4it$ and $-10t^3 + 20it^2 + 16t - 4i$ as remainders, which are linearly independent. The general case follows from the system (1.23). Indeed, one can extend Definition 1.1 to the case of systems and prove that a system of m equations with Dirichlet boundary conditions satisfies this extended condition, see [5].

As far as we are concerned with the so-called Navier boundary conditions, for which the polyharmonic operator is actually a power of the Laplacian and classical reduction methods apply, we have as byproduct of Theorem 1.19 the following

Corollary 1.7. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j},\ j = 1, \ldots, m-1, & \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m}, & \text{in } B_1, \\ \Delta^k u_j = 0,\ k = 0, \ldots, \alpha_j - 1,\ j = 1, \ldots, m & \text{on } \partial B_1 \end{cases}$$

with $p_j \geq 1$ *for any* j, $\prod_{j=1}^{m} p_j > 1$, $\alpha_j \in \mathbb{N}$, $m \geq 1$ *and* $N > 2\max\{\alpha_j\}_j$.

1.6 Open questions

We briefly recall here main results and references for a few relevant problems which are not considered in this dissertation, arising in the context of Lane–Emden type systems, as possible directions of future research.

Consider first the following system

$$\begin{cases} (-\Delta)^\alpha u = \lambda v + |v|^{q-1} v & \text{in } \Omega \subseteq \mathbb{R}^N \\ (-\Delta)^\alpha v = \mu u + |u|^{p-1} u & \\ B(u,v) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.25}$$

with critical p, q, namely belonging to the critical hyperbola (1.11). Recall that in the case of a single equation and $\alpha = 1$ [18] the existence of a nontrivial solution is guaranteed provided the perturbation term λ lies in the range (λ^*, λ_1), with a suitable λ^* and λ_1 denoting the first eigenvalue of $-\Delta$. The proof relies strongly on the explicit representation of the Talenti instantons [101]. We also mention that the polyharmonic equation is treated in [48, 72], see also [47, Section 7] and references therein. However, we do not know which are the equivalent of Talenti functions in the system case and as a consequence extending Brezis–Nirenberg results to systems turns out to be a nontrivial task. Even in the second order Lane–Emden case $\alpha = 1$ not much is actually known, and Hulshof, Mitidieri and Van der Vorst gave rather involved conditions on λ, μ such that (1.25) has a nontrivial solution [56]. Further, we recall that some partial existence results for a Lane–Emden system with a Brezis-Nirenberg perturbation in one of the two equations have been recently obtained in [73]. An interesting question would be to understand whether results in [56, 73] could be extended to the polyharmonic system (1.25).

Moreover, we point out that a related problem is given by the fractional case, namely $\alpha, \beta \in (0, 1)$. In this case, $(-\Delta)^s$, with $s \in (0, 1)$, is a non-local operator defined by

$$(-\Delta)^s u = C(N, s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy,$$

see [59, 60] for some a priori bounds and existence results, both in the variational and non-variational case. One can naturally define the following fractional higher order operators

$$(-\Delta)^{m+s} u = (-\Delta)^m (-\Delta)^s u$$

where $m \in \mathbb{N}$ and $s \in (0, 1)$. We refer to [2, 1, 38] for various properties and applications of these operators. In view of our results above and [59, 60], one could wonder whether similar existence results can be obtained in the context of fractional higher order operators. The main difficulty turns out to be the proof of suitable Liouville-type results, which would require a combination of the techniques used for the polyharmonic and for the fractional case.

Chapter 2

Variational higher order elliptic systems

In this chapter, which is an extended transcription of [88], we prove some existence and non-existence results for the following variational higher order elliptic system

$$\begin{cases} (-\Delta)^\alpha u = \partial_v H(u,v) & \text{in } \Omega \subset \mathbb{R}^N, N > 2\alpha \\ (-\Delta)^\alpha v = \partial_u H(u,v) & \\ \frac{\partial^r u}{\partial \nu^r} = 0, \ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega \\ \frac{\partial^r v}{\partial \nu^r} = 0, \ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

where $\alpha \geq 1$, Ω is a $C^{2\alpha,\gamma}$ bounded domain, and $H \in C^1(\mathbb{R}^2; \mathbb{R})$ satisfies suitable growth assumptions. We exploit a variational approach first developed in [43, 57] in the case $\alpha = 1$, which consists of an application of the Linking Theorem [11] in the context of fractional order Sobolev spaces [3, 64]. As we recalled in Section 1.3 above, power-like nonlinearities in (2.1) can be studied by means of a reduction by inversion, see [28]. The method we present here, see Section 2.1 below, allows one to treat more general nonlinearities. Moreover, in Section 2.2 we get a non-existence result on balls as a corollary of the Pucci–Serrin identity [83].

2.1 Existence results for system (2.1)

In the present section we prove Theorem 1.12, which we recall below for the reader's convenience.

Theorem 2.1. *Let $N > 2\alpha$ and let p, q be such that $pq > 1$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2\alpha}{N}, \quad \max\{(N-4\alpha)p, (N-4\alpha)q\} < N + 4\alpha. \tag{2.2}$$

Let $H\colon \mathbb{R}^2 \to \mathbb{R}$ be of class C^1 such that

(H1) $H(u,v) \geq 0$ *for all* $(u,v) \in \mathbb{R}^2$

(H2) *There exists* $R > 0$ *such that, if* $(u,v) \in \mathbb{R}^2$ *satisfies* $|(u,v)| \geq R$, *then*

$$\frac{1}{p+1}\partial_u H(u,v) \cdot u + \frac{1}{q+1}\partial_v H(u,v) \cdot v \geq H(u,v) > 0$$

(H3) *There exist* $r > 0$ *and* $a > 0$ *such that, if* $(u,v) \in \mathbb{R}^2$ *satisfies* $|(u,v)| \leq r$, *then*

$$H(u,v) \leq a(|u|^{p+1} + |v|^{q+1})$$

(H4) *There exists* $b > 0$ *such that*

$$\begin{aligned} |\partial_u H(u,v)| &\leq b(|u|^p + |v|^{p(q+1)/(p+1)} + 1) \\ |\partial_v H(u,v)| &\leq b(|v|^q + |u|^{q(p+1)/(q+1)} + 1). \end{aligned}$$

Then, there exists a nontrivial solution

$$(u,v) \in W^{2\alpha,\frac{q+1}{q}} \cap W_0^{\alpha,\frac{q+1}{q}} \times W^{2\alpha,\frac{p+1}{p}} \cap W_0^{\alpha,\frac{p+1}{p}}$$

to problem (2.1).

Remark 2.1. Due to *(H2)*, there exist $c_1, c_2 > 0$ such that

$$H(u,v) \geq c_1(|u|^{p+1} + |v|^{q+1}) - c_2, \tag{2.3}$$

see [40, Lemma 1.1]. Moreover, by *(H4)* there exist $a_1, a_2 > 0$ such that

$$H(u,v) \leq a_1(|u|^{p+1} + |v|^{q+1}) + a_2, \tag{2.4}$$

see [43, p.105].

Remark 2.2. Notice that with standard bootstrap arguments one can infer regularity of solutions to (2.1) under some additional regularity conditions, precisely in the case $H \in C^2(\mathbb{R}^2)$. We refer to [17, Lemma 5.16] for the case $\alpha = 1$, see also [95, Theorem 1]. Let us consider the model case with power-like non-linearities. The regularity assumption $H \in C^2$ is therefore equivalent to $p, q > 1$. Let us take a nontrivial solution

$$(u,v) \in W^{2\alpha,\frac{q+1}{q}} \cap W_0^{\alpha,\frac{q+1}{q}} \times W^{2\alpha,\frac{p+1}{p}} \cap W_0^{\alpha,\frac{p+1}{p}}.$$

If for instance $(p+1)/p \geq N/(2\alpha)$, then $v \in C^{0,\gamma}$ for some $\gamma > 0$, and since $q > 1$ also $v^q \in C^{0,\gamma}$. We now apply Theorem 1.3 to get $u, v \in C^{2\alpha,\gamma}$, namely (u,v) is classical.

Assume $(p+1)/p, (q+1)/q < N/(2\alpha)$. By Sobolev embeddings we know that $u \in L^{\frac{N(q+1)/q}{N-2\alpha(q+1)/q}}$, and as a consequence $u^p \in L^{\frac{N(q+1)}{(Nq-2\alpha q-2\alpha)p}}$. By elliptic regularity, see Theorem 1.2, we conclude that $v \in W^{2\alpha,\frac{N(q+1)}{(Nq-2\alpha q-2\alpha)p}}$. Since (p,q) is subcritical,

$$\frac{N(q+1)}{(Nq-2\alpha q-2\alpha)p} > \frac{p+1}{p},$$

thus we have gained some summability of v, and similarly for u. Call

$$p_1 = \frac{N(p+1)}{Np-2\alpha(p+1)}, \quad q_1 = \frac{N(q+1)}{Nq-2\alpha(q+1)}.$$

Again, if one among p_1/q and q_1/p is bigger than $N/(2\alpha)$ then the conclusion follows by Theorem 1.3. Therefore we assume both of them to be $< N/(2\alpha)$. Notice that $q_1 > q+1$ or $p_1 > p+1$ since we are assuming

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2\alpha}{N}.$$

Let for instance $q_1 > q+1$. We now iterate the reasoning above, and get two sequences

$$p_{n+1} = \frac{Nq_n}{Nq-2\alpha q_n}, \quad q_{n+1} = \frac{Np_{n+1}}{Np-2\alpha p_{n+1}},$$

such that $u \in W^{2\alpha,q_n/q}$, $v \in W^{2\alpha,p_n/p}$ for any $n \geq 2$. Notice that $p_{n+1} > p_n$ for any $n \geq 2$, and $q_{n+1} > q_n$ for any $n \geq 1$.
Assume that $p_n/p, q_n/q < N/2\alpha$ for any $n \geq 2$. Hence both the sequences $\{p_n\}$ and $\{q_n\}$ must have a finite limit. Let us call these limits l_1 and l_2 respectively. Thus

$$l_1 = \frac{Nl_2}{Nq-2\alpha l_2}, \quad l_2 = \frac{Nl_1}{Np-2\alpha l_1},$$

which implies

$$q_1 \leq l_2 = \frac{N(pq-1)}{2\alpha(p+1)} < q+1,$$

whence the contradiction follows.
Therefore, there exist k such that for instance $p_k/p > N/(2\alpha)$. By Sobolev embedding, $v \in C^{0,\gamma}$ for some $\gamma > 0$, and since $q > 1$ also $v^q \in C^{0,\gamma}$. We now apply Theorem 1.3 to get $u \in C^{2\alpha,\gamma}$, which implies $v \in C^{2\alpha,\gamma}$, namely (u,v) is classical.
The same argument applies for $H \in C^2(\mathbb{R}^2)$ satisfying the hypotheses of Theorem 2.1.

The proof of Theorem 2.1 relies on a critical point theorem due to Felmer [40], see Proposition 2.1 below, which extends the classical Linking Theorem of Benci and Rabinowitz [11]. As in [43], this allows one to treat the more natural super linearity condition $pq > 1$ in place of $p, q > 1$.

Definition 2.1. Let X be a Banach space and $I \in C^1(X;\mathbb{R})$. We say that $\{u_n\} \subset X$ is a Palais-Smale sequence for I if $|I(u_n)| \leq c$ uniformly in n and $\|I'(u_n)\| \to 0$ as $n \to \infty$. If any Palais-Smale sequence has a strongly convergent subsequence, then we say that I satisfies the Palais-Smale condition.

Remark 2.3. In the sequel, the Palais-Smale condition will be denoted by (PS).

Proposition 2.1 (Theorem 3.1 in [40]). *Let $(E, \langle\cdot,\cdot\rangle_E)$ be a Hilbert space such that $E = E^+ \oplus E^-$. Suppose that $I \in C^1(E;\mathbb{R})$ satisfies (PS) and $I(z) = \frac{1}{2}\langle Lz, z\rangle_E - J(z)$, where:*

(i) $L : E \to E$ is bounded, linear, self-adjoint and maps E^+ into E^-;

(ii) $J' : E \to \mathbb{R}$ is compact.

Assume that there exist two linear, bounded, invertible operators $B_1, B_2\colon E \to E$ such that the next condition holds:

(iii) define $\hat{B}_\tau = P_2 B_1^{-1} e^{\tau L} B_2\colon E^- \to E^-$, where $P_2\colon E \to E^-$ is the projection onto E^- and $e^{\tau L} = \sum_{n=0}^{+\infty} \frac{(\tau L)^n}{n!}$; then $\hat{B}_\tau$ is invertible for any given $\tau \geq 0$.

Let $e_1 \in E^+$ with $\|e_1\|_E = 1$. Choose $\rho > 0$, $R_1 > \rho / \left\|B_1^{-1} B_2 e_1\right\|_E$ and $R_2 > \rho$ and define

$$S = \{\, B_1 z_1 : z_1 \in E^+, \|z_1\|_E = \rho \,\}$$
$$Q = \{\, B_2(te_1 + z_2) : 0 \leq t \leq R_1, z_2 \in E^-, \|z_2\|_E \leq R_2 \,\}\,.$$

Let us assume

(iv) $I(z) \geq \sigma > 0$ on S

(v) $I(z) \leq 0$ on ∂Q.

Then, I has a critical point z_0 such that $I(z_0) \geq \sigma$.

Remark 2.4. Actually, the critical level has an explicit representation, see [40, Theorem 3.1]. Indeed, let

$$\Gamma = \{h \in C(E \times [0,1]; E) : h \text{ satisfies } (C_1), (C_2), (C_3)\},$$

where:

(C_1) h is given by $h(z,t) = \exp(\nu(z,t)L)z + K(z,t)$ where $\nu : E \times [0,1] \to \mathbb{R}^+$ is continuous and transforms bounded sets into bounded sets, and $K : E \times [0,1] \to E$ is compact.

(C_2) $h(z,t) = z$, for all $z \in \partial Q$

(C_3) $h(z,0) = z$ for any $z \in Q$.

Then the critical level is given by

$$I(z_0) = \inf_{h\in\Gamma} \sup_{z\in Q} I(h(z,1)).$$

In what follows we define a Hilbert space E as product of suitable fractional order spaces, and a functional $I : E \to \mathbb{R}$ whose critical points are solutions to (2.1). Then, we prove that I satisfies all the hypotheses of Proposition 2.1, whence Theorem 2.1 will follow immediately.

2.1.1 Definition of a suitable Hilbert space

Let us first recall some well-known facts about spectral properties of the polyharmonic operator: the proof of the following simple lemma is based on the spectral theorem for compact and self-adjoint operators, see for instance [84, Chapter 7], and Theorem 1.2.

Lemma 2.1. *Let $\alpha \in \mathbb{N}$, $\alpha \geq 1$, $N > 2\alpha$. There exists a orthonormal basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $(-\Delta)^\alpha$ subject to Dirichlet boundary conditions. These eigenfunctions are in L^s for any $s \geq 1$ and correspond to a diverging sequence of positive eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots .$$

Proof. Let $K_\alpha \colon L^2 \to H^{2\alpha} \cap H_0^\alpha$ be the solution operator of

$$\begin{cases} (-\Delta)^\alpha v = u & \text{in } \Omega \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

namely $K_\alpha u = v$ in the weak sense, and let T represent the compact embedding $H^{2\alpha} \cap H_0^\alpha \hookrightarrow L^2$. Then $T \circ K_\alpha \colon L^2 \to L^2$ is well defined and compact. Moreover, since $(-\Delta)^\alpha$ with Dirichlet boundary conditions is symmetric with respect to the scalar product in L^2, then $T \circ K_\alpha$ is symmetric as well (hence self-adjoint). By the spectral theorem for compact and self adjoint operators one concludes that there exists a orthonormal basis of L^2 made of eigenfunctions of $T \circ K_\alpha$. Therefore, there exists a orthonormal basis of L^2 made of eigenfunctions of $(-\Delta)^\alpha$ with Dirichlet boundary conditions. The eigenvalues are bounded away from 0 as if u is an eigenfunction with corresponding eigenvalue λ then

$$0 < \int u^2 \leq C \int \left|\Delta^{\alpha/2} u\right|^2 = C\lambda \int u^2$$

where α is even, and similarly for α odd. Finally, the eigenfunctions are in L^s for any s since

$$\|u\|_{H^k} \leq C \|u\|_{H^{k-2\alpha}}$$

for any $k \geq 2\alpha$, see Theorem 1.2, therefore a bootstrap argument yields the claim. □

Remark 2.5. Recall that, differently from the Navier case, no maximum principle for

$$\begin{cases} (-\Delta)^\alpha u = f & \text{in } \Omega \\ \frac{\partial^r u}{\partial \nu^r},\ r = 0, \ldots \alpha - 1 & \text{on } \partial\Omega \end{cases}$$

on a general bounded domain is known to hold, except for the ball and small deformations of the ball, see [47, Sections 3.1, 5.1, 5.2 and Chapter 6]. In particular, we cannot conclude in general that the first eigenfunction of the polyharmonic operator is positive. However, this assumption is not required in the proof we give below and this allows us to deal with general sufficiently smooth bounded domains.

Let us define the real interpolation space E^s as follows:

$$E^s := [L^2, H^{2\alpha} \cap H_0^\alpha]_{s/(2\alpha)}$$

with $0 < s < 2\alpha$ [3, Section 7.7]. In [63], see also [66], it is proved that E^s can be written explicitly in terms of Fourier series by

$$E^s = \left\{ u = \sum_{n=1}^{\infty} u_n \Phi_n \in L^2 \mid \sum_{n=1}^{\infty} \lambda_n^{s/\alpha} u_n^2 < \infty \right\},$$

where $u_n = \langle u, \Phi_n \rangle$, and Φ_n is the orthonormal basis for L^2 given by Lemma 2.1, composed of eigenfunctions of $(-\Delta)^\alpha$ with Dirichlet boundary conditions and corresponding to eigenvalues λ_n. Define for $u \in E^s$

$$A^s u = A^s \left(\sum_{n=1}^{\infty} u_n \Phi_n \right) = \sum_{n=1}^{\infty} \lambda_n^{\frac{s}{2\alpha}} u_n \Phi_n.$$

Notice that in the case $s = 2\alpha$, there holds $A^s = (-\Delta)^\alpha$. We stress that the space E^s endowed with the scalar product

$$\langle u, v \rangle_{E^s} := \int_\Omega A^s u A^s v$$

is Hilbert. Indeed

$$\|u\|_{E^s} = \left(\int_\Omega |A^s u|^2 \right)^{1/2} = \|A^s u\|_2$$

is a norm, since $\|A^s u\|_2 = 0$ implies $u = 0$ due to the Poincaré–type inequality

$$\|A^s u\|_2^2 \geq \lambda_1^{\frac{2s}{\alpha}} \|u\|_2^2$$

and the space $(E^s, \|\cdot\|_{E^s})$ is Banach: if u_n is a Cauchy sequence in E^s, then $A^s u_n$ is a Cauchy sequence in L^2, therefore there exists $v \in L^2$ such that $A^s u_n \to v$ in L^2; however,

$$v = \sum_{k=1}^{\infty} v_k \Phi_k = \sum_{k=1}^{\infty} \lambda_k^{\frac{s}{2\alpha}} w_k \Phi_k, \text{ where } w_k = \frac{v_k}{\lambda_k^{s/(2\alpha)}},$$

namely $v = A^s u$ with $u = \sum_{k=1}^{\infty} w_k \Phi_k$. Hence, $A^s u_n \to A^s u$ in L^2 and $u_n \to u$ in E^s.

Remark 2.6. Notice that in general the operators A^s do not coincide with the fractional Laplace operators

$$(-\Delta)^s u(x) = C \lim_{\varepsilon \to 0^+} \int_{\mathcal{C}B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

and also the space E^s does not coincide with the fractional Sobolev space of order s

$$W^{s,2}(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{N/2+s}} \in L^2(\Omega \times \Omega) \right\}.$$

However, if $\alpha = 1$, then $E^s = W^{s,2}$ for any s such that $0 \le s < 1/2$. Moreover, $E^s = \{u \in W^{s,2} : u = 0 \text{ on } \partial\Omega\}$ if $1/2 < s \le 2$, and $s \ne 3/2$. We refer the reader to [37, 45].

