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MULTIGRID METHODS
and
STATIONARY SUBDIVISION

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Era un buon piano, niente da dire. Bartleboom se lo rigirò in mente per tutto il viaggio, e questo fa riflettere sulla complessità delle menti di certi grandi uomini di studio e di pensiero - qual era il prof Bartleboom, fuori da ogni dubbio - ai quali la facoltà sublime di concentrarsi su un'idea con abnorme acutezza e profondità arreca l'incerto corollario di rimuovere istantaneamente, e in modo singolarmente completo, tutte le altre idee limitrofe, parenti e collimanti. Teste matte insomma.

Alessandro Baricco,
Oceano mare

CHAPTER 1

Introduction

The aim of this thesis is to deeply investigate the link between multigrid methods, fast iterative solvers for sparse ill-conditioned linear systems, and subdivision schemes, simple iterative algorithms for generation of smooth curves and surfaces. The main goal is to improve the convergence rate and the computational cost of multigrid methods taking advantage of the reproduction and regularity properties of underlying subdivision.

Multigrid methods. Multigrid methods are fast iterative solvers for sparse and ill-conditioned linear systems

$$A_n \mathbf{x} = \mathbf{b}_n, \qquad A_n \in \mathbb{C}^{n\times n}, \qquad \mathbf{b}_n \in \mathbb{C}^n, \tag{1.1}$$

where usually $A_n \in \mathbb{C}^{n\times n}$ is assumed to be symmetric and positive definite. A basic two-grid method (TGM) combines the action of a *smoother* and a *coarse grid correction*: the smoother is usually a simple iterative method such as Gauss-Seidel, weighted Jacobi or weighted Richardson [3, 70]; the coarse grid correction amounts to solving the residual equation exactly on a coarser grid. A V-cycle multigrid method solves the residual equation approximately within the recursive application of the two-grid method, until the coarsest level is reached and there the resulting small system of equations is solved exactly [8, 9, 48, 71].

The algebraic multigrid (AMG) method has been designed for the solution of linear systems of equations (1.1) whose system matrices are symmetric and positive definite [63]. The AMG method exploits the algebraic properties of the system matrix A_n in (1.1) and constructs the coarser system matrices preserving the algebraic properties of A_n. Recently [2, 3, 5, 6, 15, 34, 42, 69], AMG methods have been defined for the d-level circulant matrix algebra, and extended to other matrix algebras and to the class of d-level Toeplitz matrices.

The grid transfer operators, called *prolongation* $\mathcal{P}$ and *restriction* $\mathcal{R}$, define the coarse grid correction and they are an essential part of any multigrid method. The choice of these operators is crucial for the definition of a convergent and optimal [1] multigrid method and becomes cumbersome especially for severally ill-conditioned problems or on complex domains. A common choice is to define the prolongation and the restriction from the coefficients of the d-variate trigonometric polynomials p and r, respectively. Usually, $\mathcal{P}$ and $\mathcal{R}$ are defined as d-level circulant or d-level Toeplitz matrices of appropriate order generated by p and r, respectively. This way, the properties of the grid transfer operators are encoded in the trigonometric polynomials p, r and, thus, the effectiveness of the coarse grid correction can be analyzed in terms of the properties of p and r.

In the case of algebraic TGMs, the Galerkin approach, namely $r = cp$, $c \in (0, +\infty)$, is usually used. The sufficient conditions for the convergence of the TGM are then defined in terms of the order of the zeros of the trigonometric polynomial p [5, 31, 34, 42, 43, 67, 69]. More precisely, let f be the d-variate trigonometric polynomial associated to the d-level circulant or Toeplitz system matrix A_n in (1.1). Suppose that f vanishes only at $\mathbf{x}_0 \in [0, 2\pi)^d$ and it is strictly positive everywhere else. Then, a sufficient condition for the convergence of the TGM is that

$$\begin{aligned}
&(i) \quad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|p(\mathbf{y})|^2}{f(\mathbf{x})} < +\infty \qquad && \forall\, \mathbf{y} \in \Omega(\mathbf{x}) \setminus \{\mathbf{x}\},\\
&(ii) \quad \sum_{\mathbf{y}\in\Omega(\mathbf{x})} |p(\mathbf{x})|^2 > 0, && \forall \mathbf{x} \in [0, 2\pi)^d,
\end{aligned}$$

where $\Omega(\mathbf{x}) \subset [0, 2\pi)^d$ is a finite set depending on $\mathbf{x} \in \mathbb{C}^d$ whose structure is defined accordingly to the selected projection strategy. The symbol approach is a generalization of the LFA [31], which is a classical tool for the convergence analysis of multigrid method applied to linear systems of equations (1.1) derived via discretization of a constant-coefficients differential operator.

For the algebraic V-cycle method, different grid transfer operators can be defined at each recursive step. Thus, we have a set of trigonometric polynomials p_j, r_j. Due to the Galerkin approach, namely $r_j = c_j p_j$, $c_j \in (0, +\infty)$, the coarser matrices are still d-level circulant or d-level Toeplitz matrices. Under the hypothesis that the associated symbol f_j vanishes only at $\mathbf{x}_j \in [0, 2\pi)^d$ and all f_j are strictly positive everywhere else, a sufficient condition for the convergence of the V-cycle method is that the trigonometric polynomials p_j satisfy

$$\begin{aligned}
&(i) \quad \lim_{\mathbf{x}\to\mathbf{x}_j} \frac{|p_j(\mathbf{y})|}{f_j(\mathbf{x})} < +\infty \qquad && \forall\, \mathbf{y} \in \Omega(\mathbf{x}) \setminus \{\mathbf{x}\},\\
&(ii) \quad \sum_{\mathbf{y}\in\Omega(\mathbf{x})} |p_j(\mathbf{x})|^2 > 0, && \forall \mathbf{x} \in [0, 2\pi)^d.
\end{aligned} \tag{1.2}$$

We remark that the V-cycle conditions *(i)* and *(ii)* are well-known in the multigrid community for standard up-sampling strategy with the factor 2 in each coordinate direction [2, 3]. The generalization of the V-cycle conditions *(i)* and *(ii)* in the case of arbitrary up-sampling strategy is presented in chapter 3, subsection 3.5.3.

[1] I.e. the convergence rate is linear.

Subdivision schemes. Subdivision schemes are efficient recursive tools for generating smooth curves and surfaces and they are used in many fields ranging from computer graphics to signal and image processing. The starting point of the subdivision algorithm is a scalar sequence of control points in $\ell(\mathbb{Z}^d)$. At each recursive step, a new sequence is defined by applying simple, linear and local refinement rules to the sequence defined at the previous recursive step. The structure of the refinement rules depends on an expansive dilation matrix $M \in \mathbb{Z}^{d\times d}$, $\rho(M^{-1}) < 1$, which determines also the rate of the refinement process. In this thesis, we consider refinement rules whose coefficients are the same at each subdivision level. Thus, the refinement coefficients can be stored in a Laurent polynomial a, called *subdivision symbol*.

Attaching the data refined at the k-th subdivision level to the grid $M^{-k}\mathbb{Z}^d$, we can interpreter the subdivision process as a generation of "denser" data sequences. If the dilation M and the refinement coefficients are chosen appropriately, these increasingly denser data sequences will approach a continuous (or smoother) d-variate function.

The limit of a convergent subdivision scheme applied to the specific starting data called the delta sequence, defines the so-called *basic limit function*. It is well-known that the basic limit function satisfies a refinement equation defined by the dilation M and its integer shifts build the foundation of a multiresolution analysis [11,39].

If the starting data are sampled from a function π belonging to the space Π_q of d-variate polynomials of total degree less than or equal to q, it is natural to ask if the subdivision limit is an element of the same space and, if yes, under which assumptions this limit coincides with π. Such properties of subdivision are called *generation* and *reproduction* of the functional space Π_q, respectively. Certain algebraic properties of the symbol a characterize the properties of generation and reproduction of the functional space Π_q. A convergent stationary subdivision scheme generates the space Π_q if the associated symbol a satisfies ([11,59]) the zero conditions

$$D^{\boldsymbol{\mu}} a(\boldsymbol{\varepsilon}) = 0, \qquad \forall\, \boldsymbol{\varepsilon} \in E_M \setminus \{\mathbf{1}\}, \qquad \boldsymbol{\gamma} \in \mathbb{N}_0^d, \qquad |\boldsymbol{\mu}| \le q,$$

where we denote by $D^{\boldsymbol{\mu}}$ the $\boldsymbol{\mu}$-th directional derivative and $|\boldsymbol{\mu}| := \mu_1 + \ldots + \mu_d$. The finite set E_M is defined accordingly to the dilation M. Moreover, in order to guarantee the reproduction of polynomials of total degree less than or equal to $s \le q$, the symbol a should additionally satisfy ([12,20])

$$D^{\boldsymbol{\mu}} a(\mathbf{1}) = |\det M| \prod_{i=1}^{d} \prod_{\ell_i=0}^{\gamma_i-1} (\tau_i - \ell_i), \quad \boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d, \quad \boldsymbol{\gamma} \in \mathbb{N}_0^d, \quad |\boldsymbol{\mu}| \le s.$$

The parameter $\boldsymbol{\tau} \in \mathbb{R}^d$ is the *shift-parameter* appearing in the parametrization associated with the subdivision scheme

$$\mathbf{t}^{(k)}(\boldsymbol{\alpha}) = \mathbf{t}^{(k)}(\mathbf{0}) + M^{-k}\boldsymbol{\alpha}, \quad \mathbf{t}^{(k)}(\mathbf{0}) = \mathbf{t}^{(k-1)}(\mathbf{0}) - M^{-k}\boldsymbol{\tau}, \quad \mathbf{t}^{(0)}(\mathbf{0}) = \mathbf{0}, \quad \boldsymbol{\alpha} \in \mathbb{Z}^d, \quad k \ge 0.$$

The choice of $\boldsymbol{\tau} \in \mathbb{R}^d$ affects neither the convergence nor the polynomial generation property of the scheme, but its choice is crucial to guarantee the maximum degree of polynomial reproduction s.

Thesis overview. The thesis is organized as follows:

Chapter 1. We introduce the notation used throughout the thesis. Then, we describe circulant, d-level circulant, Toeplitz and d-level Toeplitz matrices and we recall a few of their well-known properties. Indeed, due to the matrix algebra structure, AMG methods have been defined for circulant and d-level circulant matrices (see chapter 3); nevertheless, they can be generalized to the class of Toeplitz and d-level Toeplitz matrices for our numerical experiments (see chapters 5 and 6). Finally, we present the Richardson, Jacobi and Gauss-Seidel iterative methods, which are used to define the smoother in the multigrid procedure (see chapter 3).

Chapter 2. First, we define the algebraic multigrid method (AMG) and report the well-known results about its convergence and optimality using the approach of Ruge-Stüben in [63]. Then, we define the AMG method for circulant [15, 34] and d-level circulant [5] matrix algebra with general up-sampling strategy. We relax the necessary conditions for the convergence of the TGM, see Theorem 3.7, and provide the first result, see Theorem 3.8, about the convergence of the V-cycle method generalizing the approach in [2, 3] for the standard up-sampling strategy. We conclude the chapter providing some well-known examples of grid transfer operators which already hint to a possible connection between AMG and subdivision.

Chapter 3. We introduce stationary subdivision and we describe some of its important properties, such as interpolation, generation and reproduction of polynomials. We also provide the tools and methods proposed in the literature for the analysis of stationary subdivision schemes. Then, we present the first analysis of subdivision based multigrid. Especially, we construct grid transfer operators in the multigrid procedure from the symbols of certain subdivision schemes and we analyze the subdivision properties which guarantee the convergence and optimality of a multigrid method. We highlight that polynomial generation property plays a fundamental role in our analysis, see Theorem 4.4, together with the stability of the basic limit function, see Theorem 4.5, and zero conditions of the subdivision symbol, see Theorem 4.7.

Chapter 4. We construct univariate and bivariate grid transfer operators from the symbols of well-known subdivision schemes and we test their efficacy on several numerical examples. In the univariate setting, we consider the symbols of binary and ternary primal pseudo-splines. In the bivariate setting, we consider well known approximating and interpolating subdivision schemes with dilation $M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $m = 2, 3$, such as symmetric 2-directional box splines and binary and ternary Kobbelt subdivision schemes. Due to the symmetry of this dilation matrix, our numerical tests for the anisotropic Laplacian problem fail (see subsection 5.4.3), leading to the need of grid transfer operators defined from the symbols of anisotropic subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m > 2$.

Chapter 5. We focus on bivariate subdivision schemes with anisotropic dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m > 2$. We construct a family of interpolatory subdivision schemes which are optimal in terms of the size of their support versus their polynomial generation properties. Then, we construct a new family of approximating subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ depending on parameters (J, L) characterized by a fixed degree of polynomial generation $2J-1$ and an increasing degree of polynomial reproduction $2L+1$, $L = 0, \ldots, J-1$. Finally, we define grid transfer operators from the symbols of our new families of anisotropic subdivision schemes and test their applicability for AMG on several numerical examples. Especially, we consider the anisotropic Laplacian problem and overcome the convergence problems observed in the symmetric context in chapter 5.

CHAPTER 2

Notation and Background

The aim of this introductory chapter is to fix the notation used through the thesis and to introduce the tools that will be widely used in the next chapters for the definition and the analysis of subdivision based multigrid. Especially, in section 2.1, we fix the key notation about matrices, sequences and functional spaces. Section 2.2 concerns multi-index notation and it exploits the multi-index operations that will be used in the next chapters. In section 2.3, we define unilevel and multilevel circulant and Toeplitz matrices and we recall the basic results about their properties. Finally, in section 2.4, we recall the definition of standard iterative methods from literature.

2.1 Notation

- $\mathbb{N}$ is the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,
- For $\mathcal{K} \in \{\mathbb{Z}, \mathbb{R}, \mathbb{C}\}$, $\mathcal{K}^{n\times n}$ is the linear space of $n \times n$ matrices with coefficients in the field $\mathcal{K}$.
- For $\mathcal{K} \in \{\mathbb{Z}, \mathbb{R}, \mathbb{C}\}$, $\mathcal{K}^{n}$ is the linear space of $n \times 1$ column vectors with coefficients in the field $\mathcal{K}$.
- Given $A \in \mathcal{K}^{n\times n}$, we denote by
  - A^T the transpose of A,
  - A^* the conjugate transpose of A,
  - $\lambda_j(A)$, $j = 1, \dots, n$, the eigenvalues of A,
  - $\rho(A) = \max_{j=1,\dots,n} |\lambda_j(A)|$ the spectral radius of A,

    - rank(A) the rank of A,
    - det(A) the determinant of A.

- Given $A \in \mathcal{K}^{n\times n}$, we say that
    - A is Hermitian if $A = A^*$,
    - A is symmetric if $A = A^T$,
    - A is unitary if $AA^* = A^*A = I_n$, where $I_n \in \mathbb{Z}^{n\times n}$ is the identity matrix of order n,
    - A is positive definite if A is Hermitian and $\mathbf{x}^* A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathcal{K}^n$,
    - A is positive semi-definite if A is Hermitian and $\mathbf{x}^* A\mathbf{x} \ge 0$ for all $\mathbf{x} \in \mathcal{K}^n$.

- Given $A, B \in \mathcal{K}^{n\times n}$, we say that
    - $A \prec B$ if $B - A$ is positive definite,
    - $A \preceq B$ if $B - A$ is positive semi-definite.

- If $A \in \mathcal{K}^{n\times n}$ is Hermitian positive semi-definite, $A^{\frac{1}{2}}$ is the nonnegative square root of A and
$$\|\mathbf{x}\|_A = \|A^{\frac{1}{2}}\mathbf{x}\|_2 = \sqrt{\mathbf{x}^*\left(A^{\frac{1}{2}}\right)^* A^{\frac{1}{2}}\mathbf{x}}, \qquad \forall\, \mathbf{x} \in \mathcal{K}^n.$$

- We denote by
    - $\ell(\mathbb{Z}^d)$ the space of complex sequences indexed by $\mathbb{Z}^d$, namely
    $$\ell(\mathbb{Z}^d) = \left\{\mathbf{p} = \{\mathrm{p}(\boldsymbol{\alpha}) \in \mathbb{C} : \boldsymbol{\alpha} \in \mathbb{Z}^d\}\right\},$$
    where $\mathrm{p}(\boldsymbol{\alpha})$ denotes the $\boldsymbol{\alpha}$-th element of the sequence $\mathbf{p}$,
    - $\ell_0(\mathbb{Z}^d) \subset \ell(\mathbb{Z}^d)$ the space of sequences with finite support,
    - $\ell_\infty(\mathbb{Z}^d) \subset \ell(\mathbb{Z}^d)$ the space of bounded sequences equipped with the norm
    $$\|\mathbf{p}\|_\infty = \sup_{\boldsymbol{\alpha}\in\mathbb{Z}^d} |\mathrm{p}(\boldsymbol{\alpha})|, \qquad \forall\, \mathbf{p} \in \ell_\infty(\mathbb{Z}^d).$$

- $\mathcal{C}(\mathbb{R}^d)$, $\mathcal{C}([0,2\pi)^d)$ are the spaces of continuous functions defined over $\mathbb{R}^d$ and $[0,2\pi)^d$, respectively.

- $\mathrm{i} \in \mathbb{C}$ is the imaginary unit, namely $\mathrm{i}^2 = -1$.

- $\otimes$ is the Kronecker product.

- Given $\mathbf{n} \in \mathbb{N}_0^d$, we define $N(\mathbf{n}) = \prod_{i=1}^{d} n_i \in \mathbb{N}_0$.

2.2 Multi-index notation

Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$. We say that $\boldsymbol{\mu}$ is a d-index of length $|\boldsymbol{\mu}| = \mu_1 + \dots + \mu_d$. Let $\mathbf{e}_i \in \mathbb{R}^d$, $i = 1, \dots, d$, be the i-th unit vector of $\mathbb{R}^d$, namely

$$\mathrm{e}_i(k) = \delta_{i,k}, \qquad k = 1, \dots, d.$$

We denote by $D^{\boldsymbol{\mu}}$ the mixed partial derivative $D_{\mathbf{e}_1}^{\mu_1} \dots D_{\mathbf{e}_d}^{\mu_d}$, where $D_{\mathbf{e}_i}^{\mu_i}$, $i = 1, \dots, d$, is the derivative of order μ_i along the direction $\mathbf{e}_i$, namely

$$D_{\mathbf{e}_i}^{\mu_i} f = D_{\mathbf{e}_i}\left(D_{\mathbf{e}_i}^{\mu_i - 1} f\right), \qquad D_{\mathbf{e}_i} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x}) - f(\mathbf{x} - h\mathbf{e}_i)}{h}, \qquad \mathbf{x} \in \mathbb{R}^d, \qquad f \in \mathcal{C}^{|\boldsymbol{\mu}|}(\mathbb{R}^d).$$

Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$ be two d-indexes. The d-indexes operations $\boldsymbol{\mu}^{\boldsymbol{\nu}}$, $\boldsymbol{\mu}\boldsymbol{\nu}$ and $\boldsymbol{\mu}/\boldsymbol{\nu}$ are always intended component-wise, namely

$$\boldsymbol{\mu}^{\boldsymbol{\nu}} = \left(\mu_1^{\nu_1}, \dots, \mu_d^{\nu_d}\right) \in \mathbb{N}_0^d, \qquad \boldsymbol{\mu}\boldsymbol{\nu} = (\mu_1\nu_1, \dots, \mu_d\nu_d) \in \mathbb{N}_0^d, \qquad \boldsymbol{\mu}/\boldsymbol{\nu} = \left(\mu_1/\nu_1, \dots, \mu_d/\nu_d\right) \in \mathbb{Q}^d.$$

Let $\mathbf{z} = (z_1, \dots, z_d) \in (\mathbb{C} \setminus \{0\})^d$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathbb{Z}^d$. We define

$$\mathbf{z}^{\boldsymbol{\mu}} = z_1^{\mu_1} \cdots z_d^{\mu_d} \in \mathbb{C}.$$

2.3 Unilevel and multilevel circulant and Toeplitz matrices

In this section, in subsections 2.3.1 and 2.3.2, we shortly introduce circulant and Toeplitz matrices, respectively. Then, in subsections 2.3.3 and 2.3.4, we briefly describe d-level circulant and d-level Toeplitz matrices, respectively. Our interest is motivated by theoretical and numerical needs. Indeed, in chapter 3 we define an algebraic multigrid method for circulant and d-level circulant matrices. Circulant and d-level circulant matrices build a matrix algebra, and the definition and the analysis of the algebraic multigrid exploit the matrix algebra structure. Even if the theoretical analysis of multigrid is done in the case of circulant and d-level circulant matrices, the resulting multigrid methods are applicable also for solving more general linear systems of equations, in particular, those with Toeplitz and d-level Toeplitz system matrices (see chapters 5 and 6). We recall that the discretization of ODEs and PDEs with constant coefficients define Toeplitz and d-level Toeplitz matrices.

2.3.1 Circulant matrices

Let $n \in \mathbb{N}$. We say that a matrix $A \in \mathbb{C}^{n \times n}$ is *circulant* if it satisfies

$$A = \begin{pmatrix} \mathrm{b}(0) & \mathrm{b}(n-1) & \cdots & \mathrm{b}(2) & \mathrm{b}(1) \\ \mathrm{b}(1) & \mathrm{b}(0) & \mathrm{b}(n-1) & & \mathrm{b}(2) \\ \vdots & \mathrm{b}(1) & \mathrm{b}(0) & \ddots & \vdots \\ \mathrm{b}(n-2) & & \ddots & \ddots & \mathrm{b}(n-1) \\ \mathrm{b}(n-1) & \mathrm{b}(n-2) & \cdots & \mathrm{b}(1) & \mathrm{b}(0) \end{pmatrix}, \qquad \mathrm{b}(\alpha) \in \mathbb{C}, \qquad \alpha = 0, \dots, n-1.$$

A circulant matrix A is determined by a finite sequence of coefficients

$$\mathbf{b} = \{\mathrm{b}(\alpha) \in \mathbb{C} : \alpha = 0, \dots, n-1\} \in \ell_0(\mathbb{Z}).$$

More precisely, the entries of the matrix A satisfy

$$A = \Big(\mathrm{b}\big((\alpha-\beta) \bmod n\big)\Big)_{\alpha,\beta=0,\dots,n-1}.$$

Let $f\colon [0,2\pi) \to \mathbb{C}$ be a trigonometric polynomial of degree $c \in \mathbb{N}$, namely

$$f(x) = \sum_{\substack{\alpha\in\mathbb{Z}\\ |\alpha|\le c}} \mathrm{a}(\alpha)\mathrm{e}^{\mathrm{i}\alpha x}, \qquad x \in [0,2\pi).$$

The finite sequence $\mathbf{a} = \{\mathrm{a}(\alpha) \in \mathbb{C} : \alpha \in \mathbb{Z}, |\alpha| \le c\} \in \ell_0(\mathbb{Z})$ collects the Fourier coefficients of f defined by

$$\mathrm{a}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x, \qquad \alpha \in \mathbb{Z}, \qquad |\alpha| \le c.$$

From the finite sequence $\mathbf{a}$ of the Fourier coefficients of the trigonometric polynomial f, it is possible to define a circulant matrix $A = C_n(f) \in \mathbb{C}^{n\times n}$ by

$$C_n(f) = \Big(\mathrm{a}\big((\alpha-\beta) \bmod n\big) + \mathrm{a}\big(((\alpha-\beta) \bmod n)-n\big)\Big)_{\alpha,\beta=0,\dots,n-1},$$

where we assume $\mathrm{a}(\alpha) = 0$, $\forall\, \alpha \in \mathbb{Z}$ such that $|\alpha| > c$. We say that the matrix $C_n(f)$ is the circulant matrix of order n generated by f. For instance, let

$$f(x) = 5 + 2\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{\mathrm{i}3x} + 4\mathrm{e}^{-\mathrm{i}2x}, \qquad x \in [0,2\pi), \tag{2.1}$$

be a trigonometric polynomial of degree $c = 3$. The non-zero Fourier coefficients of f are

$$\mathrm{a}(0) = 5, \quad \mathrm{a}(1) = 2, \quad \mathrm{a}(3) = -1, \quad \mathrm{a}(-2) = 4.$$

Thus, the matrices $C_5(f)$, $C_6(f)$ become

$$C_5(f) = \begin{pmatrix} 5 & 0 & 3 & 0 & 2\\ 2 & 5 & 0 & 3 & 0\\ 0 & 2 & 5 & 0 & 3\\ 3 & 0 & 2 & 5 & 0\\ 0 & 3 & 0 & 2 & 5 \end{pmatrix} \in \mathbb{C}^{5\times 5}, \qquad C_6(f) = \begin{pmatrix} 5 & 0 & 4 & -1 & 0 & 2\\ 2 & 5 & 0 & 4 & -1 & 0\\ 0 & 2 & 5 & 0 & 4 & -1\\ -1 & 0 & 2 & 5 & 0 & 4\\ 4 & -1 & 0 & 2 & 5 & 0\\ 0 & 4 & -1 & 0 & 2 & 5 \end{pmatrix} \in \mathbb{C}^{6\times 6}.$$

We define the cycling permutation matrix $Z_n \in \mathbb{C}^{n\times n}$ of order n by

$$Z_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \ddots & 1 & 0 & 0\\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{2.2}$$

Then, the circulant matrix $C_n(f)$ satisfies

$$C_n(f) = \sum_{\substack{\alpha\in\mathbb{Z}\\|\alpha|\le c}} \mathrm{a}(\alpha) Z_n^\alpha, \qquad Z_n^\alpha = \underbrace{Z_n \cdots Z_n}_{\alpha \text{ times}}.$$

Due to $\mathrm{a}(\alpha) = \overline{\mathrm{a}(-\alpha)}$, $|\alpha| \le c$, the matrix $C_n(f)$ is Hermitian.
We denote by $F_n \in \mathbb{C}^{n\times n}$ the Fourier matrix of order n, i.e.

$$F_n = \frac{1}{\sqrt{n}}\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi\alpha\beta}{n}}\right)_{\alpha,\beta=0,\dots,n-1}. \tag{2.3}$$

The matrix F_n is symmetric and unitary. Any circulant matrix $C_n(f)$ satisfies ([26])

$$C_n(f) = F_n \Delta_n(f) F_n^*, \qquad \Delta_n(f) = \operatorname*{diag}_{\alpha=0,\dots,n-1} f\left(x_\alpha^{(n)}\right) \in \mathbb{C}^{n\times n}, \qquad x_\alpha^{(n)} = \frac{2\pi\alpha}{n}. \tag{2.4}$$

Thus, $f(x_\alpha^{(n)})$, $\alpha = 0,\dots,n-1$, are the eigenvalues of $C_n(f)$.
Most of the properties of $C_n(f)$ are encoded in the trigonometric polynomial f. Due to (2.4),

- ⋆ if f is real, then $C_n(f)$ is symmetric,
- ⋆ if f is real and $f \ge 0$, then $C_n(f)$ is positive semi-definite,
- ⋆ if f is real, $f \ge 0$ and $f(x_\alpha^{(n)}) \ne 0$, $\alpha = 0,\dots,n-1$, then $C_n(f)$ is non-singular.

2.3.2 Toeplitz matrices

Let $n \in \mathbb{N}$. We say that a matrix $A \in \mathbb{C}^{n\times n}$ is *Toeplitz* if it satisfies

$$A = \begin{pmatrix} \mathrm{b}(0) & \mathrm{b}(-1) & \cdots & \mathrm{b}(2-n) & \mathrm{b}(1-n) \\ \mathrm{b}(1) & \mathrm{b}(0) & \mathrm{b}(-1) & & \mathrm{b}(2-n) \\ \vdots & \mathrm{b}(1) & \mathrm{b}(0) & \ddots & \vdots \\ \mathrm{b}(n-2) & & \ddots & \ddots & \mathrm{b}(-1) \\ \mathrm{b}(n-1) & \mathrm{b}(n-2) & \cdots & \mathrm{b}(1) & \mathrm{b}(0) \end{pmatrix}, \qquad \mathrm{b}(\alpha) \in \mathbb{C}, \qquad \alpha = 1-n,\dots,0,\dots,n-1.$$

A Toeplitz matrix A is determined by a finite sequence of coefficients

$$\mathbf{b} = \{\mathrm{b}(\alpha) \in \mathbb{C} : \alpha = 1-n,\dots,0,\dots,n-1\} \in \ell_0(\mathbb{Z}).$$

More precisely, the entries of the matrix A satisfy

$$A = \left(\mathrm{b}(\alpha-\beta)\right)_{\alpha,\beta=0,\dots,n-1}.$$

As for the circulant case, it is possible to establish a relationship between Toeplitz matrices and trigonometric polynomials. More precisely, let f be a trigonometric polynomial of degree

$c \in \mathbb{N}$. Let $\mathbf{a} = \{\mathrm{a}(\alpha) \in \mathbb{C} : \alpha \in \mathbb{Z}, |\alpha| \le c\} \in \ell_0(\mathbb{Z})$ be the collection of the Fourier coefficients of f. For $n > c$, it is possible to define the Toeplitz matrix $A = T_n(f)$ by

$$T_n(f) = \left(\mathrm{a}(\alpha - \beta)\right)_{\alpha,\beta=0,\ldots,n-1} \in \mathbb{C}^{n\times n},$$

where we assume $\mathrm{a}(\alpha) = 0$, $\forall\, \alpha \in \mathbb{Z}$ such that $|\alpha| > c$. We say that the matrix $T_n(f)$ is the Toeplitz matrix of order n generated by f. For instance, let f be defined as in (2.1). Then, the matrix $T_5(f)$ becomes

$$T_5(f) = \begin{pmatrix} 5 & 0 & 4 & 0 & 0 \\ 2 & 5 & 0 & 4 & 0 \\ 0 & 2 & 5 & 0 & 4 \\ -1 & 0 & 2 & 5 & 0 \\ 0 & -1 & 0 & 2 & 5 \end{pmatrix} \in \mathbb{C}^{5\times 5}.$$

For $\alpha \in \mathbb{Z}$, $|\alpha| < n$, we define the matrix $J_n^{(\alpha)} \in \mathbb{Z}^{n\times n}$ by

$$J_n^{(\alpha)}(\beta,\gamma) = \begin{cases} 1, & \beta - \gamma = \alpha, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta,\gamma = 0,\ldots,n-1. \tag{2.5}$$

Then, the Toeplitz matrix $T_n(f)$ satisfies

$$T_n(f) = \sum_{\substack{\alpha\in\mathbb{Z} \\ |\alpha|\le c}} \mathrm{a}(\alpha) J_n^{(\alpha)}.$$

Similarly to circulant matrices, most of the properties of $T_n(f)$ are dictated by the behavior of the trigonometric polynomial f. Indeed, the following properties are satisfied ([72]):

- ⋆ if f is real, then $T_n(f)$ is Hermitian,
- ⋆ if f is real and even, then $T_n(f)$ is symmetric,
- ⋆ if f is real and $f \ge 0$, then $T_n(f)$ is positive semi-definite,
- ⋆ if f is real, $f \ge 0$ and f vanishes on a set of Lebesgue measure 0, then $T_n(f)$ is positive definite.

2.3.3 d-LEVEL CIRCULANT MATRICES

Let $d \in \mathbb{N}$, $d > 1$. A d-level circulant matrix can be "recursively" defined as a circulant block matrix with $(d-1)$-level circulant blocks. For instance, let $d = 2$ and $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$. The bi-level circulant matrix $A_{\mathbf{n}} \in \mathbb{C}^{n_1 n_2 \times n_1 n_2}$ is $n_1 \times n_1$ block circulant with circulant blocks of size $n_2 \times n_2$, namely

$$A = \begin{pmatrix} A_0 & A_{n_1-1} & \cdots & A_2 & A_1 \\ A_1 & A_0 & A_{n_1-1} & & A_2 \\ \vdots & A_1 & A_0 & \ddots & \vdots \\ A_{n_1-2} & & \ddots & \ddots & A_{n_1-1} \\ A_{n_1-1} & A_{n_1-2} & \cdots & A_1 & A_0 \end{pmatrix},$$

where $A_\alpha \in \mathbb{C}^{n_2\times n_2}$, $\alpha = 0,\ldots,n_1-1$, are circulant matrices.
Let $f\colon [0,2\pi)^d \to \mathbb{C}$ be a d-variate trigonometric polynomial of total degree $c \in \mathbb{N}$, namely

$$f(\mathbf{x}) = \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{Z}^d\\ |\boldsymbol{\alpha}|\le c}} \mathrm{a}(\boldsymbol{\alpha})\,\mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}^T\cdot\mathbf{x}}, \qquad \boldsymbol{\alpha}^T\cdot\mathbf{x} = \sum_{i=i}^{d}\alpha_i x_i, \qquad \mathbf{x}\in[0,2\pi)^d.$$

Let $\mathbf{a} = \{\mathrm{a}(\boldsymbol{\alpha})\in\mathbb{C} : \boldsymbol{\alpha}\in\mathbb{Z}^d, |\boldsymbol{\alpha}|\le c\} \in \ell_0(\mathbb{Z}^d)$ be the finite sequence of the Fourier coefficients of f, namely

$$\mathrm{a}(\boldsymbol{\alpha}) = \frac{1}{(2\pi)^d}\int_{[0,2\pi)^d} f(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}^T\cdot\mathbf{x}}\mathrm{d}x, \qquad \boldsymbol{\alpha}\in\mathbb{Z}^d, \qquad |\boldsymbol{\alpha}|\le c.$$

Then, the d-level circulant matrix of order $\mathbf{n} = (n_1,\ldots,n_d)\in\mathbb{N}^d$ generated by f is defined by

$$C_{\mathbf{n}}(f) = \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{Z}^d\\ |\boldsymbol{\alpha}|\le c}} \mathrm{a}(\boldsymbol{\alpha})\,Z_{\mathbf{n}}^{(\boldsymbol{\alpha})} \in \mathbb{C}^{N\times N}, \qquad N = N(\mathbf{n}) = \prod_{i=1}^{d} n_i,$$

where $Z_{\mathbf{n}}^{(\boldsymbol{\alpha})} = (Z_{n_1})^{\alpha_1}\otimes\ldots\otimes(Z_{n_d})^{\alpha_d}\in\mathbb{C}^{N\times N}$ and $Z_{n_i}\in\mathbb{C}^{n_i\times n_i}$, $i=1,\ldots,d$, is the cycling permutation matrix of order n_i defined by (2.2). Due to $\mathrm{a}(\boldsymbol{\alpha}) = \overline{\mathrm{a}(-\boldsymbol{\alpha})}$, $|\boldsymbol{\alpha}|\le c$, the matrix $C_{\mathbf{n}}(f)$ is Hermitian.
Defining the d-dimensional Fourier matrix $F_{\mathbf{n}}$ of order $\mathbf{n}$ by

$$F_{\mathbf{n}} = F_{n_1}\otimes\ldots\otimes F_{n_d}\in\mathbb{C}^{N\times N}, \tag{2.6}$$

where $F_{n_i}\in\mathbb{C}^{n_i\times n_i}$, $i=1,\ldots,d$, is the unidimensional Fourier matrix of order n_i defined by (2.3), the matrix $C_{\mathbf{n}}(f)$ satisfies

$$C_{\mathbf{n}}(f) = F_{\mathbf{n}}\Delta_{\mathbf{n}}(f)F_{\mathbf{n}}^*, \qquad \Delta_{\mathbf{n}}(f) = \operatorname*{diag}_{\mathbf{0}\le\boldsymbol{\alpha}\le\mathbf{n}-\mathbf{1}} f\left(\mathbf{x}_{\boldsymbol{\alpha}}^{(\mathbf{n})}\right)\in\mathbb{C}^{N\times N}, \tag{2.7}$$

where

$$\mathbf{x}_{\boldsymbol{\alpha}}^{(\mathbf{n})} = \left(x_{\alpha_1}^{(n_1)},\ldots,x_{\alpha_d}^{(n_d)}\right)\in\mathbb{C}^d, \qquad x_{\alpha_i}^{(n_i)} = \frac{2\pi\alpha_i}{n_i}, \qquad i=1,\ldots,d$$

and $\mathbf{0}\le\boldsymbol{\alpha}\le\mathbf{n}-\mathbf{1}$ means that $0\le\alpha_i\le n_i-1$ for $i=1,\ldots,d$ (assuming the standard lexicographic ordering).
The properties of $C_{\mathbf{n}}(f)$ are encoded in the trigonometric polynomial f. Especially, if f is real, $f\ge 0$, and $f\left(\mathbf{x}_{\boldsymbol{\alpha}}^{(\mathbf{n})}\right)\ne 0$, $\mathbf{0}\le\boldsymbol{\alpha}\le\mathbf{n}-\mathbf{1}$, then $C_{\mathbf{n}}(f)$ is positive definite.

2.3.4 d-level Toeplitz matrices

Let $d\in\mathbb{N}$, $d>1$. Similarly to the d-level circulant case, a d-level Toeplitz matrix is a Toeplitz block matrix with $(d-1)$-level Toeplitz blocks. For instance, let $d=2$ and $\mathbf{n}=(n_1,n_2)\in\mathbb{N}^2$. The bi-level Toeplitz matrix $A_{\mathbf{n}} \in \mathbb{C}^{n_1 n_2 \times n_1 n_2}$ is $n_1 \times n_1$ block Toeplitz with Toeplitz blocks of size $n_2 \times n_2$, namely

$$A = \begin{pmatrix} A_0 & A_{-1} & \cdots & A_{2-n_1} & A_{1-n_1} \\ A_1 & A_0 & A_{-1} & & A_{2-n_1} \\ \vdots & A_1 & A_0 & \ddots & \vdots \\ A_{n_1-2} & & \ddots & \ddots & A_{-1} \\ A_{n_1-1} & A_{n_1-2} & \cdots & A_1 & A_0 \end{pmatrix},$$

where $A_\alpha \in \mathbb{C}^{n_2 \times n_2}$, $\alpha = 1-n_1, \ldots, 0, \ldots, n_1-1$, are Toeplitz matrices.
Let f be a d-variate trigonometric polynomial of total degree $c \in \mathbb{N}$, and let

$$\mathbf{a} = \{\mathrm{a}(\boldsymbol{\alpha}) \in \mathbb{C} : \boldsymbol{\alpha} \in \mathbb{Z}^d, |\boldsymbol{\alpha}| \le c\} \in \ell_0(\mathbb{Z}^d),$$

be the finite sequence of the Fourier coefficients of f. For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $\min\limits_{i=1,\ldots,d} n_i > c$, the d-level Toeplitz matrix of order $\mathbf{n}$ generated by f is defined by

$$T_{\mathbf{n}}(f) = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{Z}^d \\ |\boldsymbol{\alpha}| \le c}} \mathrm{a}(\boldsymbol{\alpha})\, J_{\mathbf{n}}^{(\boldsymbol{\alpha})} \in \mathbb{C}^{N \times N}, \qquad N = N(\mathbf{n}) = \prod_{j=i}^{d} n_i,$$

where $J_{\mathbf{n}}^{(\boldsymbol{\alpha})} = J_{n_1}^{(\alpha_1)} \otimes \ldots \otimes J_{n_d}^{(\alpha_d)} \in \mathbb{C}^{N \times N}$ and $J_{n_i}^{(\alpha_i)} \in \mathbb{C}^{n_i \times n_i}$, $i = 1, \ldots, d$, is the Toeplitz matrix of order n_i defined by (2.5).
The properties of $T_n(f)$ stated at the end of subsection 2.3.2 hold also for $T_{\mathbf{n}}(f)$. Especially, if f is real, $f \ge 0$, and f vanishes on a set of Lebesgue measure 0, then $T_{\mathbf{n}}(f)$ is positive definite.

2.4 Iterative methods

In this section, we shortly explain the main idea behind the construction of iterative methods. Then, in subsections 2.4.1, 2.4.2 and 2.4.3, we present Richardson, Jacobi and Gauss-Seidel methods, respectively. These three methods are well-known stationary iterative methods from literature.

Iterative methods are widely used for the solution of large systems of equations

$$A_n \mathbf{x} = \mathbf{b}_n, \qquad A_n \in \mathbb{C}^{n \times n}, \qquad \mathbf{b}_n \in \mathbb{C}^n, \qquad n \in \mathbb{N}.$$

They generate a sequence of approximate solutions which, under appropriate hypothesis on the system matrix A_n, converges to the exact solution $\overline{\mathbf{x}} = A_n^{-1} \mathbf{b}_n \in \mathbb{C}^n$. We underline two critical aspects of iterative methods:

- they involve the system matrix A_n only in the context of matrix-vector multiplication,
- their efficacy focuses on how quickly they converge to the exact solution.

A large class of iterative methods is based on the matrix splitting

$$A_n = M_n - N_n, \qquad M_n, N_n \in \mathbb{C}^{n\times n},$$

where M_n is non-singular and "easily" invertible. Indeed, we have

$$\begin{aligned} A_n\bar{\mathbf{x}} = \mathbf{b}_n \quad &\Longleftrightarrow \quad (M_n - N_n)\bar{\mathbf{x}} = \mathbf{b}_n \\ &\Longleftrightarrow \quad M_n\bar{\mathbf{x}} = N_n\bar{\mathbf{x}} + \mathbf{b}_n \\ &\Longleftrightarrow \quad \bar{\mathbf{x}} = M_n^{-1}N_n\bar{\mathbf{x}} + M_n^{-1}\mathbf{b}_n. \end{aligned} \tag{2.8}$$

Using (2.8), an iterative method generates a sequence of iterates $\{\mathbf{x}^{(k)} \in \mathbb{C}^n : k \in \mathbb{N}\}$, defined by

$$\mathbf{x}^{(k+1)} = M_n^{-1}N_n\mathbf{x}^{(k)} + M_n^{-1}\mathbf{b}_n = V_n\mathbf{x}^{(k)} + M_n^{-1}\mathbf{b}_n, \qquad V_n = M_n^{-1}N_n \in \mathbb{C}^{n\times n}, \qquad \mathbf{x}^{(0)} \in \mathbb{C}^n. \tag{2.9}$$

The convergence of an iterative method depends on the spectral properties of the iteration matrix V_n. Indeed, for $k \in \mathbb{N}$, we define $\mathbf{e}^{(k)} = \bar{\mathbf{x}} - \mathbf{x}^{(k)} \in \mathbb{C}^n$ the error at the k-th iterate. Then, we have

$$\begin{aligned} \mathbf{e}^{(k+1)} &= \bar{\mathbf{x}} - \mathbf{x}^{(k+1)} \\ &= V_n\bar{\mathbf{x}} + M_n^{-1}\mathbf{b}_n - \left(V_n\mathbf{x}^{(k)} + M_n^{-1}\mathbf{b}_n\right) \\ &= V_n\left(\bar{\mathbf{x}} - \mathbf{x}^{(k)}\right) \\ &= V_n\mathbf{e}^{(k)} = \cdots = V_n^{k+1}\mathbf{e}^{(0)}, \qquad \mathbf{e}^{(0)} = \bar{\mathbf{x}} - \mathbf{x}^{(0)}. \end{aligned}$$

Thus, $\forall\, \mathbf{e}^{(0)} \in \mathbb{C}^n$, we get

$$\lim_{k\to\infty} \|\mathbf{e}^{(k)}\| = 0 \quad \Longleftrightarrow \quad \rho(V_n) < 1.$$

Next, we introduce the well-known iterative methods of Richardson, Jacobi and Gauss-Seidel. We assume that the system matrix A_n is positive definite, thus $0 < \lambda_1(A_n) \le \cdots \le \lambda_n(A_n)$ and $\rho(A_n) = \lambda_n(A_n)$. We define

$$K(A_n) := \|A_n\|\,\|A_n^{-1}\| = \frac{\lambda_n(A_n)}{\lambda_1(A_n)},$$

where the last equality is due to A_n being positive definite. The value $K(A_n)$ is called *condition number* of the system matrix A_n and gives a bound on how inaccurate the solution will be after approximation. If $K(A_n) >> 1$, that is $\lambda_1(A_n) << 1$ or $\lambda_n(A_n) >> 1$, the linear system $A_n\mathbf{x} = \mathbf{b}_n$ is said to be bad conditioned. In this case, whenever the data A_n or $\mathbf{b}_n$ are effected by errors, it is impossible to find an accurate approximation of the solution $\bar{\mathbf{x}}$.

2.4.1 Richardson

The Richardson method [61] is defined by the matrix splitting $A_n = M_n - N_n$, where $M_n = I_n$ is the identity matrix of order n and $N_n = I_n - A_n$. Given the starting guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$, the

Richardson method defines

$$\begin{aligned}\mathbf{x}^{(k+1)} &= M_n^{-1}N_n\mathbf{x}^{(k)} + M_n^{-1}\mathbf{b}_n\\ &= (I_n - A_n)\mathbf{x}^{(k)} + \mathbf{b}_n\\ &= \mathbf{x}^{(k)} + \left(\mathbf{b}_n - A_n\mathbf{x}^{(k)}\right)\\ &= \mathbf{x}^{(k)} + \mathbf{r}^{(k)},\end{aligned}$$

where we denote by $\mathbf{r}^{(k)} = \mathbf{b}_n - A_n\mathbf{x}^{(k)}$ the residual of the k-th iterate.

Introducing a relaxation parameter $\omega \in \mathbb{R}^+$, it is possible to define the weighted Richardson method by the splitting $A_n = M_n^{\omega} - N_n^{\omega}$, where $M_n^{\omega} = \omega^{-1}I_n$ and $N_n^{\omega} = \omega^{-1}I_n - A_n$. Thus, we have

$$\begin{aligned}\mathbf{x}^{(k+1)} &= (M_n^{\omega})^{-1}N_n^{\omega}\mathbf{x}^{(k)} + (M_n^{\omega})^{-1}\mathbf{b}_n\\ &= \omega(\omega^{-1}I_n - A_n)\mathbf{x}^{(k)} + \omega\mathbf{b}_n\\ &= \mathbf{x}^{(k)} + \omega\left(\mathbf{b}_n - A_n\mathbf{x}^{(k)}\right)\\ &= \mathbf{x}^{(k)} + \omega\mathbf{r}^{(k)}.\end{aligned}$$

The relaxation parameter $\omega \in \mathbb{R}^+$ weights the contribution of the residual $\mathbf{r}^{(k)}$ to the definition of the iterate $\mathbf{x}^{(k+1)}$. The iteration matrix becomes

$$V_n^{\omega} = (M_n^{\omega})^{-1}N_n^{\omega} = I_n - \omega A_n. \tag{2.10}$$

The parameter ω should be chosen in order to guarantee the convergence of the method. More precisely, ω should ensure that

$$\rho(V_n^{\omega}) = \rho(I_n - \omega A_n) = |1 - \omega\lambda_1(A_n)| < 1.$$

Taking $\omega = \lambda_n(A_n)^{-1} > 0$, we get

$$1 > \rho(V_n^{\omega}) = 1 - \frac{\lambda_1(A_n)}{\lambda_n(A_n)} = 1 - K(A_n)^{-1}.$$

Thus, the worse is the condition number of A_n, the slower is the Richardson method.

2.4.2 Jacobi

The Jacobi method [51] is defined by the matrix splitting $A_n = M_n - N_n$, where $M_n = D_n$ is the diagonal matrix with diagonal entries of A_n, namely

$$D_n = \begin{pmatrix} \mathrm{a}(1,1) & & \\ & \ddots & \\ & & \mathrm{a}(n,n)\end{pmatrix} \in \mathbb{C}^{n\times n} \tag{2.11}$$

and $N_n = D_n - A_n = -(L_n + U_n)$, where L_n and U_n are the lower and upper triangular parts of A_n respectively, namely

$$L_n = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \mathrm{a}(2,1) & 0 & \cdots & & \vdots \\ \mathrm{a}(3,1) & \mathrm{a}(3,2) & \ddots & & 0 \\ \vdots & & & 0 & 0 \\ \mathrm{a}(n,1) & \mathrm{a}(n,2) & \cdots & \mathrm{a}(n,n-1) & 0 \end{pmatrix} \in \mathbb{C}^{n\times n}, \tag{2.12}$$

$$U_n = \begin{pmatrix} 0 & \mathrm{a}(1,2) & \cdots & \mathrm{a}(1,n-1) & \mathrm{a}(1,n) \\ 0 & 0 & & & \vdots \\ 0 & & \ddots & \mathrm{a}(n-2,n-1) & \mathrm{a}(n-2,n) \\ \vdots & & \cdots & 0 & \mathrm{a}(n-1,n) \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{n\times n}. \tag{2.13}$$

Notice that $A_n = D_n + L_n + U_n$.

Let A_n be positive definite, thus the diagonal matrix D_n is invertible. Given the starting guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$, the Jacobi method defines

$$\begin{aligned} \mathbf{x}^{(k+1)} &= M_n^{-1} N_n \mathbf{x}^{(k)} + M_n^{-1}\mathbf{b}_n \\ &= D_n^{-1}(D_n - A_n)\mathbf{x}^{(k)} + D_n^{-1}\mathbf{b}_n \\ &= (I_n - D_n^{-1}A_n)\mathbf{x}^{(k)} + D_n^{-1}\mathbf{b}_n. \end{aligned}$$

Similarly to the Richardson method, introducing the relaxation parameter $\omega \in \mathbb{R}^+$, it is possible to define the weighted Jacobi method by the splitting $A_n = M_n^{\omega} - N_n^{\omega}$, where $M_n^{\omega} = \omega^{-1}D_n$ and $N_n^{\omega} = \omega^{-1}D_n - A_n$. Thus, we have

$$\begin{aligned} \mathbf{x}^{(k+1)} &= (M_n^{\omega})^{-1} N_n^{\omega}\mathbf{x}^{(k)} + (M_n^{\omega})^{-1}\mathbf{b}_n \\ &= \omega D_n^{-1}(\omega^{-1}D_n - A_n)\mathbf{x}^{(k)} + \omega D_n^{-1}\mathbf{b}_n \\ &= (I_n - \omega D_n^{-1}A_n)\mathbf{x}^{(k)} + \omega D_n^{-1}\mathbf{b}_n \\ &= (1-\omega)\mathbf{x}^{(k)} + \omega \underbrace{\left((I_n - D_n^{-1}A_n)\mathbf{x}^{(k)} + D_n^{-1}\mathbf{b}_n\right)}_{\mathbf{y}^{(k+1)}}. \end{aligned}$$

From the latter, we can interpreter the iterate $\mathbf{x}^{(k+1)}$ as a mean with weight ω between the previous iterate $\mathbf{x}^{(k)}$ and the iterate $\mathbf{y}^{(k+1)}$ defined by the standard Jacobi method. The iteration matrix becomes

$$V_n^{\omega} = (M_n^{\omega})^{-1} N_n^{\omega} = I_n - \omega D_n^{-1}A_n. \tag{2.14}$$

If the system matrix A_n is strictly diagonally dominant, i.e.

$$|\mathrm{a}(\alpha,\alpha)| > \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} |\mathrm{a}(\alpha,\beta)|, \qquad \alpha = 1,\ldots,n,$$

and $\omega \in [0,1]$, then the weighted Jacobi method is convergent. Indeed, we have

$$\rho(V_n^{\omega}) \leq \|V_n^{\omega}\|_\infty = \|I_n - \omega D_n^{-1} A_n\|_\infty = \max_{\alpha=1,\ldots,n}\left(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \omega \left|\frac{\mathrm{a}(\alpha,\beta)}{\mathrm{a}(\alpha,\alpha)}\right| + |1-\omega|\right) < \omega + 1 - \omega = 1.$$

Let $\alpha \in \mathbb{N}$. The α-th entry of the new iterate $\mathbf{x}^{(k+1)}$ can be computed by

$$\mathrm{x}^{(k+1)}(\alpha) = \frac{1}{\mathrm{a}(\alpha,\alpha)}\left(\mathrm{b}_n(\alpha) - \sum_{\beta=1}^{\alpha-1} \mathrm{a}(\alpha,\beta)\mathrm{x}^{(k)}(\beta) - \sum_{\beta=\alpha+1}^{n} \mathrm{a}(\alpha,\beta)\mathrm{x}^{(k)}(\beta)\right).$$

Notice that the computation of $\mathrm{x}^{(k+1)}(\alpha)$ does not use the most recently available information $\mathrm{x}^{(k+1)}(\beta)$, $\beta = 1,\ldots,\alpha-1$. To overcome this issue, the Gauss-Seidel method has been introduced.

2.4.3 Gauss-Seidel

The Gauss-Seidel method [45] is defined by the matrix splitting $A_n = M_n - N_n$, where $M_n = D_n + L_n$ and $N_n = D_n + L_n - A_n = -U_n$, with D_n in (2.11) and L_n, U_n in (2.12). Given the starting guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$, the Gauss-Seidel method defines

$$\begin{aligned}\mathbf{x}^{(k+1)} &= M_n^{-1} N_n \mathbf{x}^{(k)} + M_n^{-1}\mathbf{b}_n\\ &= -(D_n + L_n)^{-1} U_n \mathbf{x}^{(k)} + (D_n + L_n)^{-1}\mathbf{b}_n.\end{aligned}$$

The iteration matrix becomes

$$V_n = M_n^{-1} N_n = -(D_n + L_n)^{-1} U_n. \tag{2.15}$$

Let A_n be positive definite. Then, the Gauss-Seidel method is convergent. We omit the proof and refer e.g. to [46].

Let $\alpha \in \mathbb{N}$. The α-th entry of the new iterate $\mathbf{x}^{(k+1)}$ can be computed by

$$\mathrm{x}^{(k+1)}(\alpha) = \frac{1}{\mathrm{a}(\alpha,\alpha)}\left(\mathrm{b}_n(\alpha) - \sum_{\beta=1}^{\alpha-1} \mathrm{a}(\alpha,\beta)\mathrm{x}^{(k+1)}(\beta) - \sum_{\beta=\alpha+1}^{n} \mathrm{a}(\alpha,\beta)\mathrm{x}^{(k)}(\beta)\right).$$

Notice that the computation of $\mathrm{x}^{(k+1)}(\alpha)$ uses the most current estimation of $\mathbf{x}^{(k+1)}$, namely $\mathrm{x}^{(k+1)}(\beta)$, $\beta = 1,\ldots,\alpha-1$.

CHAPTER 3

Algebraic multigrid

In this chapter, we define and analyze the algebraic multgrid for circulant and d-level circulant matrices. First, in sections 3.1 and 3.2, we introduce the algebraic two-grid method and V-cycle method, respectively, for the solution of linear systems of equations

$$A_n\mathbf{x} = \mathbf{b}_n, \qquad A_n \in \mathbb{C}^{n\times n}, \qquad \mathbf{b}_n \in \mathbb{C}^n, \tag{3.1}$$

where the system matrix A_n is positive definite. We give a proper description of the algebraic multgrid methods and recall the well-known results about their convergence and optimality following the theory in [63]. Then, in sections 3.3 and 3.4, we focus on algebraic multigrid methods for circulant and d-level circulant matrices, respectively. The description we are going to present is characterized by a downsampling/upsampling strategy with the factor $m \in \mathbb{N}$, $m \geq 2$, for the circulant matrix algebra, and with the factor $\mathbf{m} \in \mathbb{N}^d$, $m_i \geq 2$, $i = 1,\ldots,d$, for the d-level circulant matrix algebra. For the "standard" cases $m = 2$ and $\mathbf{m} = (2,\ldots,2) \in \mathbb{N}^d$, the definition of the algebraic multgrid methods is well-understood and the corresponding convergence theory has been fully investigated, see [2, 3, 69]. In our general setting, the definition of the algebraic multgrid methods is extremely recent and the convergence analysis is still incomplete. In [6, 34], the authors provide sufficient conditions for the convergence and optimality of the two-grid method. In section 3.5, we slightly relax these conditions, see *(ii)* of Theorem 3.7, as preparatory result for the analysis carried out in chapter 4. Then, we provide sufficient conditions for the convergence of the V-cycle method, see Theorem 3.8. At the best of our knowledge, Theorem 3.8 is the first result concerning the convergence of the V-cycle method for circulant and d-level circulant matrices with general downsampling/upsampling strategies. Finally, in subsection 3.5.4, we give a few examples from literature of grid transfer operators which already hint at the possibility of a connection between algebraic multigrid and stationary subdivision.

3.1 Algebraic Two-grid method

A basic two-grid method (TGM) combines the action of a *smoother* and a *coarse grid correction operator*: the smoother is often a simple iterative method such as Gauss-Seidel, weighted Jacobi or weighted Richardson [3, 70]; the coarse grid correction operator amounts to solving exactly the residual equation associated to the approximation computed by the smoother on a coarser space where the smoother is ineffective.

Let $n_0 \in \mathbb{N}$ be a positive integer. To design two-grid methods for solving linear systems of the form (3.1) with positive definite system matrices A_{n_0}, we define

- ⋆ $n_1 \in \mathbb{N}$, $n_1 < n_0$ the dimension of the coarse space at which we project our problem,
- ⋆ the grid transfer operator $P_{n_0} \in \mathbb{C}^{n_0 \times n_1}$, usually called *prolongation*, which is a full-rank rectangular matrix, $\mathrm{rank}(P_{n_0}) = n_1$, and
- ⋆ a class $\mathcal{V}(\cdot)$ of iterative methods of the form (2.9).

In the following, we define the coarse grid correction operator by the *Galerkin approach*, which is characterized by the following two conditions:

- ⋆ the *restriction* is the conjugate transpose of the prolongation, i.e., $P^*_{n_0}$,
- ⋆ the coarser matrix is defined by $A_{n_1} = P^*_{n_0} A_{n_0} P_{n_0}$.

Let $\mathcal{V}_{n,\mathrm{pre}}$, $\mathcal{V}_{n,\mathrm{post}}$ be some iterative methods from $\mathcal{V}(\cdot)$ and $\nu_{\mathrm{pre}}, \nu_{\mathrm{post}} \in \mathbb{N}_0$, be the numbers of pre- and post-smoothing steps, respectively. The TGM method determines a sequence of iterates

$$\{\mathbf{x}^{(k)}_{n_0} \in \mathbb{C}^{n_0} \,:\, k \in \mathbb{N}\}, \qquad \mathbf{x}^{(k+1)}_{n_0} = \mathrm{TGM}\big(\mathcal{V}^{\nu_{\mathrm{pre}}}_{n_0,\mathrm{pre}}, \mathcal{V}^{\nu_{\mathrm{post}}}_{n_0,\mathrm{post}}, P_{n_0}\big)\big(\mathbf{x}^{(k)}_{n_0}\big), \qquad \mathbf{x}^{(0)}_{n_0} \in \mathbb{C}^{n_0},$$

where the mapping $\mathrm{TGM} : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$ is defined by

$$\mathrm{TGM}\big(\mathcal{V}^{\nu_{\mathrm{pre}}}_{n_0,\mathrm{pre}}, \mathcal{V}^{\nu_{\mathrm{post}}}_{n_0,\mathrm{post}}, P_{n_0}\big)\big(\mathbf{x}^{(k)}_{n_0}\big)$$

1.	Pre-smoother:	$\mathbf{x}^{(k)}_{n_0} = \mathcal{V}^{\nu_{\mathrm{pre}}}_{n_0,\mathrm{pre}}\big(\mathbf{x}^{(k)}_{n_0}\big)$
2.	Residual on the fine grid:	$\mathbf{r}^{(k)}_{n_0} = \mathbf{b}_{n_0} - A_{n_0}\mathbf{x}^{(k)}_{n_0}$
3.	Projection of residual on the coarse grid:	$\mathbf{r}^{(k)}_{n_1} = P^*_{n_0}\mathbf{r}^{(k)}_{n_0}$
4.	Error equation:	$\mathbf{x}^{(k+1)}_{n_1} = A^{-1}_{n_1}\mathbf{r}^{(k)}_{n_1}$
5.	Correction of the previous smoothed iterate:	$\mathbf{x}^{(k+1)}_{n_0} = \mathbf{x}^{(k)}_{n_0} + P_{n_j}\mathbf{x}^{(k+1)}_{n_1}$
6.	Post-smoother:	$\mathbf{x}^{(k+1)}_{n_0} = \mathcal{V}^{\nu_{\mathrm{post}}}_{n_0,\mathrm{post}}\big(\mathbf{x}^{(k+1)}_{n_0}\big)$

For the sake of clarity, we depict the action of steps 2. – 5. in the following diagram.

$$\begin{tikzcd}[column sep=small]
\mathbb{C}^{n_0}: & \mathbf{r}_{n_0}^{(k)} = \mathbf{b}_{n_0} - A_{n_0}\mathbf{x}_{n_0}^{(k)} \arrow[dr, "P_{n_0}^*"] & & \mathbf{x}_{n_0}^{(k+1)} = \mathbf{x}_{n_0}^{(k)} + P_{n_0}\mathbf{x}_{n_1}^{k+1} \\
\mathbb{C}^{n_1}: & & \begin{cases} \mathbf{r}_{n_1}^{(k)} = P_{n_0}^*\mathbf{r}_{n_0}^{(k)} \\ \mathbf{x}_{n_1}^{(k+1)} = A_{n_1}^{-1}\mathbf{r}_{n_1}^{(k)} \end{cases} \arrow[ur, "P_{n_0}"] &
\end{tikzcd}$$

Steps 2. – 5. in the above algorithm define the *coarse grid correction* (CGC) operator on $\mathbb{C}^{n_0}$ by

$$\mathbf{x}_{n_0}^{(k+1)} = CGC_{n_0}\mathbf{x}_{n_0}^{(k)}, \quad CGC_{n_0} = I_{n_0} - P_{n_0}A_{n_1}^{-1}P_{n_0}^*A_{n_0} = I_{n_0} - P_{n_0}\left(P_{n_0}^*A_{n_0}P_{n_0}\right)^{-1}P_{n_0}^*A_{n_0}, \quad k \in \mathbb{N}_0.$$

Define $V_{n_0,\text{pre}}$, $V_{n_0,\text{post}}$ the iteration matrices of $\mathcal{V}_{n_0,\text{pre}}^{\nu_{\text{pre}}}$, $\mathcal{V}_{n_0,\text{post}}^{\nu_{\text{pre}}}$, respectively. The global iteration matrix of the TGM is then given by

$$TGM = \left(V_{n_0,\text{post}}\right)^{\nu_{\text{post}}} CGC_{n_0}\left(V_{n_0,\text{pre}}\right)^{\nu_{\text{pre}}}. \tag{3.2}$$

Theorem 3.1 is a well-known result from [3, 63], which provides sufficient conditions for convergence of TGM. To formulate Theorem 3.1, we define $D_{n_0} \in \mathbb{C}^{n_0\times n_0}$ to be the diagonal matrix with the diagonal entries of A_{n_0}.

Theorem 3.1. *Let $n_0, n_1 \in \mathbb{N}$, $n_0 > n_1$, $A_{n_0} \in \mathbb{C}^{n_0\times n_0}$ positive definite, $\mathcal{V}_{n_0,pre}, \mathcal{V}_{n_0,post} \in \mathcal{V}(\cdot)$, $\nu_{pre}, \nu_{post} \in \mathbb{N}_0$ and $P_{n_0} \in \mathbb{C}^{n_0\times n_1}$, $\operatorname{rank}(P_{n_0}) = n_1$. If*

(i) $\exists\, \alpha_{pre} > 0$ independent of n_0 such that

$$\|\left(V_{n_0,pre}\right)^{\nu_{pre}}\mathbf{x}_{n_0}\|_{A_{n_0}}^2 \le \|\mathbf{x}_{n_0}\|_{A_{n_0}}^2 - \alpha_{pre}\|\left(V_{n_0,pre}\right)^{\nu_{pre}}\mathbf{x}_{n_0}\|_{A_{n_0}^2}^2, \qquad \forall \mathbf{x}_{n_0} \in \mathbb{C}^{n_0}, \tag{3.3}$$

(ii) $\exists\, \alpha_{post} > 0$ independent of n_0 such that

$$\|\left(V_{n_0,post}\right)^{\nu_{post}}\mathbf{x}_{n_0}\|_{A_{n_0}}^2 \le \|\mathbf{x}_{n_0}\|_{A_{n_0}}^2 - \alpha_{post}\|\mathbf{x}_{n_0}\|_{A_{n_0}^2}^2, \qquad \forall \mathbf{x}_{n_0} \in \mathbb{C}^{n_0}, \tag{3.4}$$

(iii) $\exists\, \gamma > 0$ independent of n_0 such that

$$\min_{\mathbf{y}\in\mathbb{C}^{n_1}} \|\mathbf{x}_{n_0} - P_{n_0}\mathbf{y}\|_{D_{n_0}}^2 \le \gamma\|\mathbf{x}_{n_0}\|_{A_{n_0}}^2, \qquad \forall \mathbf{x}_{n_0} \in \mathbb{C}^{n_0}, \tag{3.5}$$

defined

$$\delta_{pre} = \frac{\alpha_{pre}}{\gamma}, \qquad \delta_{post} = \frac{\alpha_{post}}{\gamma},$$

then $\delta_{post} \le 1$ and

$$\|TGM\|_{A_{n_0}} \le \sqrt{\frac{1-\delta_{post}}{1+\delta_{pre}}} < 1.$$

Conditions (3.3) and (3.4) are called *pre-smoothing* and *post-smoothing* properties, respectively, while condition (3.5) is called *approximation* property.

Theorem 3.1 defines sufficient conditions for the convergence of the two-grid methods, since the norm of the iteration matrix of the latter is less than 1. The beauty of Theorem 3.1 displays in the following two properties:

Optimality: due to α_{pre}, α_{post}, γ being independent of n_0, the number of iterations needed to reach a given accuracy $\epsilon \in \mathbb{R}^+$ is bounded from above by a constant independent of n_0 (but, possibly depending on ϵ).

Simplicity: pre-smoothing, post-smoothing and approximation properties depend exclusively on the choice of pre-smoother, post-smoother and grid transfer operator, respectively. The possibility to analyze smoothers and coarse grid correction separately simplifies the convergence analysis of the two-grid method, whose iteration matrix (3.2) is simultaneously defined by them.

3.2 Algebraic V-cycle method

If n_1 is large, then the numerical solution of the linear system at Step 4. in the TGM could be computationally expensive. In this case, one usually applies a multigrid method based on several, possibly different, grid transfer operators. A V-cycle multigrid method solves the residual equation approximately within the recursive application of the two-grid method, until the coarsest level is reached and there the resulting small system of equations is solved exactly [8, 9, 48, 71].

Let $\ell \in \mathbb{N}$ be the depth of the multigrid method. We define a strictly decreasing sequence $n_0 > n_1 > \cdots > n_{\ell-1} > n_\ell > 0$ of integers $n_j \in \mathbb{N}$, $j = 1, \ldots, \ell$. For each n_j, $j = 0, \ldots, \ell - 1$, one chooses the grid transfer operator $P_{n_j} \in \mathbb{C}^{n_j \times n_{j+1}}$, $\text{rank}(P_{n_j}) = n_{j+1}$. Using the Galerkin approach, we define the projected matrix at level j of the multigrid method by

$$A_{n_{j+1}} = P_{n_j}^* A_{n_j} P_{n_j}, \qquad j = 0, \ldots, \ell - 1.$$

Let $\mathcal{V}_{n_j,\text{pre}}$ and $\mathcal{V}_{n_j,\text{post}}$, $j = 0, \ldots, \ell - 1$, be some iterative methods from $\mathcal{V}(\cdot)$ and $\nu_{\text{pre}}, \nu_{\text{post}} \in \mathbb{N}_0$, be the numbers of pre- and post-smoothing steps, respectively. For a fixed $s \in \mathbb{N}$, the Multigrid method (MGM) generates a sequence of iterates $\{\mathbf{x}_{n_0}^{(k)} \in \mathbb{C}^{n_0} : k \in \mathbb{N}\}$ defined by

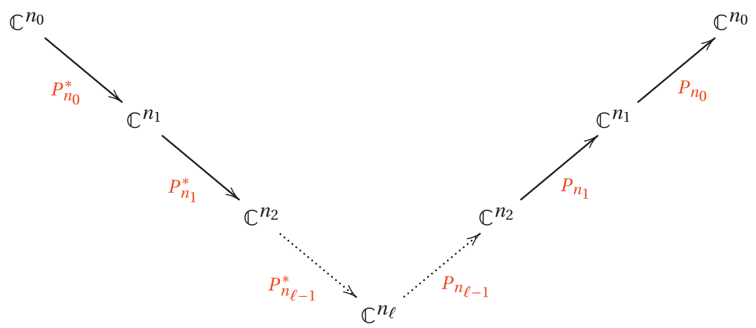
$$\mathbf{x}_{n_0}^{(k+1)} = \text{MGM}\big(P_{n_0}, A_{n_0}, \mathbf{b}_n, s, 0\big)\big(\mathbf{x}_{n_0}^{(k)}\big), \qquad \mathbf{x}_{n_0}^{(0)} \in \mathbb{C}^{n_0},$$

where the mapping MGM : $\mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ is defined iteratively by

$$\mathrm{MGM}\big(P_{n_j}, A_{n_j}, \mathbf{b}_{n_j}, s, j\big)\big(\mathbf{x}_{n_j}^{(k)}\big)$$

$$\begin{array}{lll}
\text{If } j=\ell & \text{then} \quad \mathbf{x}_{n_\ell}^{(k+1)} = A_{n_\ell}^{-1}\mathbf{b}_{n_\ell} & \\
\text{Else} & & \\
1. & \text{Pre-smoother:} & \mathbf{x}_{n_j}^{(k)} = \mathcal{V}_{n_j,\mathrm{pre}}^{\nu_{\mathrm{pre}}}\big(\mathbf{x}_{n_j}^{(k)}\big) \\
2. & \text{Residual on the } j\text{-th grid:} & \mathbf{r}_{n_j}^{(k)} = \mathbf{b}_{n_j} - A_{n_j}\mathbf{x}_{n_j}^{(k)} \\
3. & \text{Projection of residual on the } (j+1)\text{-th grid:} & \mathbf{r}_{n_{j+1}}^{(k)} = P_{n_j}^{*}\mathbf{r}_{n_j}^{(k)} \\
4. & \text{Recursion:} & \\
 & \quad \mathbf{x}_{n_{j+1}}^{(k+1)} = 0 & \\
 & \quad \text{for } r=1 \text{ to } s & \\
 & \qquad \mathbf{x}_{n_{j+1}}^{(k+1)} = \mathrm{MGM}\big(P_{n_{j+1}}, A_{n_{j+1}}, \mathbf{r}_{n_{j+1}}^{(k)}, s, j+1\big)\big(\mathbf{x}_{n_{j+1}}^{(k+1)}\big) & \\
5. & \text{Correction of the previous smoothed iterate:} & \mathbf{x}_{n_j}^{(k+1)} = \mathbf{x}_{n_j}^{(k)} + P_{n_j}\mathbf{x}_{n_{j+1}}^{(k+1)} \\
6. & \text{Post-smoother:} & \mathbf{x}_{n_j}^{(k+1)} = \mathcal{V}_{n_j,\mathrm{post}}^{\nu_{\mathrm{post}}}\big(\mathbf{x}_{n_j}^{(k+1)}\big)
\end{array} \tag{3.6}$$

The choice $s = 1$ corresponds to the well-known V-cycle method [71]. The iterative structure of V-cycle is depicted in the following figure.



If $\ell = 1$, then the V-cycle reduces to the TGM, since the structure consists of one fine space $\mathbb{C}^{n_0}$ and one coarse space $\mathbb{C}^{n_1}$. Similarly to the TGM, at each level $j = 0,\ldots,\ell-1$ of the V-cycle method, one defines the corresponding coarse grid transfer operator on $\mathbb{C}^{n_j}$ by

$$CGC_{n_j} = I_{n_j} - P_{n_j}A_{n_{j+1}}^{-1}P_{n_j}^{*}A_{n_j} = I_{n_j} - P_{n_j}\big(P_{n_j}^{*}A_{n_j}P_{n_j}\big)^{-1}P_{n_j}^{*}A_{n_j}. \tag{3.7}$$

For $j = 0,\ldots,\ell-1$, define $V_{n_j,\mathrm{pre}}$, $V_{n_j,\mathrm{post}}$ the iteration matrices of $\mathcal{V}_{n_j,\mathrm{pre}}$, $\mathcal{V}_{n_j,\mathrm{post}}$, respectively. The global iteration matrix of the V-cycle method is $MGM = MGM_0$, where

$$MGM_\ell = 0 \in \mathbb{C}^{n_\ell \times n_\ell},$$

$$MGM_j = \big(V_{n_j,\mathrm{post}}\big)^{\nu_{\mathrm{post}}}\Big(I_{n_j} - P_{n_j}\big(I_{n_{j+1}} - MGM_{j+1}\big)A_{n_{j+1}}^{-1}P_{n_j}^{*}A_{n_j}\Big)\big(V_{n_j,\mathrm{pre}}\big)^{\nu_{\mathrm{pre}}}, \quad j = \ell-1,\ldots,0.$$

The following result is the analogous of Theorem 3.1 for the V-cycle method. We refer to [63] for more details.

Theorem 3.2. *Let* $\ell \in \mathbb{N}$, $n_0 > n_1 > \cdots > n_{\ell-1} > n_\ell > 0$, $n_j \in \mathbb{N}$, $j = 0,\ldots,\ell$, $A_{n_0} \in \mathbb{C}^{n_0\times n_0}$ *positive definite and* $\nu_{pre}, \nu_{post} \in \mathbb{N}_0$. *Let, for* $j = 0,\ldots,\ell-1$, $\mathcal{V}_{n_j,pre}, \mathcal{V}_{n_j,post} \in \mathcal{V}(\cdot)$, $P_{n_j} \in \mathbb{C}^{n_j\times n_{j+1}}$, $\mathrm{rank}(P_{n_j}) = n_{j+1}$ *and* CGC_{n_j} *from* (3.7). *If, for* $j = 0,\ldots,\ell-1$,

(i) $\exists\, \alpha_{j,pre} > 0$ *independent of* n_j *such that*

$$\|(V_{n_j,pre})^{\nu_{pre}}\mathbf{x}_{n_j}\|^2_{A_{n_j}} \le \|\mathbf{x}_{n_j}\|^2_{A_{n_j}} - \alpha_{j,pre}\|(V_{n_j,pre})^{\nu_{pre}}\mathbf{x}_{n_j}\|^2_{A^2_{n_j}}, \qquad \forall \mathbf{x}_{n_j} \in \mathbb{C}^{n_j}, \tag{3.8}$$

(ii) $\exists\, \alpha_{j,post} > 0$ *independent of* n_j *such that*

$$\|(V_{n_j,post})^{\nu_{post}}\mathbf{x}_{n_j}\|^2_{A_{n_j}} \le \|\mathbf{x}_{n_j}\|^2_{A_{n_j}} - \alpha_{post}\|\mathbf{x}_{n_j}\|^2_{A^2_{n_j}}, \qquad \forall \mathbf{x}_{n_j} \in \mathbb{C}^{n_j}, \tag{3.9}$$

(iii) $\exists\, \gamma_j > 0$ *independent of* n_j *such that*

$$\|CGC_{n_j}\mathbf{x}_{n_j}\|^2_{A_{n_j}} \le \gamma_j\|\mathbf{x}_{n_j}\|^2_{A^2_{n_j}}, \qquad \forall\, \mathbf{x}_{n_j} \in \mathbb{C}^{n_j}, \tag{3.10}$$

defined

$$\delta_{pre} = \min_{j=0,\ldots,\ell-1} \frac{\alpha_{j,pre}}{\gamma_j}, \qquad \delta_{post} = \min_{j=0,\ldots,\ell-1} \frac{\alpha_{j,post}}{\gamma_j},$$

then $\delta_{post} \le 1$ *and*

$$\|MGM\|_{A_{n_0}} \le \sqrt{\frac{1-\delta_{post}}{1+\delta_{pre}}} < 1.$$

Conditions (3.8) and (3.9) are called *pre-smoothing* and *post-smoothing* properties, respectively, while condition (3.10) is called *approximation* property.