Let r_1, r_2 be such that $\frac{1}{r_1} = \frac{1}{2} - \frac{2\alpha}{N}$ and $\frac{1}{r_2} = \frac{1}{2} - \frac{s}{N}$. Then by [66, Chapter 7, Theorem 1.1] we have $L^{r_2} = [L^2, L^{r_1}]_{s/(2\alpha)}$, since

$$\frac{N-2s}{2N} = \left(1 - \frac{s}{2\alpha}\right)\frac{1}{2} + \frac{s}{2\alpha}\frac{N-4\alpha}{2N}$$

by direct computation. Moreover, $H^{2\alpha} \cap H_0^\alpha \hookrightarrow L^{r_1}$, and hence $E^s \hookrightarrow L^{p+1}$ if $\frac{1}{p+1} \ge \frac{1}{r_2}$ and the embedding is compact provided the strict inequality holds (see [80], see also [64, Chapter 1, Theorem 5.1] and [3, Sections 7.22, 7.23]).

Assume (2.2). Then there exist suitable $s, t \in (0, 2\alpha)$ such that $E^s \times E^t \hookrightarrow L^{p+1} \times L^{q+1}$ compactly and $s + t = 2\alpha$. Indeed, (2.2) implies:

(i) $N\left(\frac{1}{2} - \frac{1}{p+1}\right) < N\left(\frac{1}{q+1} - \frac{N-4\alpha}{2N}\right)$;

(ii) $N\left(\frac{1}{2} - \frac{1}{p+1}\right) < 2\alpha$;

(iii) $N\left(\frac{1}{q+1} - \frac{N-4\alpha}{2N}\right) > 0$,

and we can fix $s \in (0, 2\alpha)$ such that $N\left(\frac{1}{2} - \frac{1}{p+1}\right) < s < N\left(\frac{1}{q+1} - \frac{N-4\alpha}{2N}\right)$. Notice that here we used the technical assumption

$$\max\{(N-4\alpha)p, (N-4\alpha)q\} < N + 4\alpha$$

in order to have $s \in (0, 2\alpha)$.

From now on, s is such that $E^s \times E^{2\alpha - s} \hookrightarrow L^{p+1} \times L^{q+1}$ compactly. Let us define $E = E^s \times E^{2\alpha-s}$, which can be decomposed as $E = E^+ \oplus E^-$, where

$$E^+ = \{ (u, A^{-2\alpha+2s}u), u \in E^s \}$$
$$E^- = \{ (u, -A^{-2\alpha+2s}u), u \in E^s \}$$

are orthogonal subspaces of E. Indeed if $z = (u, v) \in E$ one has

$$z = z^+ + z^-,$$

where

$$z^+ = (u^+, v^+) = \left(\frac{u + A^{2\alpha-2s} v}{2}, \frac{v + A^{-2\alpha+2s} u}{2} \right) \in E^+$$
$$z^- = (u^-, v^-) = \left(\frac{u - A^{2\alpha-2s} v}{2}, \frac{v - A^{-2\alpha+2s} u}{2} \right) \in E^-.$$

Note that

$$\frac{1}{2} \|z\|_E^2 = \int_\Omega A^s u^+ A^{2\alpha-s} v^+ \, dx - \int_\Omega A^s u^- A^{2\alpha-s} v^- \, dx. \tag{2.6}$$

2.1.2 The energy functional

In this context a natural choice for the energy functional related to (2.1) turns out to be the following:

$$I(u, v) = \int_\Omega A^s u A^{2\alpha-s} v \, dx - \int_\Omega H(u, v) \, dx, \tag{2.7}$$

where s is such that $E = E^s \times E^{2\alpha-s} \hookrightarrow L^{p+1} \times L^{q+1}$ compactly. Note that I is well defined on E by (2.4). The functional (2.7) may be written also as

$$I(u, v) = \frac{1}{2} \langle L(u, v), (u, v) \rangle_E - J(u, v),$$

where

$$J(u, v) = \int_\Omega H(u, v) \, dx \tag{2.8}$$

and $L \colon E \to E$ is given by

$$\frac{1}{2} \langle L(u, v), (u, v) \rangle_E = \int_\Omega A^s u A^{2\alpha-s} v \, dx,$$

namely

$$L(u, v) = (A^{2\alpha-2s} v, A^{-2\alpha+2s} u).$$

Remark 2.7. Note that $J' : E \to E'$ is compact. Indeed, the inclusion $E \hookrightarrow L^{p+1} \times L^{q+1}$ is compact (thus $(L^{p+1} \times L^{q+1})' \hookrightarrow E'$ is compact as well), whereas $J' \colon L^{p+1} \times L^{q+1} \to (L^{p+1} \times L^{q+1})'$ maps bounded sets into bounded sets, since by *(H4)* one has

$$\begin{aligned} \left| J'(u, v)(\varphi, \psi) \right| \le C(&\|u\|_{p+1}^p \|\varphi\|_{p+1} + \|v\|_{q+1}^{p(q+1)/(p+1)} \|\varphi\|_{p+1} + \|\varphi\|_{p+1} \\ &+ \|v\|_{q+1}^q \|\psi\|_{q+1} + \|u\|_{p+1}^{q(p+1)/(q+1)} \|\psi\|_{q+1} + \|\psi\|_{q+1}). \end{aligned}$$

As a consequence,

$$J' : E \hookrightarrow L^{p+1} \times L^{q+1} \to (L^{p+1} \times L^{q+1})' \hookrightarrow E'$$

is compact. Moreover, L is bounded (actually, $\|L(u,v)\|_E = \|(u,v)\|_E$, hence the operator norm of L is 1), linear and symmetric, namely self-adjoint since E is Hilbert. Finally, L is invariant on E^+, since

$$L(u, A^{-2\alpha+2s}u) = (A^{2\alpha-2s}A^{-2\alpha+2s}u, A^{-2\alpha+2s}u) = (u, A^{-2\alpha+2s}u),\ u \in E^s.$$

Next we prove that critical points of I are weak solutions to (2.1). This will be done by extending Theorem 1.2 in [43].

Proposition 2.2. *Let* $(u,v) \in E$ *be a critical point of* I *and* p,q *satisfy* (2.2). *Then* $u \in W^{2\alpha,\frac{q+1}{q}} \cap W_0^{\alpha,\frac{q+1}{q}}$, $v \in W^{2\alpha,\frac{p+1}{p}} \cap W_0^{\alpha,\frac{p+1}{p}}$ *and* (u,v) *is a weak solution to* (2.1).

Proof. Since (u,v) is a critical point of I, one has

$$I'(u,v)(\varphi,\psi) = 0 \tag{2.9}$$

and in particular for $\varphi = 0$ and any $\psi \in E^{2\alpha-s}$ in (2.9) one has

$$\int_\Omega A^s u A^{2\alpha-s}\psi\, dx = \int_\Omega \partial_v H(u,v)\psi\, dx. \tag{2.10}$$

If $\psi \in H^{2\alpha} \cap H_0^\alpha$ then

$$\int_\Omega A^s u A^{2\alpha-s}\psi\, dx = \int_\Omega u A^{2\alpha}\psi\, dx = \int_\Omega u(-\Delta)^\alpha\psi\, dx. \tag{2.11}$$

Since $v \in L^{q+1}$ and $u \in L^{p+1}$, by *(H4)* and the Minkowski inequality one has $\partial_v H(u,v) \in L^{\frac{q+1}{q}}$ and hence by elliptic regularity (see Theorem 1.2) there exists $w \in W^{2\alpha,\frac{q+1}{q}} \cap W_0^{\alpha,\frac{q+1}{q}}$ such that

$$\int_\Omega (-\Delta)^\alpha w\psi = \int_\Omega \partial_v H(u,v)\psi. \tag{2.12}$$

Furthermore, by (2.2)

$$\frac{1}{2} \geq \frac{q}{q+1} - \frac{2\alpha}{N}$$

and by the Sobolev embedding theorem we get $w \in L^2$. Thus

$$\int_\Omega \partial_v H(u,v)\psi\, dx = \int_\Omega (-\Delta)^\alpha w\psi\, dx = \int_\Omega w(-\Delta)^\alpha\psi\, dx. \tag{2.13}$$

Hence, by combining (2.10), (2.11), (2.13) one has:

$$\int_\Omega (u-w)(-\Delta)^\alpha\psi\, dx = 0$$

for any $\psi \in H^{2\alpha} \cap H^{\alpha}_0$, so that $u = w$. Finally, by (2.12)

$$\int_\Omega (-\Delta)^\alpha u\psi = \int_\Omega \partial_v H(u,v)\psi.$$

A similar argument applies to v, therefore (u, v) satisfies the regularity conditions in the statement and it is a weak solution to (2.1). □

2.1.3 Palais Smale condition

Let us recall the following preliminary lemma, see for instance [99, Proposition 2.2].

Lemma 2.2. *Let X be a Banach space and $I \in C^1(X)$ such that:*

(i) any Palais-Smale sequence is bounded;

(ii) $I'(z) = S(z)+K(z)$ where $S\colon X \to X'$ is a homeomorphism and $K\colon X \to X'$ is a compact map.

Then I satisfies (PS).

Proof. Let z_n be a Palais-Smale sequence; by hypotheses, it is bounded and $S(z_n) + K(z_n) = I'(z_n) \to 0$. Take $w_n = K(z_n)$. By compactness, there exists a subsequence w_{n_k} such that $w_{n_k} \to w$ for some w. Hence

$$z_{n_k} = S^{-1}(I'(z_{n_k}) - w_{n_k}) \to S^{-1}(-w),$$

thus z_{n_k} is a strongly convergent subsequence of z_n and I satisfies (PS). □

Proposition 2.3. *Assume condition* (2.2) *and let* $pq > 1$. *Then the functional* (2.7) *satisfies the (PS) condition.*

Proof. Let (u_n, v_n) be a Palais-Smale sequence for I and assume $\|(u_n, v_n)\|_E \geq R$, with R as in *(H2)*. Let us prove that it is bounded. Since the operator norm of $I'(u_n, v_n)$ satisfies $\|I'(u_n, v_n)\| \to 0$, one has for any choice of test functions φ, ψ

$$\left|I'(u_n, v_n)(\varphi,\psi)\right| \leq \varepsilon_n \|(\varphi,\psi)\|_E\,, \tag{2.14}$$

with $\varepsilon_n \to 0$ as $n \to \infty$. Moreover,

$$I'(u,v)(\varphi,\psi) = \int_\Omega (A^s u A^{2\alpha-s}\psi + A^s\varphi A^{2\alpha-s}v - \partial_u H(u,v)\varphi - \partial_v H(u,v)\psi)\, dx,$$

thus by *(H2)*

$$\begin{aligned} 0 &< \frac{pq-1}{p+q+2}\int_\Omega H(u_n,v_n)dx \\ &\leq I(u_n,v_n) - I'(u_n,v_n)\left(\frac{q+1}{p+q+2}u_n, \frac{p+1}{p+q+2}v_n\right) \\ &\leq C_0(1+\varepsilon_n \|(u_n,v_n)\|_E). \end{aligned}$$

By (2.3) there exist constants $c_1, c_2 > 0$ such that

$$H(u,v) \geq c_1(|u|^{p+1} + |v|^{q+1}) - c_2,$$

thus

$$C_1(1 + \varepsilon_n \|(u_n, v_n)\|_E) \geq \|u_n\|_{p+1}^{p+1} + \|v_n\|_{q+1}^{q+1}. \tag{2.15}$$

However, by (2.14) with $\psi = 0$,

$$\left| \int_\Omega A^s \varphi A^{2\alpha - s} v_n \, dx \right| \leq \int_\Omega |\partial_u H(u_n, v_n)\varphi| \, dx + \varepsilon_n \|\varphi\|_{E^s} \tag{2.16}$$

and by Hölder inequality and *(H4)*

$$\int_\Omega |\partial_u H(u_n, v_n)| \, |\varphi| \, dx \leq C_3(\|u_n\|_{p+1}^p \|\varphi\|_{E^s} + \|v_n\|_{q+1}^{p(q+1)/(p+1)} \|\varphi\|_{E^s} + \|\varphi\|_{E^s}). \tag{2.17}$$

Moreover, one can apply the Riesz Lemma (E^s is a Hilbert space) to the functional $T_{v_n} \colon E^s \to \mathbb{R}$, $T_{v_n}(\varphi) := \langle \varphi, A^{2\alpha - 2s} v_n \rangle_{E^s} = \int_\Omega A^s \varphi A^{2\alpha - s} v_n$ to obtain:

$$\|v_n\|_{E^{2\alpha - s}} = \left\|A^{2\alpha - 2s} v_n\right\|_{E^s} = \|T_{v_n}\| = \sup_{\|\varphi\|_{E^s} = 1} \left| \int_\Omega A^s \varphi A^{2\alpha - s} v_n \, dx \right|.$$

Combining (2.16) and (2.17), one has

$$\|v_n\|_{E^{2\alpha - s}} \leq C_4(\|u_n\|_{p+1}^p + \|v_n\|_{q+1}^{p(q+1)/(p+1)} + 1).$$

Analogously

$$\|u_n\|_{E^s} \leq C_5(\|v_n\|_{q+1}^q + \|u_n\|_{p+1}^{q(p+1)/(q+1)} + 1),$$

hence by (2.15)

$$\|(u_n, v_n)\|_E \leq C_6(1 + \varepsilon_n \|(u_n, v_n)\|_E)$$

and (u_n, v_n) turns out to be bounded.

Next we apply Lemma 2.2. Indeed,

$$I'(u,v) = A'(u,v) - J'(u,v),$$

where $A(u,v) = \int_\Omega A^s u A^{2\alpha - s} v \, dx$ and J as in (2.8); $A' \colon E \to E'$ is an homeomorphism, whereas J' is compact, as pointed out in Remark 2.7. □

2.1.4 Linking geometry

In the sequel, set $z = (u, v) \in E$ and $I(z) = \frac{1}{2} \langle Lz, z \rangle_E - J(z)$ as defined in (2.7).

Proposition 2.4. *Assume condition* (2.2) *holds and let* $pq > 1$. *Then, there exist two linear, bounded, invertible operators* $B_1, B_2 \colon E \to E$ *such that, given* $\tau \geq 0$, *then* $\hat{B}_\tau = P_2 B_1^{-1} e^{\tau L} B_2 \colon E^- \to E^-$ *is invertible, where* P_2 *is the projection of* E *onto* E^-.

Moreover, let $e^+ = (e_1^+, e_2^+) \in E^+$ *with* $\|e^+\|_E = 1$ *and* $e_1^+ \in E^s$ *eigenfunction of* $(-\Delta)^\alpha$ *with associated eigenvalue* $\lambda > 0$. *Then, there exist constants* $\rho > 0$, $R_1 > \rho / \left\|B_1^{-1} B_2 e^+\right\|_E$ *and* $R_2 > \rho$ *such that, setting*

$$S := \{ B_1 z^+ : z^+ \in E^+, \left\|z^+\right\|_E = \rho \}$$

and

$$Q := \{ B_2(te^+ + z^-) : 0 \leq t \leq R_1, z^- \in E^-, \left\|z^-\right\|_E \leq R_2 \},$$

the following conditions hold true:

(G1) $I(z) \geq \sigma > 0$ *on* S

(G2) $I(z) \leq 0$ *on* ∂Q.

Define

$$B_1(u, v) = (\rho^{\mu-1} u, \rho^{\nu-1} v)$$

and

$$B_2(u, v) = (R_1^{\mu-1} u, R_1^{\nu-1} v),$$

where ρ and R_1 will be chosen in the sequel and $\mu, \nu \geq 1$ satisfy

$$\frac{1}{p+1} < \frac{\mu}{\mu+\nu}, \quad \frac{1}{q+1} < \frac{\nu}{\mu+\nu}. \tag{2.18}$$

We claim that $\hat{B}_\tau$ is invertible, and more precisely $\hat{B}_\tau z^- = mz^-$ with $m > 0$ constant if one assumes that $R_1 > 1$ and $\rho < 1$. See [43, Proposition 3.1] for the details.

Note that with our choice of B_1, B_2 one has:

$$\begin{aligned}
S =& \{ (\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) : \left\|(u^+, v^+)\right\|_E = \rho,\ z^+ = (u^+, v^+) \in E^+ \};\\
Q =& \{ t(R_1^{\mu-1} e_1^+, R_1^{\nu-1} e_2^+) + (R_1^{\mu-1} u^-, R_1^{\nu-1} v^-) \ :\ 0 \leq t \leq R_1,\\
&\ z^- = (u^-, v^-) \in E^-, \left\|(u^-, v^-)\right\|_E \leq R_2 \}.
\end{aligned}$$

Proof of (G1). For any $(\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) \in S$ one has by *(H3)* and ρ small enough

$$\begin{aligned}
I(\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) \geq \rho^{\mu+\nu-2} \int_\Omega A^s u^+ A^{2\alpha-s} v^+ \, dx - a\rho^{(\mu-1)(p+1)} \int_\Omega \left|u^+\right|^{p+1} dx \\
- a\rho^{(\nu-1)(q+1)} \int_\Omega \left|v^+\right|^{q+1} dx,
\end{aligned}$$

hence recalling (2.6) and the continuous embedding $E \hookrightarrow L^{p+1} \times L^{q+1}$,

$$I(\rho^{\mu-1}u^+, \rho^{\nu-1}v^+) \geq \frac{1}{2}\rho^{\mu+\nu-2} \left\|z^+\right\|_E^2 - b_1\rho^{(\mu-1)(p+1)} \left\|z^+\right\|_E^{p+1} - b_2\rho^{(\nu-1)(q+1)} \left\|z^+\right\|_E^{q+1},$$

for suitable constants b_1, b_2. Considering $\|z^+\|_E = \rho$,

$$I(\rho^{\mu-1}u^+, \rho^{\nu-1}v^+) \geq \frac{1}{2}\rho^{\mu+\nu} - b_1\rho^{\mu(p+1)} - b_2\rho^{\nu(q+1)}$$

and by (2.18) this quantity is positive for ρ small enough. □

Proof of (G2). Let us split the boundary into three parts, namely $Q \cap \{\, t = 0 \,\}$, $Q \cap \{\, t = R_1 \,\}$ and $Q \cap \{\, \|z^-\|_E = R_2 \,\}$.

Let $z \in Q \cap \{\, t = 0 \,\}$. By direct computation, $I(z) \leq 0$. Indeed,

$$\begin{aligned} I(R_1^{\mu-1}u^-, R_1^{\nu-1}v^-) &\leq R_1^{\mu+\nu-2} \int_\Omega A^s u^- A^{2\alpha-s} v^- \\ &= -R_1^{\mu+\nu-2} \int_\Omega \left|A^s u^-\right|^2 \leq 0. \end{aligned}$$

Let us now consider $z \in Q \cap \{\, t = R_1 \,\}$. Fix $R_2 > 0$ arbitrary and choose

$$z^- = (u^-, v^- = -A^{-2\alpha+2s}u^-) \in E^-$$

such that $\|z^-\|_E \leq R_2$, thus

$$z = t(R_1^{\mu-1}e_1^+, R_1^{\nu-1}e_2^+) + (R_1^{\mu-1}u^-, R_1^{\nu-1}v^-) = (u, v) \in Q.$$

We can write $u^- = re_1^+ + w$ where $w \in E^s$ is orthogonal to e_1^+ in L^2 and $r \in \mathbb{R}$. Suppose $r \geq 0$. One has

$$(r+t)\int_\Omega \left|e_1^+\right|^2 = \int_\Omega (te_1^+ + u^-)e_1^+ \leq \left\|te_1^+ + u^-\right\|_{p+1} \left\|e_1^+\right\|_{(p+1)/p}$$

and

$$(r+t) \leq C_1 \left\|te_1^+ + u^-\right\|_{p+1}.$$

By (2.3)

$$\begin{aligned} J(z) \geq c_1 {R_1}^{(p+1)(\mu-1)} &\int_\Omega \left|te_1^+ + u^-\right|^{p+1} \\ &+ c_1 {R_1}^{(q+1)(\nu-1)} \int_\Omega \left|te_2^+ + v^-\right|^{q+1} - c_2, \end{aligned}$$

thus

$$J(z) \geq C_2 {R_1}^{(p+1)(\mu-1)}(r+t)^{p+1} - c_2$$

and

$$J(z) \geq C_2 {R_1}^{(p+1)(\mu-1)} t^{p+1} - c_2.$$

Similarly, if $r \leq 0$, since

$$e_2^+ = A^{-2\alpha+2s} e_1^+ = \lambda^{\frac{-2\alpha+2s}{2\alpha}} e_1^+$$

and $v^- = -A^{-2\alpha+2s} u^-$, we get

$$\begin{aligned}\langle v^-, e_1^+ \rangle &= \langle -A^{-2\alpha+2s} u^-, e_1^+ \rangle = \langle -A^{-2\alpha+2s}(r e_1^+ + w), e_1^+ \rangle \\ &= -r \lambda^{\frac{-2\alpha+2s}{2\alpha}} \int_\Omega \left|e_1^+\right|^2 - \langle w, A^{-2\alpha+2s} e_1^+ \rangle = -r \lambda^{\frac{-2\alpha+2s}{2\alpha}} \int_\Omega \left|e_1^+\right|^2,\end{aligned}$$

hence

$$\begin{aligned}\lambda^{\frac{-2\alpha+2s}{2\alpha}} (-r+t) \int_\Omega \left|e_1^+\right|^2 &= \int_\Omega (t e_2^+ + v^-) e_1^+ \\ &\leq \left\|t e_2^+ + v^-\right\|_{q+1} \left\|e_1^+\right\|_{(q+1)/q}\end{aligned}$$

and

$$\lambda^{\frac{-2\alpha+2s}{2\alpha}} (-r+t) \leq C_3 \left\|t e_2^+ + v^-\right\|_{q+1}.$$

As a consequence,

$$J(z) \geq C_4 {R_1}^{(q+1)(\nu-1)} t^{q+1} - c_2.$$

Concluding, we have that either

$$J(z) \geq C_2 {R_1}^{(p+1)(\mu-1)} t^{p+1} - c_2$$

or

$$J(z) \geq C_4 {R_1}^{(q+1)(\nu-1)} t^{q+1} - c_2.$$

Thus by (2.6) either

$$I(z) \leq R_1^{\mu+\nu-2} \frac{t^2}{2} - R_1^{\mu+\nu-2} \frac{1}{2} \left\|z^-\right\|_E^2 - C_2 {R_1}^{(p+1)(\mu-1)} t^{p+1} + c_2 \tag{2.19}$$

or

$$I(z) \leq R_1^{\mu+\nu-2} \frac{t^2}{2} - R_1^{\mu+\nu-2} \frac{1}{2} \left\|z^-\right\|_E^2 - C_4 {R_1}^{(q+1)(\nu-1)} t^{q+1} + c_2. \tag{2.20}$$

Therefore, by (2.18) one can choose $t = R_1$ such that the right-hand sides of both (2.19) and (2.20) are negative.