3.3 Algebraic multigrid for circulant matrix algebra

In this section, we assume that the system matrix $A_n \in \mathbb{C}^{n\times n}$ in (3.1) is circulant. It is well-known that the analysis of multigrid for circulant matrices depicts well the properties of multigrid in the case of positive definite Toeplitz system matrices and allows to use the matrix algebra structure.

Let $n = m^k$ with $m \in \mathbb{N}$, $m \ge 2$ and $k \in \mathbb{N}$. Suppose that $A_n = C_n(f)$ is the circulant matrix of order n defined using the Fourier coefficients

$$\mathrm{a}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x, \qquad \alpha \in \mathbb{Z}, \qquad |\alpha| \le c,$$

of the trigonometric polynomial $f\colon [0,2\pi) \to \mathbb{C}$

$$f(x) = \sum_{\substack{\alpha\in\mathbb{Z}\\ |\alpha|\le c}} \mathrm{a}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha x}, \qquad x \in [0,2\pi),$$

of degree $c < n$ (see chapter 2, subsection 2.3.1). Let us suppose that the trigonometric polynomial f is real, $f \geq 0$ and

$$f\left(x_\alpha^{(n)}\right) \neq 0, \qquad x_\alpha^{(n)} = \frac{2\pi\alpha}{n}, \qquad \alpha = 0,\dots,n-1.$$

Hence, $C_n(f)$ is positive definite.

Remark 3.1. If $f(x_\alpha^{(n)}) = 0$ for some $x_\alpha^{(n)}$, $\alpha \in \{0,\dots,n-1\}$, the matrix $C_n(f)$ is singular. In this case, the matrix A_n can be defined as a sum of $C_n(f)$ and a rank-one correction such that A_n is positive definite. Such correction, due to Strang, has been considered in the convergence analysis in [3]. However, it leads only to unnecessary complication of the notation, since the convergence results are not affected by such rank-one correction. Moreover, in applications, A_n is usually positive definite due to incorporated boundary conditions. Therefore, similarly to the analysis based on the LFA [31], the successive papers on the convergence analysis of multigrid methods for circulant matrices have neglected such a correction (see e.g. [2]). We follow this standard approach and refer the interested reader to [3] for more details on rank-one corrections.

In the case of circulant system matrices $A_n = C_n(f)$, the grid transfer operators also have a special structure. Let $\ell \in \mathbb{N}$, $1 \leq \ell \leq k-1$, be the depth of the V-cycle method. We define

$$P_{n_j} = C_{n_j}(p_j)\, K_{n_j,m}^T \in \mathbb{C}^{n_j \times n_{j+1}}, \qquad n_j = m^{k-j}, \qquad j = 0,\dots,\ell-1, \tag{3.11}$$

where p_j is a certain trigonometric polynomial and $K_{n_j,m} \in \mathbb{C}^{n_{j+1}\times n_j}$ is the *downsampling matrix* of factor m

$$K_{n_j,m} = \begin{pmatrix} 1 & 0_{m-1} & & & & & \\ & & 1 & 0_{m-1} & & & \\ & & & & \ddots & & \\ & & & & & 1 & 0_{m-1} \end{pmatrix}, \qquad 0_{m-1} = (0,\dots,0) \in \mathbb{N}_0^{1\times m-1}. \tag{3.12}$$

For $j = 0,\dots,\ell-1$, the operator $K_{n_j,m}$ allows to express the Fourier matrix of order n_{j+1}, i.e. $F_{n_{j+1}}$ in (2.3), in terms of the Fourier matrix of order n_j, i.e. F_{n_j} in (2.3), see [34]. Indeed, F_{n_j} and $K_{n_j,m}$ satisfy the following packaging property

$$K_{n_j,m} F_{n_j} = \frac{1}{\sqrt{m}} \Big(\underbrace{F_{n_{j+1}} \;\mid\; \dots \;\mid\; F_{n_{j+1}}}_{m \text{ times}}\Big) \in \mathbb{C}^{n_{j+1}\times n_j}. \tag{3.13}$$

This simple relation is the key step in defining multigrid methods for circulant matrices, since it allows us to obtain circulant matrices $A_{n_{j+1}}$ at the lower levels. Indeed, we denote the set of *m-corners* of $x \in [0,2\pi)$ by

$$\Omega_m(x) = \left\{ x + \frac{2\pi\alpha}{m} \pmod{2\pi} : \alpha = 0,\dots,m-1 \right\}, \qquad \#\Omega_m(x) = m.$$

The set $\Omega_m(x)$, $x \in [0,2\pi)$, is the set of frequencies on the fine grid which correspond to the same frequency on the coarse grid. It has been proved in [34] that

$$A_{n_{j+1}} = P_{n_j}^* A_{n_j} P_{n_j} = C_{n_{j+1}}(f_{j+1}), \qquad f_{j+1}(x) = \frac{1}{m} \sum_{y \in \Omega_m(\frac{x}{m})} f_j(y)|p_j(y)|^2 \qquad x \in [0,2\pi), \quad (3.14)$$

where f_j are the trigonometric polynomials associated with the circulant matrices $A_{n_j} = C_{n_j}(f_j)$, $j = 0,\dots,\ell$, and $f_0 = f$.

3.4 Algebraic multigrid for d-level circulant matrix algebra

In this section, we define an algebraic multigrid for d-level circulant matrices generalizing, via a Kronecker product argument, the definition of the algebraic multigrid for circulant matrices in section 3.3.

Let $\mathbf{n} = \mathbf{m}^{\mathbf{k}}$ with $\mathbf{m} = (m_1,\dots,m_d) \in \mathbb{N}^d$, $m_i \geq 2$, $i = 1,\dots,d$, and $\mathbf{k} = (k_1,\dots,k_d) \in \mathbb{N}^d$, $k_i > 0$, $i = 1,\dots,d$. We recall that the multi-index operation is intended component-wise, namely

$$\mathbf{m}^{\mathbf{k}} = \left(m_1^{k_1},\dots,m_d^{k_d}\right) \in \mathbb{N}^d.$$

We assume that the system matrix in (3.1) is the d-level circulant matrix

$$A_{\mathbf{n}} = C_{\mathbf{n}}(f) \in \mathbb{C}^{N\times N}, \qquad N = N(\mathbf{n}) = \prod_{i=1}^{d} n_i,$$

of order $\mathbf{n}$ generated by the d-variate trigonometric polynomial $f\colon [0,2\pi)^d \to \mathbb{C}$ of total degree $c < \min_{i=1,\dots,d} n_i$. See chapter 2, subsection 2.3.3, for all the details.

We assume that f is real, $f \geq 0$ and

$$f\left(\mathbf{x}_{\boldsymbol{\alpha}}^{(\mathbf{n})}\right) \neq 0, \qquad \mathbf{x}_{\boldsymbol{\alpha}}^{(\mathbf{n})} = \left(x_{\alpha_1}^{(n_1)},\dots,x_{\alpha_d}^{(n_d)}\right), \qquad x_{\alpha_i}^{(n_i)} = \frac{2\pi\alpha_i}{n_i}, \qquad \alpha_i = 0,\dots,n_i-1, \qquad i = 1,\dots,d.$$

Hence, $C_{\mathbf{n}}(f)$ is positive definite. If the latter property is not satisfied for some $\alpha_i \in \{0,\dots,n_i-1\}$, $i = 1,\dots,d$, we remand to the observations in Remark 3.1.

Similarly to the circulant case, when the system matrix in (3.1) is d-level circulant, the grid transfer operators are defined accordingly. Let $\ell \in \mathbb{N}$, $1 \leq \ell \leq \min_{i=1,\dots,d} k_i - 1$, be the depth of the multigrid procedure. We define the coarser spaces of V-cycle as

$$\mathbf{n}_j = \left(m_1^{k_1-j},\dots,m_d^{k_d-j}\right) \in \mathbb{N}^d, \qquad j = 0,\dots,\ell. \quad (3.15)$$

We can interpreter the definition of the coarser spaces in (3.15) as follows: for $j = 0,\dots,\ell-1$, the coarser space $\mathbf{n}_{j+1}$ is obtained from the finer space $\mathbf{n}_j$ reducing the dimension in each coordinate direction i by a factor m_i, $i = 1,\dots,d$. For $j = 0,\dots,\ell-1$, the grid transfer operator $P_{\mathbf{n}_j}$ at the j-th level of V-cycle is defined by

$$P_{\mathbf{n}_j} = C_{\mathbf{n}_j}(p_j)\, K_{\mathbf{n}_j,\mathbf{m}}^T \in \mathbb{C}^{N_j\times N_{j+1}}, \qquad N_j = N(\mathbf{n}_j) = \prod_{i=1}^{d} (n_j)_i, \quad (3.16)$$

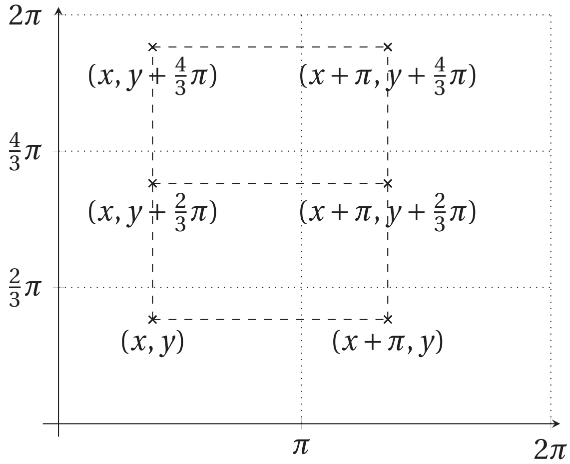


Figure 3.1: *Example of* $\Omega(\mathbf{x}) \subset [0,2\pi)^2$ *with* $\mathbf{m} = (2,3)$.

where p_j is a certain trigonometric polynomial. The matrix $K_{\mathbf{n}_j,\mathbf{m}} \in \mathbb{C}^{N_{j+1}\times N_j}$ in (3.16) is the d-level *downsampling matrix* of factor $\mathbf{m}$

$$K_{\mathbf{n}_j,\mathbf{m}} = K_{(n_j)_1,m_1} \otimes \ldots \otimes K_{(n_j)_d,m_d}, \tag{3.17}$$

where $K_{(n_j)_i,m_i}$, $i = 1,\ldots,d$, is the "univariate" downsampling matrix of factor m_i in (3.12). Similarly to the circulant case, for $j = 0,\ldots,\ell-2$, the d-level downsampling matrix $K_{\mathbf{n}_j,\mathbf{m}}$ in (3.17) satisfies the following packaging property (see [5])

$$K_{\mathbf{n}_j,\mathbf{m}} F_{\mathbf{n}_j} = \frac{1}{\sqrt{\prod_{i=1}^{d} m_i}} F_{\mathbf{n}_{j+1}} \Sigma_{\mathbf{n}_{j+1},\mathbf{m}}, \tag{3.18}$$

where $F_{\mathbf{n}_j}$, $F_{\mathbf{n}_{j+1}}$ are the d-dimensional Fourier matrix of order $\mathbf{n}_j$, $\mathbf{n}_{j+1}$ defined in (2.6) and

$$\Sigma_{\mathbf{n}_{j+1},\mathbf{m}} = \Big(\underbrace{I_{(n_{j+1})_1} \;\big|\; \ldots \;\big|\; I_{(n_{j+1})_1}}_{m_1 \text{ times}}\Big) \otimes \ldots \otimes \Big(\underbrace{I_{(n_{j+1})_d} \;\big|\; \ldots \;\big|\; I_{(n_{j+1})_d}}_{m_d \text{ times}}\Big) \in \mathbb{C}^{N_{j+1}\times N_j}.$$

Property (3.18) links the space of frequencies at the coarser level $j+1$ and the space of frequencies at the finer level j, $j = 0,\ldots,\ell-2$. Moreover, it allows us to obtain multilevel circulant matrices $A_{\mathbf{n}_{j+1}}$ at the lower levels. In order to properly explain the latter statement, we define the set of $\mathbf{m}$-*corners* of $\mathbf{x} \in [0,2\pi)^d$ by

$$\Omega_{\mathbf{m}}(\mathbf{x}) = \left\{ \mathbf{y} \in [0,2\pi)^d \, : \, y_i = x_i + \frac{2\pi\alpha_i}{m_i} \pmod{2\pi},\, \alpha_i = 0,\ldots,m_i-1, i = 1,\ldots,d \right\}. \tag{3.19}$$

The set $\Omega_{\mathbf{m}}(\mathbf{x})$, $\mathbf{x} \in [0,2\pi)^d$, determines the frequencies on the finer grid which correspond to the same frequency on the coarser grid. Notice that $\#\Omega_{\mathbf{m}}(\mathbf{x}) = \prod_{i=1}^{d} m_i$. Figure 3.1 illustrates a bivariate example of $\Omega_{\mathbf{m}}(\mathbf{x})$ in (3.19).

It is well-known [3, 6, 69] that

$$A_{\mathbf{n}_{j+1}} = P^*_{\mathbf{n}_j} A_{\mathbf{n}_j} P_{\mathbf{n}_j} = C_{\mathbf{n}_{j+1}}(f_{j+1}), \qquad f_{j+1}(\mathbf{x}) = \frac{1}{\prod_{i=1}^d m_i} \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\frac{\mathbf{x}}{\mathbf{m}})} f_j(\mathbf{y})|p_j(\mathbf{y})|^2, \qquad \mathbf{x}\in[0,2\pi)^d, \tag{3.20}$$

where f_j are the trigonometric polynomials associated with the d-level circulant matrices $A_{\mathbf{n}_j} = C_{\mathbf{n}_j}(f_j)$, $j = 0,\dots,\ell-1$, and $f_0 = f$.

3.5 Convergence and optimality analysis

In this section, we provide a convergence analysis of the algebraic multigrid methods defined for circulant and d-level circulant matrices with general downsampling/upsampling strategy. Our aim is to translate pre-smoothing, post-smoothing and approximation properties in Theorem s 3.1 and 3.2 in the circulant and d-level circulant matrix algebra context. We state the results only in the case of d-level circulant matrix algebra. By slight abuse of notation, the case $d = 1$ corresponds to the case of circulant matrix algebra.

The following analysis is structured as follows. In subsection 3.5.1, we recall the main results about pre- and post-smoothing properties for both two-grid method and V-cycle method. In subsections 3.5.2 and 3.5.3, we focus on the approximation property for two-grid method and V-cycle method, respectively. We split the analysis between two-grid method and V-cycle method since the approximation property (3.5) for the two-grid method and the approximation property (3.10) for the V-cycle method are not equivalent. Indeed, condition (3.10) defines grid transfer operators for the V-cycle method which are more powerful than those defined by condition (3.5) for the two-grid method. Here and in the following, we adopt the notation of section 3.4.

3.5.1 Smoothing property

In this subsection, we focus on pre- and post-smoothing properties in Theorem 3.1 and Theorem 3.2. We notice that if the smoothers $\mathcal{V}_{n,\mathrm{pre}}, \mathcal{V}_{n,\mathrm{post}} \in \mathcal{V}(\cdot)$ satisfy (3.3) and (3.4) for $\nu_{\mathrm{pre}}, \nu_{\mathrm{post}} \in \mathbb{N}_0$ and $A_{\mathbf{n}} = C_{\mathbf{n}}(f) \in \mathbb{C}^{N\times N}$, then they satisfy also (3.8) and (3.9) for $\nu_{j,\mathrm{pre}} = \nu_{\mathrm{pre}}$, $\nu_{j,\mathrm{post}} = \nu_{\mathrm{post}}$ and $A_{\mathbf{n}_j} = C_{\mathbf{n}_j}(f_j) \in \mathbb{C}^{N_j\times N_j}$, $j = 0,\dots,\ell-1$. This means that a "good" smoother for the two-grid method is also a "good" smoother for the V-cycle method. Thus, we need to focus only on the smoothing properties (3.3) and (3.4).

First, we focus on the weighted Richardson method in $\mathcal{V}(\cdot)$, see chapter 2, subsection 2.4.1. As the following result shows, using appropriate weights, weighted Richardson smoother fulfills pre- and post-smoothing properties [2, 70].

Proposition 3.3. *Let $\mathbf{n}_0 \in \mathbb{N}^d$, $f\colon [0,2\pi)^d \to \mathbb{C}$ non-negative and not identically zero, $A_{\mathbf{n}_0} = C_{\mathbf{n}_0}(f) \in \mathbb{C}^{N_0\times N_0}$, $\omega_{pre}, \omega_{post} \in \mathbb{R}^+$, $V^{\omega_{pre}}_{N_0,pre}$, $V^{\omega_{post}}_{N_0,post}$ in* (2.10)*, $\nu_{pre}, \nu_{post} \in \mathbb{N}_0$. If*

$$0 < \omega_{pre}, \omega_{post} < \frac{2}{\|f\|_{L^\infty}},$$

then there exist $\alpha_{pre}, \alpha_{post} > 0$ independent of N_0 such that the pre-smoothing property (3.3) *and the post-smoothing property* (3.4) *are satisfied.*

Now, we focus on the weighted Jacobi method in $\mathcal{V}(\cdot)$, see chapter 2, subsection 2.4.2. Let $A_{\mathbf{n}_0} = C_{\mathbf{n}_0}(f) \in \mathbb{C}^{N_0 \times N_0}$ be the d-level circulant matrix of order $\mathbf{n}_0 \in \mathbb{N}^d$ generated by the d-variate trigonometric polynomial $f\colon [0,2\pi)^d \to \mathbb{C}$. Let $D_{\mathbf{n}_0} \in \mathbb{C}^{N_0 \times N_0}$ be the diagonal matrix with diagonal entries of $A_{\mathbf{n}_0}$. Since $A_{\mathbf{n}_0}$ is d-level circulant, we have $D_{\mathbf{n}_0} = \mathrm{a}(\mathbf{0}) I_{N_0}$, where $\mathrm{a}(\mathbf{0}) \in \mathbb{C}$ is the $\mathbf{0}$-th Fourier coefficient of f and I_{N_0} is the identity matrix of order N_0. If f is non-negative and not identically zero, then $\mathrm{a}(\mathbf{0}) > 0$. Thus, the iteration matrix of the weighted Jacobi method in (2.14) becomes

$$V_{N_0}^{\omega} = I_{N_0} - \omega D_{\mathbf{n}_0}^{-1} A_{\mathbf{n}_0} = I_{N_0} - \frac{\omega}{\mathrm{a}(\mathbf{0})} A_{\mathbf{n}_0}.$$

Thus, weighted Jacobi method with weight $\omega \in \mathbb{R}^+$ is equivalent to weighted Richardson method with weight $\frac{\omega}{\mathrm{a}(\mathbf{0})} \in \mathbb{R}^+$. The following result is a direct consequence of Proposition 3.3.

Proposition 3.4. *Let $\mathbf{n}_0 \in \mathbb{N}^d$, $f\colon [0,2\pi)^d \to \mathbb{C}$ non-negative and not identically zero, $A_{\mathbf{n}_0} = C_{\mathbf{n}_0}(f) \in \mathbb{C}^{N_0 \times N_0}$, $\omega_{pre}, \omega_{post} \in \mathbb{R}^+$, $V_{N_0,pre}^{\omega_{pre}}$, $V_{N_0,post}^{\omega_{post}}$ in* (2.14)*, $\nu_{pre}, \nu_{post} \in \mathbb{N}_0$. If*

$$0 < \omega_{pre}, \omega_{post} < \frac{2\mathrm{a}(\mathbf{0})}{\|f\|_{L^\infty}},$$

then there exist $\alpha_{pre}, \alpha_{post} > 0$ independent of N_0 such that the pre-smoothing property (3.3) *and the post-smoothing property* (3.4) *are satisfied.*

Finally, we focus on the Gauss-Seidel method in $\mathcal{V}(\cdot)$, see chapter 2, subsection 2.4.3.

Proposition 3.5 ([63])**.** *Let $\mathbf{n}_0 \in \mathbb{N}^d$, $f\colon [0,2\pi)^d \to \mathbb{C}$ non-negative and not identically zero, $A_{\mathbf{n}_0} = C_{\mathbf{n}_0}(f) \in \mathbb{C}^{N_0 \times N_0}$, $V_{N_0,pre}$, $V_{N_0,post}$ in* (2.15)*, $\nu_{pre}, \nu_{post} \in \mathbb{N}_0$. Then, there exist $\alpha_{pre}, \alpha_{post} > 0$ independent of N_0 such that the pre-smoothing property* (3.3) *and the post-smoothing property* (3.4) *are satisfied.*

3.5.2 Approximation property for two-grid method

In this subsection, we focus on the approximation property (3.5) in Theorem 3.1 for the two-grid method. In [5, 34, 69], the authors provide well-known sufficient conditions for the convergence of the two-grid method for the solution of linear systems of equations (3.1) whose system matrices are d-level circulant. We report those conditions in Theorem 3.6. To do so, for $\mathbf{x} \in [0,2\pi)^d$, we define the set of $\mathbf{m}$-mirror points of $\mathbf{x}$ by

$$\Omega'_{\mathbf{m}}(\mathbf{x}) := \Omega_{\mathbf{m}}(\mathbf{x}) \setminus \{\mathbf{x}\},$$

where $\Omega(\mathbf{x})$ is defined by (3.19).

Theorem 3.6. *Let f and p be real d-variate trigonometric polynomials such that $f(\mathbf{x}_0) = 0$ and $f(\mathbf{x}) > 0$, $\mathbf{x} \in [0,2\pi)^d \setminus \{\mathbf{x}_0\}$. If p satisfies*

$$(i) \quad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|p(\mathbf{y})|^2}{f(\mathbf{x})} < +\infty \qquad \forall\, \mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{x}_0),$$

$$(ii) \quad \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2 > 0, \qquad \forall\, \mathbf{x} \in [0,2\pi)^d,$$

then $P_{\mathbf{n}} = C_{\mathbf{n}}(p)\, K^T_{\mathbf{n},\mathbf{m}}$ satisfies the approximation property (3.5).

For the definition of the grid transfer operator $P_{\mathbf{n}}$, condition *(ii)* in Theorem 3.6 appears unnecessary if we admit that $P_{\mathbf{n}}$ is rank deficient. We slightly relax the assumptions of Theorem 3.6 following the analysis in [6]. These conditions are easy to check for any given grid transfer operator $P_{\mathbf{n}}$. Our simplification in Theorem 3.7 replaces *(ii)* in Theorem 3.6 by an even simpler condition, see *(ii)* in Theorem 3.7.

Theorem 3.7. *Let f and p be real d-variate trigonometric polynomials such that $f(\mathbf{x}_0) = 0$ and $f(\mathbf{x}) > 0$, $\mathbf{x} \in [0,2\pi)^d \setminus \{\mathbf{x}_0\}$. If p satisfies*

$$(i) \quad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|p(\mathbf{y})|^2}{f(\mathbf{x})} < +\infty \qquad \forall\, \mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{x}_0),$$

$$(ii) \quad |p(\mathbf{x}_0)|^2 > 0,$$

then $P_{\mathbf{n}} = C_{\mathbf{n}}(p)\, K^T_{\mathbf{n},\mathbf{g}}$ satisfies the approximation property (3.5).

Proof. The proof consists of three steps. The first and second steps are borrowed from [69] and [5], thus we only state them shortly. We present in detail the proof of the main step, *Step 3*.

Step 1: Let $\mathrm{a}(\mathbf{0}) = \dfrac{1}{(2\pi)^d} \displaystyle\int_{[0,2\pi)^d} f(\mathbf{x})\mathrm{d}x > 0$. By Theorem 3.4 in [5], (3.5) is equivalent to

$$\exists\, \gamma > 0 \quad \text{independent of } \mathbf{n} = (n_1,\dots,n_d) \text{ such that} \quad I_N - P_{\mathbf{n}}(P^*_{\mathbf{n}} P_{\mathbf{n}})^{-1} P^*_{\mathbf{n}} \preceq \frac{\gamma}{\mathrm{a}(\mathbf{0})} C_{\mathbf{n}}(f), \tag{3.21}$$

with $N = N(\mathbf{n}) = \prod_{i=1}^d n_i$.

Step 2: Define $\tilde{m} = \prod_{i=1}^d m_i$. For $\mathbf{x} \in [0,2\pi)^d$, let $\mathbf{y}_\alpha = \mathbf{y}_\alpha(\mathbf{x}) \in [0,2\pi)^d$, $\alpha = 0,\dots,\tilde{m}-1$, be the elements of the $\mathbf{m}$-corner set $\Omega_{\mathbf{m}}(\mathbf{x})$ in (3.19). Define the row vectors $p[\mathbf{x}], f[\mathbf{x}] \in \mathbb{C}^{1\times\tilde{m}}$ by

$$p[\mathbf{x}] := \big(p(\mathbf{y}_\alpha)\big)_{0\le\alpha\le\tilde{m}-1} \quad \text{and} \quad f[\mathbf{x}] := \big(f(\mathbf{y}_\alpha)\big)_{0\le\alpha\le\tilde{m}-1}.$$

By Theorem 3.4 in [5], (3.21) is equivalent to

$$\exists\, \gamma > 0 \quad \text{independent of } \mathbf{n} = (n_1,\dots,n_d) \text{ such that} \quad I_{\tilde{m}} - \frac{p[\mathbf{x}]^* \cdot p[\mathbf{x}]}{\|p[\mathbf{x}]\|_2^2} \preceq \frac{\gamma}{\mathrm{a}(\mathbf{0})} \operatorname{diag}\big(f[\mathbf{x}]\big), \tag{3.22}$$

for all $\mathbf{x} \in [0,2\pi)^d$.

Step 3: We follow the approach of Bolten et alt. in [6]. To prove the claim, we show that assumptions *(i)* and *(ii)* imply (3.22). To do so, we need to show that the $\tilde{m} \times \tilde{m}$ matrix

$$R[\mathbf{x}] := \left(\operatorname{diag}\left(f[\mathbf{x}]\right)\right)^{-\frac{1}{2}} \left(I_{\tilde{m}} - \frac{p[\mathbf{x}]^* \cdot p[\mathbf{x}]}{\|p[\mathbf{x}]\|_2^2}\right) \left(\operatorname{diag}\left(f[\mathbf{x}]\right)\right)^{-\frac{1}{2}}$$

is well-defined, i.e. we can bound the modulus of its entries $R[\mathbf{x}](\alpha,\beta)$ by

$$\left|R[\mathbf{x}](\alpha,\beta)\right| \leq \frac{\gamma}{\mathrm{a}(\mathbf{0})} < \infty \qquad \forall \mathbf{x} \in [0,2\pi)^d, \qquad \alpha,\beta = 0,\ldots,\tilde{m}-1. \tag{3.23}$$

Note that, for $\alpha, \beta = 0,\ldots,\tilde{m}-1$, the entries $R[\mathbf{x}](\alpha,\beta)$ of $R[\mathbf{x}]$ are given by

$$\begin{aligned} R[\mathbf{x}](\alpha,\beta) &= -\frac{p(\mathbf{y}_\alpha)\overline{p(\mathbf{y}_\beta)}}{\sqrt{f(\mathbf{y}_\alpha) f(\mathbf{y}_\beta)} \sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2}, && \alpha \neq \beta, \\ R[\mathbf{x}](\beta,\beta) &= \frac{\sum\limits_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)} |p(\mathbf{y})|^2}{f(\mathbf{y}_\beta) \sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2}, && \alpha = \beta. \end{aligned} \tag{3.24}$$

In the following, we consider two cases: $\mathbf{x} \in \Omega_{\mathbf{m}}(\mathbf{x}_0)$ and $\mathbf{x} \not\in \Omega_{\mathbf{m}}(\mathbf{x}_0)$.

Case 3.a: If $\mathbf{x} \in \Omega_{\mathbf{m}}(\mathbf{x}_0)$, then by (3.19), $\Omega_{\mathbf{m}}(\mathbf{x}_0) = \Omega_{\mathbf{m}}(\mathbf{x})$. Moreover, if $\mathbf{x} \in \Omega_{\mathbf{m}}(\mathbf{x}_0)$, then $\exists \beta \in \{0,\ldots,\tilde{m}-1\}$ such that $\mathbf{y}_\beta = \mathbf{x}_0$ and $f(\mathbf{y}_\beta) = 0$. If $\alpha \neq \beta$, then by *(i)*, the order of the zero of $\sqrt{f}$ at $\mathbf{y}_\beta$ matches the order of the zero of p at $\mathbf{y}_\alpha$. If $\alpha = \beta$, then again by *(i)* with

$$\sum_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)} |p(\mathbf{y})|^2 = \sum_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{x}_0)} |p(\mathbf{y})|^2,$$

the order of the zero of f at $\mathbf{y}_\beta$ matches the order of the zero of $\sum\limits_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)} |p(\mathbf{y})|^2$. It is left to show that $\sum\limits_{\mathbf{x}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2 > 0$. Then all the entries of $R[\mathbf{x}]$, $\mathbf{x} \in \Omega_{\mathbf{m}}(\mathbf{x}_0)$, are well-defined. The identity $\Omega_{\mathbf{m}}(\mathbf{x}_0) = \Omega_{\mathbf{m}}(\mathbf{x})$ and *(i)* imply that $p(\mathbf{y}) = 0$ for all $\mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{x}_0)$. Thus, by *(ii)*, we get

$$\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2 = \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x}_0)} |p(\mathbf{y})|^2 = |p(\mathbf{x}_0)|^2 > 0.$$

Case 3.b: We assume next that $\mathbf{x} \notin \Omega_{\mathbf{m}}(\mathbf{x}_0)$. First, we notice that if $\mathbf{x} \notin \Omega_{\mathbf{m}}(\mathbf{x}_0)$, then $\mathbf{x}_0 \notin \Omega_{\mathbf{m}}(\mathbf{x})$ and $f(\mathbf{y}_\beta) \neq 0$, $\beta = 0,\ldots,\tilde{m}-1$, since f has a unique zero at $\mathbf{x}_0$ by hypothesis. Thus, we only need to study the properties of $\sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2$.

If $\sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2 > 0$, then $|R[\mathbf{x}](\alpha,\beta)| < \infty$ for any $\alpha,\beta = 0,\ldots,\tilde{g}-1$.

If $\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2 = 0$, then we need to study the behavior of its zeros. To do that, we first define, for a d-variate trigonometric polynomial h, the function $\theta_h\colon [0,2\pi)^d \to \mathbb{N}$ such that $\theta_h(\overline{\mathbf{x}}) = q$, $\overline{\mathbf{x}} \in [0,2\pi)^d$, $q \in \mathbb{N}$, if and only if

$$D^{\boldsymbol{\mu}} h(\overline{\mathbf{x}}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}^d, \quad |\boldsymbol{\mu}| \le q-1, \quad \text{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}^d, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} h(\overline{\mathbf{x}}) \neq 0, \tag{3.25}$$

i.e. $q \in \mathbb{N}$ is the order of the zero of h at $\overline{\mathbf{x}}$. We rewrite the entries $R[\mathbf{x}](\alpha,\beta)$ $\alpha,\beta = 0,\dots,\tilde{m}-1$, of $R[\mathbf{x}]$ in (3.24) and get

$$\begin{aligned} R[\mathbf{x}](\alpha,\beta) &= -\frac{h_{\alpha,\beta}(\mathbf{x})}{\sqrt{f(\mathbf{y}_\alpha)f(\mathbf{y}_\beta)}\,h(\mathbf{x})}, && \alpha \neq \beta, \\ R[\mathbf{x}](\beta,\beta) &= \frac{h_\beta(\mathbf{x})}{f(\mathbf{x}_\beta)\,h(\mathbf{x})}, && \alpha = \beta, \end{aligned}$$

where

$$\begin{aligned} h(\mathbf{x}) &:= \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p(\mathbf{y})|^2, \\ h_\beta(\mathbf{x}) &:= \sum_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)} |p(\mathbf{y})|^2, \\ h_{\alpha,\beta}(\mathbf{x}) &:= p(\mathbf{y}_\alpha)\overline{p(\mathbf{y}_\beta)}. \end{aligned}$$

To prove the boundedness of $R[\mathbf{x}](\alpha,\beta)$, $\alpha,\beta = 0,\dots,\tilde{m}-1$, we show that

$$\theta_{h_\beta}(\mathbf{x}) \ge \theta_h(\mathbf{x}), \qquad \text{and} \qquad \theta_{h_{\alpha,\beta}}(\mathbf{x}) \ge \theta_h(\mathbf{x}).$$

Recall that we consider the case when $h(\mathbf{x}) = 0$, then $p(\mathbf{y}) = 0$ for all $\mathbf{y} \in \Omega_{\mathbf{m}}(\mathbf{x})$. Thus, for $\Theta := \min_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} \theta_p(\mathbf{y})$, we have $\theta_h(\mathbf{x}) = 2\Theta$. Due to $\Omega'_{\mathbf{m}}(\mathbf{y}_\beta) \subset \Omega_{\mathbf{m}}(\mathbf{x})$ we get $\theta_{h_\beta}(\mathbf{x}) \ge 2\Theta$. Similarly, $\theta_{h_{\alpha,\beta}}(\mathbf{x}) \ge 2\Theta$. And, thus, the claim follows. ■

Theorem 3.7 deals with a d-level circulant matrix $C_{\mathbf{n}}(f)$ whose generating function f has a single zero at $\mathbf{x}_0 \in [0,2\pi)^d$. If f has an additional zero at some point $\mathbf{x}_1 \notin \Omega'_{\mathbf{m}}(\mathbf{x}_0)$, then we choose a trigonometric polynomial p which satisfies *(i)* of Theorem 3.7 for $\mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{x}_0) \cup \Omega'_{\mathbf{m}}(\mathbf{x}_1)$ and *(ii)* of Theorem 3.7 for $\mathbf{x}_0$ and $\mathbf{x}_1$. Then, the corresponding $P_{\mathbf{n}} = C_{\mathbf{n}}(p)\,K^T_{\mathbf{n},\mathbf{m}}$ also satisfies the *approximation property* (3.5). The proof of the latter is a straightforward generalization of the proof of Theorem 3.7 and is omitted. If f has an additional zero at some point $\mathbf{x}_1 \in \Omega'_{\mathbf{m}}(\mathbf{x}_0)$, then we choose a different downsampling factor $\overline{\mathbf{m}} \in \mathbb{N}^d$, $\overline{\mathbf{m}} \neq \mathbf{m}$, so that $\mathbf{x}_1 \notin \Omega'_{\overline{\mathbf{m}}}(\mathbf{x}_0)$.

3.5.3 Approximation property for V-cycle method

For the V-cycle, according to the convergence and optimality results in [3], the assumptions of Theorem 3.7 should be strengthen to guarantee that the corresponding coarse grid correction operators satisfy the approximation property (3.10). The appropriate modifications of the assumptions of Theorem 3.7 were given in [69] for the circulant case $d=1$ with $m=2$ and in [2] for the multilevel circulant case with $\mathbf{m}=(2,\ldots,2)\in\mathbb{N}^d$. The following Theorem 3.8 is the generalization of Theorem 3.7 to the case of generic $\mathbf{m}\in\mathbb{N}^d$.

Theorem 3.8. *Let f_0, p_j, $j=0,\ldots,\ell-1$, be real d-variate trigonometric polynomials such that $f_0(\mathbf{x}_0)=0$ and $f_0(\mathbf{x})>0, \mathbf{x}\in[0,2\pi)^d\setminus\{\mathbf{x}_0\}$. Let f_j, $j=1,\ldots,\ell-1$, be real trigonometric polynomials defined as in* (3.20) *and such that $f_j(\mathbf{x}_j)=0$, $f_j(\mathbf{x})>0, \mathbf{x}\in[0,2\pi)^d\setminus\{\mathbf{x}_j\}$. If, for $j=0,\ldots,\ell-1$, p_j satisfy*

$$
\begin{array}{llll}
(i) & \displaystyle\lim_{\mathbf{x}\to\mathbf{x}_j}\frac{|p_j(\mathbf{y})|}{f_j(\mathbf{x})} & < \; +\infty & \forall\,\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{x}_j),\\[2ex]
(ii) & \displaystyle\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})}|p_j(\mathbf{y})|^2 & > \; 0 & \forall\,\mathbf{x}\in[0,2\pi)^d,
\end{array}
$$

then $A_{\mathbf{n}_j}$ in (3.20) *and $CGC_{\mathbf{n}_j}$ in* (3.7) *satisfy the approximation property* (3.10).

Before proving Theorem 3.8, we would like to comment on its hypothesis. Let $j\in\{0,\ldots,\ell-1\}$. If $f_j(\mathbf{x}_j)=0$, $f_j(\mathbf{x})>0$ for $\mathbf{x}\in[0,2\pi)^d\setminus\{\mathbf{x}_j\}$ and p_j satisfies *(i)* and *(ii)* of Theorem 3.8, then [6, Remark 3.3] guarantees that $f_{j+1}(\mathbf{x})=0$ if and only if

$$\mathbf{x}=\mathbf{x}_{j+1}:=\mathbf{m}\mathbf{x}_j(\mathrm{mod}2\pi)=\big(m_1x_1(\mathrm{mod}2\pi),\ldots,m_dx_d(\mathrm{mod}2\pi)\big)^T.$$

Moreover, the order of the zero of f_{j+1} at $\mathbf{x}_{j+1}$ coincides with the order of the zero of f_j at $\mathbf{x}_j$ and $f_{j+1}(\mathbf{x})>0$ for $\mathbf{x}\in[0,2\pi)^d\setminus\{\mathbf{x}_{j+1}\}$.

Proof. The proof consists of two steps: the first one is borrowed from [3], the second one is similar to *Step 3* of the proof of Theorem 3.7. Let $j\in\{0,\ldots,\ell-1\}$.

Step 1: By [3, Proposition 16], $A_{\mathbf{n}_j}$ in (3.20) and $CGC_{\mathbf{n}_j}$ in (3.7) satisfy the approximation property (3.10) if and only if $\exists\,\gamma_j>0$ independent of $\mathbf{n}_j=((n_j)_1,\ldots,(n_j)_d)$ such that

$$I_{N_j}-\hat P_{\mathbf{n}_j}(\hat P^*_{\mathbf{n}_j}\hat P_{\mathbf{n}_j})^{-1}\hat P^*_{\mathbf{n}_j}\preceq\gamma_jC_{\mathbf{n}_j}(f),\qquad N_j=N(\mathbf{n}_j)=\prod_{i=1}^d(n_j)_i,\tag{3.26}$$

where

$$\hat P_{\mathbf{n}_j}:=C_{\mathbf{n}_j}(\hat p_j)\,K^T_{\mathbf{n}_j,\mathbf{m}}\in\mathbb{C}^{N_j\times N_{j+1}},\qquad \hat p_j(\mathbf{x}):=p_j(\mathbf{x})\sqrt{f_j(\mathbf{x})},\qquad \mathbf{x}\in[0,2\pi)^d.$$

Step 2: To prove the claim, we show that *(i)* and *(ii)* imply (3.26). As shown in *Step 3* of the proof of Theorem 3.7, (3.26) holds true if and only if the entries of the matrix $R[\mathbf{x}]$ in (3.24) are bounded in modulus, where, for $\mathbf{y}_\alpha,\mathbf{y}_\beta\in\Omega_{\mathbf{m}}(\mathbf{x})$, $\mathbf{x}\in[0,2\pi)^d$, $\alpha,\beta=0,\ldots,\tilde m-1$,

$$
\begin{aligned}
R[\mathbf{x}](\alpha,\beta)&=-\frac{\hat p_j(\mathbf{y}_\alpha)\overline{\hat p_j(\mathbf{y}_\beta)}}{\sqrt{f_j(\mathbf{y}_\alpha)f_j(\mathbf{y}_\beta)}\displaystyle\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})}|\hat p_j(\mathbf{y})|^2}, && \alpha\neq\beta,\\
R[\mathbf{x}](\beta,\beta)&=\frac{\displaystyle\sum_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)}|\hat p_j(\mathbf{y})|^2}{f_j(\mathbf{y}_\beta)\displaystyle\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})}|\hat p_j(\mathbf{y})|^2}, && \alpha=\beta.
\end{aligned}\tag{3.27}
$$

Substituting the definition of $\hat{p}_j$ into (3.27), we get

$$
\begin{aligned}
R[\mathbf{x}](\alpha,\beta) &= -\frac{p_j(\mathbf{y}_\alpha)\overline{p_j(\mathbf{y}_\beta)}}{\sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y})}, & \alpha \neq \beta,\\
R[\mathbf{x}](\beta,\beta) &= \frac{\sum\limits_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)} |p_j(\mathbf{y})|^2 f_j(\mathbf{y})}{f_j(\mathbf{y}_\beta)\sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y})}, & \alpha = \beta.
\end{aligned}
\tag{3.28}
$$

We split the analysis of quantities in (3.28) into two cases: $\mathbf{x}\in\Omega_{\mathbf{m}}(\mathbf{x}_j)$ and $\mathbf{x}\notin\Omega_{\mathbf{m}}(\mathbf{x}_j)$.

Case 2.a: If $\mathbf{x}\in\Omega_{\mathbf{m}}(\mathbf{x}_j)$, then by (3.19), $\Omega_{\mathbf{m}}(\mathbf{x}_j)=\Omega_{\mathbf{m}}(\mathbf{x})$. Thus, the hypothesis $f_j(\mathbf{x}_j)=0$ and *(i)* imply that

$$
\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}) = \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x}_j)} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}) = |p_j(\mathbf{x}_j)|^2 f_j(\mathbf{x}_j) + \sum_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{x}_j)} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}) = 0. \tag{3.29}
$$

We define

$$
\begin{aligned}
h(\mathbf{x}) &:= \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}),\\
h_\beta(\mathbf{x}) &:= \sum_{\mathbf{y}\in\Omega'_{\mathbf{m}}(\mathbf{y}_\beta)} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}),\\
h_{f_j,\beta}(\mathbf{x}) &:= f_j(\mathbf{y}_\beta)\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}),\\
h_{\alpha,\beta}(\mathbf{x}) &:= p_j(\mathbf{y}_\alpha)\overline{p_j(\mathbf{y}_\beta)}.
\end{aligned}
$$

Then, we can rewrite $R[\mathbf{x}](\alpha,\beta)$, $\alpha,\beta = 0,\ldots,\tilde{m}-1$, as

$$
R[\mathbf{x}](\alpha,\beta) = -\frac{h_{\alpha,\beta}(\mathbf{x})}{h(\mathbf{x})},\quad \alpha\neq\beta,\qquad \text{and}\qquad R[\mathbf{x}](\beta,\beta) = \frac{h_\beta(\mathbf{x})}{h_{f_j,\beta}(\mathbf{x})}.
$$

To prove the boundedness of $R[\mathbf{x}](\alpha,\beta)$ $\alpha,\beta = 0,\ldots,\tilde{m}-1$, we show, for θ as in (3.25), that

$$
\theta_{h_{\alpha,\beta}}(\mathbf{x}) \geq \theta_h(\mathbf{x}) \qquad \text{and} \qquad \theta_{h_\beta}(\mathbf{x}) \geq \theta_{h_{f_j,\beta}}(\mathbf{x}).
$$

Note first that *(i)* and (3.29) guarantee that the order of the zero of h at $\mathbf{x}$ is the same as the order of the zero of f_j at $\mathbf{x}_j$. Namely, for $\Theta := \theta_{f_j}(\mathbf{x}_j)$, we have $\theta_h(\mathbf{x}) = \Theta$. Due to *(i)*, $\theta_{h_{\alpha,\beta}}(\mathbf{x}) \geq \Theta$. Thus, $\theta_{h_{\alpha,\beta}}(\mathbf{x}) \geq \theta_h(\mathbf{x})$. Since $\mathbf{x}\in\Omega_{\mathbf{m}}(\mathbf{x}_j)$, there exists $\bar{\beta}\in\{0,\ldots,\tilde{m}-1\}$ such that $\mathbf{y}_{\bar{\beta}} = \mathbf{x}_j$. If $\beta=\bar{\beta}$, then, by *(i)* and (3.29), $\theta_{h_\beta}(\mathbf{x}) \geq \theta_{h_{f_j,\beta}}(\mathbf{x}) = 2\Theta$. Otherwise, $\theta_{h_\beta}(\mathbf{x}) = \theta_{h_{f_j,\beta}}(\mathbf{x}) = \Theta$.

Case 2.b: We assume next that $\mathbf{x}\notin\Omega_{\mathbf{m}}(\mathbf{x}_j)$. First, we notice that, if $\mathbf{x}\notin\Omega_{\mathbf{m}}(\mathbf{x}_j)$, then $\mathbf{x}_j\notin\Omega_{\mathbf{m}}(\mathbf{x})$. Since f_j has a unique zero at $\mathbf{x}_j$ by hypothesis, we have $f_j(\mathbf{y}_\beta)\neq 0$, $\beta = 0,\ldots,\tilde{m}-1$. Thus, we only need to study the properties of $\sum\limits_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y})$. Since f_j has a unique zero at $\mathbf{x}_j$ by hypothesis, by *(ii)*, we obtain

$$
\sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\mathbf{x})} |p_j(\mathbf{y})|^2 f_j(\mathbf{y}) > 0.
$$

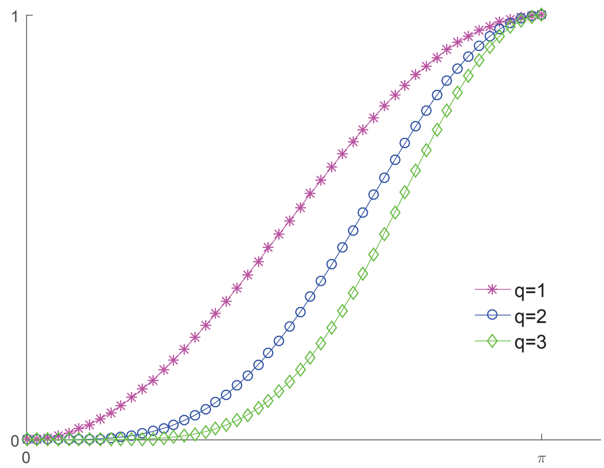


Figure 3.2: *Plot of the univariate generating functions* $f^{(q)}/\|f^{(q)}\|_\infty$ *in* (3.31) *with* $q = 1,2,3$ *in the reference interval* $[0,\pi]$.

And, thus, the claim follows. ■

3.5.4 Examples of grid transfer operators

In this subsection, we present some well-known examples [3, 9, 31, 48, 71] of grid transfer operators designed for the solution of linear systems of equations (3.1) derived from the discretization of elliptic PDEs. More precisely, let $d \in \mathbb{N}$ and $q \in \mathbb{N}$. Consider the $2q$ elliptic d-variate problem

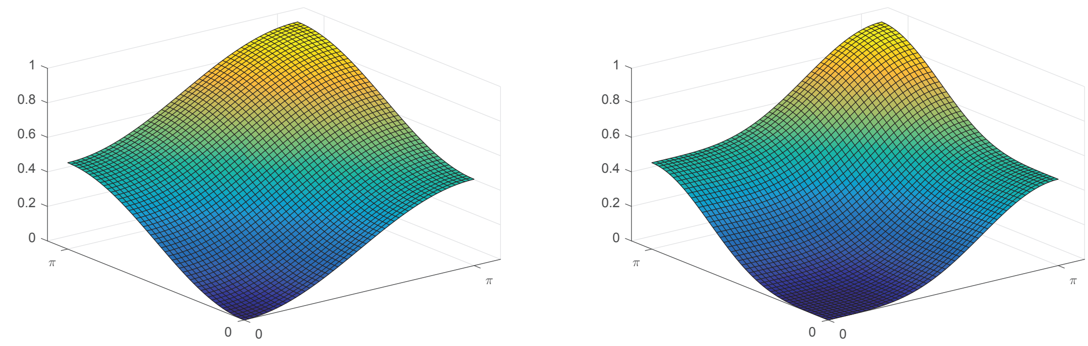
$$\begin{cases} (-1)^q \sum_{i=1}^{d} \frac{\partial^{2q}}{\partial x_i^{2q}} \psi(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega = (0,1)^d, \\ \text{periodic boundary conditions on} & \partial\Omega. \end{cases} \tag{3.30}$$

The system matrix in (3.1) is obtained via finite difference discretization of order $2q$ of (3.30) on a grid on $[0,1]^d$ of n_i subintervals of size h_i in each coordinate direction $i = 1,\dots,d$. It is well-known [68], that in this case the system matrix in (3.1) is the d-level circulant matrix $C_{\mathbf{n}}(f^{(q)})$, of order $\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{N}^d$, generated by the d-variate trigonometric polynomial

$$f^{(q)}(\mathbf{x}) = \sum_{i=1}^{d} (2-2\cos(x_i))^q, \qquad \mathbf{x} = (x_1,\dots,x_d) \in [0,2\pi)^d. \tag{3.31}$$

The generating function $f^{(q)}$ in (3.31) vanishes at $\mathbf{x}_0 = \mathbf{0}$ with order $2q$ and it is positive on $(0,2\pi)^d$. See Figures 3.2 ($d = 1$) and 3.3 ($d = 2$).

In the case $d = 1$ [9, 48, 71], the first univariate grid transfer operator defined for the solution of the univariate Laplacian problem, i.e. $q = 1$ in (3.30), is the so-called *linear interpolation*.



(a) *Case* $q = 1$. (b) *Case* $q = 2$.

Figure 3.3: *Plot of the bivariate generating functions* $f^{(q)}/\|f^{(q)}\|_\infty$ *in* (3.31) *with* $q = 1$ (a) *and* $q = 2$ (b) *in the reference interval* $[0,\pi]^2$.

The linear interpolation grid transfer operator P_{n_j} in (3.11) with the downsampling factor $m = 2$ is defined by the trigonometric polynomial

$$p^{(1)}(x) = 1 + \cos(x), \qquad x \in [0,2\pi), \tag{3.32}$$

or, equivalently, by its Fourier coefficients

$$\mathbf{p}^{(1)} = \{\tfrac{1}{2} \quad 1 \quad \tfrac{1}{2}\}.$$

For $j = 0,\ldots,\ell-1$, given the coarser error $\mathbf{e}_{n_{j+1}} \in \mathbb{C}^{n_{j+1}}$, the components of the finer error $\mathbf{e}_{n_j} = P_{n_j}\mathbf{e}_{n_{j+1}} \in \mathbb{C}^{n_j}$ are computed by

$$\begin{cases} \mathrm{e}_{n_j}(2\alpha) &= \mathrm{e}_{n_{j+1}}(\alpha), \\ \mathrm{e}_{n_j}(2\alpha+1) &= \frac{1}{2}\big(\mathrm{e}_{n_{j+1}}(\alpha) + \mathrm{e}_{n_{j+1}}(\alpha+1)\big), \end{cases} \qquad \alpha \in \{0,\ldots,n_{j+1}-1\},$$

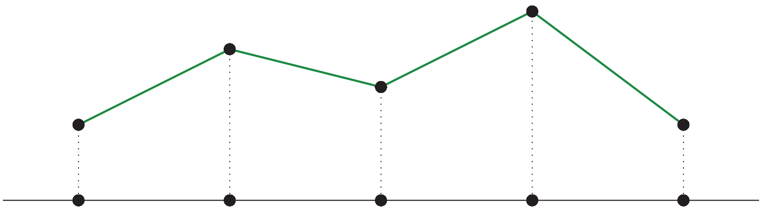
where we assume that $\mathrm{e}_{n_{j+1}}(n_{j+1}) = 0$. Thus, all the entries of the coarser error $\mathbf{e}_{n_{j+1}}$ are also present in the finer error $\mathbf{e}_{n_j}$. See Figure 3.4.

The trigonometric polynomial $p^{(1)}$ in (3.32) belongs to a family $\{p^{(J)} : J \in \mathbb{N}\}$ of trigonometric polynomials

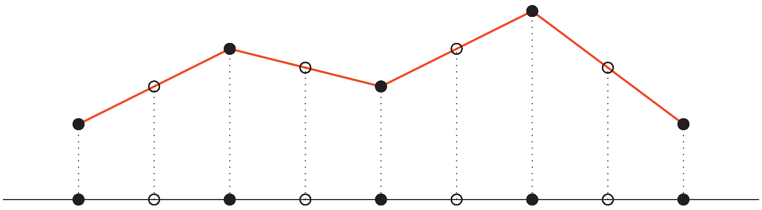
$$p^{(J)}(x) = 2\left(\frac{1+\cos(x)}{2}\right)^J, \qquad x \in [0,2\pi), \tag{3.33}$$

each of which defines the grid transfer operators P_{n_j} in (3.11) with the downsampling factor $m = 2$.

The trigonometric polynomial $p^{(J)}$ in (3.33) has a zero of order $2J$ at $\pi = \Omega_2'(0)$ and it is positive in $[0,2\pi)\setminus\{\pi\}$. See Figure 3.5. We highlight that, for $J \geq q$, the trigonometric polynomial $p^{(J)}$ in (3.33) satisfies *(i)* and *(ii)* of Theorem 3.8 with $d = 1$ and with respect to the generating function $f_0 = f^{(q)}$ in (3.31). Indeed, since $f^{(q)}(0) = 0$ and $f^{(q)}(x) > 0$, $x \in (0,2\pi)$,



(a) *Coarser error* $\mathbf{e}_{n_{j+1}} \in \mathbb{C}^{n_{j+1}}$



(b) *Finer error* $\mathbf{e}_{n_j} \in \mathbb{C}^{n_j}$

Figure 3.4: *Action of the linear interpolation grid transfer operator on the coarser error* $\mathbf{e}_{n_{j+1}}$ *(black dots in* (a)*) for the definition of the finer error* $\mathbf{e}_{n_j}$ *(black and white dots in* (b)*).*

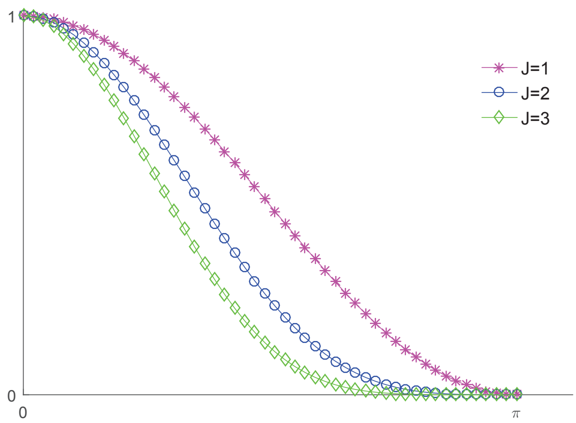
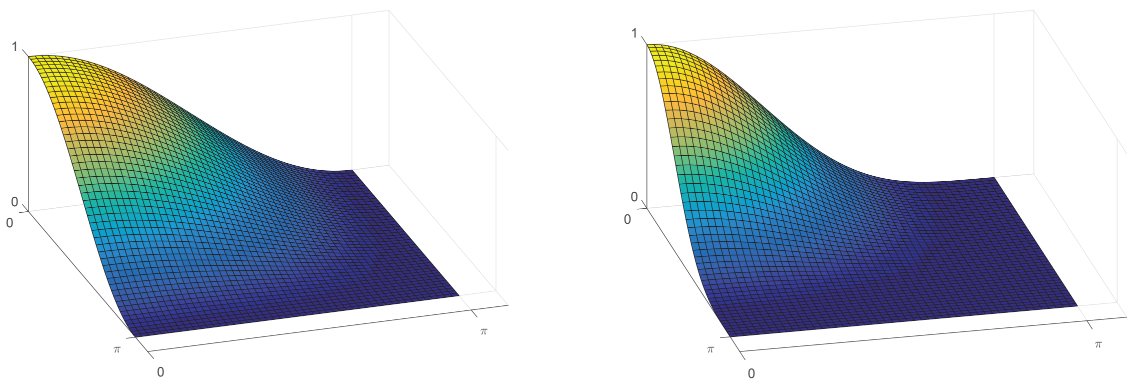


Figure 3.5: *Plot of the trigonometric polynomials* $p^{(J)}/\|p^{(J)}\|_\infty$ *in* (3.33) *with* $J = 1,2,3$ *in the reference interval* $[0,\pi]$.



(a) *Case* $J = 1$.   (b) *Case* $J = 2$.

Figure 3.6: *Plot of the trigonometric polynomials* $p^{(J,J)}/\|p^{(J,J)}\|_\infty$ *in* (3.34) *with* $J = 1$ (a) *and* $J = 2$ (b) *in the reference interval* $[0,\pi]^2$.

then [34, Proposition 4.1] guarantees that every f_j, $j = 1,\dots,\ell-1$, in (3.14) vanishes only at 0 with the same order as the one of the zero of f_0, i.e. $2q$. Thus, we set $p_j = p^{(J)}$, $j = 0,\dots,\ell-1$.

In the bivariate setting ($d = 2$), a family of grid transfer operators P_{n_j} in (3.16) with the downsampling factor $\mathbf{m} = (2,2)$ has been defined [3, 31]. The associated trigonometric polynomial $p^{(J,J)}$ is the tensor product of the univariate trigonometric polynomial $p^{(J)}$ in (3.33) with itself, namely

$$p^{(J,J)}(x_1,x_2) = 4\left(\frac{(1+\cos(x_1))(1+\cos(x_2))}{4}\right)^J, \qquad \mathbf{x} = (x_1,x_2) \in [0,2\pi)^2. \tag{3.34}$$

The trigonometric polynomial $p^{(J,J)}$ in (3.34) vanishes at $\Omega'_{(2,2)}(\mathbf{0}) = \{(0,\pi),(\pi,0),(\pi,\pi)\}$ with order $2J$ and it is positive in $[0,2\pi)^2 \setminus \{x_1 = \pi, x_2 = \pi\}$. See Figure 3.6. Notice that, for $J \geq q$, the trigonometric polynomial $p^{(J,J)}$ in (3.34) satisfies *(i)* and *(ii)* of Theorem 3.8 with respect to the generating function $f_0 = f^{(q)}$ in (3.31). Indeed, since $f^{(q)}(\mathbf{0}) = 0$ and $f^{(q)}(\mathbf{x}) > 0$, $\mathbf{x} \in (0,2\pi)^2$, then [6, Lemma 3.2] guarantees that every f_j, $j = 1,\dots,\ell-1$, in (3.20) vanishes only at $\mathbf{0}$ with the same order as the one of the zero of f_0, i.e. $2q$. Thus, we set $p_j = p^{(J,J)}$, $j = 0,\dots,\ell-1$.

The grid transfer operators defined in this subsection are actually well-known stationary subdivision schemes, see chapter 5. Indeed, Step 3. in (3.6) can be interpreted as the lowpass branch of a wavelet decomposition. For $j = 0,\dots,\ell-1$, at the j-th level of V-cycle, the convolution with the lowpass filter is the multiplication by the matrix $C_{\mathbf{n}_j}(p_j)^*$ and the downsampling by $\mathbf{m}$ is done via multiplication by the matrix $K_{\mathbf{n}_j,\mathbf{m}}$. If the smoother works well, then the residual is smooth and the highpass branches of the wavelet decomposition contain no additional information and are omitted. The reconstruction is done as usual by upsampling via multiplication by $K^T_{\mathbf{n}_j,\mathbf{m}}$ and by convolution via multiplication by $C_{\mathbf{n}_j}(p_j)$. It is well-known that upsampling and convolution amount to one step of subdivision scheme with the corresponding subdivision matrix $P_{\mathbf{n}_j}$. It is then natural to study conditions on the corresponding subdivision symbols p_j that will guarantee convergence and optimality of the corresponding multigrid methods.

CHAPTER 4

Stationary subdivision and algebraic multigrid

In this chapter, we analyze the link between algebraic multigrid and stationary subdivision. As already mentioned at the end of chapter 3 in subsection 3.5.4, it is possible to identify the d-variate grid transfer operators which appear in the definition of the algebraic two-grid method and V-cycle method with d-variate subdivision schemes. In section 4.1, we introduce d-variate stationary subdivision and list the well-known results on their convergence, interpolation and polynomial generation properties. The link between multigrid and subdivision is presented in section 4.2. We highlight that the definition and analysis of subdivision based multigrid appear for the first time in [14, 15].

4.1 Stationary subdivision

Let $M \in \mathbb{Z}^{d\times d}$ be a dilation matrix, namely all its eigenvalues are in the absolute value greater than 1. Let $\mathbf{p} = \{\mathrm{p}(\boldsymbol{\alpha}) \in \mathbb{R} : \boldsymbol{\alpha} \in \mathbb{Z}^d\} \in \ell_0(\mathbb{Z}^d)$ be a finite sequence of real numbers. The *dilation* M and the *mask* $\mathbf{p}$ are used to define the subdivision operator $\mathcal{S}_\mathbf{p} : \ell(\mathbb{Z}^d) \to \ell(\mathbb{Z}^d)$, which is a linear operator such that

$$\left(\mathcal{S}_\mathbf{p}\mathbf{c}\right)(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^d} \mathrm{p}(\boldsymbol{\alpha} - M\boldsymbol{\beta})\mathrm{c}(\boldsymbol{\beta}), \quad \alpha \in \mathbb{Z}^d, \quad \mathbf{c} \in \ell(\mathbb{Z}^d). \tag{4.1}$$

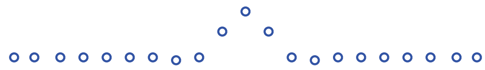
A subdivision scheme $S_\mathbf{p}$ with dilation M and mask $\mathbf{p}$ is the recursive application of the subdivision operator $\mathcal{S}_\mathbf{p}$ in (4.1) to some initial sequence $\mathbf{c}^{(0)} = \{\mathrm{c}^{(0)}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbb{Z}^d\} \in \ell(\mathbb{Z}^d)$ of real numbers, namely

$$\mathbf{c}^{(k+1)} = \mathcal{S}_\mathbf{p}\mathbf{c}^{(k)}, \quad k \in \mathbb{N}_0. \tag{4.2}$$

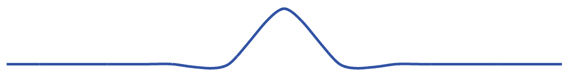
Notice that $\mathbf{c}^{(k+1)} = \mathcal{S}_\mathbf{p}\mathbf{c}^{(k)} = \cdots = (\mathcal{S}_\mathbf{p})^{k+1}\mathbf{c}^{(0)}$.

Since the subdivision scheme $S_\mathbf{p}$ generates sequences $\mathbf{c}^{(k)} \in \ell(\mathbb{Z}^d)$, $k \geq 0$, a natural way to define a notion of its convergence is to attach the data $\mathbf{c}^{(k)} = \{\mathrm{c}^{(k)}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbb{Z}^d\}$, $k \geq 0$, to the

(a) *Starting sequence* $\mathbf{c}^{(0)} = \boldsymbol{\delta}$



(b) *After one step of subdivision* $\mathbf{c}^{(1)} = S_{\mathbf{p}}\boldsymbol{\delta}$



(c) *Basic limit function* $\phi = S_{\mathbf{p}}^{\infty}\boldsymbol{\delta}$

Figure 4.1: *Basic limit function of the binary 4-point Dubuc-Deslauries subdivision scheme*

parameter values $\mathbf{t}^{(k)} = \{M^{-k}\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \mathbb{Z}^d\}$, $k \geq 0$, and to require that there exists a continuous function $F_{\mathbf{c}^{(0)}} : \mathbb{R}^d \to \mathbb{R}$ depending on the starting sequence $\mathbf{c}^{(0)}$ such that, for sufficiently large k, the values of $F_{\mathbf{c}^{(0)}}$ at the parameter values $\mathbf{t}^{(k)}$ are "close" enough to the data $\mathbf{c}^{(k)}$.

Definition 4.1. A subdivision scheme $S_{\mathbf{p}}$ is *convergent* if for any initial data $\mathbf{c} \in \ell^{\infty}(\mathbb{Z}^d)$ there exist a uniformly continuous function $F_{\mathbf{c}} \in \mathcal{C}(\mathbb{R}^d)$ such that

$$\lim_{k\to\infty} \sup_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \left| F_{\mathbf{c}}\left(M^{-k}\boldsymbol{\alpha}\right) - \left(\mathcal{S}_{\mathbf{p}}^k \mathbf{c}\right)(\boldsymbol{\alpha}) \right| = 0.$$

The particular choice of the initial data $\boldsymbol{\delta} = \left\{\delta_{\boldsymbol{\alpha},0} : \boldsymbol{\alpha} \in \mathbb{Z}^d\right\}$ defines the so-called *basic limit function* $\phi = F_{\boldsymbol{\delta}}$. Figure 4.1 shows the basic limit function of the univariate binary 4-point Dubuc-Deslauriers subdivision scheme [37] defined by

$$m = 2, \qquad \mathbf{p} = \left\{ -\tfrac{1}{16} \quad 0 \quad \tfrac{9}{16} \quad 1 \quad \tfrac{9}{16} \quad 0 \quad -\tfrac{1}{16} \right\}. \tag{4.3}$$

Figure 4.2 shows the basic limit function of the bivariate anisotropic linear subdivision scheme [14] defined by

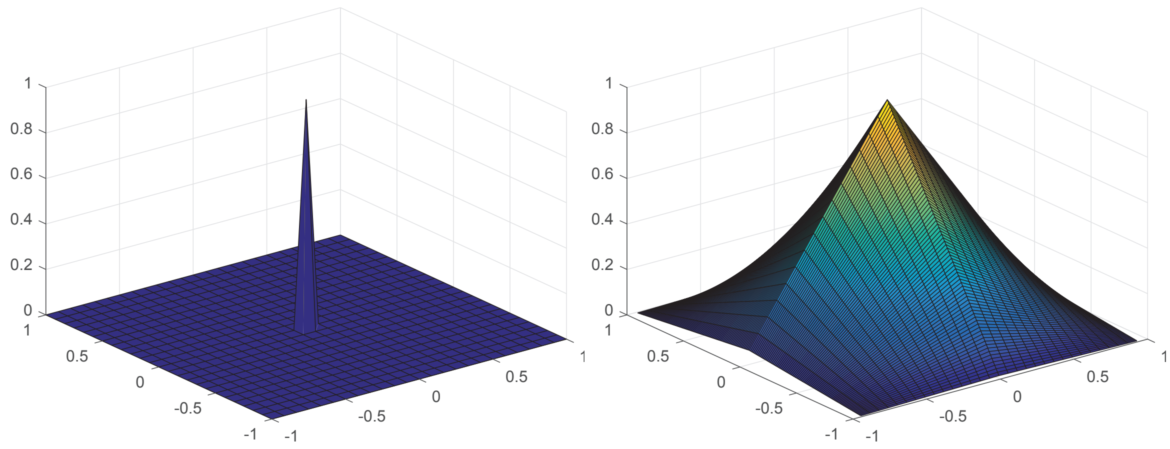
$$M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in \mathbb{Z}^{2\times 2}, \qquad \mathbf{p} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}. \tag{4.4}$$

Since the mask $\mathbf{p} \in \ell_0(\mathbb{Z}^d)$ is a finite sequence, ϕ is compactly supported [39]. It is well-known [39] that the basic limit function ϕ satisfies the refinement equation

$$\phi = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \mathrm{p}(\boldsymbol{\alpha})\phi(M\cdot - \boldsymbol{\alpha}). \tag{4.5}$$

Thus, due to the linearity of $\mathcal{S}_{\mathbf{p}}$, for any initial data $\mathbf{c} \in \ell^{\infty}(\mathbb{Z}^d)$, $\mathbf{c} = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \mathrm{c}(\boldsymbol{\alpha})\boldsymbol{\delta}(\cdot - \boldsymbol{\alpha})$, we have

$$F_{\mathbf{c}} = \lim_{k\to\infty} \mathcal{S}_{\mathbf{p}}^k \mathbf{c} = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \mathrm{c}(\boldsymbol{\alpha}) \lim_{k\to\infty} \mathcal{S}_{\mathbf{p}}^k \boldsymbol{\delta}(\cdot - \boldsymbol{\alpha}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \mathrm{c}(\boldsymbol{\alpha})\phi(\cdot - \boldsymbol{\alpha}).$$



(a) *Starting sequence* $\mathbf{c}^{(0)} = \boldsymbol{\delta}$

(b) *Basic limit function* $\phi = S_{\mathbf{p}}^{\infty}\boldsymbol{\delta}$

Figure 4.2: *Basic limit function of the anisotropic linear subdivision scheme*

For more details on the properties of the basic limit function, see the seminal work of Cavaretta et al. [11] and the survey by Dyn and Levin [39].

Most of the properties of the subdivision scheme $S_{\mathbf{p}}$ can be investigated studying the Laurent polynomial

$$p(\mathbf{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \mathrm{p}(\boldsymbol{\alpha})\mathbf{z}^{\boldsymbol{\alpha}}, \qquad \mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1}\cdot\ldots\cdot z_d^{\alpha_d}, \qquad \mathbf{z} = (z_1,\ldots,z_d) \in (\mathbb{C}\setminus\{0\})^d, \tag{4.6}$$

called the *symbol* of the subdivision scheme. For instance, a well-known necessary condition for the convergence of $S_{\mathbf{p}}$ is $p(\mathbf{1}) = |\det M|$. With respect to the previous examples, the symbol of the univariate binary 4-point Dubuc-Deslauriers subdivision scheme defined by (4.3) is given by

$$p(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3 = -\frac{1}{16z^3}(1+z)^4(1-4z+z^2), \qquad z \in \mathbb{C}\setminus\{0\}, \tag{4.7}$$

and the symbol of the bivariate anisotropic linear subdivision scheme defined by (4.4) is given by

$$\begin{aligned} p(\mathbf{z}) &= \frac{1}{6}z_1^{-1}z_2^{-2} + \frac{1}{3}z_1^{-1}z_2^{-1} + \frac{1}{2}z_1^{-1} + \frac{1}{3}z_1^{-1}z_2 + \frac{1}{6}z_1^{-1}z_2^2 \\ &+ \frac{1}{3}z_2^{-2} + \frac{2}{3}z_2^{-1} + 1 + \frac{2}{3}z_2 + \frac{1}{3}z_2^2 \\ &+ \frac{1}{6}z_1z_2^{-2} + \frac{1}{3}z_1z_2^{-1} + \frac{1}{2}z_1 + \frac{1}{3}z_1z_2 + \frac{1}{6}z_1z_2^2 \\ &= \frac{1}{6z_1z_2^2}(1+z_1)^2(1+z_2+z_2^2)^2, \qquad \mathbf{z} = (z_1,z_2) \in (\mathbb{C}\setminus\{0\})^2. \end{aligned} \tag{4.8}$$

4.1.1 Interpolatory subdivision

In this subsection, we shortly describe interpolatory subdivision. We say that a subdivision scheme $S_{\mathbf{p}}$ with dilation M and mask $\mathbf{p}$ is *interpolatory* if, given any starting sequence $\mathbf{c}^{(0)} \in \ell(\mathbb{Z}^d)$, all the entries of the k-th refined sequence $\mathbf{c}^{(k)} \in \ell(\mathbb{Z}^d)$ are also entries of the $(k+1)$-th refined sequence $\mathbf{c}^{(k+1)} \in \ell(\mathbb{Z}^d)$, $k \geq 0$. More precisely,

$$\mathrm{c}^{(k+1)}(M\boldsymbol{\alpha}) = \left(\mathcal{S}_{\mathbf{p}}\mathbf{c}^{(k)}\right)(M\boldsymbol{\alpha}) = \mathrm{c}^{(k)}(\boldsymbol{\alpha}), \qquad \forall\, \boldsymbol{\alpha} \in \mathbb{Z}^d.$$

The interpolation property can be interpreted in terms of the mask of the subdivision scheme [39, 55]. A subdivision scheme $S_{\mathbf{p}}$ with dilation M and mask $\mathbf{p}$ is interpolatory if its mask $\mathbf{p}$ satisfies

$$\mathrm{p}(\mathbf{0}) = 1 \qquad \text{and} \qquad \mathrm{p}(M\boldsymbol{\alpha}) = 0, \qquad \forall\, \boldsymbol{\alpha} \in (\mathbb{Z} \setminus \{0\})^d. \tag{4.9}$$

In [19], in the case of dilation matrix $2I_d$, where I_d is the identity matrix of order d, the authors characterize the interpolation property of subdivision in terms of the corresponding subdivision symbol. Their result can be easily extended to the case of diagonal anisotropic dilation matrix

$$M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_d \end{pmatrix} \in \mathbb{Z}^{d\times d}, \qquad m_i \geq 2, \qquad i = 1, \dots, d. \tag{4.10}$$

We denote by Γ, $\#\Gamma = |\det M|$, the complete set of representatives of the distinct cosets of $\mathbb{Z}^d / M\mathbb{Z}^d$ containing $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^d$ and we define the set

$$E_M = \left\{ \mathrm{e}^{-\mathrm{i}2\pi M^{-T}\boldsymbol{\gamma}} : \boldsymbol{\gamma} \text{ is a coset representative of } \mathbb{Z}^d / M^T\mathbb{Z}^d \right\}, \qquad \#E_M = |\det M|, \tag{4.11}$$

containing $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^d$. If the dilation M is diagonal (4.10), then the set Γ is defined by

$$\Gamma = \left\{ \boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}^d : \gamma_i \in \{0, \dots, m_i - 1\},\, i = 1, \dots, d \right\}, \tag{4.12}$$

and, due to $M = M^T$, the set E_M becomes

$$\begin{aligned} E_M &= \left\{ \mathrm{e}^{-\mathrm{i}2\pi M^{-1}\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \Gamma \right\}, \\ &= \left\{ \boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d : \xi_i = \mathrm{e}^{-\mathrm{i}2\pi \frac{\gamma_i}{m_i}},\, \gamma_i \in \{0, \dots, m_i - 1\},\, i = 1, \dots, d \right\}. \end{aligned} \tag{4.13}$$

The result of Theorem 4.1 is well-known in case of dilation matrix $2I_d$, see [19]. We include the prove for dilation (4.10) for reader's convenience.