Finally, let $z \in Q \cap \{ \|z^-\|_E = R_2 \}$. Choose R_2 such that the quantities in (2.19) and (2.20) are negative for any $t \leq R_1$.

Therefore, I is negative on ∂Q taking R_1, R_2 sufficiently large and the proof is complete. □

Remark 2.8. Note that if $\mu = \nu = 1$, then Proposition 2.1 coincides with the classical Linking Theorem of Benci and Rabinowitz [11], and moreover conditions (2.18) imply $p, q > 1$. The possibility of choosing different values for μ and ν allows to deal with p, q not necessarily both bigger than 1, namely such that $pq > 1$.

Remark 2.9. Note that in the proof we have used both the fact that there exists a strictly positive eigenvalue λ and that the eigenfunctions of $(-\Delta)^\alpha$ are in L^s for any s, and these properties hold true by Lemma 2.1.

2.1.5 Proof of Theorem 2.1

We apply Proposition 2.1 where E, E^+, E^- are defined as in Subsection 2.1.1, and I as in (2.7). Indeed, L and J satisfy the hypotheses of Proposition 2.1 due to Remark 2.7, I satisfies (PS) by Proposition 2.3 and one can choose constants ρ, R_1 and R_2 and operators B_1 and B_2 such that the hypotheses on S and Q are satisfied, as shown in Proposition 2.4. Thus, one finds a critical point (u, v) of I such that $I(u, v) > 0$, and by Proposition 2.2 (u, v) is a solution to (2.1), nontrivial since $I(0, 0) = 0$.

Remark 2.10. Notice that the proof above can also be adapted easily to treat the Navier case

$$\begin{cases} (-\Delta)^\alpha u = \partial_v H(u,v) & \\ & \text{in } \Omega \subset \mathbb{R}^N, N > 2\alpha \\ (-\Delta)^\alpha v = \partial_u H(u,v) & \\ \Delta^r u = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega \\ \Delta^r v = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega, \end{cases}$$

just taking $E^s = [L^2, H^{2\alpha} \cap H^\alpha_\theta]_{s/(2\alpha)}$, see also [57].

2.2 Non-existence results for system (2.1)

Let us recall the following Pohozaev type identity, see [83, Section 5], see also [76].

Theorem 2.2. *Let $a, b \in \mathbb{R}$, and (u, v) be a solution to (2.1). Then if α is even*

$$\int_{\partial\Omega} \Delta^{\alpha/2} u \Delta^{\alpha/2} v (x \cdot \nu) = \int_\Omega (NH(u,v) - au\partial_u H(u,v) - (N - a - b - 2\alpha)\Delta^{\alpha/2} u \Delta^{\alpha/2} v - bv\partial_v H(u,v)). \quad (2.21)$$

Similarly, for odd α one has

$$\int_{\partial\Omega} \nabla\Delta^{(\alpha-1)/2} u \nabla\Delta^{(\alpha-1)/2} v (x \cdot \nu) = \int_\Omega (NH(u,v) - au\partial_u H(u,v) - (N - a - b - 2\alpha)\nabla\Delta^{(\alpha-1)/2} u \nabla\Delta^{(\alpha-1)/2} v - bv\partial_v H(u,v)).$$

As a corollary we have

Theorem 2.3. *Let* Ω *be a ball in* $\mathbb{R}^N$ *and* $N > 2\alpha$. *Assume that there exists* $a \in \mathbb{R}$ *such that*

$$NH(u,v) - au\partial_u H(u,v) - (N - 2\alpha - a)v\partial_v H(u,v)) \leq 0.$$

Then, no classical positive solutions to (2.1) *do exist.*

Proof. Let (u, v) be a positive classical solution to (2.1). In the sequel, we consider the case of even α, the odd case being similar. Recall that the Green function of the problem

$$\begin{cases} (-\Delta)^\alpha u = f & \text{in } B_1 \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, & \text{on } \partial B_1 \end{cases}$$

is positive; therefore, due to Theorem 1.5, if (u, v) is a positive classical solution to (2.1), then $\Delta^{\alpha/2}u, \Delta^{\alpha/2}v > 0$ on ∂B_1. Hence, by choosing $b = N - 2\alpha - a$ in (2.21), one has

$$\begin{aligned} 0 &< \int_{\partial B_1} \Delta^{\alpha/2}u\Delta^{\alpha/2}v(x \cdot \nu) \\ &= \int_{B_1} (NH(u,v) - au\partial_u H(u,v) - (N - 2\alpha - a)v\partial_v H(u,v)) \leq 0, \end{aligned}$$

which is a contradiction. □

Remark 2.11. We recall that in [68] a different Pohozaev type identity is exploited to treat the Navier case.

Chapter 3

Non variational Lane–Emden systems on a ball

In this chapter we recall and slightly extend results we proved in [89]. We consider the following system

$$
\begin{cases}
(-\Delta)^{\alpha} u = |v|^{q} & \text{in } B_1 \subset \mathbb{R}^N \\
(-\Delta)^{\beta} v = |u|^{p} & \\
\frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, & \text{on } \partial B_1 \\
\frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \ldots, \beta - 1, & \text{on } \partial B_1.
\end{cases}
\tag{3.1}
$$

where B_1 is the unitary ball in $\mathbb{R}^N$. Notice that if $\alpha \neq \beta$ we cannot apply the approach in Chapter 2, nor the critical point theorem in [28]. We prove some existence results by means of a continuation method as in [8], together with a priori estimates obtained through the moving planes technique, see [13, 49], and [12], [47, Section 7] for equations with polyharmonic operators. A key role is played by Liouville–type results for

$$
\begin{cases}
(-\Delta)^{\alpha} u = |v|^{q} \\
(-\Delta)^{\beta} v = |u|^{p}
\end{cases}
\text{in } \mathbb{R}^N,
\tag{3.2}
$$

see [76, 69]. We also prove a non-existence result for (3.2) in the spirit of [29], see Subsection 3.1.4 below, which allows us to treat the natural superlinearity condition $pq > 1$. Lastly, in Section 3.2 and Section 3.3 we show how to extend the proofs to systems of the form (3.1) with $m > 2$ equations, see also [23], and to the Navier case. We stress that to the best of our knowledge these are the first results concerning the non-variational system (3.1).

3.1 Main result: existence to (3.1)

This section is devoted to the proof of Theorem 1.14, which we recall below.

Theorem 3.1. *If the only solution to* (3.1) *is the trivial one, and if* $pq > 1$, *then there exists a classical, nontrivial, nonnegative and radially symmetric solution to* (3.2).

We remark that Theorem 3.1 plays a pivotal role in proving existence results for (3.1), by means of a contradiction argument and non existence results for (3.2). In particular, one can exploit Theorem 1.9 and Theorem 1.11 above to get

Corollary 3.1. *Assume* $p, q > 1$. *Assume further that one of the following is satisfied:*

(i) $2\beta q + N + 2\alpha pq - Npq \geq 0$ *or* $2\alpha p + N + 2\beta pq - Npq \geq 0$*;*

(ii) $p, q < \min\{\frac{N+2\alpha}{N-2\beta}, \frac{N+2\beta}{N-2\alpha}\}$.

Then there exists a positive classical solution to (3.1).

Here, we further extend this existence result to $pq > 1$ as follows:

Corollary 3.2. *Assume* $pq > 1$. *Assume further that one of the following is satisfied:*

(i) $2\beta q + N + 2\alpha pq - Npq \geq 0$ *or* $2\alpha p + N + 2\beta pq - Npq \geq 0$*;*

(ii) $p, q < \min\{\frac{N+2\alpha}{N-2\beta}, \frac{N+2\beta}{N-2\alpha}\}$.

Then there exists a positive classical solution to (3.1).

We split the argument of the proof of Theorem 3.1 into four steps, each corresponding to a different subsection in the sequel:

Step 1. By exploiting the Leray–Schauder degree [71] and a continuation argument [8, 7], one proves that if the only solution to (3.1) is the trivial one, then there exists an unbounded sequence of solutions to a system S_t depending on a scaling parameter t.

Step 2. One performs a blow-up analysis [8]; more precisely, by assuming the existence of a sequence of functions as in Step 1 and suitable geometric requirements on the position of global maxima of these functions, one gets the existence of a classical radial nontrivial solution to system (3.2).

Step 3. The moving planes procedure [13], see also [47, 12], together with Gidas–Ni–Nirenberg arguments [49], is used to derive information on the location of global maxima of solutions to S_t in order to establish the geometric assumptions required in Step 2. Actually, solutions to S_t turn out to be radially symmetric and strictly decreasing in the radial variable.

Step 4. Theorem 3.1 now follows simply by combining Steps 1–3. In order to prove Corollary 3.2, we give a non-existence result for system (3.2) adapting ideas in [29] and we combine it with [96, Proposition 2].

3.1.1 An auxiliary system

Proposition 3.1. *Let p, q be such that $pq > 1$. Denote with $\mathcal{C}$ the component in $\mathbb{R}^+ \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ of solutions (t, u, v) to*

$$\begin{cases} (-\Delta)^\alpha u = (t + |v|)^q & \text{in } B_1 \\ (-\Delta)^\beta v = (t^\vartheta + |u|)^p & \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial B_1 \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \dots, \beta - 1, & \text{on } \partial B_1 \end{cases} \tag{3.3}$$

containing $(0, 0, 0)$, where $\vartheta \in (1/p, q)$. If

$$\mathcal{C} \cap (\{0\} \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)) = \{(0, 0, 0)\}$$

then $\mathcal{C}$ is unbounded in $\mathbb{R}^+ \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$.

The proof of Proposition 3.1 needs a few preliminary results.

Lemma 3.1. *Let us consider the following*

$$\begin{cases} (-\Delta)^\alpha u = f & \text{in } \Omega \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 2\alpha$. If $f \in C^r(\bar{\Omega})$ with $r \geq 1$ then the map $K_\alpha \colon C^r(\bar{\Omega}) \to C^{2\alpha}(\bar{\Omega})$ defined as $K_\alpha(f) = u$ is continuous and compact.

Proof. By elliptic regularity, see Theorem 1.2, we know that if $f \in W^{k-2\alpha,p}(\Omega)$ for $k \geq 2\alpha$ then there exists a unique nontrivial solution to (3.4) such that

$$\|u\|_{W^{k,p}} \leq C \|f\|_{W^{k-2\alpha,p}} . \tag{3.5}$$

Assume $f \in C^r(\bar{\Omega})$ with $r \geq 1$. Then $f \in W^{r,p}(\Omega)$ for any p. Hence, by (3.5) one has

$$\|u\|_{W^{r+2\alpha,p}} \leq C \|f\|_{W^{r,p}}$$

for any p. By taking p large enough, one has $W^{r+2\alpha,p}(\Omega) \hookrightarrow C^{r+2\alpha-1,\gamma}(\bar{\Omega})$ compactly for any $\gamma \in (0, 1)$ (see [3, Theorem 6.3]) and since $r \geq 1$ then $u \in C^{2\alpha,\gamma}(\bar{\Omega}) \subset C^{2\alpha}(\bar{\Omega})$. □

Lemma 3.2 (Lemma A.2 in [7])**.** *Let $(E, \|\cdot\|)$ be a real Banach space. Let $G \colon \mathbb{R}^+ \times E \to E$ be continuous and compact. Suppose, moreover, G satisfies*

(a) $G(0, 0) = 0$

(b) there exists $R > 0$ such that

  (i) $u \in E$, $\|u\| \leq R$ *and* $u = G(0, u)$ *implies* $u = 0$

(ii) $deg(Id - G(0,\cdot), B_R^E, 0) = 1$, *where* B_R^E *represents the ball in* E *centered in 0 of radius* R.

Let J *denote the set of solutions to the problem* $u = G(t,u)$ *in* $\mathbb{R}^+ \times E$. *Let* $\mathcal{C}$ *denote the component of* J *containing* $(0,0)$. *If*

$$\mathcal{C} \cap (\{0\} \times E) = \{(0,0)\}$$

then $\mathcal{C}$ *is unbounded in* $\mathbb{R}^+ \times E$.

Lemma 3.3. *Let* p, q *such that* $pq > 1$. *Then there exists a real number* $R > 0$ *such that if* $(\lambda, u, v) \in [0,1] \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ *is a solution to*

$$\begin{cases} u = K_\alpha(\lambda\,|v|^q) \\ v = K_\beta(\lambda\,|u|^p) \\ u \neq 0 \text{ or } v \neq 0 \end{cases} \tag{3.6}$$

then $\|u\|_\infty > R$ *and* $\|v\|_\infty > R$.

Proof. By Theorem 1.4 one has $K_\alpha\left(\left(\frac{|v|}{\|v\|_\infty}\right)^q\right) \leq K_\alpha(1)$, hence

$$|u| = |K_\alpha(\lambda\,|v|^q)| \leq \left| K_\alpha\left(\|v\|_\infty^q \left(\frac{|v|}{\|v\|_\infty}\right)^q\right)\right| \leq \|v\|_\infty^q\,|K_\alpha(1)| \leq C_1\,\|v\|_\infty^q$$

thus

$$\|u\|_\infty \leq C_1\,\|v\|_\infty^q$$

and similarly

$$\|v\|_\infty \leq C_2\,\|u\|_\infty^p\,.$$

Then $\|u\|_\infty \leq C_1 C_2^q\,\|u\|_\infty^{pq}$ and therefore $\|u\|_\infty \geq R$; similarly for v. □

Proof of Proposition 3.1. We apply Lemma 3.2: let us define

$$G\colon [0,+\infty) \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1) \to C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$$

as follows

$$G(t,u,v) = (K_\alpha(t + |v|)^q, K_\beta(t^\vartheta + |u|)^p).$$

The operator G is continuous and compact since K_α, K_β have these properties by Lemma 3.1. Note that $v \in C^{2\beta}(\bar{B}_1)$ implies $(v+t)^q \in C^{2\beta}(\bar{B}_1)$ and thus by Lemma 3.1 with $r = 2\beta > 1$ one has $K_\alpha(v+t)^q \in C^{2\alpha}(\bar{B}_1)$. Moreover $G(0,0,0) = (0,0)$. Hypothesis $(b)(i)$ of Lemma 3.2 is satisfied due to Lemma 3.3 with $\lambda = 1$.
Let $B_R^{\alpha,\beta}$ be the ball in $C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ centered in 0 of radius R. Let us define an homotopy

$$h\colon [0,1] \times \overline{B_R^{\alpha,\beta}} \to C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$$

by

$$h(\lambda, u, v) \mapsto (K_\alpha(\lambda\,|v|^q), K_\beta(\lambda\,|u|^p)).$$

By Lemma 3.1 the map h is continuous and compact, $h(1,\cdot,\cdot) = G(0,\cdot,\cdot)$, $h(0,\cdot,\cdot) = (0,0)$ and by Lemma 3.3 one has $h(\lambda,u,v) \neq (u,v)$ for all $(u,v) \in \partial B_R^{\alpha,\beta}$. Therefore hypothesis $(b)(ii)$ of Lemma 3.2 is also satisfied. Indeed, one uses the homotopy invariance of the Leray–Schauder degree (see e.g. Theorem 2.1(iii) in [71]):

$$\begin{aligned} deg(Id - G(0,\cdot,\cdot), B_R^{\alpha,\beta}, 0) &= deg(Id - h(1,\cdot,\cdot), B_R^{\alpha,\beta}, 0) \\ &= deg(Id - h(0,\cdot,\cdot), B_R^{\alpha,\beta}, 0) = deg(Id, B_R^{\alpha,\beta}, 0) = 1. \quad \square \end{aligned}$$

3.1.2 Blow-up analysis

The main result of this subsection is the following

Proposition 3.2. *Let (t_n, u_n, v_n) be a sequence of solutions to* (3.3) *in $\mathbb{R}^+ \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ with $pq > 1$ and $\vartheta \in (1/p, q)$ fixed such that*

$$t_n + \|u_n\|_\infty + \|v_n\|_\infty \to \infty. \tag{3.7}$$

Suppose that there exist $\rho > 0$ and $\{x_n\}, \{x'_n\} \in B_1$ satisfying $u_n(x_n) = \|u_n\|_\infty$, $v_n(x'_n) = \|v_n\|_\infty$ and such that

$$dist(x_n, \partial B_1) \geq \rho,\ dist(x'_n, \partial B_1) \geq \rho.$$

Then there exists $(u,v) \in C^{2\alpha}(\mathbb{R}^N) \times C^{2\beta}(\mathbb{R}^N)$ nontrivial nonnegative solution to

$$\begin{cases} (-\Delta)^\alpha u = |v|^q \\ (-\Delta)^\beta v = |u|^p \end{cases} \text{in } \mathbb{R}^N. \tag{3.8}$$

Moreover, if u_n, v_n are radially symmetric and $x_n = x'_n = 0$ for any n, then there exists a nontrivial nonnegative radial solution to (3.8).