Theorem 4.1. *A convergent subdivision scheme $S_{\mathbf{p}}$ with dilation* (4.10) *is interpolatory if and only if*

$$\sum_{\boldsymbol{\xi} \in E_M} p(\boldsymbol{\xi} \times \boldsymbol{z}) = |det\, M|, \qquad \boldsymbol{\xi} \times \boldsymbol{z} = (\xi_1 z_1, \dots, \xi_d z_d), \qquad \boldsymbol{z} \in (\mathbb{C} \setminus \{0\})^d.$$

Proof. Let $S_{\mathbf{p}}$ be a convergent subdivision scheme with dilation M and mask $\mathbf{p}$. By [13], the subdivision symbol p satisfies

$$p(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \mathrm{p}(\boldsymbol{\alpha})\mathbf{z}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})\mathbf{z}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}}, \qquad \boldsymbol{z}\in(\mathbb{C}\setminus 0)^d.$$

Thus, we get

$$\begin{aligned}
\sum_{\boldsymbol{\xi}\in E_M} p\big(\boldsymbol{\xi}\times\boldsymbol{z}\big) &= \sum_{\boldsymbol{\xi}\in E_M}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})(\boldsymbol{\xi}\times\mathbf{z})^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}},\\
&= \sum_{\boldsymbol{\xi}\in E_M}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})(\xi_1 z_1)^{(M\boldsymbol{\alpha}+\boldsymbol{\gamma})_1}\cdots(\xi_d z_d)^{(M\boldsymbol{\alpha}+\boldsymbol{\gamma})_d},\\
&= \sum_{\boldsymbol{\xi}\in E_M}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})\big(z_1^{(M\boldsymbol{\alpha}+\boldsymbol{\gamma})_1}\cdots z_d^{(M\boldsymbol{\alpha}+\boldsymbol{\gamma})_d}\big)\big(\xi_1^{(M\boldsymbol{\alpha}+\boldsymbol{\gamma})_1}\cdots \xi_d^{(M\boldsymbol{\alpha}+\boldsymbol{\gamma})_d}\big),\\
&= \sum_{\boldsymbol{\xi}\in E_M}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})\mathbf{z}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}}\boldsymbol{\xi}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}},\\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})\mathbf{z}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}}\sum_{\boldsymbol{\xi}\in E_M}\boldsymbol{\xi}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}},\\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\sum_{\boldsymbol{\gamma}\in\Gamma} \mathrm{p}(M\boldsymbol{\alpha}+\boldsymbol{\gamma})\mathbf{z}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}} s_{E_M,\boldsymbol{\gamma}}, \qquad s_{E_M,\boldsymbol{\gamma}} = \sum_{\boldsymbol{\xi}\in E_M}\boldsymbol{\xi}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}}, \qquad \boldsymbol{z}\in(\mathbb{C}\setminus 0)^d.
\end{aligned} \tag{4.14}$$

We focus on $s_{E_M,\boldsymbol{\gamma}}$ in (4.14). Let M be diagonal (4.10). By (4.13), $\boldsymbol{\xi}\in E_M$ is of the form

$$\boldsymbol{\xi} = \Big(\mathrm{e}^{-\mathrm{i}2\pi\frac{\beta_1}{m_1}},\ldots,\mathrm{e}^{-\mathrm{i}2\pi\frac{\beta_d}{m_d}}\Big), \qquad \beta_i\in\{0,\ldots,m_i-1\}, \qquad i=1,\ldots,d.$$

Due to

$$\boldsymbol{\xi}^{M\boldsymbol{\alpha}+\boldsymbol{\gamma}} = \xi_1^{m_1\alpha_1+\gamma_1}\cdots\xi_d^{m_d\alpha_d+\gamma_d} = \xi_1^{\gamma_1}\cdots\xi_d^{\gamma_d} = \boldsymbol{\xi}^{\boldsymbol{\gamma}},$$

we have $s_{E_M,\mathbf{0}} = \#E_M = |\det M|$ and

$$s_{E_M,\boldsymbol{\gamma}} = \sum_{\boldsymbol{\xi}\in E_M}\boldsymbol{\xi}^{\boldsymbol{\gamma}} = \sum_{\beta_1=0}^{m_1-1}\Big(\mathrm{e}^{-\mathrm{i}2\pi\frac{\beta_1}{m_1}}\Big)^{\gamma_1}\cdots\sum_{\beta_d=0}^{m_d-1}\Big(\mathrm{e}^{-\mathrm{i}2\pi\frac{\beta_d}{m_d}}\Big)^{\gamma_d} = 0, \qquad \boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_d)\in\Gamma\setminus\{\mathbf{0}\}, \tag{4.15}$$

where the last equality holds true due to (4.12), $\boldsymbol{\gamma}\neq\mathbf{0}$ and

$$\sum_{\beta_i=0}^{m_i-1}\Big(\mathrm{e}^{-\mathrm{i}2\pi\frac{\beta_i}{m_i}}\Big)^{\gamma_i} = 0, \qquad \gamma_i=1,\ldots,m_i-1, \qquad i=1,\ldots,d.$$

Then, (4.14) becomes

$$\sum_{\boldsymbol{\xi}\in E_M} p\big(\boldsymbol{\xi}\cdot\boldsymbol{z}\big) = |\det M|\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\mathrm{p}(M\boldsymbol{\alpha})\mathbf{z}^{M\boldsymbol{\alpha}}.$$

The claim follows by the fact that a subdivision scheme $S_{\mathbf{p}}$ is interpolatory if and only if its mask $\mathbf{p}$ satisfies the interpolation property (4.9).

■

Let us consider the univariate binary 4-point Dubuc-Deslauriers subdivision scheme. Its mask in (4.3) satisfies the interpolation property, namely

$$\mathrm{p}(0)=1, \qquad \mathrm{p}(2\alpha)=0, \qquad \alpha\in\mathbb{Z}\setminus\{0\}.$$

Moreover, its symbol (4.7) satisfies the identity in Theorem 4.1. Indeed, we have

$$\Gamma=\{0,1\}, \qquad E_2=\{1,-1\}, \qquad p(z)+p(-z)=2.$$

Let us consider the bivariate anisotropic interpolatory subdivision scheme. Its mask in (4.4) satisfies the interpolation property, namely

$$\mathrm{p}(0,0)=1, \qquad \mathrm{p}(M\boldsymbol{\alpha})=\mathrm{p}(2\alpha_1,3\alpha_2)=0, \qquad \boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in(\mathbb{Z}\setminus\{0\})^2.$$

Its symbol (4.8) satisfies the identity in Theorem 4.1. Indeed, we have

$$\Gamma=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}, \qquad \#\Gamma=6,$$

$$E_M=\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\\mathrm{e}^{-\mathrm{i}2/3\pi}\end{pmatrix},\begin{pmatrix}1\\\mathrm{e}^{-\mathrm{i}4/3\pi}\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\\mathrm{e}^{-\mathrm{i}2/3\pi}\end{pmatrix},\begin{pmatrix}-1\\\mathrm{e}^{-\mathrm{i}4/3\pi}\end{pmatrix}\right\}, \qquad \#E_M=6,$$

and

$$p(z_1,z_2)+p(z_1,\mathrm{e}^{-\mathrm{i}2/3\pi}z_2)+p(z_1,\mathrm{e}^{-\mathrm{i}4/3\pi}z_2)+p(-z_1,z_2)+p(-z_1,\mathrm{e}^{-\mathrm{i}2/3\pi}z_2)+p(-z_1,\mathrm{e}^{-\mathrm{i}4/3\pi}z_2)=6.$$

4.1.2 Generation and reproduction properties

We now introduce the concepts of polynomial generation and reproduction. The property of generation of polynomials of degree q is the capability of a subdivision scheme to generate the full space of polynomials up to degree q, while the property of reproduction of polynomials of degree q is the capability of a subdivision scheme to produce in the limit exactly the same polynomial from which the initial data is sampled. It is easy to see that reproduction of polynomials of degree q implies generation of polynomials of degree q.

Definition 4.2. Let $q\in\mathbb{N}_0$. A convergent subdivision scheme $S_{\mathbf{p}}$ *generates* polynomials up to degree q if

$$\text{for any } \pi\in\Pi_q, \quad \exists\mathbf{c}\in\ell(\mathbb{Z}^d) \quad \text{such that} \quad \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\mathrm{c}(\boldsymbol{\alpha})\phi(\cdot-\boldsymbol{\alpha})=\pi\in\Pi_q.$$

The property of polynomial generation has been studied e.g. by Cabrelli et al. in [10], Cavaretta et al. in [11], Jetter and Plonka in [52], Jia in [53, 54], Levin in [59]. Definition 4.2 can be interpreted as follows: a convergent subdivision scheme $S_{\mathbf{p}}$ *generates* polynomials up to degree q if the space Π_q is contained in the span of the integer shifts of its basic limit function $\{\phi(\cdot-\boldsymbol{\alpha}) : \boldsymbol{\alpha}\in\mathbb{Z}^d\}$.

Algebraic properties of the symbol p characterize the polynomial generation property of subdivision.

Theorem 4.2 ([13]). *Let $q \in \mathbb{N}_0$. A convergent subdivision scheme $S_{\mathbf{p}}$ generates polynomials up to degree q if and only if*

$$D^{\boldsymbol{\mu}} p(\boldsymbol{\varepsilon}) = 0, \qquad \forall\, \boldsymbol{\varepsilon} \in E_M \setminus \{\mathbf{1}\}, \qquad \boldsymbol{\mu} \in \mathbb{N}_0^d, \qquad |\boldsymbol{\mu}| \le q. \tag{4.16}$$

Thus, the property of polynomial generation of a convergent subdivision scheme $S_{\mathbf{p}}$ is strictly related to the behavior of the subdivision symbol $p(\mathbf{z})$ and of its derivatives at the "special" points $E_M \setminus \{\mathbf{1}\}$. Conditions in (4.16) are also known as *zero conditions of order* $q+1$. We say that a subdivision symbol $p(\mathbf{z})$ satisfies the zero conditions of order $q+1$ if and only if the associated mask $\mathbf{p}$ satisfies the *sum rules of order* $q+1$, namely

$$\sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} \mathrm{p}(M\boldsymbol{\alpha})\pi(M\boldsymbol{\alpha}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} \mathrm{p}(\boldsymbol{\gamma} + M\boldsymbol{\alpha})\pi(\boldsymbol{\gamma} + M\boldsymbol{\alpha}), \qquad \forall\, \boldsymbol{\gamma} \in \Gamma \setminus \{\mathbf{0}\}, \pi \in \Pi_q. \tag{4.17}$$

In the univariate setting, Theorem 4.2 is equivalent to requiring that the symbol $p(z)$ of the subdivision scheme $S_{\mathbf{p}}$ of dilation $m \in \mathbb{N}$, $m \ge 2$, has the following factorization

$$p(z) = \left(1 + z + z^2 + \cdots + z^{m-1}\right)^{q+1} b(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.18}$$

for some Laurent polynomial $b(z)$ such that $b(1) = m^{-q}$, i.e. $p(1) = m$. For instance, the univariate binary 4-point Dubuc-Deslauriers subdivision scheme defined by (4.3) generates polynomials up to degree $q = 3$. Indeed, its symbol in (4.7) satisfies

$$p(z) = (1+z)^4\, b(z), \qquad b(z) = -\frac{1}{16z^3}(1 - 4z + z^2), \qquad b(1) = \frac{1}{8} = 2^{-3}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

In addition, we consider the binary cubic Bspline subdivision scheme defined by

$$m = 2, \qquad \mathbf{p} = \left\{\ \tfrac{1}{8} \quad \tfrac{1}{2} \quad \tfrac{3}{4} \quad \tfrac{1}{2} \quad \tfrac{1}{8}\ \right\}. \tag{4.19}$$

Its symbol satisfies

$$p(z) = \frac{1}{8}z^{-2} + \frac{1}{2}z^{-1} + \frac{3}{4} + \frac{1}{2}z + \frac{1}{8}z^2 = (1+z)^4\, b(z), \quad b(z) = \frac{1}{8z^2}, \quad b(1) = \frac{1}{8} = 2^{-3}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.20}$$

thus it generates polynomials up to degree $q = 3$.

In the bivariate setting, we lose the factorization property (4.18). Nevertheless, in case of diagonal dilation matrix M (4.10), Theorem 4.2 can be reformulated in terms of ideals [64], leading to an equivalent decomposition property. Let $q \in \mathbb{N}_0$ and define

$$\mathcal{J}_q := < (1 - z_1^{m_1})^{\mu_1} (1 - z_2^{m_2})^{\mu_2} : \boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{N}_0^2, |\boldsymbol{\mu}| = q+1 > .$$

$\mathcal{J}_q$ is the ideal of all bivariate polynomials $p(z_1, z_2)$ which satisfy

$$D^{\boldsymbol{\mu}} p(\boldsymbol{\varepsilon}) = 0, \qquad \forall \boldsymbol{\varepsilon} \in E_M, \qquad \forall \boldsymbol{\mu} \in \mathbb{N}_0^2, \qquad |\boldsymbol{\mu}| \le q.$$

We recall that, given two ideals $\mathcal{J}, \mathcal{I}$ of the commutative ring $\mathbb{C}[z_1, z_2]$, their quotient ideal is the set $\mathcal{J} : \mathcal{I} = \{ p \in \mathbb{C}[z_1, z_2] : p \cdot \mathcal{I} \subset \mathcal{J} \}$ and it is itself an ideal of $\mathbb{C}[z_1, z_2]$. Moreover, if $\mathcal{J} = \mathcal{J}(V)$

and $\mathcal{I} = \mathcal{I}(W)$ are the ideal associated to two affine varieties $V, W \subset \mathbb{C}^2$, the quotient ideal $\mathcal{J} : \mathcal{I}$ is the ideal associated to the difference of varieties $V \setminus W$ ([23]). Thus, the quotient ideal

$$\mathcal{I}_q \quad = \quad \mathcal{J}_q \quad : \quad < (1-z_1)^{\mu_1}(1-z_2)^{\mu_2} : \boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{N}_0^2, |\boldsymbol{\mu}| = q+1 >$$

is the ideal of all bivariate polynomials $p(z_1, z_2)$ which satisfy (4.16). Consequently, a convergent subdivision scheme generates polynomials up to degree q if and only if its symbol $p \in \mathcal{I}_q$. For instance, the bivariate anisotropic interpolatory subdivision scheme defined by (4.4) generates polynomials up to degree $q = 1$. Indeed, we have

$$\begin{aligned} \mathcal{J}_1 &= < \left(1-z_1^2\right)^{\mu_1}\left(1-z_2^3\right)^{\mu_2} : \boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{N}_0^2, |\boldsymbol{\mu}| = 2 > \\ &= < \left((1-z_1)(1+z_1)\right)^{\mu_1}\left((1-z_2)(1+z_2+z_2^2)\right)^{\mu_2} : \boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{N}_0^2, |\boldsymbol{\mu}| = 2 > . \end{aligned}$$

Thus, the symbol (4.8) of the bivariate anisotropic interpolatory subdivision scheme belongs to $\mathcal{I}_1$. Finally, if for $q, r \in \mathbb{N}_0$, $p_1 \in \mathcal{I}_q$ and $p_2 \in \mathcal{I}_r$, then $p_1 \cdot p_2 \in \mathcal{I}_{q+r+1}$.

The definition of the polynomial reproduction property differs from the definition of the polynomial generation property as the before mentioned property depends on the so-called sequence of *parameter values*. Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d$. The parameter values $\mathbf{t}^{(k)} = \{\mathbf{t}^{(k)}(\boldsymbol{\alpha}) \in \mathbb{R}^d : \boldsymbol{\alpha} \in \mathbb{Z}^d\}$, $k \geq 0$, are defined recursively by

$$\mathbf{t}^{(k)}(\boldsymbol{\alpha}) = \mathbf{t}^{(k)}(\mathbf{0}) + M^{-k}\boldsymbol{\alpha}, \qquad \mathbf{t}^{(k)}(\mathbf{0}) = \mathbf{t}^{(k-1)}(\mathbf{0}) - M^{-k}\boldsymbol{\tau}, \qquad \mathbf{t}^{(0)}(\mathbf{0}) = \mathbf{0}, \qquad \boldsymbol{\alpha} \in \mathbb{Z}^d, \qquad k \geq 0. \tag{4.21}$$

Definition 4.3. Let $q \in \mathbb{N}_0$. A convergent subdivision scheme $S_{\mathbf{p}}$ *reproduces* polynomials up to degree q with respect to the parameter values (4.21) if

$$\text{for any } \pi \in \Pi_q \text{ and } \mathbf{c} = \left\{\pi\left(\mathbf{t}^{(0)}(\boldsymbol{\alpha})\right) : \boldsymbol{\alpha} \in \mathbb{Z}^d\right\} \in \ell(\mathbb{Z}^d), \quad \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} \mathrm{c}(\boldsymbol{\alpha})\phi(\cdot - \boldsymbol{\alpha}) = \pi \in \Pi_q.$$

Definition 4.3 is more restrictive than Definition 4.2 since we require that the subdivision limit is exactly the same polynomial π from which the initial data $\mathbf{c}$ is sampled. Conti and Hormann [20] and Charina et al. [13] proved that the property of polynomial reproduction is characterized in terms of the subdivision symbol.

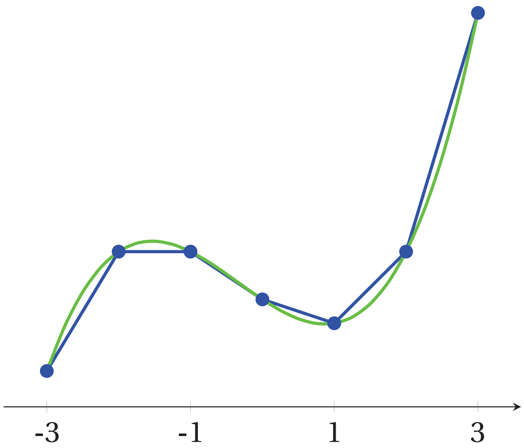
Theorem 4.3. *Let $q \in \mathbb{N}_0$. A convergent subdivision scheme $S_{\mathbf{p}}$ with parameter values* (4.21) *reproduces polynomials up to degree q if and only if*

$$D^{\boldsymbol{\mu}} p(\mathbf{1}) = |det\, M| \prod_{i=1}^{d} \prod_{\ell_i=0}^{\mu_i-1} (\tau_i - \ell_i) \quad and \quad D^{\boldsymbol{\mu}} p(\boldsymbol{\varepsilon}) = 0, \quad \forall \boldsymbol{\varepsilon} \in E_M \setminus \{\mathbf{1}\}, \quad \boldsymbol{\mu} \in \mathbb{N}_0^d, \quad |\boldsymbol{\mu}| \leq q. \tag{4.22}$$

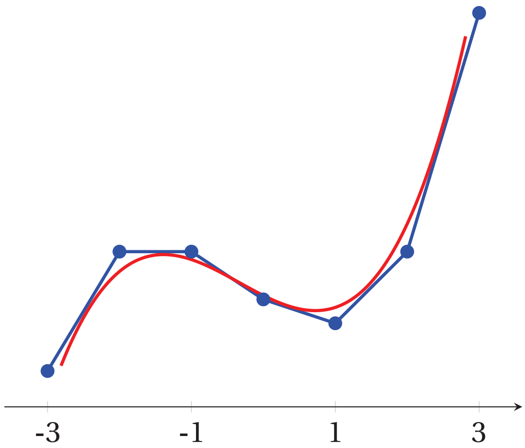
Theorem 4.3 implies that, in order to have the maximum degree of polynomial reproduction, it is necessary to choose the parameter $\boldsymbol{\tau} \in \mathbb{R}^d$ in (4.21) carefully.

In the univariate case, if the subdivision mask $\mathbf{p}$ is symmetric, i.e. $\mathrm{p}(\alpha) = \mathrm{p}(-\alpha)$, or interpolatory, then $\tau = 0$ is the optimal choice ([20]). Thus, (4.22) becomes

$$\begin{aligned} &p(1) = m, \\ &D^{\mu} p(1) = 0, \quad \mu \in \mathbb{N}_0, \quad 1 \leq \mu \leq q, \\ &D^{\mu} p(\varepsilon) = 0, \quad \forall \varepsilon \in E_M \setminus \{1\}, \quad \mu \in \mathbb{N}_0, \quad 0 \leq \mu \leq q. \end{aligned}$$



(a) *4-point Dubuc-Deslauriers scheme*



(b) *Cubic Bspline scheme*

Figure 4.3: *Subdivision limits of the univariate binary 4-point Dubuc-Deslauriers subdivision scheme* (a) *and of the univariate binary cubic Bspline subdivision scheme* (b). *The starting data* $\mathbf{c}^{(0)}$ *(blue) is sampled from the cubic polynomial* $\pi(\alpha) = \alpha^3 + \alpha^2 - 4\alpha - 8 \in \Pi_3$, $\alpha \in \mathbb{Z}$.

Therefore, in the univariate symmetric or interpolatory setting, Theorem 4.3 is equivalent to requiring that the symbol $p(z)$ of the subdivision scheme $S_{\mathbf{p}}$ of dilation $m \in \mathbb{Z}$, $m \geq 2$, has the following decomposition [20, 38]

$$p(z) = m + (1-z)^{q+1} c(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.23}$$

for a suitable Laurent polynomial $c(z)$. For instance, the univariate binary 4-point Dubuc-Deslauriers subdivision scheme defined by (4.3) reproduces polynomials up to degree $q = 3$. Indeed, its mask is interpolatory and symmetric and its symbol (4.7) satisfies

$$p(z) = 2 + (1-z)^4 c(z), \qquad c(z) = -\frac{1+4z+z^2}{16z^3}, \qquad z \in \mathbb{C} \setminus \{0\},$$

where the Laurent polynomial $c(z)$ is not divisible by $(1-z)$. The binary cubic B-spline subdivision scheme defined by (4.19), instead, reproduces only polynomials up to degree $q = 1$. Indeed, its mask is symmetric and its symbol (4.20) satisfies

$$p(z) = 2 + (1-z)^2 c(z), \qquad c(z) = \frac{1+6z+z^2}{8z^2}, \qquad z \in \mathbb{C} \setminus \{0\},$$

where the Laurent polynomial $c(z)$ is not divisible by $(1-z)$. Figure 4.3 shows the subdivision limits of the univariate binary 4-point Dubuc-Deslauriers subdivision scheme and of the univariate binary cubic B-spline subdivision scheme applied to a starting sequence $\mathbf{c}^{(0)}$ sampled from a cubic polynomial. Figure 4.3 illustrates that the univariate binary 4-point Dubuc-Deslauriers subdivision scheme reproduces cubic polynomials, while the univariate binary cubic Bspline subdivision scheme generates cubic polynomials.

In the bivariate case, if the subdivision mask $\mathbf{p}$ is symmetric, i.e.

$$\mathrm{p}(\alpha_1,\alpha_2)=\mathrm{p}(\alpha_1,-\alpha_2)=\mathrm{p}(-\alpha_1,\alpha_2)=\mathrm{p}(-\alpha_1,-\alpha_2),$$

or interpolatory, then $\boldsymbol{\tau}=\mathbf{0}$ is the optimal choice ([13]) and (4.22) becomes

$$\begin{aligned}
&p(\mathbf{1})=|\det M|,\\
&D^{\boldsymbol{\mu}}p(\mathbf{1})=0,\quad \boldsymbol{\mu}\in\mathbb{N}_0^d,\quad 1\le|\boldsymbol{\mu}|\le q,\\
&D^{\boldsymbol{\mu}}p(\boldsymbol{\varepsilon})=0,\quad \forall\,\boldsymbol{\varepsilon}\in E_M\setminus\{\mathbf{1}\},\quad \boldsymbol{\mu}\in\mathbb{N}_0^d,\quad 0\le|\boldsymbol{\mu}|\le q.
\end{aligned}$$

Thus, in the bivariate symmetric or interpolatory setting, Theorem 4.3 is equivalent to requiring that

$$p(\mathbf{z})-|\det M|\quad\in\quad <(1-z_1)^{\mu_1}(1-z_2)^{\mu_2}:\boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathbb{N}_0^2,\,|\boldsymbol{\mu}|\ge q+1>,$$

or, equivalently, that the symbol $p(\mathbf{z})$ of the subdivision scheme $S_{\mathbf{p}}$ with dilation M has the following decomposition

$$\begin{aligned}
&p(z_1,z_2)=|\det M|+\sum_{h=0}^{H}(1-z_1)^{\alpha_h}(1-z_2)^{\beta_h}c_h(z_1,z_2),\qquad (z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,\\
&\alpha_h,\beta_h\in\mathbb{N}_0,\qquad \alpha_h+\beta_h\ge q+1,\qquad h=0,\dots,H,
\end{aligned}\tag{4.24}$$

for suitable Laurent polynomials c_h, $h=0,\dots,H$ (we require in (4.24) that at least one pair $\alpha_h,\beta_h\in\mathbb{N}_0$ satisfies $\alpha_h+\beta_h=q+1$). Identity (4.24) is a natural generalization of the univariate identity (4.23).

Remark 4.1. We are interested in symmetric subdivision schemes due to the use of vertex centered discretization for our multigrid numerical examples in chapters 5 and 6.

4.2 SUBDIVISION BASED MULTIGRID

In this section, we analyze the connection between algebraic multigrid and subdivision. To do so, we assume that the dilation matrix $M\in\mathbb{Z}^{d\times d}$ is diagonal (4.10) and, for $j=0,\dots,\ell$, we define the coarser spaces $\mathbf{n}_j$ of V-cycle in (3.15) accordingly. Let $j\in\{1,\dots,\ell\}$ and $\mathbf{c}\in\ell_0(\mathbb{Z}^d)$ be a finite sequence with support contained in the d-dimensional "hypercube"

$$\underbrace{\left\{-\left\lfloor\frac{(n_j)_1-1}{2}\right\rfloor,\dots,\left\lfloor\frac{(n_j)_1}{2}\right\rfloor\right\}}_{(n_j)_1\text{ entries}}\times\cdots\times\underbrace{\left\{-\left\lfloor\frac{(n_j)_d-1}{2}\right\rfloor,\dots,\left\lfloor\frac{(n_j)_d}{2}\right\rfloor\right\}}_{(n_j)_d\text{ entries}}.$$

Define the vector $\overline{\mathbf{c}}\in\mathbb{R}^{N_j}$, $N_j=N(\mathbf{n}_j)=\prod_{i=1}^{d}(n_j)_i$, by

$$\overline{\mathbf{c}}=\left(\cdots\left(\left(\mathrm{c}(\alpha_1,\dots,\alpha_d)\right)_{\alpha_d=-\left\lfloor\frac{(n_j)_d-1}{2}\right\rfloor,\dots,\left\lfloor\frac{(n_j)_d}{2}\right\rfloor}\right)\cdots\right)_{\alpha_1=-\left\lfloor\frac{(n_j)_1-1}{2}\right\rfloor,\dots,\left\lfloor\frac{(n_j)_1}{2}\right\rfloor}.$$

The vector $\bar{\mathbf{c}}$ contains all the entries of the finite sequence $\mathbf{c}$. For instance, let $d = 2$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{n}_j = (2^3, 3^2) = (8,9)$. Let

$$\mathbf{c} = \{c(\alpha_1, \alpha_2) \in \mathbb{R} : \alpha_1 = -3,\dots,4,\ \alpha_2 = -4,\dots,4\} \in \ell_0(\mathbb{Z}^2), \quad \mathrm{supp}(\mathbf{c}) = \{-3,\dots,4\} \times \{-4,\dots,4\}.$$

Then, the vector $\bar{\mathbf{c}} \in \mathbb{R}^{72}$ is defined by

$$\bar{\mathbf{c}} = \begin{pmatrix} c(-3,-4) & \cdots & c(-3,4) & \cdots & \cdots & c(4,-4) & \cdots & c(4,4) \end{pmatrix}^T.$$

The action of a subdivision operator $\mathcal{S}_{\mathbf{p}}$ in (4.1) with dilation M and mask $\mathbf{p}$ on the sequence $\mathbf{c}$ is "equivalent" to the action of the grid transfer operator $P_{\mathbf{n}_j}$ in (3.16) on the vector $\bar{\mathbf{c}}$. The corresponding $P_{\mathbf{n}_j}$ in (3.16) is defined by the d-variate trigonometric polynomial p whose Fourier coefficients are the entries of the mask $\mathbf{p}$. Indeed, the action of both $\mathcal{S}_{\mathbf{p}}$ and $P_{\mathbf{n}_j}$ can be interpreted as upsampling with the factor M and $\mathbf{m}$, respectively, and convolution with the mask $\mathbf{p}$. The new refined sequence $\mathbf{e} = \mathcal{S}_{\mathbf{p}}\mathbf{c} \in \ell_0(\mathbb{Z}^d)$ is a finite sequence with support contained in the d-dimensional "hypercube"

$$\underbrace{\left\{-\left\lfloor \frac{(n_{j-1})_1 - 1}{2} \right\rfloor, \dots, \left\lfloor \frac{(n_{j-1})_1}{2} \right\rfloor\right\}}_{(n_{j-1})_1 \text{ entries}} \times \cdots \times \underbrace{\left\{-\left\lfloor \frac{(n_{j-1})_d - 1}{2} \right\rfloor, \dots, \left\lfloor \frac{(n_{j-1})_d}{2} \right\rfloor\right\}}_{(n_{j-1})_d \text{ entries}}.$$

Then, the vector $\bar{\mathbf{e}} \in \mathbb{R}^{N_{j-1}}$, $N_{j-1} = N(\mathbf{n}_{j-1}) = \prod_{i=1}^{d} (n_{j-1})_i$, which contains all the entries of $\mathbf{e}$, satisfies $\bar{\mathbf{e}} = P_{\mathbf{n}_j}\bar{\mathbf{c}}$. Thus, it is natural to ask if and how the reproduction and regularity properties of a stationary subdivision scheme, or equivalently the algebraic properties of the associated symbol, define a convergent two-grid method and V-cycle method. Under appropriate hypothesis, we answer these questions in subsections 4.2.1 and 4.2.2, respectively.

Here and in the following, we use the notation introduced so far for algebraic multigrid, see chapter 3, sections 3.4 and 3.5, and for stationary subdivision, see section 4.1, with diagonal dilation matrix $M \in \mathbb{Z}^{d\times d}$ (4.10).

4.2.1 Subdivision for algebraic two-grid method

In this section, we relate the algebraic properties of the trigonometric polynomial f defining the system matrix A_n, see chapter 3, section 3.4, with the algebraic properties of the subdivision symbol p in (4.6). Under the assumption that the trigonometric polynomial f has a zero at $\mathbf{x}_0 = \mathbf{0}$, Theorem 3.7 has an equivalent subdivision formulation, see Theorem 4.4. We, thus, focus on the case $\mathbf{x}_0 = \mathbf{0}$, since it is of practical interest, see e.g. chapter 5. To state Theorem 4.4, we use Laurent polynomial formalism and talk about the subdivision symbol p in (4.6).

Theorem 4.4. *Let f be a real d-variate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0, 2\pi)^d$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad |\boldsymbol{\mu}| \le q-1, \quad \textit{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}^d, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0.$$

Assume that the subdivision scheme $S_{\mathbf{p}}$ with dilation M and symbol p as in (4.6) *is convergent. If $S_{\mathbf{p}}$ generates polynomials up to degree $\lceil \frac{q}{2} \rceil - 1$, then the corresponding grid transfer operator $P_{\mathbf{n}}$ satisfies the approximation property* (3.5).

Proof. By Theorem 4.2 and due to convergence of $S_{\mathbf{p}}$, the symbol p satisfies

$$\begin{aligned} &(i) \quad D^{\boldsymbol{\mu}} p(\boldsymbol{\varepsilon}) = 0 \qquad |\boldsymbol{\mu}| \le \lceil \tfrac{q}{2} \rceil - 1, \qquad \forall\, \boldsymbol{\varepsilon} \in E_M \setminus \{\mathbf{1}\} = \left\{ \mathrm{e}^{-\mathrm{i}2\pi M^{-1}\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \Gamma \setminus \{\mathbf{0}\} \right\}, \\ &(ii) \quad p(\mathbf{1}) = |\mathrm{det} M| = \prod_{i=1}^{d} m_i. \end{aligned} \tag{4.25}$$

To prove the claim, we show that *(i)* and *(ii)* in (4.25) imply conditions *(i)* and *(ii)* of Theorem 3.7. Indeed, for

$$\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1}, \dots, \mathrm{e}^{-\mathrm{i}x_d} \right) \in (\mathbb{C} \setminus \{0\})^d, \qquad \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

the Laurent polynomial p is a 2π-periodic trigonometric polynomial. Thus, we write $p(\mathbf{y}) := p(\mathrm{e}^{-\mathrm{i}\mathbf{y}})$, $\mathbf{y} \in [0, 2\pi)^d$. From (4.11) and (3.19),

$$E_M \setminus \{\mathbf{1}\} = \left\{ \mathrm{e}^{-\mathrm{i}\mathbf{y}} : \mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{0}) \right\},$$
$$\Omega'_{\mathbf{m}}(\mathbf{0}) = \Omega_{\mathbf{m}}(\mathbf{0}) \setminus \{\mathbf{0}\} = \left\{ \mathbf{y} \in [0, 2\pi)^d \,:\, y_i = \frac{2\pi \alpha_i}{m_i},\ \alpha_i = 1, \dots, m_i - 1, i = 1, \dots, d \right\},$$

conditions *(i)* and *(ii)* in (4.25) become

$$\begin{aligned} &(i) \quad D^{\boldsymbol{\mu}} p(\mathbf{y}) = 0 \qquad |\boldsymbol{\mu}| \le \lceil \tfrac{q}{2} \rceil - 1, \qquad \forall\, \mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{0}), \\ &(ii) \quad p(\mathbf{0}) = |\mathrm{det} M|, \end{aligned}$$

which imply assumptions *(i)* and *(ii)* of Theorem 3.7. ■

4.2.2 Subdivision for algebraic V-cycle method

If $f_0(\mathbf{0}) = 0$ and $f_0(\mathbf{x}) > 0$, $\mathbf{x} \in (0, 2\pi)^d$, then [5, Lemma 3.2] guarantees that every f_j, $j = 1, \dots, \ell - 1$, vanishes only at $\mathbf{0}$ with the same order as the one of the zero of f_0. Thus, we use $p_j = p$, $j = 0, \dots, \ell - 1$. If p satisfies

$$\lim_{\mathbf{x} \to 0} \frac{|p(\mathbf{y})|}{f_0(\mathbf{x})} < +\infty \qquad \forall\, \mathbf{y} \in \Omega'_{\mathbf{m}}(\mathbf{0}),$$

then condition *(i)* of Theorem 3.8 is satisfied.

Recall, from (3.7), that one of the main ingredients in the definition of $CGC_{\mathbf{n}_j}$ are the grid transfer operators $P_{\mathbf{n}_j} = C_{\mathbf{n}_j}(p) K^T_{\mathbf{n}_j, \mathbf{m}}$. We view again p as the symbol of a convergent subdivision scheme $S_{\mathbf{p}}$. Our goal is to identify subdivision schemes $S_{\mathbf{p}}$ whose symbols p satisfy assumptions of Theorem 3.8 for $\mathbf{x}_0 = \mathbf{0}$.

Theorem 4.5. *Let f be a real d-variate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0, 2\pi)^d$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad |\boldsymbol{\mu}| \le q-1, \quad \text{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}^d, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0.$$

Assume that the subdivision scheme $S_{\mathbf{p}}$ with dilation M and symbol p as in (4.6) *is convergent. If*

(i) *$S_{\mathbf{p}}$ generates polynomials up to degree $q-1$,*

(ii) *the basic limit function ϕ of $S_{\mathbf{p}}$ is ℓ^∞-stable, i.e. there exist constants $0 < A \le B < \infty$ such that*

$$A\|\mathbf{c}\|_\infty \le \left\| \sum_{\alpha \in \mathbb{Z}^d} \mathrm{c}(\alpha)\phi(\cdot - \alpha) \right\|_{L^\infty(\mathbb{R}^d)} \le B\|\mathbf{c}\|_\infty, \qquad \forall\, \mathbf{c} \in \ell^\infty(\mathbb{Z}^d),$$

then the approximation property (3.10) *is satisfied.*

Proof. To prove the claim we show that condition *(i)* is equivalent to *(i)* of Theorem 3.8 and that property *(ii)* implies *(ii)* of Theorem 3.8. The equivalence of *(i)* follows by the same argument as in the proof of Theorem 4.4. Next we show that, if the basic limit function ϕ is ℓ^∞-stable, then condition *(ii)* of Theorem 3.8 is satisfied. Define the Fourier transform of a continuous, compactly supported function ϕ by

$$\hat{\phi}(\mathbf{x}) = \int_{\mathbb{R}^d} \phi(\mathbf{t}) \mathrm{e}^{-\mathrm{i}\mathbf{t}^T \cdot \mathbf{x}} \mathrm{d}t, \qquad \mathbf{t}^T \cdot \mathbf{x} = \sum_{i=1}^d t_i x_i, \qquad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Define also

$$\Pi_\phi(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} |\hat{\phi}(\mathbf{x} + 2\pi\boldsymbol{\alpha})|^2, \qquad \mathbf{x} \in \mathbb{R}^d.$$

Note that, due to the Poisson summation formula, we have

$$\Pi_\phi(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} \mathrm{d}(\boldsymbol{\alpha}) \mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}^T \cdot \mathbf{x}}, \quad \mathrm{d}(\boldsymbol{\alpha}) = \int_{\mathbb{R}^d} \phi(\mathbf{t})\phi(\mathbf{t} - \boldsymbol{\alpha}) \mathrm{d}t, \quad \boldsymbol{\alpha}^T \cdot \mathbf{x} = \sum_{i=1}^d \alpha_i x_i, \quad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

The compact support of ϕ implies that Π_ϕ is a trigonometric polynomial. Next, we take the Fourier transforms of both sides of the refinement equation (4.5) and obtain

$$\hat{\phi}(\mathbf{x}) = \frac{1}{|\det M|} p\left(\mathrm{e}^{-\mathrm{i}M^{-1}\mathbf{x}}\right) \hat{\phi}\left(M^{-1}\mathbf{x}\right), \qquad \mathbf{x} \in \mathbb{R}^d.$$

Then, by (4.12) and following the steps in [73], we write $\boldsymbol{\alpha} = \boldsymbol{\gamma} + M\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \Gamma$, $\boldsymbol{\beta} \in \mathbb{Z}^d$, and get

$$\begin{aligned}
\Pi_\phi(\mathbf{x}) &= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} |\hat{\phi}(\mathbf{x}+2\pi\boldsymbol{\alpha})|^2 \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \frac{1}{|\det M|^2} \left|p\left(\mathrm{e}^{-\mathrm{i}M^{-1}(\mathbf{x}+2\pi\boldsymbol{\alpha})}\right)\right|^2 \left|\hat{\phi}\left(M^{-1}(\mathbf{x}+2\pi\boldsymbol{\alpha})\right)\right|^2 \\
&= \sum_{\boldsymbol{\gamma}\in\Gamma} \frac{1}{|\det M|^2} \left|p\left(\mathrm{e}^{-\mathrm{i}M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})}\right)\right|^2 \sum_{\boldsymbol{\beta}\in\mathbb{Z}^d} \left|\hat{\phi}\left(M^{-1}(\mathbf{x}+2\pi(\boldsymbol{\gamma}+M\boldsymbol{\beta}))\right)\right|^2 \\
&= \sum_{\boldsymbol{\gamma}\in\Gamma} \frac{1}{|\det M|^2} \left|p\left(\mathrm{e}^{-\mathrm{i}M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})}\right)\right|^2 \sum_{\boldsymbol{\beta}\in\mathbb{Z}^d} \left|\hat{\phi}\left(M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})+2\pi\boldsymbol{\beta}\right)\right|^2 \\
&= \sum_{\boldsymbol{\gamma}\in\Gamma} \frac{1}{|\det M|^2} \left|p\left(\mathrm{e}^{-\mathrm{i}M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})}\right)\right|^2 \Pi_\phi\left(M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})\right), \qquad \mathbf{x}\in\mathbb{R}^d.
\end{aligned}$$

It was proved in [56] that a continuous, compactly supported function ϕ is ℓ^∞-stable if and only if

$$\sup_{\boldsymbol{\alpha}\in\mathbb{Z}^d} |\hat{\phi}(\mathbf{x}+2\pi\boldsymbol{\alpha})| > 0, \qquad \forall\, \mathbf{x}\in\mathbb{R}^d. \tag{4.26}$$

This is equivalent to $\Pi_\phi(\mathbf{x}) > 0$, $\forall\, \mathbf{x}\in\mathbb{R}^d$. Thus, we have

$$\sum_{\boldsymbol{\gamma}\in\Gamma} \left|p\left(\mathrm{e}^{-\mathrm{i}M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})}\right)\right|^2 > 0, \qquad \forall\, \mathbf{x}\in\mathbb{R}^d.$$

Since, for

$$\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1},\ldots,\mathrm{e}^{-\mathrm{i}x_d}\right) \in (\mathbb{C}\setminus\{0\})^d, \qquad \mathbf{x} = (x_1,\ldots,x_d)\in\mathbb{R}^d,$$

the Laurent polynomial p is a 2π-periodic trigonometric polynomial, we write

$$p(\mathbf{x}) := p(\mathrm{e}^{-\mathrm{i}\mathbf{x}}), \qquad \mathbf{x}\in[0,2\pi)^d.$$

Thus, the claim follows by the definition of the set of $\mathbf{m}$-corners $\Omega_{\mathbf{m}}$ (3.19),

$$\sum_{\boldsymbol{\gamma}\in\Gamma} \left|p\left(M^{-1}(\mathbf{x}+2\pi\boldsymbol{\gamma})\right)\right| = \sum_{\boldsymbol{\gamma}\in\Gamma} \left|p\left(M^{-1}\mathbf{x}+2\pi M^{-1}\boldsymbol{\gamma}\right)\right| = \sum_{\mathbf{y}\in\Omega_{\mathbf{m}}(\frac{\mathbf{x}}{\mathbf{m}})} |p(\mathbf{y})|^2 > 0, \qquad \forall\, \mathbf{x}\in\mathbb{R}^d.$$

Therefore, *(ii)* of Theorem 3.8 is also satisfied. ■

If ϕ is not given explicitly or *(ii)* of Theorem 4.5 is difficult to check, one can use an alternative criterion which guarantees the validity of condition *(ii)* of Theorem 3.8.

Proposition 4.6. *Let p be a d-variate trigonometric polynomial and $M\in\mathbb{Z}^{d\times d}$ in* (4.10)*. If*

$$\left|p\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right)\right| > 0, \qquad \forall\, \mathbf{x}\in M^{-1}[-\pi,\pi]^d,$$

then

$$\sum_{\boldsymbol{\gamma}\in\Gamma} \left|p\left(\mathrm{e}^{-\mathrm{i}(\mathbf{x}+2\pi M^{-1}\boldsymbol{\gamma})}\right)\right|^2 > 0, \qquad \forall\, \mathbf{x}\in[0,2\pi)^d. \tag{4.27}$$

Proof. To simplify the argument, we first rewrite (4.27) in an equivalent way. For $\mathbf{x} \in [0,2\pi)^d$, we have

$$\sum_{\boldsymbol{\gamma}\in\Gamma}\left|p\left(\mathrm{e}^{-\mathrm{i}(\mathbf{x}+2\pi M^{-1}\boldsymbol{\gamma})}\right)\right|^2 = \sum_{\gamma_1=0}^{m_1-1}\cdots\sum_{\gamma_d=0}^{m_d-1}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\left|\mathrm{p}(\boldsymbol{\alpha})\mathrm{e}^{-\mathrm{i}\alpha_1\left(x_1+\frac{2\pi\gamma_1}{m_1}\right)}\cdots\mathrm{e}^{-\mathrm{i}\alpha_d\left(x_d+\frac{2\pi\gamma_d}{m_d}\right)}\right|^2 > 0.$$

Using the substitution $\gamma_i' = \gamma_i + 1$, $i = 1,\ldots,d$, the latter inequality is equivalent to

$$\sum_{\gamma_1'=1}^{m_1}\cdots\sum_{\gamma_d'=1}^{m_d}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\left|\mathrm{p}(\boldsymbol{\alpha})\mathrm{e}^{-\mathrm{i}\alpha_1\left(x_1+\frac{2\pi\gamma_1'}{m_1}\right)}\cdots\mathrm{e}^{-\mathrm{i}\alpha_d\left(x_d+\frac{2\pi\gamma_d'}{m_d}\right)}\right|^2 > 0,$$
$$\forall\mathbf{x}\in\left[-\frac{2\pi}{m_1},\frac{(2m_1-2)\pi}{m_1}\right)\times\ldots\times\left[-\frac{2\pi}{m_d},\frac{(2m_d-2)\pi}{m_d}\right).$$

Straightforwardly, due to the 2π-periodicity of p, the previous inequality is equivalent to

$$\sum_{\gamma_1=1}^{m_1}\cdots\sum_{\gamma_d=1}^{m_d}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\left|\mathrm{p}(\boldsymbol{\alpha})\mathrm{e}^{-\mathrm{i}\alpha_1\left(x_1+\frac{2\pi\gamma_1}{m_1}\right)}\cdots\mathrm{e}^{-\mathrm{i}\alpha_d\left(x_d+\frac{2\pi\gamma_d}{m_d}\right)}\right|^2 > 0,$$
$$\forall\mathbf{x}\in\left[-\frac{\pi}{m_1},\frac{(2m_1-1)\pi}{m_1}\right)\times\ldots\times\left[-\frac{\pi}{m_d},\frac{(2m_d-1)\pi}{m_d}\right).$$

Let

$$\mathbf{x}\in\left[-\frac{\pi}{m_1},\frac{(2m_1-1)\pi}{m_1}\right)\times\ldots\times\left[-\frac{\pi}{m_d},\frac{(2m_d-1)\pi}{m_d}\right).$$

There exists $\mathbf{k} = (k_1,\ldots,k_d)\in\mathbb{Z}^d$, $k_i\in\{0,\ldots,m_i-1\}$, $i = 1,\ldots,d$, such that

$$\mathbf{x}\in\left[\frac{(2k_1-1)\pi}{m_1},\frac{(2k_1+1)\pi}{m_1}\right)\times\ldots\times\left[\frac{(2k_1-1)\pi}{m_1},\frac{(2k_1+1)\pi}{m_1}\right).$$

Define $\boldsymbol{\ell} = \mathbf{m}-\mathbf{k} = (m_1-k_1,\ldots,m_d-k_d)\in\mathbb{Z}^d$. Then $\ell_i\in\{1,\ldots,m_i\}$, $i = 1,\ldots,d$, and $x_i+\frac{2\pi\ell_i}{m_i}\in\left[-\frac{\pi}{m_i},\frac{\pi}{m_i}\right)$, $i = 1,\ldots,d$, due to

$$\frac{(2k_i-1)\pi}{m_i}\le x_i<\frac{(2k_i+1)\pi}{m_i}$$
$$\frac{(2m_i-1)\pi}{m_i}=\frac{(2k_i-1)\pi}{m_i}+\frac{2\pi\ell_i}{m_i}\le x_i+\frac{2\pi\ell_i}{m_i}<\frac{(2k_i+1)\pi}{m_i}+\frac{2\pi\ell_i}{m_i}=\frac{(2m_i+1)\pi}{m_i}$$
$$-\frac{\pi}{m_i}\le x_i+\frac{2\pi\ell_i}{m_i}<\frac{\pi}{m_i}\qquad(\mathrm{mod}\ 2\pi).$$

Thus, $\mathbf{x}+2\pi M^{-1}\boldsymbol{\ell}\in M^{-1}[-\pi,\pi]^d$. By hypothesis, we get

$$\left|p\left(\mathrm{e}^{-\mathrm{i}(\mathbf{x}+2\pi M^{-1}\boldsymbol{\ell})}\right)\right|^2 = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d}\left|\mathrm{p}(\boldsymbol{\alpha})\mathrm{e}^{-\mathrm{i}\alpha_1\left(x_1+\frac{2\pi\ell_1}{m_1}\right)}\cdots\mathrm{e}^{-\mathrm{i}\alpha_d\left(x_d+\frac{2\pi\ell_d}{m_d}\right)}\right|^2 > 0,$$

which yields the claim

$$\sum_{\gamma_1=1}^{m_1} \cdots \sum_{\gamma_d=1}^{m_d} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \left| \mathrm{p}(\boldsymbol{\alpha}) \mathrm{e}^{-\mathrm{i}\alpha_1\left(x_1+\frac{2\pi\gamma_1}{m_1}\right)} \cdots \mathrm{e}^{-\mathrm{i}\alpha_d\left(x_d+\frac{2\pi\gamma_d}{m_d}\right)} \right|^2 > 0, \qquad \forall \mathbf{x} \in [0,2\pi)^d.$$

∎

Hypothesis of Proposition 4.6 is a simplified version of the so-called *Cohen's condition*. This condition was first introduced by Cohen in [18] and then it was analyzed in depth regarding wavelets and orthonormality by Daubechies in [24].

Definition 4.4. Let $M \in \mathbb{Z}^{d\times d}$ be a dilation matrix. We say that a d-variate trigonometric polynomial p satisfies the Cohen's condition with respect to M if there exists a compact set $K \subset \mathbb{R}^d$ satisfying

(i) K contains a neighborhood of $\mathbf{0}$,

(ii) $\bigcup_{\boldsymbol{\alpha}\in\mathbb{Z}^d} (K + 2\pi\boldsymbol{\alpha}) = \mathbb{R}^d$,

(iii) $(K + 2\pi\boldsymbol{\alpha}) \cap K = \emptyset$ whenever $\boldsymbol{\alpha} \neq \mathbf{0}$,

and such that there exists $k_0 > 0$ for which

$$\left| p\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) \right| > 0, \qquad \forall \mathbf{x} \in \bigcup_{j=1}^{k_0} (M^T)^{-j} K.$$

Remark 4.2. If a compact set $K \subset \mathbb{R}^d$ satisfies conditions *(ii)* and *(iii)* in Definition 4.4, we say that K is congruent to $[-\pi,\pi]^d$ modulo 2π. In Proposition 4.6, we require that the trigonometric polynomial p satisfies Cohen's condition with the special choices $K = [-\pi,\pi]^d$, $k_0 = 1$. Indeed, the dilation M in (4.10) satisfies $M = M^T$.

Finally, using the result of Proposition 4.6, we get the following Theorem .

Theorem 4.7. *Let f be a real d-variate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0,2\pi)^d$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad |\boldsymbol{\mu}| \le q-1, \quad \text{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}^d, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0.$$

If the symbol p in (4.6) *satisfies*

(i) zero conditions of order q,

(ii) $\left| p\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) \right| > 0, \quad \forall \mathbf{x} \in M^{-1}[-\pi,\pi]^d,$

then the approximation property (3.10) *is satisfied.*

Proof. We have already shown in the proof of Theorem 4.5 that assumption *(i)* is equivalent to condition *(i)* of Theorem 3.8. By Proposition 4.6, Cohen's condition implies (4.27). Note that (4.27) is equivalent to *(ii)* in Theorem 3.8. ∎

CHAPTER 5

Grid transfer operators from stationary subdivision schemes

This chapter is dedicated to the construction of univariate and bivariate grid transfer operators from stationary subdivision symbols.

In section 5.1, we define two classes of univariate grid transfer operators from the well-known symbols of symmetric binary ($m = 2$) and ternary ($m = 3$) pseudo-splines, see subsections 5.1.1 and 5.1.2 respectively. We test the efficiency of our univariate pseudo-splines grid transfer operators in section 5.2.

In section 5.3, we define bivariate grid transfer operators from the symbols of approximating and interpolatory subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. The numerical examples in section 5.4 test the validity of our bivariate subdivision based grid transfer operators. Especially, in subsection 5.4.3, we highlight a critical drawback of our bivariate subdivision based grid transfer operators that we will overcome in chapter 6.

5.1 UNIVARIATE GRID TRANSFER OPERATORS FROM PRIMAL PSEUDO-SPLINES

In this section, we define grid transfer operators from well-known subdivision symbols of pseudo-splines introduced in [25]. Recall that we only consider symmetric symbols, i.e. we restrict our attention to primal pseudo-splines. This is due to the use of vertex centered discretization in section 5.2.

5.1.1 BINARY PRIMAL PSEUDO-SPLINES

We start our discussion by introducing the family of binary primal pseudo-spline schemes.

Definition 5.1 ([25]). Let $J \in \mathbb{N}$ and $L \in \{0,\dots,J-1\}$. The *binary primal pseudo-spline scheme* $S_{\mathbf{p}_{J,L}}$ of order (J,L) is defined by its symbol

$$p_{J,L}(z) = 2\sigma^J(z)\, q_{J,L}(z), \qquad q_{J,L}(z) = \sum_{k=0}^{L} \binom{J-1+k}{k} \delta^k(z), \qquad z \in \mathbb{C}\setminus\{0\},$$

where

$$\sigma(z) = \frac{(1+z)^2}{4z} \qquad \text{and} \qquad \delta(z) = -\frac{(1-z)^2}{4z}.$$

These pseudo-spline schemes range from B-splines to Dubuc-Deslauries schemes. When $L=0$, the symbol in Definition 5.1 is the symbol of the B-spline subdivision scheme of order $2J-1$ and, when $L=J-1$, one gets the symbol of the $(2J)$-point Dubuc-Deslauries interpolatory subdivision scheme. For more details on binary pseudo-splines see [25, 35, 36, 60].

Next, we give several examples of grid transfer operators derived from symbols of binary primal pseudo-splines of order $(J,0)$, namely binary B-splines of order $2J-1$. The symbols $p_{1,0}$ and $p_{2,0}$ have already been used in multigrid literature [31, 69] as well as the classical cubic interpolation $p_{2,1}$ (see [71]).

***Example* 5.1.** Let $J \in \mathbb{N}$ and $L=0$. Then, we have $q_{J,0}(z) \equiv 1$. Thus, from Definition 5.1, we get

$$p_{J,0}(z) = 2\left(\frac{(1+z)^2}{4z}\right)^J, \qquad z \in \mathbb{C}\setminus\{0\}.$$

Set $z = \mathrm{e}^{-\mathrm{i}x}$, $x \in \mathbb{R}$. Then, the symbols $p_{J,0}$ become trigonometric polynomials

$$p_{J,0}(x) = 2\left(\frac{1+\cos x}{2}\right)^J, \qquad x \in [0,2\pi),$$

that are used to define the grid transfer operators in (3.11). Notice that the trigonometric polynomial $p_{J,0}$ coincides with the trigonometric polynomial $p^{(J)}$ defined in (3.33). For readers convenience, we also present the corresponding masks. For $J=1,2,3$, they are given by

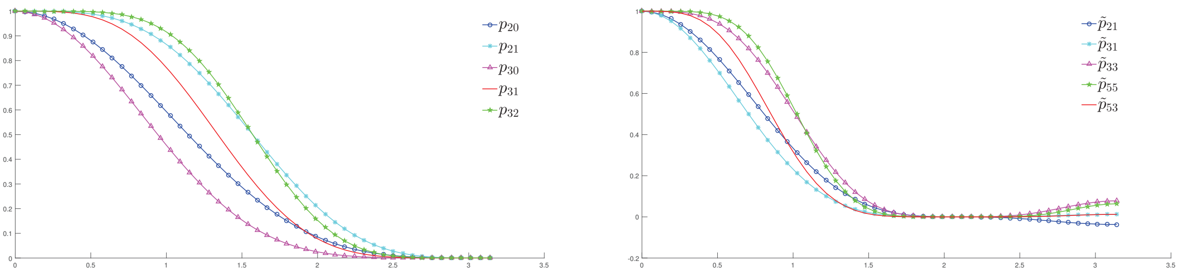
$$\mathbf{p}_{1,0} = \frac{1}{2}\{1 \quad 2 \quad 1\}, \quad \mathbf{p}_{2,0} = \frac{1}{8}\{1 \quad 4 \quad 6 \quad 4 \quad 1\}, \quad \mathbf{p}_{3,0} = \frac{1}{32}\{1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1\}.$$

Note that we use the corresponding grid transfer operators for our numerical examples in Tables 5.1 and 5.3. ◀

Less known are grid transfer operators which we derive from symbols in Definition 5.1 for $L \neq 0$.

***Example* 5.2.** Let $J=2$, $L=1$, and $J=3$, $L \in \{1,2\}$. Then, from Definition 5.1, using standard trigonometric identities, we get

$$\begin{aligned}
p_{2,1}(x) &= \frac{1}{16}\big(16 + 18\cos x - 2\cos(3x)\big),\\
p_{3,1}(x) &= \frac{1}{128}\big(110 + 144\cos x + 24\cos(2x) - 16\cos(3x) - 6\cos(4x)\big),\\
p_{3,2}(x) &= \frac{1}{256}\big(256 + 300\cos x - 50\cos(3x) + 6\cos(5x)\big), \qquad x \in [0,2\pi).
\end{aligned}$$



(a) *Binary pseudo-splines* (b) *Ternary pseudo-splines*

Figure 5.1: *Symbols of the grid transfer operators defined in* (a) *by primal binary pseudo-splines and in* (b) *by primal ternary pseudo-splines in the reference interval* $[0,\pi]$.

The corresponding masks are

$$\begin{aligned}
\mathbf{p}_{2,1} &= \frac{1}{16}\{-1 \quad 0 \quad 9 \quad 16 \quad 9 \quad 0 \quad -1\},\\
\mathbf{p}_{3,1} &= \frac{1}{128}\{-3 \quad -8 \quad 12 \quad 72 \quad 110 \quad 72 \quad 12 \quad -8 \quad -3\},\\
\mathbf{p}_{3,2} &= \frac{1}{256}\{3 \quad 0 \quad -25 \quad 0 \quad 150 \quad 256 \quad 150 \quad 0 \quad -25 \quad 0 \quad 3\}.
\end{aligned}$$

Note that the corresponding grid transfer operators also appear in Tables 5.1 and 5.3. ◀

The symbols of the binary grid transfer operators proposed in Examples 5.1 and 5.2 are plotted in Figure 5.1 (a) for the reference interval $[0,\pi]$.

The justification that primal pseudo-spline symbols define good grid transfer operators is given by Theorem 4.5. The convergence of the corresponding subdivision schemes has been proved by Dong and Shen in [36]. The special structure of the symbols in Definition 5.1, i.e. the presence of the factor $(1+z)^{2J}$, implies that the corresponding schemes of order (J,L) generate polynomials up to degree $2J-1$ for every $J\in\mathbb{N}$, $L=0,\dots,J-1$, see Theorem 4.2. Thus, *(i)* of Theorem 4.5 is satisfied. Therefore, it is left to show that the corresponding basic limit functions are ℓ^∞-stable. In [35], the authors addressed this issue. We present an alternative proof of ℓ^∞-stability of primal pseudo splines for completeness. To do that, we first recall that in [60], the author showed the following.

Lemma 5.1. *Let $S_{\mathbf{p}}$ be a convergent binary subdivision scheme with associated symbol*

$$p(z) = 2\left(\frac{1+z}{2}\right)^r z^{-\lfloor r/2\rfloor} q(z), \qquad r\ge 1, \qquad z\in\mathbb{C}\setminus\{0\}.$$

If $q(\mathrm{e}^{-\mathrm{i}x})>0$ for all $x\in\mathbb{R}$, then the basic limit function ϕ of $S_{\mathbf{p}}$ is ℓ^∞-stable.

The result of Lemma 5.1 is used in the proof of Proposition 5.2.

Proposition 5.2. *Let $J\in\mathbb{N}$ and $L\in\{0,\dots,J-1\}$. The basic limit function ϕ of $S_{\mathbf{p}_{J,L}}$ is ℓ^∞-stable.*

Proof. By Definition 5.1, the symbol $p_{J,L}$ of the primal pseudo-spline scheme $S_{\mathbf{p}_{J,L}}$ of order (J,L) is of the form required by Lemma 5.1, with

$$q(z) := q_{J,L}(z) = \sum_{k=0}^{L} \binom{J-1+k}{k} \left(-\frac{(1-z)^2}{4z}\right)^k, \qquad z \in \mathbb{C} \setminus \{0\},$$

i.e.

$$q(\mathrm{e}^{-\mathrm{i}x}) = \sum_{k=0}^{L} \binom{J-1+k}{k} \sin^{2k}\left(\frac{x}{2}\right) = 1 + \sum_{k=1}^{L} \binom{J-1+k}{k} \sin^{2k}\left(\frac{x}{2}\right) > 0, \qquad \forall\, x \in \mathbb{R}.$$

■

Thus, *(ii)* of Theorem 4.5 is also satisfied and it implies the following result.

Proposition 5.3. *Let f be a real trigonometric polynomial such that $f(x) > 0$, $x \in (0, 2\pi)$, and*

$$D^{\mu} f(0) = 0, \qquad \mu = 0, \dots, q-1, \qquad D^{q} f(0) \neq 0, \qquad q \in \mathbb{N}.$$

The grid transfer operator derived from the symbol $p_{\lceil q/2 \rceil, L}$, $L \in \{0, \dots, \lceil q/2 \rceil - 1\}$, satisfies the approximation property (3.10).

5.1.2 Ternary primal pseudo-splines

In section 5.2, in the case of PDE discretizations via isogeometric approach with high order B-splines, we show that the grid transfer operators derived from the binary primal pseudo-spline schemes lead to computationally expensive multigrid methods. On the contrary, if we use the ternary primal pseudo-spline schemes, the number of multigrid iterations decreases drastically.

The recursive definition of ternary pseudo-splines was introduced in [20]. Their explicit form can be found in [60].

Definition 5.2. Let $J \in \mathbb{N}$ and $L \in \mathbb{N}$, $L = 2L' + 1$, $1 \le L \le J$. The *ternary primal pseudo-spline scheme* $S_{\tilde{\mathbf{p}}_{J,L}}$ of order (J,L) is defined by its symbol

$$\tilde{p}_{J,L}(z) = 3\,\tilde{\sigma}^{J+1}(z)\,\tilde{q}_{J,L}(z), \qquad \tilde{q}_{J,L}(z) = \sum_{k=0}^{L'} \binom{J+k}{k} \tilde{\delta}^{k}(z), \qquad z \in \mathbb{C} \setminus \{0\},$$

where

$$\tilde{\sigma}(z) = \frac{1+z+z^2}{3z} \qquad \text{and} \qquad \tilde{\delta}(z) = -\frac{(1-z)^2}{3z}.$$

Similarly to the binary case, when $L = 1$, the Laurent polynomial $\tilde{p}_{J,1}$ in Definition 5.2 is the symbol of the ternary B-spline subdivision scheme of order J and, when $L = J$, J odd, one gets the symbol of the ternary $(J+1)$-point Dubuc-Deslauries interpolatory subdivision scheme.

Next, we show how to derive grid transfer operators from symbols of some ternary primal pseudo-spline schemes.

***Example* 5.3.** Let $J \in \mathbb{N}$ and $L = 1$. Then, we have $\tilde{q}_{J,1}(z) \equiv 1$. Thus, from Definition 5.2, we obtain the following symbols of primal pseudo-splines of order $(J, 1)$, i.e symbols of the ternary B-splines of order J,

$$\tilde{p}_{J,1}(z) = 3\left(\frac{1+z+z^2}{3z}\right)^{J+1}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Set $z = e^{-ix}$, $x \in \mathbb{R}$. Using simple trigonometric identities, we get the trigonometric polynomials

$$\tilde{p}_{J,1}(x) = 3\left(\frac{1+2\cos x}{3}\right)^{J+1}, \qquad x \in [0, 2\pi).$$

The corresponding masks for linear ($J = 1$), quadratic ($J = 2$) and cubic ($J = 3$) ternary B-splines are

$$\tilde{\mathbf{p}}_{1,1} = \frac{1}{3}\{1 \quad 2 \quad 3 \quad 2 \quad 1\}, \qquad \tilde{\mathbf{p}}_{2,1} = \frac{1}{9}\{1 \quad 3 \quad 6 \quad 7 \quad 6 \quad 3 \quad 1\},$$
$$\tilde{\mathbf{p}}_{3,1} = \frac{1}{27}\{1 \quad 4 \quad 10 \quad 16 \quad 19 \quad 16 \quad 10 \quad 4 \quad 1\}.$$

Note that we use the corresponding grid transfer operators to obtain results in Tables 5.2 and 5.4. ◀

Further examples are obtained for $J \in \mathbb{N}$, J odd, and $L = J$, in the following example. These correspond to the ternary $(J + 1)$-point Dubuc-Deslauries interpolatory subdivision schemes.

***Example* 5.4.** Let $J = 3, 5$ and $L = J$. From Definition 5.2, we derive the trigonometric polynomials

$$\tilde{p}_{3,3}(x) = \frac{1}{81}\big(81 + 120\cos x + 60\cos(2x) - 10\cos(4x) - 8\cos(5x)\big),$$
$$\tilde{p}_{5,5}(x) = \frac{1}{729}\big(729 + 1120\cos x + 560\cos(2x) - 140\cos(4x) - 112\cos(5x) + 16\cos(7x) + 14\cos(8x)\big),$$
$x \in [0, 2\pi)$.

The corresponding masks are

$$\tilde{\mathbf{p}}_{3,3} = \frac{1}{81}\{-4 \quad -5 \quad 0 \quad 30 \quad 60 \quad 81 \quad 60 \quad 30 \quad 0 \quad -5 \quad -4\},$$
$$\tilde{\mathbf{p}}_{5,5} = \frac{1}{729}\{7 \quad 8 \quad 0 \quad -56 \quad -70 \quad 0 \quad 280 \quad 560 \quad 729 \quad 560 \quad 280 \quad 0 \quad -70 \quad -56 \quad 0 \quad 8 \quad 7\}.$$

Note that the corresponding grid transfer operators are also used in Tables 5.2 and 5.4. ◀

The symbols of the ternary grid transfer operators proposed in Examples 5.3 and 5.4 are plotted in Figure 5.1 (b) for the reference interval $[0, \pi]$. We use Theorem 4.7 to show that ternary pseudo-splines lead to appropriate grid transfer operators. Note that we could also use Theorem 4.7 in subsection 5.1.1. To check the assumptions of Theorem 4.7, we need the following auxiliary lemma.

Lemma 5.4. *Let $J \in \mathbb{N}$ and $L \in \mathbb{N}$, $L = 2L' + 1$, $1 \le L \le J$. The symbol $\tilde{p}_{J,L}$ of the ternary primal pseudo spline scheme of order (J, L) in Definition 5.2 satisfies*

$$\left|\tilde{p}_{J,L}\left(\mathrm{e}^{-\mathrm{i}x}\right)\right| > 0, \qquad \forall\, x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]. \tag{5.1}$$

Proof. Consider

$$\tilde{p}_{J,L}\left(\mathrm{e}^{-\mathrm{i}x}\right) = 3\,\tilde{\sigma}^{J+1}\left(\mathrm{e}^{-\mathrm{i}x}\right)\tilde{q}_{J,L}\left(\mathrm{e}^{-\mathrm{i}x}\right) = 3\,\tilde{B}_{J+1}(x)\,\tilde{Q}(x), \qquad x \in [0, 2\pi],$$

where

$$\tilde{B}_{J+1}(x) := \tilde{\sigma}^{J+1}\left(\mathrm{e}^{-\mathrm{i}x}\right) = \left(\frac{1 + \mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{-2\mathrm{i}x}}{3\mathrm{e}^{-\mathrm{i}x}}\right)^{J+1}$$

and

$$\tilde{Q}(x) := \tilde{q}_{J,L}\left(\mathrm{e}^{-\mathrm{i}x}\right) = \sum_{k=0}^{L'} \binom{J+k}{k} \left(\frac{4}{3}\sin^2\left(\frac{x}{2}\right)\right)^k.$$

Note that $\tilde{B}_{J+1}(x)$ vanishes only at $-\frac{2\pi}{3}$ and $\frac{2\pi}{3}$. Thus, to check condition (5.1), it suffices to show that $\left|\tilde{Q}(x)\right| > 0$ for all $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$. The latter holds due to

$$\tilde{Q}(x) = \sum_{k=0}^{L'} \binom{J+k}{k} \left(\frac{4}{3}\sin^2\left(\frac{x}{2}\right)\right)^k = 1 + \sum_{k=1}^{L'} \binom{J+k}{k} \left(\frac{4}{3}\right)^k \sin^{2k}\left(\frac{x}{2}\right) > 0, \quad \forall x \in \mathbb{R}.$$

■

By Definition 5.2, the presence of the factor $(1 + z + z^2)^{J+1}$ in the symbols $\tilde{p}_{J,L}$ shows that the ternary pseudo spline-schemes of order (J, L) generate polynomials up to degree J, for every $J \in \mathbb{N}$ and $L \in \mathbb{N}$, $L = 2L' + 1$, $1 \le L \le J$, see Theorem 4.2. This, together with $\tilde{p}_{J,L}(1) = 3$ by definition, implies that $\tilde{p}_{J,L}$ satisfies the zero conditions of order $J + 1$. Thus, Theorem 4.7 implies the following proposition.

Proposition 5.5. *Let f be a real trigonometric polynomial such that $f(x) > 0$, $x \in (0, 2\pi)$, and*

$$D^\mu f(0) = 0, \qquad \mu = 0, \dots, q-1, \qquad D^q f(0) \ne 0, \qquad q \in \mathbb{N}.$$

The grid transfer operator derived from the symbol $\tilde{p}_{q-1,L}$, $L = 2L' + 1 \in \{1, \dots, q-1\}$, satisfies the approximation property (3.10).

5.2 Univariate numerical examples

In this section, in subsections 5.2.1 and 5.2.2, we illustrate the univariate theoretical results of Propositions 5.3 and 5.5 with numerical examples of the algebraic Galerkin approach applied to certain Toeplitz matrices. On the other hand, in practical applications, the variable coefficient case is often of interest and the algebraic Galerkin approach could be computationally expensive. Indeed, using the algebraic Galerkin approach, the bandwidth of the coarser matrices A_{n_j} is approximately the double of the bandwidth of the grid transfer operator (see [2, Proposition

2]). Thus, it can be large for high order grid transfer operators. Therefore, in subsection 5.2.3, we present also an example of the geometric approach for PDEs with variable coefficients.

We consider sequence of starting linear systems $A_n\mathbf{x} = \mathbf{b}_n$ for different n. In subsection 5.2.1, we use a finite difference discretization of the biharmonic elliptic PDE problem. In subsection 5.2.2, we consider the isogeometric approach for the Laplacian problem. In both cases, we have PDEs with constant coefficients with homogeneous Dirichlet boundary conditions, hence the system matrices A_n are positive definite with a Toeplitz structure. According to [3, 34], we have to change the definition of the grid transfer operators in (3.11). Namely, let $n = m^k - 1$, $k \in \mathbb{N}$. For $\ell \in \mathbb{N}$, $1 \le \ell \le k-1$, define

$$n_j = m^{k-j} - 1, \qquad j = 0, \ldots, \ell. \tag{5.2}$$

and

$$P_{n_j}(p) = T_{n_j}(p)\overline{Z}_{n_j,m}^T \in \mathbb{C}^{n_j \times n_{j+1}}, \quad j = 0, \ldots, \ell-1, \tag{5.3}$$

where $T_{n_j}(p) \in \mathbb{C}^{n_j \times n_j}$ is the Toeplitz matrix of order n_j generated by the trigonometric polynomial p (see chapter 2, subsection 2.3.2) and $\overline{Z}_{n_j,m} \in \mathbb{C}^{n_{j+1} \times n_j}$ is the downsampling matrix with the factor m defined by

$$\overline{Z}_{n_j,m} = \begin{pmatrix} 0_{m-1} & 1 & 0_{m-1} & & & & & \\ & & & 1 & 0_{m-1} & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & 0_{m-1} \end{pmatrix}, \qquad 0_{m-1} = (0, \ldots, 0) \in \mathbb{N}_0^{1 \times m-1}. \tag{5.4}$$

In the binary case, we solve the coarse grid system exactly when the dimension of the coarse grid is $n_\ell = 2^2 - 1 = 3$ and, in the ternary case, when $n_\ell = 3^2 - 1 = 8$.

In subsection 5.2.3, we consider linear systems $A_n\mathbf{x} = \mathbf{b}_n$ derived via finite difference discretization from Laplacian with non-constant coefficients with homogeneous Dirichlet boundary conditions. In this case, the corresponding system matrices A_n are tridiagonal positive definite, but we lose the Toeplitz structure. Nevertheless, for $\ell \in \mathbb{N}$, $1 \le \ell \le k-1$, $n = m^k - 1$, the properties of A_n allow us to define the dimension n_j and the grid transfer operator P_{n_j} at level j as in (5.2) and (5.3), respectivelly. To solve the linear systems $A_n\mathbf{x} = \mathbf{b}_n$, we use geometric multigrid method, which we briefly describe in subsection 5.2.3. We solve the coarse grid system exactly when the dimension of the coarse grid is $n_\ell = 2^3 - 1 = 7$ in the binary case, and $n_\ell = 3^2 - 1 = 8$ in the ternary case.

In all examples, we use as pre- and post-smoother one step of Gauss-Seidel method. The zero vector is used as the initial guess and the stopping criterion is $\|\mathbf{r}_s\|_2 / \|\mathbf{r}_0\|_2 < 10^{-7}$, where $\mathbf{r}_s$ is the residual vector after s iterations and 10^{-7} is the given tolerance.