Proof. Assume without loss of generality α, β even. Let us prove first that there exists a subsequence such that

$$\frac{t_n^\vartheta}{\|u_n\|_\infty} \to 0 \text{ and } \frac{t_n}{\|v_n\|_\infty} \to 0 \tag{3.9}$$

with $\|u_n\|_\infty > 0$ and $\|v_n\|_\infty > 0$ for all n. Observe that if $u_n = 0$, then $v_n = 0$ and $t_n = 0$, therefore by (3.7) we have that $u_n = 0$ only for a finite number of indices n and similarly for v_n, namely there exists a subsequence such that $u_n \neq 0, v_n \neq 0$ for any n. Two cases may occur: if t_n is bounded, then for example $\|u_n\|_\infty \to \infty$, thus by Lemma 3.1 $\|v_n\|_\infty \to \infty$ as well and (3.9) follows. If $t_n \to \infty$, then we assume without loss of generality that $t_n > 0$. Let us introduce the following change of variable:

$$\tilde{u}_n = \frac{u_n}{t_n^{\vartheta}}, \quad \lambda_n = t_n^{q-\vartheta}$$
$$\tilde{v}_n = \frac{v_n}{t_n}, \quad \mu_n = t_n^{\vartheta p-1}.$$

Then for all n one has

$$\begin{cases} (-\Delta)^{\alpha}\tilde{u}_n = \lambda_n(1+|\tilde{v}_n|)^q \geq \lambda_n & \text{on } B_1 \\ (-\Delta)^{\beta}\tilde{v}_n = \mu_n(1+|\tilde{u}_n|)^p \geq \mu_n & \\ \frac{\partial^r \tilde{u}_n}{\partial \nu^r} = 0,\ r = 0,\dots,\alpha-1, & \text{on } \partial B_1 \\ \frac{\partial^r \tilde{v}_n}{\partial \nu^r} = 0,\ r = 0,\dots,\beta-1, & \text{on } \partial B_1. \end{cases}$$

Moreover, since $\vartheta \in (1/p, q)$, then $\lambda_n, \mu_n \to \infty$. For any fixed n, let us denote by (w_n, z_n) the solution to

$$\begin{cases} (-\Delta)^{\alpha} w_n = \lambda_n & \text{on } B_1 \\ (-\Delta)^{\beta} z_n = \mu_n & \\ \frac{\partial^r w_n}{\partial \nu^r} = 0,\ r = 0,\dots,\alpha-1, & \text{on } \partial B_1 \\ \frac{\partial^r z_n}{\partial \nu^r} = 0,\ r = 0,\dots,\beta-1, & \text{on } \partial B_1 \end{cases}$$

given by Theorem 1.2. Then one has $\tilde{u}_n \geq w_n$ and $\tilde{v}_n \geq z_n$ by the comparison principle (see Theorem 1.4). Moreover, we claim

$$\sup_n \|w_n\|_\infty = \sup_n \|z_n\|_\infty = +\infty.$$

Indeed, let us suppose by contradiction that $\sup_n \|w_n\|_\infty \leq c$. Then one has

$$\|w_n\|^2_{W^{\alpha,2}} = \int_{B_1} \left|\Delta^{\alpha/2} w_n\right|^2 = \lambda_n \int_{B_1} w_n \leq c\lambda_n. \tag{3.10}$$

However, for any $\varphi \geq 0, \not\equiv 0$ one has

$$0 < D = \int_{B_1} \varphi = \frac{1}{\lambda_n} \int_{B_1} \Delta^{\alpha/2} w_n \Delta^{\alpha/2}\varphi$$

and by (3.10),

$$0 < D \leq \frac{1}{\lambda_n} \|w_n\|_{W^{\alpha,2}} \|\varphi\|_{W^{\alpha,2}} \leq c^{\frac{1}{2}} \lambda_n^{-\frac{1}{2}} \|\varphi\|_{W^{\alpha,2}}$$

which tends to 0 as $n \to \infty$, a contradiction. Similarly for z_n. Then the claim holds and as a consequence $\sup_n \|\tilde{u}_n\|_\infty = \sup_n \|\tilde{v}_n\|_\infty = \infty$, which is equivalent to (3.9). Now, let us consider (t_n, u_n, v_n) which satisfies (3.9). Let $A_n, B_n, C_n > 0$ to be chosen in the sequel and assume first that $\|u_n\|_\infty^{1/\tau} \geq$

$\|v_n\|_\infty^{1/\sigma}$ for any n, where $\tau = \frac{2\beta q+2\alpha}{pq-1}$ and $\sigma = \frac{2\alpha p+2\beta}{pq-1}$. Define the following scaling

$$\hat{u}_n(y) = \frac{u_n(C_n^{-1}y + x_n)}{A_n}$$

and

$$\hat{v}_n(y) = \frac{v_n(C_n^{-1}y + x_n)}{B_n}$$

for any $y \in C_n(B_1 - x_n) = B(C_n x_n, C_n)$. Let $\hat{\psi}_n(x) = \psi(C_n(x - x_n))$, where $\psi \in C^\infty(\mathbb{R}^N)$. Then

$$\begin{aligned}\int_{B(C_n x_n, C_n)} \Delta^{\alpha/2}\hat{u}_n \Delta^{\alpha/2}\psi\, dy &= \int_{B_1} A_n^{-1} C_n^{N-2\alpha} \Delta^{\alpha/2} u_n \Delta^{\alpha/2}\hat{\psi}_n\, dx \\ &= \int_{B_1} A_n^{-1} C_n^{N-2\alpha} (t_n + |v_n|)^q \hat{\psi}_n\, dx \\ &= \int_{B(C_n x_n, C_n)} A_n^{-1} B_n^q C_n^{-2\alpha} \left(\frac{t_n}{B_n} + |\hat{v}_n|\right)^q \psi\, dy\end{aligned}$$

and

$$\int_{B(C_n x_n, C_n)} \Delta^{\beta/2}\hat{v}_n \Delta^{\beta/2}\psi\, dy = \int_{B(C_n x_n, C_n)} B_n^{-1} A_n^p C_n^{-2\beta} \left(\frac{t_n^\vartheta}{A_n} + |\hat{u}_n|\right)^p \psi\, dy.$$

Now choose $A_n = C_n^\tau$, $B_n = C_n^\sigma$ and $C_n = \|u_n\|_\infty^{1/\tau} + \|v_n\|_\infty^{1/\sigma}$, thus

$$\begin{aligned}\int_{B(C_n x_n, C_n)} \Delta^{\alpha/2}\hat{u}_n \Delta^{\alpha/2}\psi\, dy &= \int_{B(C_n x_n, C_n)} \left(\frac{t_n}{B_n} + |\hat{v}_n|\right)^q \psi\, dy \\ \int_{B(C_n x_n, C_n)} \Delta^{\beta/2}\hat{v}_n \Delta^{\beta/2}\psi\, dy &= \int_{B(C_n x_n, C_n)} \left(\frac{t_n^\vartheta}{A_n} + |\hat{u}_n|\right)^p \psi\, dy.\end{aligned} \tag{3.11}$$

By (3.7) and (3.9) one has $C_n \to \infty$, hence,

$$dist(0, \partial B(C_n x_n, C_n)) = C_n dist(x_n, \partial B_1) \geq C_n \rho \to \infty. \tag{3.12}$$

Moreover,

$$0 \leq \frac{t_n}{B_n} = \frac{t_n}{\left(\|u_n\|_\infty^{1/\tau} + \|v_n\|_\infty^{1/\sigma}\right)^\sigma} \leq \frac{t_n}{\|v_n\|_\infty} \to 0$$

and

$$0 \leq \frac{t_n^\vartheta}{A_n} = \frac{t_n^\vartheta}{\left(\|v_n\|_\infty^{1/\sigma} + \|u_n\|_\infty^{1/\tau}\right)^\tau} \leq \frac{t_n^\vartheta}{\|u_n\|_\infty} \to 0.$$

Let B be any closed ball. Then by (3.12) B is contained in $B(C_n x_n, C_n)$ for n large enough. Moreover, since the embedding $C^{2\alpha,\gamma}(\bar{\Omega}) \hookrightarrow C^{2\alpha}(\bar{\Omega})$ is compact, see [3], then $(\hat{u}_n, \hat{v}_n) \in C^{2\alpha,\gamma}(\bar{B}) \times C^{2\beta,\gamma}(\bar{B})$ converges up to a subsequence to nonnegative functions $(\hat{u}, \hat{v})$ in $C^{2\alpha}(\bar{B}) \times C^{2\beta}(\bar{B})$. Notice that $\|\hat{u}_n\|_\infty$ and

$\|\hat{v}_n\|_\infty$ are both bounded. By considering integrals in (3.11) on the ball B and letting $n \to \infty$, one has

$$\int_B \Delta^{\alpha/2}\hat{u}\Delta^{\alpha/2}\psi\,dy = \int_B |\hat{v}|^q\,\psi\,dy$$
$$\int_B \Delta^{\beta/2}\hat{v}\Delta^{\beta/2}\psi\,dy = \int_B |\hat{u}|^p\,\psi\,dy$$

for any $\psi \in C_0^\infty(B)$. Note that $(\hat{u}, \hat{v}) \neq (0,0)$: indeed,

$$\begin{aligned}\hat{u}_n^{1/\tau}(0) &= \frac{u_n(x_n)^{1/\tau}}{\|u_n\|_\infty^{1/\tau} + \|v_n\|_\infty^{1/\sigma}}\\ &= \frac{\|u_n\|_\infty^{1/\tau}}{\|u_n\|_\infty^{1/\tau} + \|v_n\|_\infty^{1/\sigma}} = \frac{1}{1 + \|v_n\|_\infty^{1/\sigma} / \|u_n\|_\infty^{1/\tau}} \geq \frac{1}{2}\end{aligned}$$

and therefore $\hat{u}(0) \neq 0$. Let us now take a larger ball $\tilde{B}$ and repeat the argument on the subsequence obtained at the previous step. Taking balls larger and larger and iterating the reasoning, we get two Cantor diagonal subsequences converging on all compacts of $\mathbb{R}^N$ to nontrivial functions $(\hat{u}, \hat{v}) \in C^{2\alpha}(\mathbb{R}^N) \times C^{2\beta}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} \Delta^{\alpha/2}\hat{u}\Delta^{\alpha/2}\psi\,dy = \int_{\mathbb{R}^N} |\hat{v}|^q\,\psi\,dy$$
$$\int_{\mathbb{R}^N} \Delta^{\beta/2}\hat{v}\Delta^{\beta/2}\psi\,dy = \int_{\mathbb{R}^N} |\hat{u}|^p\,\psi\,dy.$$

If $\|u_n\|_\infty^{1/\tau} \leq \|v_n\|_\infty^{1/\sigma}$ we take x_n' instead of x_n in the definition of $\hat{u}_n$ and $\hat{v}_n$ and at the end we observe

$$\hat{v}_n^{1/\sigma}(0) \geq \frac{1}{2}.$$

This concludes the proof. □

3.1.3 A priori estimates

Let us consider the problem

$$\begin{cases} (-\Delta)^\alpha u = g(v) & \text{in } B_1 \\ (-\Delta)^\beta v = f(u) & \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \ldots, \alpha - 1, & \text{on } \partial B_1 \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \ldots, \beta - 1, & \text{on } \partial B_1 \end{cases} \tag{3.13}$$

where $f, g\colon [0,\infty) \to \mathbb{R}$ are continuous, positive and non decreasing. The aim of this subsection is to obtain information on the position of global maxima of solutions to (3.13), in order to apply Proposition 3.2. In what follows, a solution (u, v) is nontrivial if both u and v are nontrivial. More precisely, we show

Proposition 3.3. *Let* $(u, v) \in C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ *be a nontrivial solution to* (3.13). *Then* u, v *are radially symmetric and strictly decreasing in the radial variable. In particular,* u *and* v *attain their maximum at* 0.

In order to prove Proposition 3.3, we apply the moving planes technique [13] and we adapt the classical symmetry result by Gidas–Ni–Nirenberg [49], by extending to the case of systems a few proofs of [12] where the case of a single equation is considered, see also [47, Section 7]. Define

$$T_{i,\lambda} = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_i = \lambda\}$$
$$\Sigma_{i,\lambda} = \{x = (x_1, \dots, x_N) \in B_1 : x_i < \lambda\}$$

where $\lambda \in [0, 1]$, and let $x^{i,\lambda}$ denote the reflection of x about $T_{i,\lambda}$.

The next Lemmas constitute the preparation to the moving planes procedure.

Lemma 3.4 (Lemma 3 and Lemma 4 in [12])**.** *For all* $x, y \in \Sigma_{i,\lambda}$ $x \neq y$, *we have*

$$G_\alpha(x, y) > \max\{G_\alpha(x, y^{i,\lambda}), G_\alpha(x^{i,\lambda}, y)\}$$
$$G_\alpha(x, y) - G_\alpha(x^{i,\lambda} y^{i,\lambda}) > \left| G_\alpha(x, y^{i,\lambda}) - G_\alpha(x^{i,\lambda}, y) \right|.$$

Moreover, for every $x \in B_1 \cap T_{i,\lambda}$ *and* $y \in \Sigma_{i,\lambda}$ *we have*

$$\partial_{x_i} G_\alpha(x, y) < 0 \quad \text{and} \quad \partial_{x_i} G_\alpha(x, y) + \partial_{x_i} G_\alpha(x, y^{i,\lambda}) \leq 0. \tag{3.14}$$

The second inequality in (3.14) *is strict if* $\lambda > 0$.

Let us define

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s > 0 \\ 0 & \text{if } s = 0 \end{cases} \qquad \tilde{g}(s) = \begin{cases} g(s) & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases}$$

Note that $f(0) \neq 0$, $g(0) \neq 0$ in general: indeed, we will apply the results of this subsection to $f(u) = (t + |u|)^p$ and $g(v) = (t + |v|)^q$. From now on, let us extend u, v out of B_1 by imposing $u = v = 0$.

Lemma 3.5. *Let* $(u, v) \in C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$, *nontrivial solution to* (3.13). *Suppose* $u(y) \geq u(y^{i,\lambda})$ *and* $v(y) \geq v(y^{i,\lambda})$ *for all* $y \in \Sigma_{i,\lambda}$. *Then the following inequalities hold:*

- $f(u(y)) \geq \tilde{f}(u(y^{i,\lambda})) \geq 0$ *and* $g(v(y)) \geq \tilde{g}(v(y^{i,\lambda})) \geq 0$ *for all* $y \in \Sigma_{i,\lambda}$
- *there exist two nonempty open sets* $\mathcal{O}^u_{i,\lambda}, \mathcal{O}^v_{i,\lambda} \subset \Sigma_{i,\lambda}$ *such that* $f(u(y)) > \tilde{f}(u(y^{i,\lambda}))$ *or* $\tilde{f}(u(y^{i,\lambda})) > 0$ *for all* $y \in \mathcal{O}^u_{i,\lambda}$ *and* $g(v(y)) > \tilde{g}(v(y^{i,\lambda}))$ *or* $\tilde{g}(v(y^{i,\lambda})) > 0$ *for all* $y \in \mathcal{O}^v_{i,\lambda}$.

Proof. First note that $u, v > 0$ in B_1 due to Theorem 1.4. The inequalities $f(u(y)) \geq \tilde{f}(u(y^{i,\lambda})) \geq 0$ and $g(v(y)) \geq \tilde{g}(v(y^{i,\lambda})) \geq 0$ follow from the monotonicity and positivity assumptions on f, g.
For the second statement it is enough to show that $f(u) \neq 0$ and $g(v) \neq 0$ in $\Sigma_{i,\lambda}$. By contradiction, if $f(u) = 0$ on $\Sigma_{i,\lambda}$ then the above inequalities imply $\tilde{f}(u(y^{i,\lambda})) = 0$, however since $u > 0$ this means $f(u) = 0$ on B_1. In turn, this implies $(-\Delta)^\beta v = 0$, thus $v = 0$, which contradicts the positivity of v. Similarly for $g(v)$. □

The following result will allow us to slide the hyperplane.

Lemma 3.6. *Let $(u, v) \in C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ nontrivial solution to (3.13). Suppose $u(x) \geq u(x^{i,\lambda})$ and $v(x) \geq v(x^{i,\lambda})$ for all $x \in \Sigma_{i,\lambda}$. Then there exists $\gamma \in (0, \lambda)$ such that $\frac{\partial u}{\partial x_i} < 0$, $\frac{\partial v}{\partial x_i} < 0$ on $T_{i,l} \cap B_1$ for all $l \in (\lambda - \gamma, \lambda]$.*

Proof. For all $x \in T_{i,\lambda} \cap B_1$

$$\begin{aligned}\frac{\partial u}{\partial x_i}(x) &= \int_{B_1} \partial_{x_i} G_\alpha(x,y) g(v(y))\, dy \\ &= \int_{\Sigma_{i,\lambda}} [\partial_{x_i} G_\alpha(x,y) g(v(y)) + \partial_{x_i} G_\alpha(x, y^{i,\lambda}) \tilde{g}(v(y^{i,\lambda}))]\, dy.\end{aligned}$$

Note that $y^{i,\lambda}$ may be outside B_1, however $g(0) \neq 0$ in general, hence one has to consider $\tilde{g}$ in place of g in the last integral above. By Lemma 3.5 two cases may occur: $g(v(y)) > \tilde{g}(v(y^{i,\lambda}))$ for all $y \in \mathcal{O}^v_{i,\lambda}$ or $\tilde{g}(v(y^{i,\lambda})) > 0$ for all $y \in \mathcal{O}^v_{i,\lambda}$. In the first case,

$$\frac{\partial u}{\partial x_i}(x) < \int_{\Sigma_{i,\lambda}} (\partial_{x_i} G_\alpha(x,y) + \partial_{x_i} G_\alpha(x, y^{i,\lambda})) \tilde{g}(v(y^{i,\lambda}))\, dy \leq 0$$

for all $x \in T_{i,\lambda} \cap B_1$. In the second case,

$$\frac{\partial u}{\partial x_i}(x) \leq \int_{\Sigma_{i,\lambda}} (\partial_{x_i} G_\alpha(x,y) + \partial_{x_i} G_\alpha(x, y^{i,\lambda})) \tilde{g}(v(y^{i,\lambda}))\, dy < 0$$

for all $x \in T_{i,\lambda} \cap B_1$. In any case,

$$\frac{\partial u}{\partial x_i}(x) < 0 \quad \text{for all } x \in T_{i,\lambda} \cap B_1. \tag{3.15}$$

We now proceed exactly as in Lemma 8 in [12]. For any $y \in \mathbb{R}^N$ and any $a > 0$ let us consider the cube centered at y, namely

$$\mathcal{U}_a(y) = \{x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i - y_i| < a\}.$$

Then by Theorem 1.5, for any $x_0 \in T_{i,\lambda} \cap \partial B_1$ we have

$$(-1)^\alpha \left(\frac{\partial}{\partial x_i}\right)^{\alpha-1} \frac{\partial u}{\partial x_i}(x_0) = \left(-\frac{\partial}{\partial x_i}\right)^\alpha u(x_0) > 0.$$

From boundary conditions we also know that $\left(\frac{\partial}{\partial x_i}\right)^k u(x_0) = 0$ for all $k = 0, \dots, \alpha - 1$, and hence there exists $a > 0$ such that

$$\frac{\partial u}{\partial x_i}(x) < 0 \quad \text{for all } x \in \mathcal{U}_a(x_0) \cap B_1.$$

By compactness of $T_{i,\lambda} \cap \partial B_1$ there exists $\bar{a} > 0$ such that

$$\frac{\partial u}{\partial x_i}(x) < 0 \quad \text{for all } x \in A = \bigcup_{x_0 \in T_{i,\lambda} \cap \partial B_1} (\mathcal{U}_{\bar{a}}(x_0) \cap B_1).$$

Let us set $K = (T_{i,\lambda} \cap B_1) \setminus A$ and for $d > 0$ consider $K_d = K - de_i$, where e_i is the unit vector in the the direction x_i. In view of (3.15) and by compactness of K, there exists $\delta > 0$ such that

$$\frac{\partial u}{\partial x_i} < 0 \text{ on } K_d \text{ for all } d \in [0, \delta].$$

Let $\gamma_u = \min\{\bar{a}, \delta\}$. Then, $\frac{\partial u}{\partial x_i} < 0$ on $T_{i,l} \cap B_1$ for all $l \in (\lambda - \gamma_u, \lambda]$. Analogously, one gets γ_v such that $\frac{\partial v}{\partial x_i} < 0$ on $T_{i,l} \cap B_1$ for all $l \in (\lambda - \gamma_v, \lambda]$. The conclusion follows by taking $\gamma = \min\{\gamma_u, \gamma_v\}$. □

The following Lemma is the starting point of the moving planes procedure, see also Figure 3.1.

Lemma 3.7. *Let* $(u, v) \in C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ *be a nontrivial solution to* (3.13). *There exists* $\varepsilon > 0$ *such that for all* $\lambda \in [1 - \varepsilon, 1)$ *one has*

$$\begin{aligned} u(x) > u(x^{i,\lambda}) \text{ for } x \in \Sigma_{i,\lambda}, \quad \frac{\partial u}{\partial x_i} < 0 \text{ on } T_{i,\lambda} \cap B_1 \\ v(x) > v(x^{i,\lambda}) \text{ for } x \in \Sigma_{i,\lambda}, \quad \frac{\partial v}{\partial x_i} < 0 \text{ on } T_{i,\lambda} \cap B_1 \end{aligned} \tag{3.16}$$

Proof. Since $x^{i,1} \in B_1^c$ for any $x \in \Sigma_{i,1} = B_1$, by Lemma 3.6 there exists ε such that $\frac{\partial u}{\partial x_i} < 0$, $\frac{\partial v}{\partial x_i} < 0$ on $T_{i,l} \cap B_1$ for all $l \in (1 - 2\varepsilon, 1)$.