5.2.1 Finite difference approximation for the biharmonic operator

The first example we present arises from the discretization of the biharmonic elliptic PDE problem with homogeneous Dirichlet boundary conditions, namely

$$\begin{cases} \dfrac{d^4}{dx^4}\psi(x) = g(x), & x \in (0,1), \\ \psi(0) = \psi(1) = 0. \end{cases}$$

Subdivision scheme	$n=2^{10}-1$ iter	conv. rate	$n=2^{11}-1$ iter	conv. rate	$n=2^{12}-1$ iter	conv. rate	generation degree
$p_{1,0}$ (Linear Bspline)	617	0.9742	744	0.9785	801	0.9800	1
$p_{2,0}$ (Cubic Bspline)	40	0.6647	43	0.6846	45	0.6979	3
$p_{2,1}$ (Interp. 4 point)	19	0.4275	23	0.4937	26	0.5351	3
$p_{3,0}$ (Quintic Bspline)	30	0.5784	35	0.6285	41	0.6741	5
$p_{3,1}$	19	0.4258	22	0.4748	24	0.5063	5
$p_{3,2}$ (Interp. 6 point)	13	0.2798	13	0.2879	14	0.3080	5

Table 5.1: *Binary subdivision schemes for biharmonic problem*

Subdivision scheme	$n=3^{6}-1$ iter	conv. rate	$n=3^{7}-1$ iter	conv. rate	$n=3^{8}-1$ iter	conv. rate	generation degree
$\tilde{p}_{1,1}$ (Linear Bspline)	462	0.9656	864	0.9815	1057	0.9841	1
$\tilde{p}_{2,1}$ (Quadratic Bspline)	72	0.7990	63	0.7742	50	0.7217	2
$\tilde{p}_{3,1}$ (Cubic Bspline)	67	0.7858	80	0.8167	87	0.8308	3
$\tilde{p}_{3,3}$ (Interp. 4-point)	46	0.7017	47	0.7090	53	0.7368	3
$\tilde{p}_{5,3}$	30	0.5824	31	0.5878	30	0.5814	5
$\tilde{p}_{5,5}$ (Interp. 6-point)	39	0.6594	39	0.6604	40	0.6644	5

Table 5.2: *Ternary subdivision schemes for biharmonic problem*

For the discretization, we use finite differences of order 4. It leads to the linear systems $A_n\mathbf{x} = \mathbf{b}_n$, where $A_n = T_n(f)$ is the Toeplitz matrix of order n generated by the trigonometric polynomial

$$f(x) = (2-2\cos x)^2, \qquad x \in [0,2\pi).$$

Note that f has a 4-fold zero at $x = 0$. Thus, by Propositions 5.3 and 5.5 with $q = 4$, the binary pseudo-spline symbols from Example 5.1 (with $J \geq 2$), the ternary pseudo-spline symbols from Example 5.3 (with $J = 3$) and all symbols from Examples 5.2 and 5.4 can be used to define the corresponding grid transfer operators. To define $\mathbf{b}_n$, we choose $\mathbf{x} = (\mathrm{x}(1),\ldots,\mathrm{x}(n)) \in \mathbb{C}^n$, $\mathrm{x}(\alpha) = \alpha/n$, $\alpha = 1,\ldots,n$ and set $\mathbf{b}_n := A_n\mathbf{x} \in \mathbb{C}^n$.

Tables 5.1 and 5.2 show how the number of iterations and convergence rates for the V-cycle change with increasing dimension n.

Tables 5.1 and 5.2 illustrate the importance of the polynomial generation property (zero conditions) that, by Theorems 4.5 and 4.7, ensures the correct choice of the grid transfer operator. The subdivision schemes with the symbols $p_{1,0}$, $\tilde{p}_{1,1}$ generate polynomials of degree 1. The lack of the appropriate degree of polynomial generation leads to dramatic increase of the number of iterations. For ternary schemes, $\tilde{p}_{2,1}$ generates polynomials of degree at most 2 and so it does not satisfy the assumptions of Theorem 4.7 for $q = 4$. Nevertheless, such conditions are only sufficient and they could be further relaxed (see e.g. [69]). Moreover, the quadratic B-splines are very effective as grid transfer operators for ternary methods as shown also in the next example.

We observe that the number of iterations necessary for convergence of the V-cycle is larger in the ternary case (see Table 5.2) than in the binary case (see Table 5.1). This happens, since, at each Coarse Grid Correction step, we downsample the data with the factor m and the larger is m the more information we lose. Thus, the number of iterations required for convergence is larger for $m=3$. Nevertheless, in our tests, the CPU time is comparable in both cases, since the length of the V-cycle iteration is shorter in the ternary case ($m=3$).

5.2.2 Isogeometric analysis for the Poisson operator

In the second example, we consider the Laplacian problem with homogeneous Dirichlet boundary conditions, namely

$$\begin{cases} -\dfrac{d^2}{dx^2}\psi(x) = g(x), & x \in (0,1), \\ \psi(0) = \psi(1) = 0. \end{cases}$$

We consider the isogeometric approach with collocation by splines for the discretization of the above problem, see [32]. We fix the integers $\nu, \mu > 0$ and define the spline space

$$\mathcal{W} = \left\{ s \in C^{\mu-1}([0,1]) : s\big|_{\left[\frac{\gamma}{\nu},\frac{\gamma+1}{\nu}\right)} \in \Pi_\mu,\, \gamma = 0,\dots,\nu-1,\, s(0) = s(1) = 0 \right\}$$

the finite dimensional approximation space of dimension $n = \dim \mathcal{W} = \nu+\mu-2$. As a basis for $\mathcal{W}$, one chooses the B-splines $B_\gamma^{[\mu]} : [0,1] \to \mathbb{R}$, $\gamma = 2,\dots,\nu+\mu-1$ of degree μ as explained in [7]. These are defined over the uniform knot sequence of length $\nu+2\mu+1$

$$t_1 = \cdots = t_{\mu+1} = 0 < t_{\mu+2} < \cdots < t_{\mu+\nu} < 1 = t_{\nu+\mu+1} = \cdots = t_{\nu+2\mu+1},$$

where

$$t_{\mu+\gamma+1} = \frac{\gamma}{\nu}, \qquad \gamma = 1,\dots,\nu-1,$$

and the extreme knots have multiplicity $\mu+1$. We recall that the B-splines $B_\gamma^{[\mu]} : [0,1] \to \mathbb{R}$ are defined recursively by

$$B_\gamma^{[0]}(x) = \begin{cases} 1 & x \in [t_\gamma, t_{\gamma+1}), \\ 0 & \text{otherwise}, \end{cases} \qquad \gamma = 1,\dots,\nu+2\mu,$$

and

$$B_\gamma^{[m]}(x) = \frac{x - t_\gamma}{t_{\gamma+m} - t_\gamma} B_\gamma^{[m-1]}(x) + \frac{t_{\gamma+m+1} - x}{t_{\gamma+m+1} - t_{\gamma+1}} B_{\gamma+1}^{[m-1]}(x),$$
$$\gamma = 1,\dots,(\nu+\mu)+\mu-m, \qquad m = 1,\dots,\mu,$$

where we set the fractions with zero denominators to be equal to zero. Next, one defines the set of collocation points, the so-called Greville abscissae,

$$\tau_\gamma = \frac{t_{\gamma+1} + t_{\gamma+2} + \cdots + t_{\gamma+\mu}}{\mu}, \qquad \gamma = 2,\dots,\nu+\mu-1.$$

Subdivision scheme	iter	$\mu=3$ conv. rate	iter	$\mu=10$ conv. rate	iter	$\mu=16$ conv. rate	generation degree
$p_{1,0}$ (Linear Bspline)	8	0.1111	16	0.3360	126	0.8798	1
$p_{2,0}$ (Cubic Bspline)	8	0.1111	13	0.2757	126	0.8799	3
$p_{2,1}$ (Interp. 4 point)	8	0.1111	13	0.2758	126	0.8799	3
$p_{3,0}$ (Quintic Bspline)	8	0.1111	13	0.2758	126	0.8798	5
$p_{3,1}$	8	0.1111	13	0.2759	126	0.8798	5
$p_{3,2}$ (Interp. 6 point)	8	0.1111	13	0.2759	126	0.8798	5

Table 5.3: *Binary subdivision schemes for isogeometric Laplacian problem*

Subdivision scheme	iter	$\mu=3$ conv. rate	iter	$\mu=10$ conv. rate	iter	$\mu=16$ conv. rate	generation degree
$\tilde{p}_{1,1}$ (Linear Bspline)	31	0.5910	25	0.5247	48	0.7078	1
$\tilde{p}_{2,1}$ (Quadratic Bspline)	30	0.5847	19	0.4271	49	0.7124	2
$\tilde{p}_{3,1}$ (Cubic Bspline)	29	0.5739	16	0.3617	49	0.7120	3
$\tilde{p}_{3,3}$ (Interp. 4-point)	30	0.5853	17	0.3731	49	0.7118	3
$\tilde{p}_{5,3}$	28	0.643	16	0.358	49	0.7137	5
$\tilde{p}_{5,5}$ (Interp. 6-point)	30	0.5831	16	0.3523	49	0.7120	5

Table 5.4: *Ternary subdivision schemes for isogeometric Laplacian problem*

This choice is crucial for the stability of the discrete problem, see [4] for more details. The solution $\psi_{\mathcal{W}} \in \mathcal{W}$ of the interpolation problem

$$-\psi''_{\mathcal{W}}(\tau_\gamma) = g(\tau_\gamma), \quad \gamma = 2, \dots, \nu+\mu-1,$$

written in the Bspline basis of $\mathcal{W}$ leads to

$$A_n = \left(-\left(B^{[\mu]}_{\beta+1} \right)'' (\tau_{\alpha+1}) \right)_{\alpha,\beta=1,\dots,n} \in \mathbb{C}^{n\times n}.$$

For $\mu \geq 2$, it is possible to split the above matrix into $A_n = T_n(f^{[\mu]}) + R_n^{[\mu]}$, where $T_n(f^{[\mu]})$ is a Toeplitz matrix with symbol

$$f^{[\mu]}(x) = (2-2\cos x)h^{[\mu]}(x), \quad h^{[\mu]}(x) = \sum_{\alpha\in\mathbb{Z}} \left(\frac{2\sin(x/2)+\alpha\pi}{x+2\alpha\pi} \right)^{\mu-1}, \quad x \in [0, 2\pi), \tag{5.5}$$

and $R_n^{[\mu]}$ is a low rank correction term, see [33]. The symbols for the grid transfer operators are chosen as in Example 5.2.1. To define $\mathbf{b}_n$, we choose the exact solution

$$\mathbf{x} = (\mathrm{x}(1), \dots, \mathrm{x}(n))^T \in \mathbb{C}^n, \quad \mathrm{x}(\alpha) = \sin\left(5\frac{\pi(\alpha-1)}{n-1}\right) + \sin\left(n\frac{\pi(\alpha-1)}{n-1}\right), \quad \alpha = 1, \dots, n,$$

and set $\mathbf{b}_n := A_n\mathbf{x} \in \mathbb{C}^n$.

Tables 5.3 and 5.4 show how the number of iterations and convergence rates for the V-cycle change with increasing μ and fixed n. The starting dimension of the linear systems are $n = 2^9 - 1$

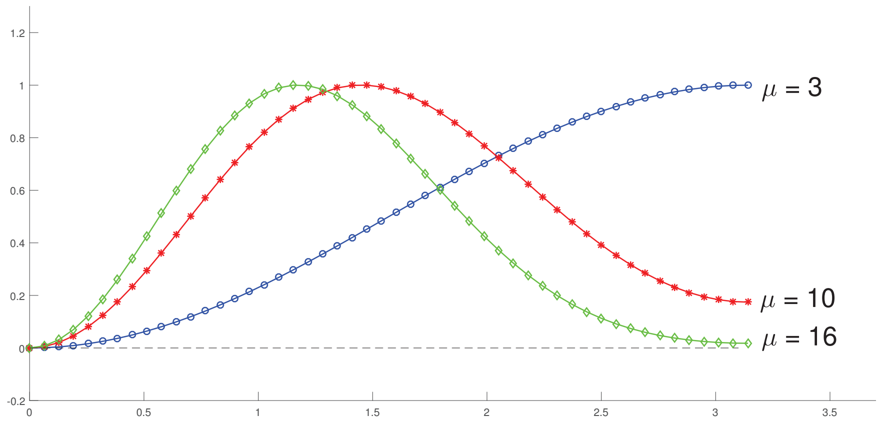
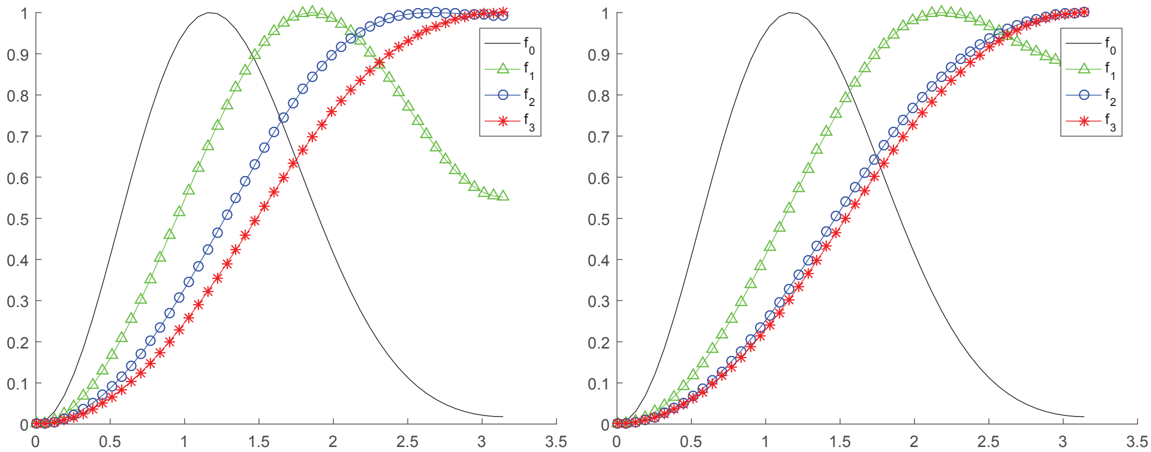


Figure 5.2: *Symbols* $f^{[\mu]}/\|f^{[\mu]}\|_\infty$ *for* $\mu \in \{3, 10, 16\}$ *in the reference interval* $[0, \pi]$.

and $n = 3^6 - 1$ in the binary and the ternary cases, respectively. For small μ, the results in Tables 5.3 and 5.4 mimic the ones from Example 5.2.1. Note that, in this case, even the grid transfer operators defined from the subdivision symbols $p_{1,0}$, $\tilde{p}_{1,1}$ and $\tilde{p}_{2,1}$ behave well, as the order of $f^{[\mu]}$ at zero is $m = 2$ in this case. Thus, Propositions 5.3 and 5.5 are also applicable for these symbols. However, when μ increases, the results in the binary and ternary cases differ. This is the case, since the symbol $f^{[\mu]}$ in (5.5) has a numerical zero at π whose order increases when μ increases, see Figure 5.2. In fact, by [33], $h^{[\mu]}(\pi)$ in (5.5) converges to 0 exponentially when μ goes to infinity. The symbols $p_{J,L}$ also vanish at π for $J \geq 1$ and $L \in \{0, \ldots, J-1\}$, which is the source of further ill-conditioning. Note that the ternary symbols $\tilde{p}_{J,L}$ do not vanish at π and, hence, lead to more stable methods for increasing μ. Figure 5.3 illustrates the symbols $f_j^{[16]}$, $j = 0, \ldots, 3$, defined by (3.14) of the Toeplitz matrices $A_{n_j} = T_{n_j}(f_j^{[16]})$ using the trigonometric polynomial p associated to the 4-point interpolatory subdivision scheme $p_{2,1}$ in the binary case $(m = 2)$ (a) and to the 4-point interpolatory subdivision scheme $\tilde{p}_{3,3}$ in the ternary case $(m = 3)$ (b). We observe that the coarse symbols $f_j^{[16]}$, $j = 1, \ldots, 3$, do not vanish at π. The grid transfer operator defined from the ternary subdivision symbol $\tilde{p}_{3,3}$ is more powerful than the grid transfer operator defined from the binary subdivision symbol $p_{2,1}$. Indeed, for the ternary subdivision symbol $\tilde{p}_{3,3}$, at level $j = 1$, the value $f_1^{[16]}(\pi)$ is close to the maximum of the coarse symbol $f_1^{[16]}(x)$, $x \in [0, \pi]$, and, at level $j = 2$, it holds $f_2^{[16]}(\pi) = \max_{x \in [0,\pi]} f_2^{[16]}(x)$. For the binary subdivision symbol $p_{2,1}$, at levels $j = 1, 2$, it holds $f_j^{[16]}(\pi) < \max_{x \in [0,\pi]} f_j^{[16]}(x)$. On the contrary, for small μ, the ternary symbols are not at all a good choice for the definition of a grid transfer operator (compare Tables 3 and 4). Indeed, the grid transfer operators associated to binary subdivision schemes reduce the ill-conditioning in the high frequencies of the error (where the smoother is ineffective) already at level $j = 1$.



(a) *Binary case.* (b) *Ternary case.*

Figure 5.3: *Symbols* $f_j^{[16]}$, $j = 0, \ldots, 3$ *defined by* (3.14) *using the binary 4-point scheme* $p_{2,1}$ (a) *and the ternary 4-point scheme* $\tilde{p}_{3,3}$ (b) *in the reference interval* $[0, \pi]$.

5.2.3 Finite difference approximation of the non-constant coefficients Poisson operator

In this example, we consider the Laplacian with non-constant coefficients with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{d}{dx}\psi(x)\right) = g(x), & x \in (0,1), \\ \psi(0) = \psi(1) = 0. \end{cases} \tag{5.6}$$

The function $a(x)$, $x \in [0,1]$, is strictly positive and its behavior influences the conditioning of the problem. The oscillatory behavior of $a(x)$, $x \in [0,1]$, affects the convergence rate of geometric multigrid. We illustrate this phenomenon with multiple examples.

In order to solve the linear system $A_n \mathbf{x} = \mathbf{b}_n$ derived from the discretization of problem (5.6), we use the geometric multigrid method. Let $n = m^k - 1$, $m \in \mathbb{N}$, $m \geq 2$, fix $\ell \in \mathbb{N}$, $1 \leq \ell \leq k-1$ and define $n_j \in \mathbb{N}$, $j = 0, \ldots, \ell$, as in (5.2). At the j-th recursion level of the V-cycle, we compute the matrix A_{n_j} by discretizing problem (5.6) using finite differences of order 2 on a uniform grid of $[0,1]$ of size $h_j = 1/(n_j + 1)$. For the fixed prolongation operator P_{n_j} in (5.3), the restriction operator is defined by $\frac{1}{m} P_{n_j}^*$ in order to preserve the right scaling. For more details, we refer to [71].

To define $\mathbf{b}_n$, we choose the exact solution

$$\mathbf{x} = (\mathrm{x}(1), \ldots, \mathrm{x}(n))^T \in \mathbb{C}^n, \quad \mathrm{x}(\alpha) = \sin\left(2\frac{\pi(\alpha-1)}{n-1}\right) + \sin\left(13\frac{\pi(\alpha-1)}{n-1}\right), \quad \alpha = 1, \ldots, n,$$

and set $\mathbf{b}_n := A_n \mathbf{x} \in \mathbb{C}^n$.

Subdivision scheme	$a(x) = x^2 + \varepsilon$ iter	conv. rate	$a(x) = e^x$ iter	conv. rate	$a(x) = 1 + \sin(5\pi x) \cdot \sin(7\pi x) + \varepsilon$ iter	conv. rate
$p_{1,0}$ (Linear Bspline)	9	0.1484	9	0.1533	44	0.6919
$p_{2,0}$ (Cubic Bspline)	10	0.1731	10	0.1753	52	0.7331
$p_{2,1}$ (Interp. 4 point)	8	0.1264	8	0.1239	40	0.6656
$p_{3,0}$ (Quintic Bspline)	12	0.2473	12	0.2545	60	0.7643
$p_{3,1}$	9	0.1454	9	0.1431	41	0.6719
$p_{3,2}$ (Interp. 6 point)	8	0.1295	8	0.1273	40	0.6647

Table 5.5: *Binary subdivision schemes for Laplacian with non-constant coefficients ($\varepsilon = 10^{-5}$)*

Subdivision scheme	$a(x) = x^2 + \varepsilon$ iter	conv. rate	$a(x) = e^x$ iter	conv. rate	$a(x) = 1 + \sin(5\pi x) \cdot \sin(7\pi x) + \varepsilon$ iter	conv. rate
$\tilde{p}_{1,1}$ (Linear Bspline)	11	0.2254	11	0.2279	19	0.4196
$\tilde{p}_{2,1}$ (Quadratic Bspline)	12	0.2491	12	0.2476	26	0.5307
$\tilde{p}_{3,1}$ (Cubic Bspline)	14	0.3007	14	0.2993	34	0.6197
$\tilde{p}_{3,3}$ (Interp. 4-point)	10	0.1991	10	0.1995	13	0.2705
$\tilde{p}_{5,3}$	12	0.2453	12	0.2427	18	0.3965
$\tilde{p}_{5,5}$ (Interp. 6-point)	11	0.2107	11	0.2104	14	0.3053

Table 5.6: *Ternary subdivision schemes for Laplacian with non-constant coefficients ($\varepsilon = 10^{-5}$)*

Tables 5.5 and 5.6 show how the number of iterations and convergence rates for the V-cycle change for different $a(x)$, $x \in [0,1]$. The starting dimension is fixed, $n = 2^{11} - 1$ and $n = 3^7 - 1$ for binary and ternary pseudo splines, respectively. Parameter $\varepsilon \in (0,1)$ guarantees the strict positivity of function $a(x)$, $x \in [0,1]$. In the numerical examples, $\varepsilon = 10^{-5}$.

The results of Tables 5.5 and 5.6 also illustrate the impact of the function $a(x)$, $x \in [0,1]$. When $a(x)$ is strictly positive and non-oscillatory, such as $a(x) = x^2 + \varepsilon$ or $a(x) = e^x$, our numerical results are equivalent to the ones for $a(x) \equiv 1$. On the other hand, the case of strictly positive and oscillatory $a(x)$, such as $a(x) = 1 + \sin(5\pi x) \cdot \sin(7\pi x) + \varepsilon$, can be better handled by ternary primal pseudo-splines. Especially, ternary interpolatory pseudo-splines define the grid transfer operators P_{n_j} as in (5.3), which lead to the best results. The most competitive grid transfer operators P_{n_j} are the ones defined from the ternary pseudo-spline $\tilde{p}_{3,3}$, which is the ternary interpolatory 4-point subdivision scheme. These grid transfer operators not only improve the convergence rate. They are also extremely competitive from the computational point of view. In fact, the small number of non-zero coefficients in their stencil decreases the computational cost of restriction and prolongation.

5.3 Bivariate grid transfer operators from symmetric subdivision schemes

In this section, we define bivariate grid transfer operators from symbols of bivariate symmetric subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (binary) and $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (ternary). We distinguish between approximating and interpolatory subdivision schemes.

5.3.1 Symmetric binary 2-directional box splines

The box splines were introduced in [28], while their several basic properties were given in [27]. Since then, many interesting and important results on box splines have been found. See [17] for detailed summary and other references. We focus on the bivariate 2-directional symmetric box splines. Indeed, in [3, 31], the authors proposed a family of grid transfer operators defined from symmetric trigonometric polynomials satisfying properties *(i)* and *(ii)* of Theorem 3.8 for standard binary cutting $\mathbf{m} = (2,2)$. These grid transfer operators are equal to the ones defined from the symbols of the bivariate 2-directional symmetric box splines. Thus, this subsection highlights that our subdivision based multigrid analysis is consistent with the algebraic multigrid analysis based on matrix algebra symbols and it shows that different approaches lead to the definition of the same grid transfer operators.

Definition 5.3. Let $J \in \mathbb{N}$. The *2-directional symmetric box spline* subdivision scheme $S_{\mathbf{P}_J}$ of order J with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is defined by its symbol

$$P_J(\mathbf{z}) = 4\left(\frac{(1+z_1)^2(1+z_2)^2}{16 z_1 z_2}\right)^J, \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2.$$

Let $J \in \mathbb{N}$. The symbol P_J in Definition 5.3 is defined as the tensor product of the symbol $p_{J,0}$ of the univariate binary B-spline subdivision scheme of order $2J-1$ in Definition 5.1 with itself. We set

$$\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1}, \mathrm{e}^{-\mathrm{i}x_2}\right), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Then, the symbols P_J become trigonometric polynomials

$$P_J(\mathbf{x}) = 4\left(\frac{(1+\cos x_1)(1+\cos x_2)}{4}\right)^J, \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2,$$

that are used to define the grid transfer operators in (3.16). Notice that, for $J \in \mathbb{N}$, the trigonometric polynomial P_J coincides with the trigonometric polynomial $p^{(J,J)}$ defined in (3.34). For readers convenience, we also present the corresponding masks. For $J = 1, 2$, they are given by

$$\mathbf{P}_1 = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

$$\mathbf{P}_2 = \frac{1}{64}\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

We use Theorem 4.7 to show that 2-directional symmetric box splines lead to appropriate grid transfer operators. Let $J \in \mathbb{N}$. The 2-directional symmetric box spline subdivision scheme $S_{\mathbf{P}_J}$ in Definition 5.3 generates polynomials up to degree $2J-1$, see Theorem 4.2, and its symbol satisfies by definition $P(\mathbf{1}) = 4$. Thus, $S_{\mathbf{P}_J}$ satisfies the zero conditions of order $2J$ and condition *(i)* of Theorem 4.7 is satisfied. Moreover,

$$P_j\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) = 4\left(\frac{(1+\cos x_1)(1+\cos x_2)}{4}\right)^J \geq \left(\frac{1}{4}\right)^{J-1} > 0, \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2.$$

Thus, also condition *(ii)* of Theorem 4.7 is satisfied. We proved the following proposition.

Proposition 5.6. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0, 2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \leq q-1, \quad \textit{and} \quad \exists\, \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad q \in \mathbb{N}.$$

The grid transfer operator derived from the symbol P_J, $J \geq \lceil q/2 \rceil$, satisfies the approximation property (3.10).

5.3.2 Bivariate binary interpolatory subdivision schemes

In this subsection, we define bivariate binary interpolatory grid transfer operators. We report a couple of well-known bivariate interpolatory subdivision schemes and define a new bivariate interpolatory subdivision scheme. The grid transfer operators defined from the symbols of the interpolatory subdivision schemes presented in this subsection are of practical interest for our numerical examples in section 5.4.

The first interpolatory subdivision scheme we present is *Kobbelt*'s subdivision scheme [58], which is a tensor product scheme based on the univariate binary 4-point interpolatory subdivision scheme with the symbol $p_{2,1}$ in Example 5.2. The symbol of Kobbelt's scheme is

$$\mathcal{K}(\mathbf{z}) = \frac{4}{z_1^3 z_2^3} \cdot \left(\frac{(1+z_1)(1+z_2)}{4}\right)^4 \frac{(1-4z_1+z_1^2)(1-4z_2+z_2^2)}{4}, \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2, \tag{5.7}$$

and the corresponding mask is

$$\mathcal{K} = \frac{1}{256} \begin{pmatrix} 1 & 0 & -9 & -16 & -9 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & 81 & 144 & 81 & 0 & -9 \\ -16 & 0 & 144 & 256 & 144 & 0 & -16 \\ -9 & 0 & 81 & 144 & 81 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -9 & -16 & -9 & 0 & 1 \end{pmatrix}. \tag{5.8}$$

The second interpolatory subdivision scheme we present is the *Butterfly* subdivision scheme [40]. The Butterfly subdivision scheme is one of the first interpolatory subdivision schemes defined for surfaces and it is the generalization of the univariate 4-point interpolatory subdivision scheme with the symbol $p_{2,1}$ in Example 5.2 to a bivariate 3-directional mesh. The symbol of the butterfly scheme is

$$\mathcal{B}(\mathbf{z}) = \frac{4}{z_1^3 z_2^3} \cdot \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right) \cdot b(\mathbf{z}), \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2, \tag{5.9}$$

where

$$\begin{aligned} b(\mathbf{z}) = \frac{1}{16}\big(& (-z_2^4 z_1^5) + (-z_2^5 + 2z_2^4 + z_2^3 - z_2^2) z_1^4 + (z_2^4 + 4z_2^3 + 4z_2^2 - z_2) z_1^3 + \\ & (-z_2^4 + 4z_2^3 + 4z_2^2 + z_2) z_1^2 + (-z_2^3 + z_2^2 + 2z_2 - 1) z_1 - z_2\big), \qquad b(1,1) = 1, \end{aligned}$$

and the corresponding mask is

$$\mathcal{B} = \frac{1}{16} \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}. \tag{5.10}$$

Finally, we construct a new bivariate interpolatory 3-directional subdivision scheme $S_{\mathcal{P}}$. This scheme has not yet appeared in either subdivision or multigrid literature. The symbol of our new subdivision scheme $S_{\mathcal{P}}$ is

$$\mathcal{P}(\mathbf{z}) = \frac{4}{z_1^3 z_2^3} \cdot \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1 z_2}{2}\right) \cdot b(\mathbf{z}), \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2, \tag{5.11}$$

where

$$\begin{aligned} b(\mathbf{z}) = \frac{1}{1592}\big(& (-184z_2^4 + 184z_2^3 - 199z_2^2) z_1^4 + \\ & (184z_2^4 + 15z_2^2 + 199z_2) z_1^3 + \\ & (-199z_2^4 + 15z_2^3 + 1562z_2^2 + 15z_2 - 199) z_1^2 + \\ & (199z_2^3 + 15z_2^2 + 184) z_1 + \\ & (-199z_2^2 + 184z_2 - 184)\big), \qquad b(1,1) = 1, \end{aligned}$$

and the corresponding mask is

$$\mathcal{P} = \frac{1}{3184} \begin{pmatrix} -184 & 0 & -15 & -199 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -15 & 0 & 1776 & 1791 & 30 & 0 & 0 \\ -199 & 0 & 1791 & 3184 & 1791 & 0 & -199 \\ 0 & 0 & 30 & 1791 & 1776 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -199 & -15 & 0 & -184 \end{pmatrix}. \tag{5.12}$$

Straightforward computation shows that the subdivision scheme associated with $\mathcal{P}$ is convergent and generates polynomials up to degree 3.

We use Theorem 4.5 to show that Kobbelt's subdivision scheme $S_{\mathcal{K}}$, Butterfly subdivision scheme $S_{\mathcal{B}}$ and our 3-directional subdivision scheme $S_{\mathcal{P}}$ lead to appropriate grid transfer operators. Indeed, $S_{\mathcal{K}}, S_{\mathcal{B}}, S_{\mathcal{P}}$ subdivision schemes generate polynomials up to degree 3. Moreover, they are interpolatory subdivision schemes and, thus, by [12, Proposition 1.3], their basic limit functions are ℓ^∞-stable. We proved the following proposition.

Proposition 5.7. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0,2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \le q-1, \quad \text{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad 0 \le q \le 4.$$

The grid transfer operators derived from the symbols $\mathcal{K}$ in (5.7)*, $\mathcal{B}$ in* (5.9) *and $\mathcal{P}$ in* (5.11) *satisfy the approximation property* (3.10).

5.3.3 Symmetric ternary 2-directional box splines

In this subsection, we present the symmetric ternary 2-directional box splines.

Definition 5.4. Let $J \in \mathbb{N}$. The ternary *2-directional symmetric box spline* subdivision scheme $S_{\tilde{\mathbf{P}}_J}$ of order J with dilation $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is defined by its symbol

$$\tilde{P}_J(\mathbf{z}) = 9\left(\frac{(1+z_1+z_1^2)^2(1+z_2+z_2^2)^2}{81 z_1^2 z_2^2}\right)^J, \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C}\setminus\{0\})^2.$$

Let $J \in \mathbb{N}$. Similarly to the binary case, the symbol $\tilde{P}_J$ in Definition 5.4 is defined as the tensor product of the symbol $\tilde{p}_{2J-1,1}$ of the univariate ternary B-spline subdivision scheme of order $2J-1$ in Definition 5.2 with itself. We set

$$\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1}, \mathrm{e}^{-\mathrm{i}x_2}\right), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Then, the symbols $\tilde{P}_J$ become trigonometric polynomials

$$\tilde{P}_J(\mathbf{x}) = 9\left(\frac{(1+2\cos x_1)(1+2\cos x_2)}{9}\right)^{2J}, \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2,$$

that are used to define the grid transfer operators in (3.16). For readers convenience, we also present the corresponding masks. For $J = 1, 2$, they are given by

$$\tilde{\mathbf{P}}_1 = \frac{1}{9}\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 4 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{pmatrix},$$

$$\tilde{\mathbf{P}}_2 = \frac{1}{729}\begin{pmatrix} 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\ 4 & 16 & 40 & 64 & 76 & 64 & 40 & 16 & 4 \\ 10 & 40 & 100 & 160 & 190 & 160 & 100 & 40 & 10 \\ 16 & 64 & 160 & 256 & 304 & 256 & 160 & 64 & 16 \\ 19 & 76 & 190 & 304 & 361 & 304 & 190 & 76 & 19 \\ 16 & 64 & 160 & 256 & 304 & 256 & 160 & 64 & 16 \\ 10 & 40 & 100 & 160 & 190 & 160 & 100 & 40 & 10 \\ 4 & 16 & 40 & 64 & 76 & 64 & 40 & 16 & 4 \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \end{pmatrix}.$$

We use Theorem 4.7 to show that ternary 2-directional symmetric box splines lead to appropriate grid transfer operators. Let $J \in \mathbb{N}$. The ternary 2-directional symmetric box spline subdivision scheme $S_{\tilde{\mathbf{P}}_J}$ in Definition 5.4 generates polynomials up to degree $2J-1$, see Theorem 4.2, and its symbol satisfies by definition $\tilde{P}(\mathbf{1}) = 9$. Thus, $S_{\mathbf{B}_J}$ satisfies the zero conditions of order $2J$ and condition *(i)* of Theorem 4.7 is satisfied. Moreover,

$$\tilde{P}_j\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) = 9\left(\frac{(1+2\cos x_1)(1+2\cos x_2)}{9}\right)^{2J} \geq 9\left(\frac{4}{9}\right)^{2J} > 0, \qquad \forall \mathbf{x} = (x_1, x_2) \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]^2.$$

Thus, also condition *(ii)* of Theorem 4.7 is satisfied. We proved the following proposition.

Proposition 5.8. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0, 2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \leq q-1, \quad \text{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad q \in \mathbb{N}.$$

The grid transfer operator derived from the symbol $\tilde{P}_J$, $J \geq \lceil q/2 \rceil$, satisfies the approximation property (3.10).

5.3.4 Bivariate ternary interpolatory subdivision schemes

In this subsection, we define bivariate ternary interpolatory grid transfer operators. We present several ternary interpolatory subdivision schemes from subdivision literature which reproduce polynomials up to degree 3. The grid transfer operators defined from their symbols are of practical interest for our numerical examples in section 5.4.