Hence, for all $\lambda \in [1 - \varepsilon, 1)$ one has

$$u(x) > u(x^{i,\lambda}), v(x) > v(x^{i,\lambda}) \text{ for } x \in \Sigma_{i,\lambda}.$$

Indeed, if $x_i \leq 1 - 2\varepsilon$ then $x^{i,\lambda} \in B_1^c$ and since $u > 0$ in B_1 and $= 0$ outside B_1, $u(x) > u(x^{i,\lambda})$. If $1 - 2\varepsilon < x_i < 1$, then two cases may occur:

- $x^{i,\lambda} \in B_1^c$ and the conclusion follows as above,
- $x^{i,\lambda} \in B_1$: in this case, it is enough to exploit the fact that $\frac{\partial u}{\partial x_i}(x) < 0$ for all $x \in T^{i,\lambda} \cap B_1$, with $\lambda \in (1 - 2\varepsilon, 1)$.

Similarly for v. Therefore, (3.16) holds for all $\lambda \in [1 - \varepsilon, 1)$. □

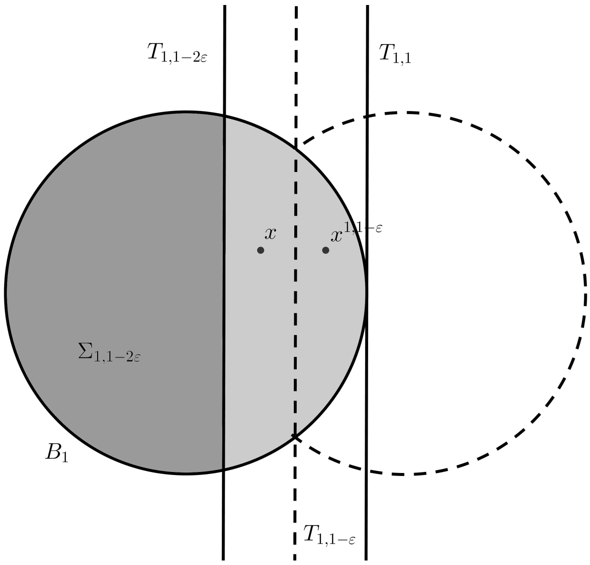


Figure 3.1: Proof of Lemma 3.7, $N = 2$, $i = 1$.

We are now ready to slide the hyperplane to the critical position $\lambda = 0$.

Proposition 3.4. *If* $(u, v) \in C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)$ *is a nontrivial solution to* (3.13) *we have*

$$\Lambda = \{\lambda \in (0,1) : u(x) > u(x^{i,\lambda}),\, v(x) > v(x^{i,\lambda}) \quad \forall\, x \in \Sigma_{i,\lambda} \\ \frac{\partial u}{\partial x_i} < 0, \frac{\partial v}{\partial x_i} < 0 \text{ on } T_{i,\lambda} \cap B_1\} = (0,1). \quad (3.17)$$

Proof. By Lemma 3.7 it turns out that $[1-\varepsilon, 1) \subset \Lambda$. Let $\bar{\lambda}$ be the smallest number such that $(\bar{\lambda}, 1) \subset \Lambda$. The proof will be complete once we show that $\bar{\lambda} = 0$. By continuity one has

$$u(x) \geq u(x^{i,\bar{\lambda}}),\, v(x) \geq v(x^{i,\bar{\lambda}}) \text{ for all } x \in \Sigma_{i,\bar{\lambda}}.$$

By contradiction assume $\bar{\lambda} > 0$. Take $x \in \Sigma_{i,\bar{\lambda}}$. Then

$$\begin{aligned} u(x) - u(x^{i,\bar{\lambda}}) &= \int_{B_1} (G_\alpha(x,y) - G_\alpha(x^{i,\bar{\lambda}}, y)) g(v(y))\, dy \\ &= \int_{\Sigma_{i,\bar{\lambda}}} (G_\alpha(x,y) - G_\alpha(x^{i,\bar{\lambda}}, y)) g(v(y))\, dy \\ &\quad + \int_{\Sigma_{i,\bar{\lambda}}} (G_\alpha(x, y^{i,\bar{\lambda}}) - G_\alpha(x^{i,\bar{\lambda}}, y^{i,\bar{\lambda}})) \tilde{g}(v(y^{i,\bar{\lambda}}))\, dy. \end{aligned}$$

By Lemma 3.5, two cases may occur: $g(v(y)) > \tilde{g}(v(y^{i,\bar{\lambda}}))$ for all $y \in \mathcal{O}^v_{i,\bar{\lambda}}$ or $\tilde{g}(v(y^{i,\bar{\lambda}})) > 0$ for all $y \in \mathcal{O}^v_{i,\bar{\lambda}}$.

In the first case, by Lemma 3.4

$$u(x) - u(x^{i,\bar\lambda}) > \int_{\Sigma_{i,\bar\lambda}} [G_\alpha(x,y) - G_\alpha(x^{i,\bar\lambda}, y) + \\ G_\alpha(x, y^{i,\bar\lambda}) - G_\alpha(x^{i,\bar\lambda}, y^{i,\bar\lambda})]\tilde g(v(y^{i,\bar\lambda}))\, dy \geq 0$$

whereas in the second case

$$u(x) - u(x^{i,\bar\lambda}) \geq \int_{\Sigma_{i,\bar\lambda}} [G_\alpha(x,y) - G_\alpha(x^{i,\bar\lambda}, y) \\ + G_\alpha(x, y^{i,\bar\lambda}) - G_\alpha(x^{i,\bar\lambda}, y^{i,\bar\lambda})]\tilde g(v(y^{i,\bar\lambda}))\, dy > 0.$$

Similar considerations hold for v. Hence

$$u(x) > u(x^{i,\bar\lambda}),\ v(x) > v(x^{i,\bar\lambda}) \text{ for all } x \in \Sigma_{i,\bar\lambda}. \tag{3.18}$$

Due to Lemma 3.6 there exists γ_1 such that

$$\frac{\partial u}{\partial x_i} < 0, \frac{\partial v}{\partial x_i} < 0 \text{ on } T_{i,l} \cap B_1 \text{ for all } l \in (\bar\lambda - 2\gamma_1, \bar\lambda]. \tag{3.19}$$

Now, by continuity, for any $x \in \bar B_1$ such that $x_i \leq \bar\lambda - \gamma_1$ there exists $\gamma(x) > 0$ such that

$$u(x) \geq u(x^{i,l}),\ v(x) \geq v(x^{i,l}),\ l \in (\bar\lambda - \gamma(x), \bar\lambda]$$

and by compactness of $C = \{x \in \bar B_1 : x_i \leq \bar\lambda - \gamma_1\}$ one can take

$$\gamma = \min\{\inf_C \gamma(x), \gamma_1\} = \min\{\min_C \gamma(x), \gamma_1\} > 0,$$

hence for all $x \in \Sigma_{i,\bar\lambda-\gamma_1}$,

$$u(x) \geq u(x^{i,l}),\ v(x) \geq v(x^{i,l}),\ l \in (\bar\lambda - \gamma, \bar\lambda]$$

and exploiting the same argument as for the proof of (3.18), if $x \in \Sigma_{i,\bar\lambda-\gamma_1}$

$$u(x) > u(x^{i,l}),\ v(x) > v(x^{i,l}),\ \text{for all } l \in (\bar\lambda - \gamma, \bar\lambda]. \tag{3.20}$$

The conclusion follows in view of (3.18), (3.19) and (3.20). □

Proof of Proposition 3.3. Follows by Proposition 3.4: if (u, v) is a solution to (3.13), then (3.17) holds true and since (3.13) is invariant by rotation and the domain is radially symmetric, this implies that u, v are radially symmetric and $u'(r), v'(r) < 0$. □

3.1.4 Proof of Theorem 3.1 and Corollary 3.2

The proof of Theorem 3.1 is a simple combination of the previous steps. Let us assume that

$$\mathcal{C} \cap (\{0\} \times C_0^{2\alpha}(\bar{B}_1) \times C_0^{2\beta}(\bar{B}_1)) = \{(0,0,0)\}$$

where $\mathcal{C}$ is defined as in Proposition 3.1. Hence by Proposition 3.1 we can find an unbounded sequence (t_n, u_n, v_n) of solutions to (3.3). Note that $t_n > 0$ and as a consequence $u_n > 0$ and $v_n > 0$ by Theorem 1.4. However, for any fixed n, in view of Proposition 3.3 with $f(u) = (t_n + |u|)^p$ and $g(v) = (t_n^\vartheta + |v|)^q$, we have that u_n, v_n are radially symmetric and the global maxima are attained at 0. Therefore, by Proposition 3.2 one concludes that there exists a nontrivial radial nonnegative solution to (3.8).

In order to prove Corollary 3.2, we first recall the following result.

Proposition 3.5 (Proposition 2 in [96])**.** *Let $u \in C^{2\alpha}$ radially symmetric and bounded such that*

$$(-\Delta)^\alpha u \geq 0 \text{ in } \mathbb{R}^N.$$

Then

$$(-\Delta)^s u \geq 0,\ ((-\Delta)^s u)' \leq 0, \tag{3.21}$$

for any $s \leq \alpha - 1$.

Notice that the blow up solution we built in Theorem 3.1 satisfies the assumptions of Proposition 3.5. Condition (3.21) is known as *polysuperharmonicity condition* [20]. We are now ready to state our non-existence result.

Theorem 3.2. *Let $pq > 1$, $\alpha, \beta \geq 1$, and assume that*

$$2\beta q + N + 2\alpha pq - Npq \leq 0, \quad 2\alpha p + N + 2\beta pq - Npq \leq 0.$$

Assume further that (u, v) is a classical radial nonnegative solution to (3.2) and that (3.21) holds for both u and v. Then $u = v = 0$.

Corollary 3.2 (i) now follows by combining Theorem 3.2 and Theorem 3.1. Moreover, notice that if (p, q) is above both the Serrin curves, and (ii) in Corollary 3.2 is satisfied, then both p and q are > 1, and we can apply Theorem 1.9. Indeed,

$$2\beta q + N < -2\alpha pq + Npq < (N + 2\beta)q$$

implies $q > 1$, and similarly for p. In the sequel, we prove Theorem 3.2, by borrowing a few ideas from [29], where an (s, t)-Laplacian system is taken into account. Let us preliminarily recall the following lemma.

Lemma 3.8 (Lemma II.3 in [29]). *Let $r_0 \geq 0$. Let $u \in C^1([r_0,\infty)) \cap C^2((r_0,\infty))$ be a nonnegative function satisfying*

$$-(r^{N-1}u'(r))' \geq 0 \quad \text{on } (r_0,\infty).$$

Then for any $r > r_0$ we have

$$u(r) \geq Cr\left|u'(r)\right|.$$

Proof of Theorem 3.2. Let us call $u_s = (-\Delta)^s u$, where $0 < s \leq \alpha - 1$, $u_0 = u$, and similarly for v. By Lemma 3.8 and (3.21), it follows

$$u_s(r) \geq Cr\left|u_s'(r)\right|, \; v_s(r) \geq Cr\left|v_s'(r)\right|. \tag{3.22}$$

Since

$$-r^{N-1}u_s'(r) = \int_0^r t^{N-1}u_{s+1} \geq u_{s+1}(r)\int_0^r t^{N-1} = \frac{1}{N}u_{s+1}(r)r^N$$

then

$$u_s(r) \geq cr^2 u_{s+1}, \; v_s(r) \geq cr^2 v_{s+1} \tag{3.23}$$

and

$$u \geq cv^q r^{2\alpha}, \; v \geq cu^p r^{2\beta}. \tag{3.24}$$

Let us call

$$x(r) = r^{N-1}\left|u_{\alpha-1}'(r)\right|, \quad y(r) = r^{N-1}\left|v_{\beta-1}'(r)\right|$$

and note that

$$x'(r) = r^{N-1}v^q(r), \quad y'(r) = r^{N-1}u^p(r).$$

Moreover, by (3.23) and (3.22)

$$u \geq cr^2u_1 \geq cr^4u_2 \geq \cdots \geq cr^{2(\alpha-1)}u_{\alpha-1} \geq cr^{2(\alpha-1)+1}\left|u_{\alpha-1}'(r)\right|.$$

Hence by (3.24)

$$x'(r) \geq cr^{N-1}u^{pq}r^{2\beta q} \geq cr^{N-1}r^{(2\alpha-1)pq}\left|u_{\alpha-1}'(r)\right|^{pq}r^{2\beta q} = cx^{pq}(r)r^{\eta}$$

where

$$\eta = N + 2\alpha pq + 2\beta q - Npq - 1.$$

Assume $\eta + 1 \geq 0$. If $\eta > -1$, then

$$\begin{aligned}\frac{1}{pq-1}x^{1-pq}(s) \geq \frac{1}{1-pq}x^{1-pq}(r) + \frac{1}{pq-1}x^{1-pq}(s) &= \int_s^r \frac{x'(t)}{x^{pq}(t)}\,dt \\ &\geq c\int_s^r t^{\eta} = c\frac{1}{\eta+1}r^{\eta+1} - c\frac{1}{\eta+1}s^{\eta+1}\end{aligned}$$

which goes to infinity as $r \to \infty$, therefore giving a contradiction. If $\eta = -1$, then

$$\frac{1}{pq-1}x^{1-pq}(s) \geq c\log(r/s)$$

which is again a contradiction. By considering y one gets the other curve. □

3.2 Extension to systems of m equations

The proofs presented in the previous section can be extended to systems of $m > 2$ equations. In the sequel, we just sketch the proof, by pointing out the main differences from the case (3.1) when necessary, see also [23].

Theorem 3.3. *Let* $\prod_{j=1}^{m} p_j > 1$, $\alpha_j \geq 1$, $j = 1, \dots, m$, *and* $N > 2\max\{\alpha_j\}_j$, *and assume that there exists* $l \in \{1, \dots, m\}$ *such that*

$$N + 2\sum_{k=1}^{m} \alpha_{k+l} \prod_{j=0}^{k-1} p_{j+l} - N \prod_{j=1}^{m} p_j \geq 0, \tag{3.25}$$

where we impose $p_{k+m} = p_k$ *and* $\alpha_{k+m} = \alpha_k$ *for any* $k = 1, \dots, m$. *Then, there exists a nontrivial solution to the following*

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j}, \; j = 1, \dots, m-1 & \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m} & \text{in } B_1 \subset \mathbb{R}^N, \\ \frac{\partial^k u_j}{\partial \nu^k} = 0, \; k = 0, \dots, \alpha_j - 1, \; j = 1, \dots, m & \text{on } \partial B_1. \end{cases}$$

Notice that in the case $m = 2$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $p_1 = q$, $p_2 = p$, (3.25) reduce to the Serrin curves

$$2\beta q + N + 2\alpha pq - Npq \geq 0, \; 2\alpha p + N + 2\beta pq - Npq \geq 0.$$

As we did for Theorem 3.1, we divide the proof into steps.

Step 1. One proves that, if $\prod_{j=1}^{m} p_j > 1$, and if the only classical solution to (3.1) is the trivial one, then there exists an unbounded sequence of solutions $(t_n, u_{1,n}, \dots, u_{m,n})$ to the following

$$\begin{cases} (-\Delta)^{\alpha_j} u_{j,n} = (t_n^{\theta_j} + |u_{j+1,n}|)^{p_j}, \; j = 1, \dots, m-1 & \\ (-\Delta)^{\alpha_m} u_{m,n} = (t_n^{\theta_m} + |u_{1,n}|)^{p_m} & \text{in } B_1 \subset \mathbb{R}^N, \\ \frac{\partial^k u_{j,n}}{\partial \nu^k} = 0, \; k = 0, \dots, \alpha_j - 1, \; j = 1, \dots, m & \text{on } \partial B_1, \end{cases}$$

where θ_j are chosen such that

$$\theta_j p_j > \theta_{j-1}, \; \forall j. \tag{3.26}$$

For instance, one can call

$$a_j = 1 + j\left(\prod_{k=1}^{m} p_k - 1\right)$$

and choose

$$\theta_j = \frac{a_{j-1}}{\prod_{k=2}^{j} p_k}$$

for $j = 2, \dots, m$, and $\theta_1 = 1$. We now apply Lemma 3.2, taking

$$G(t, u_1, \dots, u_m) = (K_{\alpha_1}(t^{\theta_1} + |u_2|)^{p_1}, \dots, K_{\alpha_m}(t^{\theta_m} + |u_1|)^{p_m}).$$

Step 2. One performs a blow up analysis. We can assume without loss of generality that

$$\frac{t_n^{\theta_{j-1}}}{\|u_{j,n}\|_\infty} \to 0,\ j = 1, \dots, m, \quad n \to \infty \tag{3.27}$$

as follows by choosing

$$\tilde{u}_{j,n} = \frac{u_{j,n}}{t_n^{\theta_{j-1}}},\ \lambda_{j,n} = t_n^{\theta_j p_j - \theta_{j-1}}$$

and applying the comparison principle. Here we exploit (3.26) to have $\lambda_{j,n} \to \infty$ as $n \to \infty$. Moreover, assume that the maximum of $u_{k,n}$ is attained in 0 for any k. We define

$$\hat{u}_{j,n}(y) = \frac{u_{j,n}(C_n^{-1} y)}{A_{j,n}},$$

where

$$A_{j,n} = C_n^{\sigma_j},$$
$$C_n = \sum_j \|u_{j,n}\|_\infty^{1/\sigma_j}$$

and moreover

$$\sigma_1 = \frac{2\sum_{k=1}^m \alpha_k \prod_{j=1}^{k-1} p_j}{\prod_{j=1}^m p_j - 1},\ \sigma_j = -2\alpha_j + p_j \sigma_{j+1}.$$

This by a limit procedure as in Subsection 3.1.2 and exploiting (3.27) gives a nontrivial solution to

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j},\ j = 1, \dots, m-1 & \text{on } \mathbb{R}^N \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m} & \text{on } \mathbb{R}^N \end{cases} \tag{3.28}$$

due to our choice of the parameters $A_{j,n}$ and σ_j. This limit solution is nontrivial since

$$\sum_{i \neq k} \|u_{i,n}\|_\infty^{1/\sigma_i} \le \|u_{k,n}\|_\infty^{1/\sigma_k} (m-1)$$

for at least one value k. Indeed, if not, then upon summation

$$(m-1) \sum_i \|u_{i,n}\|_\infty^{1/\sigma_i} > (m-1) \sum_i \|u_{i,n}\|_\infty^{1/\sigma_i},$$

a contradiction. Assume for instance that $k = 1$ and call

$$b_n = \frac{\sum_{i \neq 1} \|u_{i,n}\|_\infty^{1/\sigma_i}}{\|u_{1,n}\|_\infty^{1/\sigma_1}} \le m - 1.$$

Then,

$$(\hat{u}_{1,n})^{1/\sigma_1}(0) = \frac{\|u_{1,n}\|_\infty^{1/\sigma_1}}{\sum_i \|u_{i,n}\|^{1/\sigma_i}} = \frac{1}{1+b_n} \geq \frac{1}{m},$$

and in particular the limit is nontrivial.

Step 3. We prove that the maximum of $u_{k,n}$ is attained in 0 for any k, as the following Lemma shows.

Lemma 3.9. *Let $(u_1, \dots, u_m)$ be a nontrivial solution to*

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = f_j(u_{j+1}),\ j = 1, \dots, m-1 & \text{in } B_1 \subset \mathbb{R}^N, \\ (-\Delta)^{\alpha_m} u_m = f_m(u_1) & \\ \frac{\partial^k u_j}{\partial \nu^k} = 0,\ k = 0, \dots, \alpha_j - 1,\ j = 1, \dots, m & \text{on } \partial B_1, \end{cases}$$

where $N > 2\max\{\alpha_j\}_j$ and $f_j : [0, \infty) \to \mathbb{R}$ are continuous, positive and non decreasing. Then $u_1, \dots, u_m$ are radially symmetric and strictly decreasing in the radial variable.

The proof is completely analogous to that of Proposition 3.3.

Step 4. The combination of Steps 1–3 extends Theorem 3.1 to the case of systems of $m > 2$ equations. To conclude the proof of Theorem 3.3, we notice that one can prove in the same way as Theorem 3.2 the following result.

Theorem 3.4. *Let $\prod_{j=1}^m p_j > 1$, $\alpha_j \geq 1$, $j = 1, \dots, m$, and assume that there exists $l \in \{1, \dots, m\}$ such that*

$$N + 2\sum_{k=1}^m \alpha_{k+l} \prod_{j=0}^{k-1} p_{j+l} - N \prod_{j=1}^m p_j \geq 0,$$

where we impose $p_{k+m} = p_k$ and $\alpha_{k+m} = \alpha_k$ for any $k = 1, \dots, m$. Assume further that $(u_1, \dots, u_m)$ is a classical radial nonnegative solution to (3.28) and that (3.21) holds for u_j, $j = 1, \dots, m$. Then $u_j = 0$ for any $j = 1, \dots, m$.

3.3 The Navier problem

We point out that all the considerations above hold for the Navier case as well.

Theorem 3.5. *Let $\prod_{j=1}^m p_j > 1$, $\alpha_j \geq 1$, $j = 1, \dots, m$, and $N > 2\max\{\alpha_j\}_j$, and assume that there exists $l \in \{1, \dots, m\}$ such that*

$$N + 2\sum_{k=1}^m \alpha_{k+l} \prod_{j=0}^{k-1} p_{j+l} - N \prod_{j=1}^m p_j \geq 0,$$

where we impose $p_{k+m} = p_k$ *and* $\alpha_{k+m} = \alpha_k$ *for any* $k = 1, \dots, m$. *Then, there exists a nontrivial solution to the following*

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j}\,,\ j = 1, \dots, m-1 & \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m} & \text{in } B_1 \subset \mathbb{R}^N, \\ \Delta^k u_j = 0,\ k = 0, \dots, \alpha_j - 1,\ j = 1, \dots, m & \text{on } \partial B_1. \end{cases} \tag{3.29}$$

The proof of the first two steps (reduction to an auxiliary problem and blow up procedure) is completely analogous to that of Theorem 3.3, recalling that the polyharmonic operator with Navier boundary conditions satisfies a maximum principle on a general bounded domain. Furthermore, notice that Lemma 3.1 holds also if K_α is defined as the solution operator of

$$\begin{cases} (-\Delta)^{\alpha} u = f & \text{in } \Omega \\ \Delta^r u = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial\Omega. \end{cases}$$

As for Step 3, in the Navier case the symmetry is a consequence of the classical moving planes procedure, see [102, Theorem 1]. Combining Steps 1–3 we get the analog of Theorem 3.1 for the Navier problem. Now, existence for (3.29) follows by Theorem 3.4.

Chapter 4

Uniqueness results for higher order equations and systems

In this chapter we prove some uniqueness results for polyharmonic equations and systems up to order eight of the form

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j}, \, j = 1, \dots, m-1 & \text{in } B_1, \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m} & \text{in } B_1, \\ \frac{\partial^k u_j}{\partial \nu^k} = 0, \, k = 0, \dots, \alpha_j - 1, \, j = 1, \dots, m, & \text{on } \partial B_1, \end{cases} \tag{4.1}$$

with $p_j \geq 1$ for any j, $\prod_{j=1}^m p_j > 1$, $N > 2\max\{\alpha_j\}_j$ and $1 \leq \alpha_j \leq 4$ for any $j = 1, \dots, m$, where $m \geq 1$. The proof, see [89, 23], is based on the seminal work by Gidas–Ni–Nirenberg [49] where the Lane–Emden equation is considered. The same approach has been exploited and adapted by Dalmasso to the case of the biharmonic Lane–Emden equation [35], see also [41], and to the Lane–Emden system of order 2 [33]. We further recall that the case $m = 3$ with Navier boundary conditions was treated in [32]. We consider the associated initial value problem for the ODE, and we proceed by contradiction. Taking two different solutions u and v, suitably rescaling one of the two, and considering the difference w, we apply the maximum principle on iterated Laplacians of w in order to get information on the behavior of w at the boundary, to end up with a contradiction. Apparently, this approach works up to $\alpha_j \leq 4$, as technical difficulties arise when considering higher order operators, due to the fact that Dirichlet boundary conditions prescribe the behavior only of the first $\alpha_j - 1$ derivatives of the solution, and no information seems to be retained for higher order derivatives. We point out that these difficulties can be overcome by taking different boundary conditions, precisely Navier conditions, or imposing vanishing higher order momenta along the boundary. In what follows, we first state our results (Section 4.1) and give some preliminary lemmas (Section 4.2), then we prove uniqueness for polyharmonic Lane–Emden equations, see Section 4.3 below, whereas we consider systems of m equations in Section 4.4.

4.1 Main results

Theorem 4.1. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^\alpha u = |u|^p \,, & \text{in } B_1 \subset \mathbb{R}^N, N > 2\alpha \\ \frac{\partial^k u}{\partial \nu^k} = 0, & \text{on } \partial B_1 \,,\, k \le \alpha - 1 \end{cases} \tag{4.2}$$

with $p > 1$ *and* $1 \le \alpha \le 4$.

Theorem 4.2. *There exists at most one nontrivial solution to* (4.1) *with* $p_j \ge 1$ *for any* j, $\prod_{j=1}^m p_j > 1$, $N > 2\max\{\alpha_j\}_j$ *and* $1 \le \alpha_j \le 4$ *for any* $j = 1, \dots, m$, *where* $m \ge 1$.

As a byproduct of Theorem 4.2 we can prove uniqueness for equations of arbitrary order endowed with some new boundary conditions, which impose vanishing higher order momenta along the boundary, or Navier boundary conditions.

Corollary 4.1. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^\alpha u = |u|^p \,, & \text{in } B_1, \\ \Delta^{2k} u = 0, \; 2k \le \alpha - 1, & \text{on } \partial B_1 \\ \frac{\partial}{\partial \nu} \Delta^{2k} u = 0, \; 2k+1 \le \alpha - 1, & \text{on } \partial B_1 \end{cases} \tag{4.3}$$

with $N > 2\alpha$, $p > 1$ *and* $\alpha \in \mathbb{N}$, $\alpha \ge 1$.

Proof of Corollary 4.1. Notice that (4.3) can be written as a system of $\sum \lceil \alpha_j/2 \rceil$ equations with Dirichlet boundary conditions. Let for instance α be even, and set $u_k = \Delta^{2k} u$. Then (4.3) reads as

$$\begin{cases} \Delta^2 u_j = |u_{j+1}| \,, \; j = 1, \dots, \alpha/2 - 1, & \multirow{2}{*}{\text{in } B_1} \\ \Delta^2 u_{\alpha/2} = |u_1|^p \,, & \\ u_j = \frac{\partial u_j}{\partial \nu} = 0, \; j = 1, \dots, \alpha/2 & \text{on } \partial B_1, \end{cases}$$

which is a particular case of (4.1). □

Corollary 4.2. *There exists at most one nontrivial solution to*

$$\begin{cases} (-\Delta)^{\alpha_j} u_j = |u_{j+1}|^{p_j} \,, \; j = 1, \dots, m-1, & \text{in } B_1, \\ (-\Delta)^{\alpha_m} u_m = |u_1|^{p_m} \,, & \\ \Delta^k u_j = 0, \; k = 0, \dots, \alpha_j - 1, \; j = 1, \dots, m & \text{on } \partial B_1 \end{cases}$$

with $p_j \ge 1$ *for any* j, $\prod_{j=1}^m p_j > 1$, $\alpha_j \in \mathbb{N}$, $m \ge 1$ *and* $N > 2\max\{\alpha_j\}_j$.

Proof of Corollary 4.2. One reduces the problem to system (4.1) and thus Corollary 4.2 follows from Theorem 4.2. Let for instance $m = 1$. Then

$$\begin{cases} (-\Delta)^\alpha u = |u|^p\,, & \text{in } B_1 \\ \Delta^k u = 0,\ k \le \alpha - 1, & \text{on } \partial B_1 \end{cases}$$

becomes

$$\begin{cases} -\Delta u_j = |u_{j+1}|\,,\ j = 1, \ldots, \alpha - 1, & \multirow{2}{*}{in B_1} \\ -\Delta u_\alpha = |u_1|^p\,, & \\ u_j = 0,\ j = 1, \ldots, \alpha & \text{on } \partial B_1, \end{cases}$$

where $u_j = \Delta^j u$. □

4.2 Preliminaries

Lemma 4.1 (Theorem 7.1 in [47])**.** *Let u be a nontrivial solution to* (4.2)*. Then it is radially symmetric and strictly decreasing in the radial variable.*

We next prove a key ingredient for what follows:

Lemma 4.2. *Let u be a nontrivial solution to* (4.2)*. Then, $\Delta^s u(0) < 0$ if $1 \le s < \alpha$ is odd, and in this case $\Delta^s u$ is increasing until the first zero, $\Delta^s u(0) > 0$ if $1 \le s < \alpha$ is even, and in this case $\Delta^s u$ is decreasing up to the first zero. Moreover, if $\alpha \ge 2$ is even, then the following properties hold:*

- *$\Delta^{\alpha-j} u$ has exactly $\alpha - j + 1$ zeros (including the last one in $r = 1$) and $\alpha - j$ critical points in $(0, 1)$ if $\alpha - 1 \ge j \ge \alpha/2 + 1$, exactly j zeros and $j - 1$ critical points in $(0, 1)$ if $1 \le j \le \alpha/2$;*
- *$\Delta^s u(1) = 0$ if $s \le \alpha/2 - 1$, $\Delta^s u(1) > 0$ if $s \ge \alpha/2$, and $(\Delta^s u)'(1) = 0$ if $s \le \alpha/2 - 1$, $(\Delta^s u)'(1) \ge 0$ if $s \ge \alpha/2$.*

If $\alpha \ge 3$ is odd, then we have:

- *$\Delta^{\alpha-j} u$ has exactly $\alpha - j + 1$ zeros (including the last one in $r = 1$) and $\alpha - j$ critical points in $(0, 1)$ if $\alpha - 1 \ge j \ge (\alpha + 1)/2$, exactly j zeros and $j - 1$ critical points in $(0, 1)$ if $1 \le j \le (\alpha - 1)/2$;*
- *$\Delta^s u(1) = 0$ if $s \le (\alpha - 1)/2$, $\Delta^s u(1) < 0$ if $s \ge (\alpha + 1)/2$, and $(\Delta^s u)'(1) = 0$ if $s \le (\alpha - 3)/2$, $(\Delta^s u)'(1) \le 0$ if $s \ge (\alpha - 1)/2$.*

(See Figure 4.1).

Proof. We prove only the case in which α is even, the odd case being similar. Recall that

$$r^{N-1} (\Delta^j u)'(r) = \int_0^r s^{N-1} (\Delta^{j+1} u)(s)\, ds \tag{4.4}$$

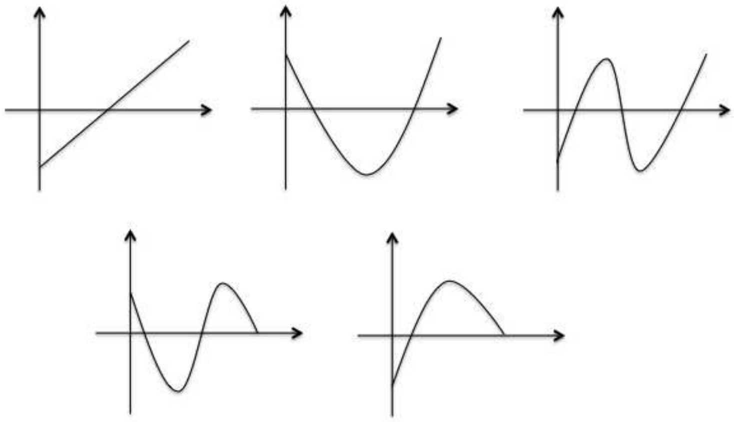


Figure 4.1: *Qualitative graphs of* $\Delta^s u(r)$ *on the interval* $[0, 1]$*, where* $s = 5, 4, 3, 2, 1$ *respectively, and* u *satisfies* (4.2) *with* $\alpha = 6$.

for any integer $j \geq 1$. By (4.4), $(\Delta^{\alpha-1}u)' > 0$ and as a consequence $\Delta^{\alpha-1}u$ has at most one zero. If $\alpha = 2$, then in view of Theorem 1.5 $\Delta u(1) > 0$, hence Δu has exactly one zero, and the proof is complete. If $\alpha \geq 4$, then we conclude that $\Delta^{\alpha-2}u$ has at most two zeros. Indeed, again by (4.4), it is decreasing up to the endpoint $r_* \geq r_0$, where r_0 is such that $\Delta^{\alpha-1}u(r_0) = 0$. Notice that if $r_* < 1$, then $(\Delta^{\alpha-2}u)'(r_*) = 0$. Therefore, it holds

$$r^{N-1}(\Delta^{\alpha-2}u)'(r) = \int_{r_*}^{r} s^{N-1}(\Delta^{\alpha-1}u)(s)\, ds,$$

and since $\Delta^{\alpha-1}u > 0$ beyond $r_* \geq r_0$, then $(\Delta^{\alpha-2}u)'(r) > 0$ for any $r \geq r_*$.

Analogously, one concludes that $\Delta^{\alpha-j}u$ has at most j zeros and $j-1$ critical points in $(0, 1)$, $j \leq \alpha - 1$. In particular, $\Delta^{\alpha/2-1}u$ has at most $\alpha/2 + 1$ zeros and $\alpha/2$ critical points in $(0, 1)$. Moreover, by Dirichlet boundary conditions, $\Delta^{\alpha/2-1}u(1) = 0$, $(\Delta^{\alpha/2-1}u)'(1) = 0$ and $(\Delta^{\alpha/2-1}u)''(1) = u^{(\alpha)}(1) = \Delta^{\alpha/2}u(1) > 0$ by Theorem 1.5. Then, $\Delta^{\alpha/2-1}u$ should be decreasing and positive near 1.

Now, assume that $\Delta^{\alpha/2-1}u$ has exactly $\alpha/2+1$ zeros and $\alpha/2$ critical points in $(0, 1)$. Then $\Delta^{\alpha/2}u$ must have exactly $\alpha/2$ zeros, and by iteration $\Delta^{\alpha-j}u$ has exactly j zeros, with $j \leq \alpha/2 + 1$. In particular, this means that $\Delta^{\alpha/2-1}u$ is positive near 0 and has a even number of zeros, if $\alpha/2 - 1$ is even; or it is negative near 0 and has a odd number of zeros, if $\alpha/2 - 1$ is odd. In any case, $\Delta^{\alpha/2-1}u$ should be increasing near 1, a contradiction. Hence $\Delta^{\alpha/2-1}u$ must have one zero less, namely at most $\alpha/2$ zeros (including also the last one in $r = 1$) and at most $\alpha/2 - 1$ critical points in $(0, 1)$.

Now, let us consider $\Delta^{\alpha/2-2}u$. Since $\Delta^{\alpha/2-1}u$ has at most $\alpha/2$ zeros, of which the last one is in $r = 1$, then it changes sing at most $\alpha/2$ times, and therefore $\Delta^{\alpha/2-2}u$ has at most $\alpha/2 - 1$ critical points, and $\alpha/2$ zeros in $(0, 1)$.

Notice that $\Delta^{\alpha/2-2}u(1) = 0$. Moreover, $(\Delta^{\alpha/2-2}u)^{(j)}(1) = 0$ for any $j \leq 3$ and $(\Delta^{\alpha/2-2}u)^{(4)}(1) = \Delta^{\alpha/2}u(1) > 0$. This means that $\Delta^{\alpha/2-2}u$ is decreasing and positive near 1. However, as above, this is possible only if $\Delta^{\alpha/2-2}u$ has at most $\alpha/2-1$ zeros (including also the last one in $r = 1$) and at most $\alpha/2-2$ critical points.

Next we iterate the procedure. Then, at each step we lose one critical point. Thus, $\Delta^{\alpha-j}u$ has at most $\alpha - j + 1$ zeros (including the last one in $r = 1$) and $\alpha - j$ critical points in $(0, 1)$ if $j \geq \alpha/2 + 1$, at most j zeros and $j - 1$ critical points in $(0, 1)$ if $j \leq \alpha/2$. In particular, Δu has at most 1 critical point. We know that $\Delta u(0) = u''(0) < 0$, as $u'(0) = 0$ and $u' < 0$ in $(0, 1)$. We have two cases: Δu is increasing and negative, reaches a positive maximum and decreases to 0, or it is always negative and has no critical points. However, we know that $\Delta u(1) = 0$ and Δu is decreasing in the last interval, as $(\Delta u)^{(j)}(1) = 0$ for any $j \leq \alpha - 3$ and $(\Delta u)^{(\alpha-2)}(1) = u^{(\alpha)}(1) = \Delta^{\alpha/2}u(1) > 0$ by Theorem 1.5. Then necessarily Δu is increasing and negative, reaches a positive maximum and decreases to 0, namely has exactly one critical point.

As a consequence, $\Delta^2 u$ has at least 2 critical points, however since it has at most 2 critical points due to what proved above, it turns out to have exactly 2 critical points. Moreover, $\Delta^2 u(0) > 0$, and it is decreasing until the first zero.

Iteratively, we conclude that $\Delta^{\alpha-j}u$ has exactly $\alpha - j + 1$ zeros (including the last one in $r = 1$) and $\alpha - j$ critical points in $(0, 1)$ if $j \geq \alpha/2+1$, exactly j zeros and $j-1$ critical points in $(0, 1)$ if $j \leq \alpha/2$. Moreover, $\Delta^s u(0) < 0$ if s is odd, and in this case it is increasing until the first zero, > 0 if s is even, and in this case it is decreasing before the first zero. Further, by boundary conditions, $\Delta^s u(1) = 0$ if $s \leq \alpha/2 - 1$, $\Delta^s u(1) > 0$ if $s \geq \alpha/2$, and $(\Delta^s u)'(1) = 0$ if $s \leq \alpha/2 - 1$, $(\Delta^s u)'(1) \geq 0$ if $s \geq \alpha/2$. □

4.3 Uniqueness for polyharmonic equations

4.3.1 Proof of Theorem 4.1 in the case $\alpha = 3$

Let u be a nontrivial solution to

$$\begin{cases} -\Delta^3 u = |u|^p, & \text{in } B_1 \\ u = \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial B_1. \end{cases} \tag{4.5}$$

By Lemma 4.1, u is positive, radially symmetric and strictly decreasing. In particular, since the maximum is attained at 0, we have $u'(0) = 0$. Moreover,

$$r^{N-1}(\Delta^2 u)'(r) = \int_0^r s^{N-1}(\Delta^3 u)(s)\, ds.$$

As a consequence,

$$(\Delta^2 u)'(0) = \lim_{r\to 0} \frac{\int_0^r s^{N-1}(\Delta^3 u)(s)\, ds}{r^{N-1}} = 0. \tag{4.6}$$

Moreover,

$$r^{N-1}(\Delta u)'(r) = \int_0^r s^{N-1}(\Delta^2 u)(s)\, ds$$

and therefore

$$(\Delta u)'(0) = \lim_{r\to 0} \frac{\int_0^r s^{N-1}(\Delta^2 u)(s)\, ds}{r^{N-1}} = 0. \tag{4.7}$$

Let w be another nontrivial solution to (4.5) and set

$$\tilde{w}(r) = \lambda^s w(\lambda r),$$

where s is chosen such that $\tilde{w}$ satisfies

$$\begin{cases} -\Delta^3 \tilde{w} = |\tilde{w}|^p\,,\ r \le 1/\lambda \\ \tilde{w}(1/\lambda) = \tilde{w}'(1/\lambda) = \tilde{w}''(1/\lambda) = 0 \end{cases}$$

namely $s = \frac{6}{p-1}$, whereas $\lambda > 0$ is such that

$$\tilde{w}(0) = u(0). \tag{4.8}$$

Claim:

$$\Delta\tilde{w}(0) = \Delta u(0),\ \Delta^2\tilde{w}(0) = \Delta^2 u(0). \tag{4.9}$$

Let us suppose for instance $\Delta^2(u-\tilde{w})(0) > 0$ and $\Delta(u-\tilde{w})(0) > 0$. Notice that by continuity $\Delta^2(u-\tilde{w}) > 0$ on $[0,\delta)$ and $\Delta(u-\tilde{w}) > 0$ on $[0,\varepsilon)$ for some δ, ε sufficiently small. Moreover $u - \tilde{w} > 0$ on $(0,\varepsilon]$: indeed, if there exists $a \le \varepsilon$ such that $u(a) - \tilde{w}(a) \le 0$, then $\Delta(u-\tilde{w}) > 0$ implies $u - \tilde{w} < 0$ on $[0,a)$, which is a contradiction.

Hence we can choose R_1 such that

$$R_1 = \sup\{r \le \min\{1, 1/\lambda\} :\ (u-\tilde{w})(s) > 0,\ \Delta(u-\tilde{w})(s) > 0,\\ \Delta^2(u-\tilde{w})(s) > 0,\ s \in (0,r)\}.$$

We have

$$(u-\tilde{w})(R_1) > 0,\ \Delta(u-\tilde{w})(R_1) > 0. \tag{4.10}$$

Indeed, let us assume by contradiction that $(u-\tilde{w})(R_1) = 0$. Then, since $\Delta(u-\tilde{w}) > 0$ on $[0,R_1)$ we would have by the maximum principle $u - \tilde{w} < 0$ on $[0,R_1)$. Analogously, if $\Delta(u-\tilde{w})(R_1) = 0$, then $\Delta(u-\tilde{w}) < 0$ on $(0,R_1)$, a contradiction. As a consequence, (4.10) holds. Moreover, either $R_1 < \min\{1,1/\lambda\}$, and in this case $\Delta^2(u-\tilde{w})(R_1) = 0$, or $R_1 = \min\{1,1/\lambda\}$.

In the first case, by applying the maximum principle to $-\Delta^3(u-\tilde{w}) = u^p - \tilde{w}^p > 0$, one has $\Delta^2(u-\tilde{w}) < 0$ on $(R_1, R_1+\delta)$ for δ sufficiently small. We can set R_2 such that

$$R_2 = \sup\{r \le \min\{1, 1/\lambda\} :\ (u-\tilde{w})(s) > 0,\ \Delta(u-\tilde{w})(s) > 0,\\ \Delta^2(u-\tilde{w})(s) < 0,\ s \in (R_1,r)\}.$$

Table 4.1: Sign of $u - \tilde{w}$, $\Delta(u - \tilde{w})$, $\Delta^2(u - \tilde{w})$.

	$(u - \tilde{w})(s)$	$\Delta(u - \tilde{w})(s)$	$\Delta^2(u - \tilde{w})(s)$
$s = 0$	$=0$	>0	>0
$s \in (0, R_1)$	>0	>0	>0
$s = R_1$	>0	>0	$=0$
$s \in (R_1, R_2)$	>0	>0	<0
$s = R_2$	>0	$=0$	<0
$s \in (R_2, R_3)$	>0	<0	<0
$s = R_3$	$=0$	<0	<0
$s \in (R_3, R_4)$	<0	<0	<0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

As above, we have

$$\Delta^2(u - \tilde{w})(R_2) < 0,\ (u - \tilde{w})(R_2) > 0$$

and either $R_2 < \min\{1, 1/\lambda\}$, which implies $\Delta(u - \tilde{w})(R_2) = 0$, or $R_2 = \min\{1, 1/\lambda\}$. Indeed, if $\Delta^2(u - \tilde{w})(R_2) = 0$, then by applying the maximum principle to $-\Delta^3(u - \tilde{w}) = u^p - \tilde{w}^p > 0$ on $B_{R_2} \setminus \overline{B_{R_1}}$ we have $\Delta^2(u - \tilde{w}) > 0$ on (R_1, R_2); on the other hand, if $(u - \tilde{w})(R_2) = 0$, then $u - \tilde{w} < 0$ on $[0, R_2)$, as $\Delta(u - \tilde{w}) > 0$.

We now apply iteratively the same reasoning as above to get a sequence (which can be finite or infinite)

$$0 = R_0 < R_1 < R_2 < \cdots \leq \min\{1, 1/\lambda\}$$

such that

$$u(R_{3k}) = \tilde{w}(R_{3k}),\ \Delta^2 u(R_{3k+1}) = \Delta^2 \tilde{w}(R_{3k+1}),\ \Delta u(R_{3k+2}) = \Delta \tilde{w}(R_{3k+2}),$$

$k \geq 0$, as long as $R_k < \min\{1, 1/\lambda\}$, see Table 4.1.

If it is infinite, then we take the limit $R_* = \lim_{i \to \infty} R_i \leq \min\{1, 1/\lambda\}$ and by continuity and differentiability, it holds

$$(u - \tilde{w})(R_*) = 0,\ \Delta(u - \tilde{w})(R_*) = 0,\ \Delta^2(u - \tilde{w})(R_*) = 0$$

and

$$(u' - \tilde{w}')(R_*) = 0,\ (\Delta(u - \tilde{w}))'(R_*) = 0,\ (\Delta^2(u - \tilde{w}))'(R_*) = 0.$$

Now, one defines

$$U(r) = (u(r), -\Delta u(r), \Delta^2 u(r)) \quad 0 \leq r \leq 1$$

and

$$W(r) = (\tilde{w}(r), -\Delta \tilde{w}(r), \Delta^2 \tilde{w}(r)) \quad 0 \leq r \leq 1/\lambda.$$

Hence, for any $0 \le r \le R_*$ one has

$$U(r) - W(r) = \int_r^{R_*} \frac{s}{N-2}\left(1 - \left(\frac{s}{r}\right)^{N-2}\right)(F(U(s)) - F(W(s)))\,ds \quad (4.11)$$

where we set $F(x, y, z) = (y, z, x^p)$. Since $p > 1$, then F is locally Lipschitz continuous, hence by the Gronwall Lemma, (4.11) implies $U = W$ on $[0, R_*]$. This is in contradiction with the assumption $\Delta^2(u - \tilde{w})(0) > 0$.

On the other hand, if the sequence stops at a maximum value R_k then on $(R_{k-1}, R_k = \min\{1, 1/\lambda\}]$ one of the following is verified, see Table 4.1:

- $u - \tilde{w}$ and $\Delta(u - \tilde{w})$ have the same sign
- $u - \tilde{w}$ and $\Delta^2(u - \tilde{w})$ have opposite sign.

Let for instance $u - \tilde{w} > 0$ and $\Delta(u - \tilde{w}) \ge 0$. Then,

$$0 < (u - \tilde{w})(\min\{1, 1/\lambda\}) = \begin{cases} u(1/\lambda) & \text{if } \lambda > 1 \\ 0 & \text{if } \lambda = 1 \\ -\tilde{w}(1) & \text{if } \lambda < 1 \end{cases}$$

which implies $\lambda > 1$, whereas by Hopf lemma

$$0 < (u' - \tilde{w}')(\min\{1, 1/\lambda\}) = (u' - \tilde{w}')(1/\lambda) = u'(1/\lambda) < 0$$

thus a contradiction.

Let now $u - \tilde{w} \ge 0$, $\Delta(u - \tilde{w}) < 0$ and $\Delta^2(u - \tilde{w}) < 0$. Hence $(u - \tilde{w})(\min\{1, 1/\lambda\}) \ge 0$, and therefore $\lambda \ge 1$. Moreover, $\Delta u(1/\lambda) = \Delta(u - \tilde{w})(1/\lambda) < 0$, whereas by Hopf Lemma and Lemma 4.2 $(\Delta u)'(1/\lambda) \le (\Delta(u - \tilde{w}))'(1/\lambda) < 0$. By Lemma 4.2, in particular we have that Δu increases until reaches a point r_0 and then decreases. Since $\Delta u(1) = 0$, Δu attains its maximum in r_0 and $(\Delta u)' < 0$, $\Delta u > 0$ on $(r_0, 1)$, whereas $(\Delta u)' > 0$ on $(0, r_0)$. Therefore, we cannot find a point such that $(\Delta u)' < 0$ and $\Delta u < 0$, hence we reach again a contradiction.

Since we get to a contradiction in all possible cases, we can not have $\Delta^2(u - \tilde{w})(0) > 0$ and $\Delta(u - \tilde{w})(0) > 0$. In a similar fashion, one proves that also the other possible choices for the sign of $\Delta^2(u - \tilde{w})(0)$ and $\Delta(u - \tilde{w})(0)$ yield a contradiction, hence the claim (4.9) holds.

Now, in view of (4.8) and (4.9), and since by (4.6) and (4.7)

$$u'(0) = \tilde{w}'(0) = (\Delta^2 u)'(0) = (\Delta^2 \tilde{w})'(0) = \tilde{z}'(0) = (\Delta u)'(0) = (\Delta \tilde{w})'(0) = 0, \quad (4.12)$$

for any $r \le \min\{1, 1/\lambda\}$ one has

$$U(r) - W(r) = \int_0^r \frac{s}{N-2}\left(1 - \left(\frac{s}{r}\right)^{N-2}\right)(F(W(s)) - F(U(s)))\,ds \quad (4.13)$$

where $F(x, y, z) = (y, z, x^p)$. Since $p > 1$, then F is locally Lipschitz continuous, hence by the Gronwall Lemma, (4.13) implies $U = W$ on $[0, \min\{1, 1/\lambda\}]$.

Finally, $0 < u(1/\lambda) = \tilde{w}(1/\lambda) = 0$ if $\lambda > 1$, whereas $0 = u(1) = \tilde{w}(1) > 0$ if $\lambda < 1$, thus $\lambda = 1$ and $u = w$. □

4.3.2 Proof of Theorem 4.1 in the case $\alpha = 4$

Let u, w be two nontrivial solutions to

$$\begin{cases} \Delta^4 u = |u|^p, & \text{in } B_1 \\ u = \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = \frac{\partial^3 u}{\partial \nu^3} = 0, & \text{on } \partial B_1. \end{cases} \quad (4.14)$$

Choose λ, s such that $\tilde{w}(r) = \lambda^s w(\lambda r)$ satisfies (4.14) on $B_{1/\lambda}$ and $u(0) = \tilde{w}(0)$. We want to prove that

$$\Delta^k u(0) = \Delta^k \tilde{w}(0), \quad k = 0, \ldots, 3. \quad (4.15)$$

For instance, assume that

$$\Delta(u - \tilde{w})(0) > 0, \quad \Delta^2(u - \tilde{w})(0) < 0, \quad \Delta^3(u - \tilde{w})(0) > 0.$$

Considerations below hold with some modifications also for other choices of the above signs. Let us define

$$R_1 = \sup\{r \le \min\{1, 1/\lambda\} : (u - \tilde{w})(s) > 0, \Delta(u - \tilde{w})(s) > 0, \\ \Delta^2(u - \tilde{w})(s) < 0, \Delta^3(u - \tilde{w})(s) > 0, s \in (0, r)\}.$$

By the maximum principle, $(u - \tilde{w})(R_1) > 0$ and $\Delta^3(u - \tilde{w})(R_1) > 0$, whereas $\Delta(u - \tilde{w})(R_1)$ and $\Delta^2(u - \tilde{w})(R_1)$ may be $= 0$. If for instance $\Delta(u - \tilde{w})(R_1) = 0$, then by considering

$$R_2 = \sup\{r \le \min\{1, 1/\lambda\} : (u - \tilde{w})(s) > 0, \Delta(u - \tilde{w})(s) < 0, \\ \Delta^2(u - \tilde{w})(s) < 0, \Delta^3(u - \tilde{w})(s) > 0, s \in (R_1, r)\}$$

we have that $\Delta(u - \tilde{w})(R_2) < 0$ and $\Delta^3(u - \tilde{w})(R_2) > 0$, whereas $(u - \tilde{w})(R_2)$ and $\Delta^2(u - \tilde{w})(R_2)$ may be $= 0$. We now iterate to get a sequence $\{R_j\}$ (finite or infinite) such that for any j one or two among $(u - \tilde{w})(R_j), \Delta(u - \tilde{w})(R_j), \Delta^2(u - \tilde{w})(R_j), \Delta^3(u - \tilde{w})(R_j)$ is $= 0$.

If $\{R_j\}$ is infinite, then we reach a contradiction as in Subsection 4.3.1 by applying the Gronwall Lemma with $F(x, y, z, w) = (y, z, w, x^p)$. Let us assume

Table 4.2: Sign of $u-\tilde{w}$, $\Delta(u-\tilde{w})$, $\Delta^2(u-\tilde{w})$, $\Delta^3(u-\tilde{w})$ in a special case.

	$(u-\tilde{w})(s)$	$\Delta(u-\tilde{w})(s)$	$\Delta^2(u-\tilde{w})(s)$	$\Delta^3(u-\tilde{w})(s)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$s \in (R_j, R_{j+1})$	<0	<0	>0	>0
$s = R_{j+1}$	<0	=0	>0	>0
$s \in (R_{j+1}, R_{j+2})$	<0	>0	>0	>0
$s = R_{j+2}$	<0	>0	>0	=0
$s \in (R_{j+2}, R_{j+3})$	<0	>0	>0	<0
$s = R_{j+3}$	<0	>0	=0	<0
$s \in (R_{j+3}, R_{j+4})$	<0	>0	<0	<0
$s = R_{j+4}$	<0	=0	<0	<0
$s \in (R_{j+4}, \min\{1, 1/\lambda\})$	<0	<0	<0	<0

that $\{R_j\}$ is finite. We want to exclude the possibility that on (R_j, R_{j+1}) for some j we have

$$(u-\tilde{w}) < 0,\ \Delta(u-\tilde{w}) > 0,\ \Delta^2(u-\tilde{w}) < 0,\ \Delta^3(u-\tilde{w}) > 0$$

(or opposite signs). In order for this to happen, since in $(0, R_1)$

$$(u-\tilde{w}) > 0,\ \Delta(u-\tilde{w}) > 0,\ \Delta^2(u-\tilde{w}) < 0,\ \Delta^3(u-\tilde{w}) > 0$$

we need that $(u-\tilde{w})(R_k) = 0$ for an odd number of $k \le j$, $\Delta(u-\tilde{w})(R_k) = 0$ for an even number of $k \le j$, $\Delta^2(u-\tilde{w})(R_k) = 0$ for an even number of $k \le j$, and $\Delta^3(u-\tilde{w})(R_k) = 0$ for an even number of $k \le j$. However, let us assume that the number of $k \le j$ such that $(u-\tilde{w})(R_k) = 0$ is n. Then, the number of zeros of $\Delta(u-\tilde{w})$ must be $\ge n$, since $u-\tilde{w}$ can be 0 only if $\Delta(u-\tilde{w})$ has been $= 0$ before. There are three possible cases:

1. The number of zeros of $\Delta(u-\tilde{w})$ is n;

2. The number of zeros of $\Delta(u-\tilde{w})$ is $n+1$ (if we stop after a zero of $\Delta(u-\tilde{w})$ and before $(u-\tilde{w})$ vanishes again);

3. The number of zeros of $\Delta(u-\tilde{w})$ is equal to $n+2$. This last case happens when $\Delta(u-\tilde{w}) = 0$ for two consecutive times, without having $(u-\tilde{w}) = 0$ in the between. Notice that such a situation may happen just once, since at the last step the four columns turn out to have the same sign and hence cannot be 0 again, see a model case in Table 4.2.

Assume n odd. In order to have an even number of zeros of $\Delta(u-\tilde{w})$ we have to consider the second case, namely the number of zeros of $\Delta(u-\tilde{w})$ must be $n+1$. Now, $\Delta(u-\tilde{w})$ might be zero in R_1 even if $\Delta^2(u-\tilde{w})$ has not vanished yet. Hence the number of zeros of $\Delta^2(u-\tilde{w})$ can be n, $n+1$ or $n+2$. Recall we need that $\Delta^2(u-\tilde{w})$ has an even number of zeros, and that

n is odd, hence we conclude that $\Delta^2(u-\tilde{w})$ has $n+1$ zeros. We deduce as above that $\Delta^3(u-\tilde{w})$ can have n, $n+1$ or $n+2$ zeros, and in turn $n+1$ since their number has to be even. However, this implies that $u-\tilde{w}$ should have at least $n+1$ zeros. This is a contradiction, since the number of zeros of $u-\tilde{w}$ is n by assumption.

As a consequence, we conclude that the following configuration is not possible

$$(u-\tilde{w})<0,\ \Delta(u-\tilde{w})>0,\ \Delta^2(u-\tilde{w})<0,\ \Delta^3(u-\tilde{w})>0,$$

and the same holds true having opposite signs. Therefore, one of the following (or reversed) is verified on $(R_k, \min\{1, 1/\lambda\})$:

- $(u-\tilde{w})>0$, $\Delta(u-\tilde{w})>0$;
- $(u-\tilde{w})>0$, $\Delta(u-\tilde{w})<0$, $\Delta^2(u-\tilde{w})<0$;
- $(u-\tilde{w})>0$, $\Delta(u-\tilde{w})<0$, $\Delta^2(u-\tilde{w})>0$, $\Delta^3(u-\tilde{w})>0$.

By Lemma 4.2, $\Delta^3 u$ is increasing. Moreover, $\Delta^3 u(0)<0$, and $\Delta^2 u$ is first positive and decreasing, then negative, reaches its minimum in this interval and then increases to a positive value $\Delta^2 u(1)$. As a consequence, $\Delta u(0)<0$, then increases, reaches a positive maximum value and then decreases to 0.

Assume that in the last interval the following holds

$$(u-\tilde{w})>0,\ \Delta(u-\tilde{w})>0.$$

If both the first and the second column have n zeros, then we apply the Hopf lemma and we obtain $0>u'(1/\lambda)=(u'-\tilde{w}')(1/\lambda)>0$, a contradiction. Otherwise, it means that the second column has $n+2$ zeros, which in turn gives that the third column has $n+1$ zeros, and the last one has n zeros, thus $\Delta^2(u-\tilde{w})>0$ and $\Delta^3(u-\tilde{w})>0$, see Table 4.2. Then, by applying Hopf lemma,

$$0<(\Delta^2(u-\tilde{w}))'(1/\lambda)\leq(\Delta^2 u)'(1/\lambda)$$

as $(\Delta^2\tilde{w})'(1/\lambda)\geq 0$, and $0<\Delta^2(u-\tilde{w})(1/\lambda)<\Delta^2 u(1/\lambda)$. Moreover, $\Delta u(1/\lambda)=\Delta(u-\tilde{w})(1/\lambda)>0$ and $(\Delta u)'(1/\lambda)=(\Delta(u-\tilde{w}))'(1/\lambda)>0$. However, by Lemma 4.2, there does not exist a point such that $\Delta^2 u>0$, $(\Delta^2 u)'>0$, $\Delta u>0$ and $(\Delta u)'>0$.

Assume that

$$(u-\tilde{w})>0,\ \Delta(u-\tilde{w})<0,\ \Delta^2(u-\tilde{w})<0.$$

Then $\lambda>1$, $0>\Delta(u-\tilde{w})(1/\lambda)=\Delta u(1/\lambda)$. If $\Delta^2(u-\tilde{w})$ does not change sign after the last zero of $\Delta(u-\tilde{w})$, then we can apply Hopf to get $(\Delta u)'(1/\lambda)=(\Delta(u-\tilde{w}))'(1/\lambda)<0$. However, it cannot exists a point such that $\Delta u(1/\lambda)<0$ and $(\Delta u)'(1/\lambda)<0$ by Lemma 4.2. If we cannot apply Hopf, then it means that the third column has $n+2$ zeros, which is not possible.

Assume finally that

$$(u - \tilde{w}) > 0,\ \Delta(u - \tilde{w}) < 0,\ \Delta^2(u - \tilde{w}) > 0,\ \Delta^3(u - \tilde{w}) > 0,$$

Again $\lambda > 1$ and $0 > \Delta u(1/\lambda)$. Moreover, $0 < \Delta^2(u - \tilde{w})(1/\lambda) < \Delta^2 u(1/\lambda)$ and by Hopf

$$0 < (\Delta^2(u - \tilde{w}))'(1/\lambda) \leq (\Delta^2 u)'(1/\lambda)$$

as $(\Delta^2 \tilde{w})'(1/\lambda) \geq 0$ by Lemma 4.2. However, such a point cannot exists, hence we have a contradiction. As in Subsection 4.3.1, we conclude that (4.15) holds, then $u = \tilde{w}$, which in turn gives $u = w$. □

4.3.3 An open problem

Consider $\alpha \geq 5$, and take two different solutions u, w. One can naturally parametrize w as $\tilde{w}(r) = \lambda^s w(\lambda r)$, where $s = \frac{2\alpha}{p-1}$, and λ is such that $\tilde{w}(0) = u(0)$. Again, it is easy to prove that the uniqueness result follows once we prove that $\Delta^k(u - \tilde{w})(0) = 0$ for any k. One builds a table as above, and gets a sequence $\{R_j\}$. If it is infinite, then one extends considerations above choosing a suitable F to apply Gronwall. The main difficulty turns out to be the proof of the contradiction in the finite case, equivalently, the extension of the following lemma to $\alpha \geq 5$.

Lemma 4.3. *Let $2 \leq \alpha \leq 4$. Then the following configuration:*

$$(-\Delta)^k(u - \tilde{w}) < 0,\ k = 0, \dots, \bar{k}$$

and

$$(-\Delta)^{\bar{k}+1}(u - \tilde{w}) > 0,$$

for some $\bar{k}$, cannot occur at the last step.

As a consequence of Lemma 4.3 we have

Lemma 4.4. *Let $2 \leq \alpha \leq 4$. Assume that $u - \tilde{w}$ has n zeros and that $u(0) = \tilde{w}(0)$ and*

$$(-\Delta)^k(u - \tilde{w})(0) < 0,\ k = 1, \dots, \alpha - 1$$

holds. Then $\Delta^{\alpha-1}(u - \tilde{w})$ must have at least $n + 1$ zeros.

Indeed, if not, then at least two consecutive columns have the same sign, and we get a contradiction.

Remark 4.1. One can prove in the same way as Lemma 4.4 that, if $\alpha \leq 4$ and

$$(-\Delta)^k(u - \tilde{w})(0) < 0,\ k = 0, \dots, \alpha - 1$$

holds, and $u - \tilde{w}$ has n zeros, then $\Delta^{\alpha-1}(u - \tilde{w})$ must have at least n zeros. This will be useful in the next section.

4.4 Uniqueness for polyharmonic systems

We first give the proof in the case $m = 2$ and then we proceed inductively. System (4.1) reads as follows

$$\begin{cases} (-\Delta)^\alpha u = |v|^q & \text{in } B_1, \\ (-\Delta)^\beta v = |u|^p & \\ \frac{\partial^r u}{\partial \nu^r} = 0,\ r = 0, \dots, \alpha - 1, & \text{on } \partial B_1, \\ \frac{\partial^r v}{\partial \nu^r} = 0,\ r = 0, \dots, \beta - 1, & \text{on } \partial B_1. \end{cases}$$

Assume without loss of generality that $\alpha \le \beta \le 4$. Recall that u, v are radially symmetric and strictly decreasing in the radial variable due to Lemma 3.9. We take two nontrivial solutions (u, v) and (w, z), and the parametrization

$$\tilde{w}(r) = \lambda^s w(\lambda r),\ \tilde{z}(r) = \lambda^t z(\lambda r)$$

where $t = \frac{2\alpha p + 2\beta}{pq-1}$, $s = \frac{2\beta q + 2\alpha}{pq-1}$. Notice that s, t are well defined if $pq \neq 1$. Moreover we build the same table as in the previous sections with columns

$$u - \tilde{w},\ \Delta(u - \tilde{w}), \dots, \Delta^{\alpha-1}(u - \tilde{w}),\ v - \tilde{z}, \dots, \Delta^{\beta-1}(v - \tilde{z})$$

if α is even, whereas

$$u - \tilde{w},\ \Delta(u - \tilde{w}), \dots, \Delta^{\alpha-1}(u - \tilde{w}),\ -v + \tilde{z}, \dots, \Delta^{\beta-1}(-v + \tilde{z})$$

if α is odd.

Assume that (for even α, and similarly for odd α)

$$\begin{aligned} (u - \tilde{w})(0) = 0,\ (-\Delta)^k(u - \tilde{w})(0) < 0,\ k = 1, \dots, \alpha - 1, \\ (-\Delta)^k(v - \tilde{z})(0) < 0,\ k = 0, \dots, \beta - 1 \end{aligned} \tag{4.16}$$

is the initial configuration of the columns. We obtain a sequence $\{R_j\}$ as in Section 4.3 and assume that this is finite.

Let n be the number of zeros of the first column, and $\alpha \ge 2$. Then by Lemma 4.4, the α-th column has at least $n + 1$ zeros, and as a consequence the next one must have n, $n + 1$ or more zeros. Knowing that the $(\alpha + \beta)$-th column has n or $n - 1$ zeros, one has (again by Lemma 4.4, see Remark 4.1) that the $(\alpha + 1)$-th column cannot have strictly more than n zeros. Hence, it has n zeros. However, $(v - \tilde{z})(s)$ has opposite sign with respect to $(u - \tilde{w})(s)$ in $(0, R_1)$, hence they have opposite sign in the last interval as well. Therefore, $(u - \tilde{w})(\min\{1, 1/\lambda\}) > 0$ implies $\lambda > 1$, whereas $(v - \tilde{z})(\min\{1, 1/\lambda\}) < 0$ gives $\lambda < 1$, a contradiction.

If $\alpha = 1$, then as above we prove that the column $\tilde{z} - v$ cannot have strictly more than n zeros. However, it must have at least n zeros, as $u(0) = \tilde{w}(0)$, thus exactly n zeros. Again, we have a contradiction.

Table 4.3: Passing from (4.16) to the configuration $u - \tilde{w} < 0$, $\Delta(u - \tilde{w}) < 0$, $(v - \tilde{z}) > 0$, $\Delta(v - \tilde{z}) > 0$, $\Delta^2(v - \tilde{z}) > 0$.

	$(u - \tilde{w})(s)$	$\Delta(u - \tilde{w})(s)$	$(v - \tilde{z})(s)$	$\Delta(v - \tilde{z})(s)$	$\Delta^2(v - \tilde{z})(s)$
$s = 0$	$=0$	>0	<0	>0	<0
$s \in (0, R_1)$	>0	>0	<0	>0	<0
$s = R_1$	>0	$=0$	$=0$	>0	<0
$s \in (R_1, R_2)$	>0	<0	>0	>0	<0
$s = R_2$	$=0$	<0	>0	>0	<0
$s \in (R_2, R_3)$	<0	<0	>0	>0	<0
$s = R_3$	<0	<0	>0	>0	$=0$
$s \in (R_3, R_4)$	<0	<0	>0	>0	>0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let us assume that for another initial configuration $\mathcal{A}$ we do not reach a contradiction as above. In $(0, R_1)$ the signs of the columns from the second to the last one are the same as in $\mathcal{A}$, and the first column must have the same sign as the second one, due to the maximum principle and the assumption $u(0) = \tilde{w}(0)$. Let us call $\mathcal{A}_1$ the configuration in $(0, R_1)$, given $\mathcal{A}$ in 0. It turns out that one can reach the configuration $\mathcal{A}_1$ starting from (4.16). Indeed, given (4.16), all the columns from the second to the second-to-last can be $= 0$ in R_1. Then, it is sufficient to impose $= 0$ in R_1 the columns which have different signs with respect to $\mathcal{A}_1$. If the first column has different sign, then it is enough to note that, once the second column has changed sign, the first column can be $= 0$ and change sign as well. Analogously, one can change the sign of the last column once the first one has been $= 0$. See Table 4.3 for an example.

Therefore, if from any other initial configuration $\mathcal{A}$ we do not have a contradiction, then this would be possible given (4.16) as well.

We have thus proved that the sequence $\{R_j\}$ has to be infinite. However, in this case we reach a contradiction as in the previous sections, as we apply Gronwall with

$$U(r) = (u(r), -\Delta u(r), \dots, (-\Delta)^{\alpha-1}u(r), v(r), \dots, (-\Delta)^{\beta-1}v(r))$$

for $0 \le r \le 1$ and

$$W(r) = (\tilde{w}(r), -\Delta\tilde{w}(r), \dots, (-\Delta)^{\alpha-1}\tilde{w}(r), \tilde{z}(r), \dots (-\Delta)^{\beta-1}\tilde{z}(r))$$

for $0 \le r \le 1/\lambda$ and

$$F(x_1, x_2, \dots, x_\alpha, y_1, \dots, y_\beta) = (x_2, x_3, \dots, x_{\alpha-1}, y_1^q, y_2, \dots, y_{\beta-1}, x_1^p) \ .$$

This proves that in 0 all the columns are zero. Therefore, again by Gronwall's Lemma, we have $u = \tilde{w}$ and $v = \tilde{z}$, which in turn gives $(u, v) = (w, z)$.

The proof in the case $m > 2$ follows by induction, once we parametrize a second solution $(w_1, \ldots, w_m)$ as follows

$$\tilde{w}_i(r) = \lambda^{s_i} w(\lambda r),\ i = 1, \ldots, m,$$

where λ is chosen such that $\tilde{w}_1(0) = u_1(0)$, whereas

$$s_1 = \frac{2\sum_{j=1}^{m} \alpha_j \prod_{k=1}^{j-1} p_k}{\prod_{k=1}^{m} p_k - 1}$$

and

$$s_{i+1} = \frac{s_i + 2\alpha_i}{p_i},\ i = 1, \ldots, m-1.$$

Assuming as induction hypothesis that the last column corresponding to the first k equations can not have less zeros than the first one, and taking $k = 1$ as the base case (see Lemma 4.4), then one proves that that property holds for $k+1$ as well, by the same arguments as above. More precisely, the induction hypothesis implies that the last column corresponding to the first k equations must have at least one zero more than the first one. By exploiting Remark 4.1, and assuming by contradiction that the last column corresponding to the first $k+1$ equations has at most the same number of zeros as the first one, one proves as above that $u_1 - \tilde{w}_1$ and $u_{k+1} - \tilde{w}_{k+1}$ must have opposite signs at the last step, which gives the contradiction due to boundary conditions. Hence, $\{R_j\}$ cannot be finite. As for the case $\{R_j\}$ infinite, the contradiction follows by applying Gronwall's lemma.

The proof of Theorem 4.2 is now complete.

Remark 4.2. Notice that the restriction $\alpha_j \leq 4$ is necessary as we need to exploit Lemma 4.4. Actually, if we could extend Lemma 4.4 to higher order operators, then it would be possible to extend Theorem 4.2 to more general operators as well.

Appendix A

Existence results on the entire space

We will focus on existence for

$$\begin{cases} \Delta^2 u = |v|^{q-1}\, v & \text{in } \mathbb{R}^N \\ -\Delta v = |u|^{p-1}\, u & \text{in } \mathbb{R}^N \end{cases} \tag{A.1}$$

where $p, q > 1$, and $N > 4$. The next result is proved in [104, Theorem 1], where also more general weighted Hardy–Littlewood–Sobolev type systems are considered.

Theorem A.1. *If for any $R > 0$, $N > \max\{2\alpha, 2\beta\}$, $p, q > 1$ the following system*

$$\begin{cases} (-\Delta)^{\alpha} u = |v|^{q-1}\, v & \textit{in } B_R \\ (-\Delta)^{\beta} v = |u|^{p-1}\, u & \textit{in } B_R \\ \Delta^s u = 0, \quad s \le \alpha - 1 & \textit{on } \partial B_R \\ \Delta^t v = 0, \quad t \le \beta - 1 & \textit{on } \partial B_R \end{cases} \tag{A.2}$$

has no classical radial positive solutions, then (A.1) *has infinitely many classical, radially symmetric and positive solutions.*

This is achieved by means of a shooting technique, see also [62, 61] for similar problems studied by this method. In view of Theorem A.1, it turns out that proving non existence results for (A.2) can be crucial in the analysis of existence of solutions to (A.1). In [68] non existence to (A.2) is established if $\alpha = \beta$ and (p, q) is above the critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N - 2\alpha}{N}.$$

However, the proof in [68] strongly depends on the variational structure of the problem, and hence the extension to the case $\alpha \neq \beta$ is not straightforward. We use here a different approach, in the spirit of [30], where the authors consider the following

$$\begin{cases} -\Delta_s u = |v|^{q-1} v & \text{in } B_R \\ -\Delta_t v = |u|^{p-1} u & \text{in } B_R \\ u = v = 0 & \text{on } \partial B_R \end{cases}$$

where

$$\Delta_s u = div(|\nabla u|^{s-2} \nabla u)$$

is the s-Laplacian operator. Their method allows one to prove a Pohozaev identity also in the case $s \neq t$. The main result of this appendix is the following

Theorem A.2. *Let $N > 4$. Assume $\alpha = 2$, $\beta = 1$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-4}{2N}. \tag{A.3}$$

Then, no positive classical radial solutions to (A.2) *exist.*

By combining Theorem A.2 and Theorem A.1 we have

Corollary A.1. *There exist infinitely many nontrivial solutions to* (A.1)*, provided $p, q > 1$, $N > 4$, and* (A.3) *is satisfied.*

A.1 Preliminaries

In order to simplify the notation, let us set $u_s = (-\Delta)^s u$, where $0 < s \leq \alpha - 1$, $u_0 = u$, and similarly for v. The following result is a special case of [29, Lemma II.2].

Lemma A.1. *Let $r_0 \geq 0$ and $R > r_0$. Let $u \in C^1([r_0, R)) \cap C^2((r_0, R))$ be a non negative function satisfying*

$$-(r^{N-1}u'(r))' \geq 0 \quad \text{on } (r_0, R).$$

If $u'(r_0) \leq 0$, then for any $r > r_0$ we have

$$u(r) \geq Cr \left|u'(r)\right|.$$

The next proposition gives estimates on the asymptotic behavior of u_s and v_s, see also the proof of Theorem 3.2 above. Notice that the superharmonicity condition (3.21) holds if Navier boundary conditions are taken into account due to [102, Theorem 1].

Proposition A.1. *If (u, v) is a positive radial solution to* (A.2)*, then one has for any $s \leq \alpha - 1$, $t \leq \beta - 1$*

$$u_s(r) \leq cr^{-\frac{2\beta q + 2\alpha}{pq-1} - 2s}, \; v_t(r) \leq cr^{-\frac{2\alpha p + 2\beta}{pq-1} - 2t}.$$

Moreover,

$$\left|u_s'(r)\right| \le cr^{-\frac{2\beta q+2\alpha}{pq-1}-2s-1},\ \left|v_t'(r)\right| \le cr^{-\frac{2\alpha p+2\beta}{pq-1}-2t-1}.$$

Proof. Due to the maximum principle, and since $u_s'(0) = v_s'(0) = 0$ by [102, Theorem 1], we can apply Lemma A.1 to obtain

$$u_s(r) \ge Cr\left|u_s'(r)\right|,\ v_t(r) \ge Cr\left|v_t'(r)\right|$$

for any $s \le \alpha - 1$, $t \le \beta - 1$. Since $u_s', v_s' < 0$ by [102, Theorem 1], then

$$-r^{N-1}u_s'(r) = \int_0^r t^{N-1}u_{s+1} \ge u_{s+1}(r)\int_0^r t^{N-1} = \frac{1}{N}u_{s+1}(r)r^N,$$

thus

$$u_s(r) \ge cr^2u_{s+1},\ v_t(r) \ge cr^2v_{t+1} \tag{A.4}$$

and

$$u \ge cv^qr^{2\alpha},\ v \ge cu^pr^{2\beta}.$$

Therefore,

$$u \le cr^{-\frac{2\beta q+2\alpha}{pq-1}},\ v \le cr^{-\frac{2\alpha p+2\beta}{pq-1}},$$

and as a consequence for any $s \le \alpha - 1$, $t \le \beta - 1$, by (A.4),

$$u_s \le cr^{-\frac{2\beta q+2\alpha}{pq-1}-2s},\ v_t \le cr^{-\frac{2\alpha p+2\beta}{pq-1}-2t}.$$ □

A.2 Proof of Theorem A.2

We borrow a few ideas from [30], where an (s,t)-Laplacian system is taken into account. Let us preliminarily notice that if (u,v) is a radial positive classical solution to

$$\begin{cases} \Delta^2 u = |v|^{q-1}v & \text{in } B_R \subset \mathbb{R}^N \\ -\Delta v = |u|^{p-1}u & \\ \Delta u = u = v = 0, & \text{on } \partial B_R \end{cases}$$

and if we call $w = -\Delta u$, then

$$\begin{cases} -(r^{N-1}u'(r))' = r^{N-1}w, \\ -(r^{N-1}w'(r))' = r^{N-1}v^q \\ -(r^{N-1}v'(r))' = r^{N-1}u^p, \\ u(R) = w(R) = v(R) = 0 \end{cases} \tag{A.5}$$

is satisfied.

Proof of Theorem A.2. Let us define the functional $E\colon [0,\infty) \to \mathbb{R}$ as

$$\begin{aligned} E(r) = r^{2N-1}u'v'w' - r^N \int_r^R s^{N-1}v'w'w - r^N \int_r^R s^{N-1}u'w'u^p \\ - r^N \int_r^R s^{N-1}u'v'v^q - \frac{N}{q+1} r^{N-1}w' \int_r^R s^{N-1}u'v' \\ - \frac{N}{2} r^{N-1}u' \int_r^R s^{N-1}v'w' - \frac{N}{p+1} r^{N-1}v' \int_r^R s^{N-1}u'w'. \end{aligned}$$

We want to verify that if u, v, w satisfy (A.5) and (A.3), then $E(0^+) = 0$, $E(R) < 0$ and $E' \geq 0$, which gives a contradiction. We point out that

$$E(R) = R^{2N-1}u'(R)v'(R)w'(R) < 0.$$

Moreover,

$$\begin{aligned} E'(r) = (2N-1)r^{2N-2}u'v'w' + r^{2N-1}u''v'w' + r^{2N-1}u'v''w' + r^{2N-1}u'v'w'' \\ - Nr^{N-1} \int_r^R s^{N-1}v'w'w + r^{2N-1}v'w'w - Nr^{N-1} \int_r^R s^{N-1}u'w'u^p \\ + r^{2N-1}u'w'u^p - Nr^{N-1} \int_r^R s^{N-1}u'v'v^q + r^{2N-1}u'v'v^q \\ + \frac{N}{q+1} r^{N-1}v^q \int_r^R s^{N-1}u'v' + \frac{N}{q+1} r^{2N-2}u'v'w' + \frac{N}{2} r^{N-1}w \int_r^R s^{N-1}v'w' \\ + \frac{N}{2} r^{2N-2}u'v'w' + \frac{N}{p+1} r^{N-1}u^p \int_r^R s^{N-1}u'w' + \frac{N}{p+1} r^{2N-2}v'u'w'. \end{aligned}$$

Hence

$$\begin{aligned} E'(r) = \left(2N - 1 - 3(N-1) + \frac{N}{p+1} + \frac{N}{q+1} + \frac{N}{2}\right) r^{2N-2}u'v'w' \\ + Nr^{N-1}(G_1 + G_2 + G_3), \end{aligned}$$

where

$$\begin{aligned} G_1 &= -\int_r^R s^{N-1}v'w'w + \frac{1}{2} w \int_r^R s^{N-1}v'w' \\ G_2 &= -\int_r^R s^{N-1}u'w'u^p + \frac{1}{p+1} u^p \int_r^R s^{N-1}u'w' \\ G_3 &= -\int_r^R s^{N-1}u'v'v^q + \frac{1}{q+1} v^q \int_r^R s^{N-1}u'v'. \end{aligned}$$

However, $G_1(R) = 0$ and

$$G_1' = r^{N-1}v'w'w + \frac{1}{2} w' \int_r^R s^{N-1}v'w' - \frac{1}{2} wr^{N-1}v'w'$$

and since $(s^{N-1}v')' = -s^{N-1}u^p \leq 0$, then

$$G_1' \leq r^{N-1}v'w'w + \frac{1}{2}w'r^{N-1}v' \int_r^R w' - \frac{1}{2}wr^{N-1}v'w' = 0.$$

As a consequence, $G_1 \geq 0$. Analogously, one gets $G_2, G_3 \geq 0$. Thus, $E' \geq 0$ if

$$2N - 1 - 3(N-1) + \frac{N}{p+1} + \frac{N}{q+1} + \frac{N}{2} \leq 0,$$

namely if (A.3) is satisfied.

Finally, by using estimates of Proposition A.1, we prove $E(0^+) = 0$. Indeed,

$$\begin{aligned}|E(r)| \leq r^{2N-6-\frac{4q+10+4p}{pq-1}} + r^{2N-6-\frac{4p+10+4q}{pq-1}} + r^{2N-4-\frac{4q+8+2pq+4p}{pq-1}} \\ + r^{2N-2-\frac{4q+6+4p+4pq}{pq-1}} + \frac{N}{q+1}r^{2N-6-\frac{4q+10+4p}{pq-1}} \\ + \frac{N}{2}r^{2N-6-\frac{4q+10+4p}{pq-1}} + \frac{N}{p+1}r^{2N-6-\frac{4q+10+4p}{pq-1}}\end{aligned}$$

and the right hand side is 0 in $r = 0$ if all the exponents are positive, that is

$$\begin{cases} 2N - 6 - \frac{4q+10+4p}{pq-1} > 0 \\ 2N - 4 - \frac{4q+8+2pq+4p}{pq-1} > 0 \\ 2N - 2 - \frac{4q+6+4p+4pq}{pq-1} > 0. \end{cases}$$

This is trivially true if $pq < 1$, whereas if $pq > 1$, then it is equivalent to

$$2N(pq-1) - 4q - 4p - 4 - 6pq > 0,$$

which is verified under (A.3). □

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