The first ternary interpolatory subdivision scheme we present is the ternary Kobbelt's subdivision scheme $S_{\mathcal{K}_3}$. The subscript 3 refers to the dilation $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and distinguish $S_{\mathcal{K}_3}$ from the "standard" Kobbelt's subdivision scheme $S_{\mathcal{K}}$ with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ defined by its subdivision symbol in (5.7). The ternary Kobbelt's subdivision scheme $S_{\mathcal{K}_3}$ is a tensor product scheme based on the univariate ternary 4-point interpolatory subdivision scheme with the symbol $\tilde{p}_{3,3}$ in Example 5.4. The symbol of $S_{\mathcal{K}_3}$ is

$$\mathcal{K}_3(\mathbf{z}) = \frac{9}{z_1^5 z_2^5} \cdot \left(\frac{(1+z_1+z_1^2)(1+z_2+z_2^2)}{9}\right)^4 \frac{(4-11z_1+4z_1^2)(4-11z_2+4z_2^2)}{9}, \quad \mathbf{z} = (z_1, z_2) \in (\mathbb{C}\setminus\{0\})^2, \tag{5.13}$$

and the corresponding mask is

$$\mathcal{K}_3 = \frac{1}{6561}\begin{pmatrix} 16 & 20 & 0 & -120 & -240 & -324 & -240 & -120 & 0 & 20 & 16 \\ 20 & 25 & 0 & -150 & -300 & -405 & -300 & -150 & 0 & 25 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & -150 & 0 & 900 & 1800 & 2430 & 1800 & 900 & 0 & -150 & -120 \\ -240 & -300 & 0 & 1800 & 3600 & 4860 & 3600 & 1800 & 0 & -300 & -240 \\ -324 & -405 & 0 & 2430 & 4860 & 6561 & 4860 & 2430 & 0 & -405 & -324 \\ -240 & -300 & 0 & 1800 & 3600 & 4860 & 3600 & 1800 & 0 & -300 & -240 \\ -120 & -150 & 0 & 900 & 1800 & 2430 & 1800 & 900 & 0 & -150 & -120 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 25 & 0 & -150 & -300 & -405 & -300 & -150 & 0 & 25 & 20 \\ 16 & 20 & 0 & -120 & -240 & -324 & -240 & -120 & 0 & 20 & 16 \end{pmatrix}. \tag{5.14}$$

The second $S_{\mathcal{H}_3}$ and the third $S_{\mathcal{H}_4}$ ternary interpolatory subdivision schemes we present are taken from [49]. The subscript $j = 3,4$ in $S_{\mathcal{H}_j}$ refers to the topology of the bivariate mesh underlying the subdivision scheme. More precisely, the interpolatory subdivision scheme $S_{\mathcal{H}_j}$ is defined over a bivariate j-directional mesh, $j = 3,4$. The symbols of $S_{\mathcal{H}_3}$ and $S_{\mathcal{H}_4}$ are

$$\mathcal{H}_j(\mathbf{z}) = \frac{9}{z_1^5 z_2^5} \cdot \frac{(1+z_1+z_1^2)(1+z_2+z_2^2)}{9} \cdot b_j(\mathbf{z}), \quad j = 3,4, \quad \mathbf{z} = (z_1,z_2) \in (\mathbb{C}\setminus\{0\})^2, \tag{5.15}$$

where

$$\begin{aligned} b_3(\mathbf{z}) = \frac{1}{81}\big(&(-2z_2^7 - 2z_2^6)z_1^8 + (-2z_2^8 + 3z_2^7 - z_2^6 + z_2^5 - 2z_2^4)z_1^7 + \\ &(-2z_2^8 - z_2^7 + 7z_2^6 + 3z_2^5 + 2z_2^4 - 4z_2^3)z_1^6 + (z_2^7 + 3z_2^6 + 10z_2^5 + 9z_2^4 + 7z_2^3 - 4z_2^2)z_1^5 + \\ &(-2z_2^7 + 2z_2^6 + 9z_2^5 + 11z_2^4 + 9z_2^3 + 2z_2^2 - 2z_2)z_1^4 + \\ &(-4z_2^6 + 7z_2^5 + 9z_2^4 + 10z_2^3 + 3z_2^2 + z_2)z_1^3 + (-4z_2^5 + 2z_2^4 + 3z_2^3 + 7z_2^2 - z_2 - 2)z_1^2 + \\ &(-2z_2^4 + z_2^3 - z_2^2 + 3z_2 - 2)z_1 + (-2z_2^2 - 2z_2)\big), \qquad b(1,1) = 1, \end{aligned}$$

and

$$\begin{aligned} b_4(\mathbf{z}) = \frac{1}{81}\big(&(-z_2^5 - 2z_2^4 - z_2^3)z_1^8 + (-z_2^5 + z_2^4 - z_2^3)z_1^7 + \\ &(2z_2^5 + z_2^4 + 2z_2^3)z_1^6 + (-z_2^8 - z_2^7 + 2z_2^6 + 7z_2^5 + 12z_2^4 + 7z_2^3 + 2z_2^2 - z_2 - 1)z_1^5 + \\ &(-2z_2^8 + z_2^7 + z_2^6 + 12z_2^5 + 5z_2^4 + 12z_2^3 + z_2^2 + z_2 - 2)z_1^4 + \\ &(-z_2^8 - z_2^7 + 2z_2^6 + 7z_2^5 + 12z_2^4 + 7z_2^3 + 2z_2^2 - z_2 - 1)z_1^3 + (2z_2^5 + z_2^4 + 2z_2^3)z_1^2 + \\ &(-z_2^5 + z_2^4 - z_2^3)z_1 + (-z_2^5 - 2z_2^4 - z_2^3)\big), \qquad b(1,1) = 1. \end{aligned}$$

The corresponding masks are

$$\mathcal{H}_3 = \frac{1}{81}\begin{pmatrix} 0 & -2 & -4 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & -4 & -1 & -4 & -1 & -2 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & 8 & 8 & 0 & -4 & -4 & 0 & 0 & 0 \\ -4 & -1 & 8 & 26 & 32 & 26 & 8 & -1 & -4 & 0 & 0 \\ -2 & -4 & 8 & 32 & 56 & 56 & 32 & 8 & -4 & -2 & 0 \\ 0 & -1 & 0 & 26 & 56 & 81 & 56 & 26 & 0 & -1 & 0 \\ 0 & -2 & -4 & 8 & 32 & 56 & 56 & 32 & 8 & -4 & -2 \\ 0 & 0 & -4 & -1 & 8 & 26 & 32 & 26 & 8 & -1 & -4 \\ 0 & 0 & 0 & -4 & -4 & 0 & 8 & 8 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & -2 & -1 & -4 & -1 & -4 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -4 & -4 & -2 & 0 \end{pmatrix}, \tag{5.16}$$

and

$$\mathcal{H}_4 = \frac{1}{81}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -4 & -3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -3 & -5 & -3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 9 & 24 & 30 & 24 & 9 & 0 & -2 & -1 \\ -3 & -3 & 0 & 24 & 42 & 60 & 42 & 24 & 0 & -3 & -3 \\ -4 & -5 & 0 & 30 & 60 & 81 & 60 & 30 & 0 & -5 & -4 \\ -3 & -3 & 0 & 24 & 42 & 60 & 42 & 24 & 0 & -3 & -3 \\ -1 & -2 & 0 & 9 & 24 & 30 & 24 & 9 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -3 & -5 & -3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -3 & -4 & -3 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.17}$$

We use Theorem 4.5 to show that the subdivision symbols of $S_{\mathcal{K}_3}$, $S_{\mathcal{H}_3}$ and $S_{\mathcal{H}_4}$ lead to appropriate grid transfer operators. Indeed, $S_{\mathcal{K}_3}$, $S_{\mathcal{H}_3}$ and $S_{\mathcal{H}_4}$ subdivision schemes generate polynomials up to degree 3. Moreover, they are interpolatory subdivision schemes and, thus, by [12, Proposition 1.3], their basic limit functions are ℓ^∞-stable. We proved the following proposition.

Proposition 5.9. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0,2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \le q-1, \quad \textit{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad 0 \le q \le 4.$$

The grid transfer operators derived from the symbols $\mathcal{K}_3$ in (5.13) *and $\mathcal{H}_j$, $j = 3,4$, in* (5.15) *satisfy the approximation property* (3.10).

5.4 Bivariate numerical examples

In this section, we illustrate the bivariate theoretical results of Propositions 5.6, 5.7, 5.8 and 5.9 with several numerical examples.

In subsection 5.4.1, we use a finite difference discretization of the biharmonic elliptic PDE problem with homogeneous Dirichlet boundary conditions. The system matrix $A_{\mathbf{n}}$ is positive definite with a bi-level Toeplitz structure. According to [3, 5], similarly to section 5.2, we have to change the definition of the grid transfer operators in (3.16). Let $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$ and $\ell \in \mathbb{N}$ such that $\ell \le \min\{k_1, k_2\} - 1$. Define

$$\mathbf{n}_j = \left(m_1^{k_1-j} - 1, m_2^{k_2-j} - 1\right), \qquad N_j = N(\mathbf{n}_j) = (n_j)_1 (n_j)_2, \qquad j = 0, \dots, \ell, \tag{5.18}$$

and

$$P_{\mathbf{n}_j}(p) = T_{\mathbf{n}_j}(p)\, \overline{Z}^T_{\mathbf{n}_j,\mathbf{m}} \in \mathbb{C}^{N_j \times N_{j+1}}, \qquad j = 0, \dots, \ell - 1, \tag{5.19}$$

where $T_{\mathbf{n}_j}(p) \in \mathbb{C}^{N_j \times N_j}$ is the bi-level Toeplitz matrix of order $\mathbf{n}_j$ generated by the bivariate trigonometric polynomial p (see chapter 2, subsection 2.3.4) and

$$\overline{Z}_{\mathbf{n}_j,\mathbf{m}} = \overline{Z}_{(n_j)_1, m_1} \otimes \overline{Z}_{(n_j)_2, m_2} \in \mathbb{C}^{N_{j+1} \times N_j},$$

is the bi-level downsampling matrix with the factor $\mathbf{m}$ with $\overline{Z}_{(n_j)_i, m_i} \in \mathbb{C}^{(n_{j+1})_i \times (n_j)_i}$ in (5.4), $i = 1, 2$.

On the other hand, in practical applications, the variable coefficient case is often of interest and the algebraic Galerkin approach could be computationally expensive. Therefore, in subsection 5.4.2, we present an example of the geometric approach for the bivariate Laplacian problem with variable coefficients and homogeneous Dirichlet boundary conditions. The system matrix $A_{\mathbf{n}}$ is derived via finite difference discretization of order 2 and minimal bandwidth. Thus, it is symmetric block tridiagonal with symmetric tridiagonal blocks and positive definite, but we lose the bi-level Toeplitz structure. Nevertheless, for $j = 0, \dots, \ell - 1$, the properties of $A_{\mathbf{n}}$ allow us to define the dimension $\mathbf{n}_j$ and the grid transfer operator $P_{\mathbf{n}_j}$ at level j as in (5.18) and (5.19), respectively.

Finally, in subsection 5.4.3, we present another example of the geometric approach for the bivariate anisotropic Laplacian problem with anisotropy along one of the coordinate axis and homogeneous Dirichlet boundary conditions. The system matrix $A_{\mathbf{n}}$ is derived via finite difference discretization of order 2 and minimal bandwidth, thus it is positive definite with a bi-level Toeplitz structure. For $j = 0, \dots, \ell - 1$, we define the dimension $\mathbf{n}_j$ and the grid transfer operator $P_{\mathbf{n}_j}$ at level j as in (5.18) and (5.19), respectively. We use the geometric approach shortly described at the beginning of subsection 5.4.2.

In all examples, we use binary $\mathbf{m} = (2, 2)$ and ternary $\mathbf{m} = (3, 3)$ coarsening, thus

$$\mathbf{n}_j = (m^{k-j} - 1, m^{k-j} - 1), \qquad m = 2, 3 \qquad N_j = N(\mathbf{n}_j) = \left(m^{k-j} - 1\right)^2, \qquad j = 0, \dots, \ell.$$

For $\mathbf{n} = (m^k - 1, m^k - 1) = (n, n)$, to define $\mathbf{b}_{\mathbf{n}}$, we choose the exact solution $X \in \mathbb{C}^{n \times n}$ as

$$X = \begin{pmatrix} \mathrm{x}(1,1) & \cdots & \mathrm{x}(1,n) \\ \vdots & \ddots & \vdots \\ \mathrm{x}(n,1) & \cdots & \mathrm{x}(n,n) \end{pmatrix} \in \mathbb{C}^{n \times n},$$

$$\mathrm{x}(\alpha,\beta) = \sin\left(5\frac{\pi(\alpha-1)}{n-1}\right) + \sin\left(5\frac{\pi(\beta-1)}{n-1}\right), \quad \alpha,\beta = 1,\dots,n,$$

we compute

$$\mathbf{x} = \begin{pmatrix}\mathrm{x}(1,1) & \cdots & \mathrm{x}(n,1) & \mathrm{x}(1,2) & \cdots & \mathrm{x}(n,2) & \cdots & \cdots & \mathrm{x}(1,n) & \cdots & \mathrm{x}(n,n)\end{pmatrix}^T \in \mathbb{C}^{n^2}$$

and set $\mathbf{b_n} := A_\mathbf{n}\mathbf{x} \in \mathbb{C}^{n^2}$. We solve the coarse grid system exactly when the dimension of the coarse grid is $N_\ell = (2^3-1)^2 = 49$ in the binary case and $N_\ell = (3^3-1)^2 = 676$ in the ternary case. For the numerical examples, the zero vector is used as the initial guess and the stopping criterion is $\|\mathbf{r}_s\|_2/\|\mathbf{r}_0\|_2 < 10^{-7}$, where $\mathbf{r}_s$ is the residual vector after s iterations and 10^{-7} is the given tolerance.

5.4.1 Biharmonic elliptic PDE

The first bivariate example we present arises from the discretization of the biharmonic elliptic PDE problem with homogeneous Dirichlet boundary conditions, namely

$$\begin{cases} \dfrac{\partial^4}{\partial x_1^4}\psi(\mathbf{x}) + \dfrac{\partial^4}{\partial x_2^4}\psi(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} = (x_1,x_2) \in \Omega = (0,1)^2, \\ \psi(\mathbf{x}) = 0 & \mathbf{x} \in \partial\Omega. \end{cases}$$

For the discretization, we use finite differences of order 4. It leads to the linear systems $A_\mathbf{n}\mathbf{x} = \mathbf{b_n}$, where $A_\mathbf{n} = T_\mathbf{n}(f)$ is the bi-Toeplitz matrix of order $\mathbf{n} \in \mathbb{N}^2$ generated by the bivariate trigonometric polynomial

$$f(\mathbf{x}) = (2-2\cos x_1)^2 + (2-2\cos x_2)^2, \qquad \mathbf{x} = (x_1,x_2) \in [0,2\pi)^2.$$

Note that f has a 4-fold zero at $\mathbf{x} = \mathbf{0}$. Thus, by Propositions 5.6, 5.7, 5.8 and 5.9 with $q = 4$, the 2-directional binary and ternary symmetric box spline symbols in Definition 5.3 and 5.4, respectively, with $J \geq 2$ and the interpolatory symbols $\mathcal{K}$ in (5.7), $\mathcal{B}$ in (5.9), $\mathcal{P}$ in (5.11), $\mathcal{K}_3$ in (5.13) and $\mathcal{H}_j$, $j = 3,4$, in (5.15) can be used to define the corresponding grid transfer operators.

Tables 5.7, 5.8 and 5.9 show how the number of iterations and convergence rates for the two-grid and V-cycle change with increasing dimension $\mathbf{n}$.

Similarly to subsection 5.2.1, Tables 5.8 and 5.9 illustrate the importance of the polynomial generation property (zero conditions) that, by Theorems 4.5 and 4.7, ensures the correct choice of the grid transfer operator. Indeed, the binary and ternary 2-directional symmetric box splines with symbols P_1 and $\tilde{P}_1$, respectively, generate polynomials of degree 1. The lack of the appropriate degree of polynomial generation leads to dramatic increase of the number of iterations. The larger is the dimension of the system matrix, the larger is the number of iterations needed to reach the required tolerance. The degree of polynomial generation which ensures convergence and optimality is lower for the two-grid method than for the V-cycle, see condition *(i)* of Theorem 3.7. Thus, for the two-grid method, the grid transfer operator associated to the binary 2-directional symmetric box spline with symbol P_1 performs as well as the other grid transfer operators associated to binary subdivision schemes generating cubic

Subdivision scheme	$\mathbf{n} = (2^6-1, 2^6-1)$ iter	conv. rate	$\mathbf{n} = (2^7-1, 2^7-1)$ iter	conv. rate	$\mathbf{n} = (2^8-1, 2^8-1)$ iter	conv. rate	generation degree
S_{P_1}	20	0.4331	19	0.4167	18	0.4076	1
S_{P_2}	13	0.2808	13	0.2782	13	0.2784	3
$S_{\mathcal{K}}$	13	0.2790	13	0.2778	13	0.2775	3
$S_{\mathcal{B}}$	16	0.3503	14	0.3077	13	0.2831	3
$S_{\mathcal{P}}$	26	0.5373	23	0.4937	20	0.4453	3

Table 5.7: *Binary bivariate subdivision schemes for biharmonic problem with two-grid method.*

Subdivision scheme	$\mathbf{n} = (2^6-1, 2^6-1)$ iter	conv. rate	$\mathbf{n} = (2^7-1, 2^7-1)$ iter	conv. rate	$\mathbf{n} = (2^8-1, 2^8-1)$ iter	conv. rate	generation degree
S_{P_1}	68	0.7873	117	0.8710	195	0.9205	1
S_{P_2}	18	0.4032	23	0.4910	27	0.5452	3
$S_{\mathcal{K}}$	15	0.329	15	0.3340	15	0.3402	3
$S_{\mathcal{B}}$	27	0.5427	31	0.5913	29	0.5698	3
$S_{\mathcal{P}}$	107	0.8602	135	0.8874	144	0.8939	3

Table 5.8: *Binary bivariate subdivision schemes for biharmonic problem with V-cycle.*

Subdivision scheme	$\mathbf{n} = (3^4-1, 3^4-1)$ iter	conv. rate	$\mathbf{n} = (3^5-1, 3^5-1)$ iter	conv. rate	$\mathbf{n} = (3^6-1, 3^6-1)$ iter	conv. rate	generation degree
$S_{\tilde{P}_1}$	47	0.7067	97	0.8459	221	0.9296	1
$S_{\tilde{P}_2}$	47	0.7058	48	0.7141	61	0.7677	3
$S_{\mathcal{K}_3}$	45	0.6988	51	0.7267	52	0.7309	3
$S_{\mathcal{H}_3}$	51	0.7282	70	0.7939	65	0.7791	3
$S_{\mathcal{H}_4}$	45	0.6986	56	0.7496	57	0.7535	3

Table 5.9: *Ternary bivariate subdivision schemes for biharmonic problem with V-cycle.*

polynomials, see Table 5.7. The best performing grid transfer operators are the ones associated to the symbols $\mathcal{K}$ in (5.7) and $\mathcal{K}_3$ in (5.13) of the binary and ternary Kobbelt's subdivision schemes $S_{\mathcal{K}}$ and $S_{\mathcal{K}_3}$, respectively. Indeed, $S_{\mathcal{K}}$ and $S_{\mathcal{K}_3}$ are both interpolatory subdivision schemes. After the smoothing steps, the error $\mathbf{e}_{\mathbf{n}_j}^{(k)} \in \mathbb{C}^{N_j}$ in the k-th iterate $\mathbf{x}_{\mathbf{n}_j}^{(k)} \in \mathbb{C}^{N_j}$ at the j-th recursive step, $k \geq 0$, $j = 0, \ldots, \ell$, is smooth. Numerical evidence shows that interpolatory grid transfer operators are more powerful than approximating grid transfer operators in the decomposition step of a smooth error $\mathbf{e}_{\mathbf{n}_j}^{(k)}$ from the finer grid $\mathbf{n}_j$ to the coarser one $\mathbf{n}_{j+1}$ and in the reconstruction step of a smooth error $\mathbf{e}_{\mathbf{n}_{j+1}}^{(k)}$ from the coarser grid $\mathbf{n}_{j+1}$ to the finer one $\mathbf{n}_j$. Tables 5.8 and 5.9 also illustrate that grid transfer operators associated to tensor product subdivision schemes, such as the binary and ternary 2-directional symmetric box splines S_{P_j} and $S_{\tilde{P}_j}$, $j = 1, 2$, and the binary and ternary Kobbelt's subdivision schemes $S_{\mathcal{K}}$ and $S_{\mathcal{K}_3}$, perform better than the other grid transfer operators. Indeed, tensor product subdivision

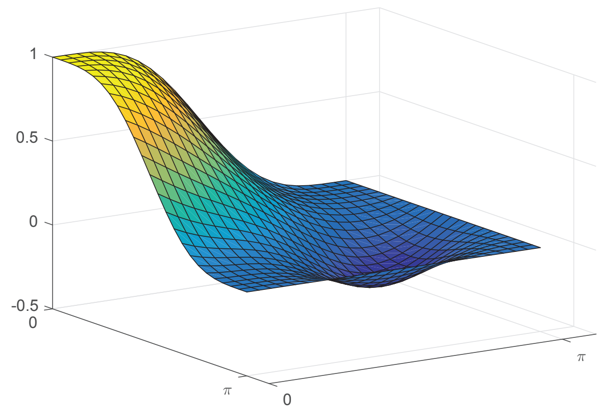


Figure 5.4: *Plot of* $\mathcal{P}(\mathrm{e}^{-\mathrm{i}\mathbf{x}})/\|\mathcal{P}\|_\infty$, $\mathbf{x} \in [0,\pi]^2$, *with* $\mathcal{P}$ *in* (5.11).

schemes are defined over a bivariate 2-directional mesh. The topology of a 2-directional mesh is equivalent to the topology of the starting rectangular grid $\mathbf{n}_0$ on which the biharmonic problem is discretized to obtain the starting linear system $A_{\mathbf{n}_0}\mathbf{x} = \mathbf{b}_{\mathbf{n}_0}$. Finally, the grid transfer operator associated to the symbol $\mathcal{P}$ in (5.11) defines an optimal V-cycle, meaning that the number of iterations needed to reach a given accuracy is bounded from above by a constant independent from the dimension of the starting linear system. Nevertheless, the convergence rate is extremely high. This phenomena relates to the behavior of the trigonometric polynomial $\mathcal{P}(\mathrm{e}^{-\mathrm{i}\mathbf{x}})$, $\mathbf{x} \in [0,2\pi)^2$, see Figure 5.4. Indeed, $\mathcal{P}(\mathrm{e}^{-\mathrm{i}\mathbf{x}})$ vanishes also on a whole curve contained in $(0,\pi)^2$. This fact leads to a further ill-conditioning of the problem on the coarser levels, as it is evident in Figure 5.5, where we can observe the increasing of the ill-conditioning in the high-frequencies going down on the coarser levels. We point out that, using the geometric multigrid, the grid transfer operator associated to the subdivision scheme $S_{\mathcal{P}}$ is extremely competitive (see next subsection).

5.4.2 Laplacian with non-constant coefficients

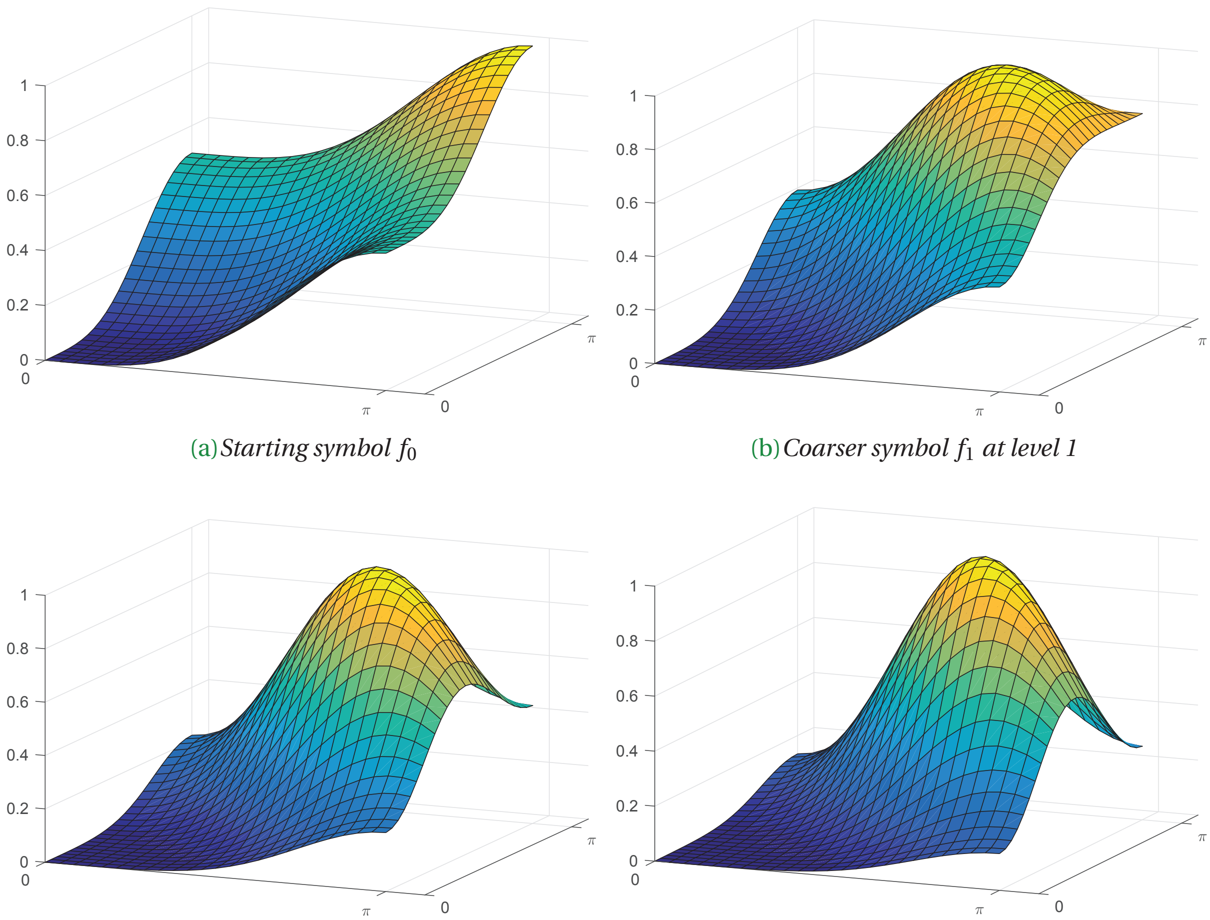
We consider the bivariate Laplacian with non-constant coefficients with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\nabla\left(a\nabla\psi\right)(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} = (x_1,x_2) \in \Omega = (0,1)^2, \\ \psi(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega. \end{cases} \tag{5.20}$$

We briefly describe the 2-dimensional geometric multigrid method for the linear systems $A_{\mathbf{n}}\mathbf{x} = \mathbf{b}_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^2$, derived via discretization of the problem (5.20). Let $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$, $1 \le \ell \le k-1$. For $j = 0,\dots,\ell$, we define

$$\mathbf{n}_j = (n_j,n_j) = (m^{k-j}-1, m^{k-j}-1), \qquad m = 2,3, \qquad N_j = N(\mathbf{n}_j) = (m^{k-j}-1)^2,$$

and $\Omega_{\mathbf{n}_j}$ as a uniform grid on $[0,1]^2$ of n_j subintervals of size $h_j = 1/(n_j+1)$ in each direction. Then, at the j-th recursion level of the V-cycle, we compute the matrix $A_{\mathbf{n}_j}$ by discretizing



(a) *Starting symbol f_0*

(b) *Coarser symbol f_1 at level 1*

(c) *Coarser symbol f_2 at level 2*

(d) *Coarser symbol f_3 at level 3*

Figure 5.5: *Symbols f_j, $j = 0, \ldots, 3$ defined by* (3.20) *using the trigonometric polynomial* $p(\mathbf{x}) = \mathcal{P}(\mathrm{e}^{-\mathrm{i}\mathbf{x}})$ *with* $\mathcal{P}$ *in* (5.11) *in the reference interval* $[0,\pi]^2$.

the problem (5.20) using finite differences of order 2 on the grid $\Omega_{\mathbf{n}_j}$. Thus, the matrices $A_{\mathbf{n}_j}$, $j = 0, \ldots, \ell$, have dimension $N_j \times N_j$. The prolongation operators are defined as in (5.19) with $\mathbf{m} = (2,2)$ in the binary case and $\mathbf{m} = (3,3)$ in the ternary case. The restriction operators are $\dfrac{1}{m^2} P^*_{\mathbf{n}_j}$, for $j = 0, \ldots, \ell - 1$.

By Propositions 5.6, 5.7, 5.8 and 5.9, the symbols of the bivariate subdivision schemes introduced in section 5.3 define suitable grid transfer operators for our problem. Tables 5.10 and 5.11 show how the number of iterations and convergence rates for the V-cycle change for different diffusion coefficients $a(x_1, x_2)$, namely

$$
\begin{aligned}
a_1(x_1,x_2) &= x_1^2 + \mathrm{e}^{x_2}, \\
a_2(x_1,x_2) &= 1 + \sin(3\pi x_1) \cdot \sin(5\pi x_2) + \varepsilon, \\
a_3(x_1,x_2) &= 1 + \sin(7\pi x_1) + (x_2 - 0.5)^4 + \varepsilon, \qquad (x_1,x_2) \in [0,1]^2.
\end{aligned}
$$

Subdivision scheme	$a_1(x_1,x_2)$ iter	$a_1(x_1,x_2)$ conv. rate	$a_2(x_1,x_2)$ iter	$a_2(x_1,x_2)$ conv. rate	$a_3(x_1,x_2)$ iter	$a_3(x_1,x_2)$ conv. rate	generation degree
S_{P_1}	9	0.1400	10	0.1770	35	0.6295	1
S_{P_2}	13	0.2770	14	0.3159	52	0.7319	3
$S_{\mathcal{K}}$	8	0.1234	9	0.1430	29	0.5688	3
$S_{\mathcal{B}}$	8	0.1252	8	0.1244	29	0.5709	3
$S_{\mathcal{P}}$	9	0.1500	9	0.1533	29	0.5726	3

Table 5.10: *Binary bivariate subdivision schemes for Laplacian with non-constant coefficients*

Subdivision scheme	$a_1(x_1,x_2)$ iter	$a_1(x_1,x_2)$ conv. rate	$a_2(x_1,x_2)$ iter	$a_2(x_1,x_2)$ conv. rate	$a_3(x_1,x_2)$ iter	$a_3(x_1,x_2)$ conv. rate	generation degree
$S_{\tilde{P}_1}$	16	0.3649	16	0.3611	29	0.4421	1
$S_{\tilde{P}_2}$	19	0.4130	18	0.4073	25	0.5120	3
$S_{\mathcal{K}_3}$	17	0.3753	17	0.3718	19	0.4191	3
$S_{\mathcal{H}_3}$	17	0.3759	17	0.3721	19	0.4193	3
$S_{\mathcal{H}_4}$	17	0.3752	17	0.3718	19	0.4191	3

Table 5.11: *Ternary bivariate subdivision schemes for Laplacian with non-constant coefficients*

Parameter $\varepsilon \in (0,1)$ guarantees the strict positivity of all these functions. In the numerical examples, $\varepsilon = 10^{-5}$. The starting grid is fixed to be $\mathbf{n}_0 = (2^7-1, 2^7-1)$ in the binary case and $\mathbf{n}_0 = (3^5-1, 3^5-1)$ in the ternary case.

The results of Tables 5.10 and 5.11 also illustrate the influence of the functions a_j, $j = 1,2,3$, on the behavior of multigrid. In both binary and ternary case, when $a(x_1,x_2) = a_3(x_1,x_2)$, $(x_1,x_2) \in [0,1]^2$, the best performing grid transfer operators are the ones associated with the interpolatory subdivision schemes, namely binary Kobbelt's subdivision scheme $S_{\mathcal{K}}$, Butterfly subdivision scheme $S_{\mathcal{B}}$ and our new subdivision scheme $S_{\mathcal{P}}$ (binary case), and ternary Kobbelt's subdivision scheme $S_{\mathcal{K}_3}$ and $S_{\mathcal{H}_j}$, $j = 3,4$, subdivision schemes (ternary case). In the binary case, the advantage of using our new scheme is the computational efficiency of the corresponding grid transfer operations. Indeed, due to the geometric approach, the matrices $A_{\mathbf{n}_j}$, $j = 0,\dots,\ell$, are independent of the grid transfer operators and the computational cost of the restriction and prolongation depends only on the number of nonzero entries of the corresponding operators. Therefore, since the mask $\mathcal{P}$ in (5.12) has 19 nonzero entries while the masks $\mathcal{K}$ in (5.8) and $\mathcal{B}$ in (5.10) have 25 nonzero entries, each iteration of the V-cycle method with the grid transfer operator associated to our new subdivision scheme $S_{\mathcal{P}}$ is cheaper than one V-cycle iteration with the grid transfer operators associated to Kobbelt's and Butterfly subdivision schemes. Similarly, in the ternary case, the mask $\mathcal{K}_3$ in (5.14) has 81 nonzero entries, the mask $\mathcal{H}_3$ in (5.16) has 79 nonzero entries and the mask $\mathcal{H}_4$ in (5.17) has 65 nonzero entries. Thus, each iteration of the V-cycle method with the grid transfer operator associated to the subdivision scheme $S_{\mathcal{H}_4}$ is cheaper than one V-cycle iteration with the grid transfer operators associated to the ternary Kobbelt's $S_{\mathcal{K}_3}$ and $S_{\mathcal{H}_3}$ subdivision schemes. Finally, similarly to the univariate case, the grid transfer operators associated to ternary subdivision schemes are robuster than the grid transfer operators associated to binary subdivision schemes when the function of the diffusion coefficients is highly oscillating (see the column for the function $a_3(x_1,x_2)$ in Tables 5.10 and 5.11).

5.4.3 Anisotropic Laplacian

The last example we present arises from the discretization of the bivariate anisotropic Laplacian problem with Dirichlet boundary conditions, namely

$$\begin{cases} -\varepsilon \dfrac{\partial^2}{\partial x_1^2}\psi(\mathbf{x}) - \dfrac{\partial^2}{\partial x_2^2}\psi(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} = (x_1,x_2) \in \Omega = (0,1)^2, \\ \psi(\mathbf{x}) = 0 & \mathbf{x} \in \partial\Omega. \end{cases} \tag{5.21}$$

The parameter $\varepsilon \in (0,1]$ in (5.21) is called *anisotropy*. If $\varepsilon = 1$, we get the bivariate isotropic Laplacian problem. If $\varepsilon << 1$, the problem becomes strongly anisotropic. We focus our attention on the latter case [71].

We use the geometric approach and the corresponding notation shortly introduced at the beginning of subsection 5.4.2. Using finite difference discretization of order 2 and minimal bandwidth, for $j = 0,\dots,\ell$, the system matrices $A_{\mathbf{n}_j} = T_{\mathbf{n}_j}(f_j^{(\varepsilon)}) \in \mathbb{C}^{N_j\times N_j}$ are the bi-level Toeplitz matrices of order $\mathbf{n}_j$ generated by the bivariate trigonometric polynomials

$$f_j^{(\varepsilon)}(\mathbf{x}) = \frac{1}{h_j^2}\big(\varepsilon(2-2\cos x_1) + (2-2\cos x_2)\big), \qquad \mathbf{x} = (x_1,x_2) \in [0,2\pi)^2. \tag{5.22}$$

We use as pre- and post-smoother one step of Gauss-Seidel method for $j = 1,\dots,\ell$, and 2 steps of Gauss-Seidel method for $j = 0$. In this example, we increase the tolerance in the stopping criterion from 10^{-7} to 10^{-5}.

Tables 5.12 and 5.13 show how the number of iterations and convergence rates for the V-cycle change when the starting grid $\mathbf{n}_0$ becomes finer and the anisotropy ε in (5.21) decreases. The number of iterations needed to reach the exact solution is large for $\varepsilon = 10^{-2}$ and drastically increases when the anisotropy decreases to 10^{-3}. Indeed, if $\varepsilon << 1$, the symbol $f_j^{(\varepsilon)}$ in (5.22) is numerically close to 0 on the entire line $x_2 = 0$, for all $j = 0,\dots,\ell$ (see Figure 5.6). Due to this pathology, when the anisotropy ε goes to 0, the number of iterations necessary for the convergence of the V-cycle method rises because the symbol vanishes on a whole curve and hence conditions *(i)* and *(ii)* of Theorem 3.8 cannot be satisfied together [44].

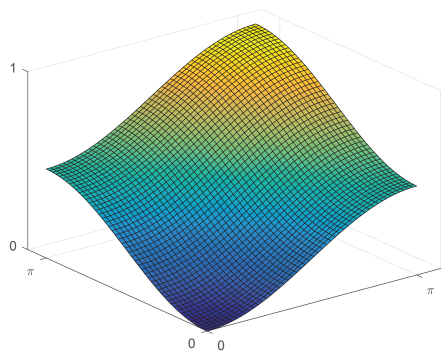
In [44], in order to handle the anisotropy along the x_1-axis, the authors propose to use semi-coarsening in the direction perpendicular to the anisotropy, i.e. in the x_2-axis direction. We propose to use a multigrid method based on anisotropic subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m \in \mathbb{N}$, $m \geq 3$ odd. In this case, the difference between the two coordinate directions is encoded in the dilation matrices M. Indeed, the coarsening strategy of such multigrid method cuts more in the x_2-axis direction than in the x_1-axis direction. Thus, it fights the anisotropy of the problem with the anisotropy of the dilation, or - equivalently - with the anisotropy of the

Anisotropy	Subdivision scheme	$\mathbf{n}_0 = (2^7-1, 2^7-1)$ iter	conv. rate	$\mathbf{n}_0 = (2^8-1, 2^8-1)$ iter	conv. rate	generation degree
$\varepsilon = 10^{-2}$	S_{P_1}	75	0.8571	80	0.8658	1
	S_{P_2}	82	0.8686	86	0.8744	3
	$S_{\mathcal{K}}$	61	0.8273	76	0.8585	3
	$S_{\mathcal{B}}$	62	0.8301	76	0.8586	3
	$S_{\mathcal{P}}$	62	0.8300	76	0.8586	3
$\varepsilon = 10^{-3}$	S_{P_2}	294	0.9616	284	0.9603	1
	S_{P_4}	295	0.9617	281	0.9599	3
	$S_{\mathcal{K}}$	253	0.9555	251	0.9551	3
	$S_{\mathcal{B}}$	253	0.9555	251	0.9552	3
	$S_{\mathcal{P}}$	253	0.9555	251	0.9552	3

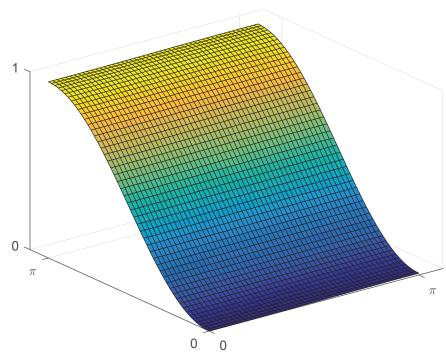
Table 5.12: *Binary bivariate subdivision schemes for the anisotropic Laplacian problem with* $\varepsilon = 10^{-2}, 10^{-3}$.

Anisotropy	Subdivision scheme	$\mathbf{n}_0 = (3^4-1, 3^4-1)$ iter	conv. rate	$\mathbf{n}_0 = (3^5-1, 3^5-1)$ iter	conv. rate	generation degree
$\varepsilon = 10^{-2}$	$S_{\tilde{P}_1}$	105	0.8958	119	0.9077	1
	$S_{\tilde{P}_2}$	101	0.8922	120	0.9083	3
	$S_{\mathcal{K}_3}$	98	0.8888	116	0.9053	3
	$S_{\mathcal{H}_3}$	98	0.8888	116	0.9053	3
	$S_{\mathcal{H}_4}$	98	0.8888	116	0.9053	3
$\varepsilon = 10^{-3}$	$S_{\tilde{P}_1}$	343	0.9670	404	0.9719	1
	$S_{\tilde{P}_2}$	332	0.9659	393	0.9711	3
	$S_{\mathcal{K}_3}$	312	0.9638	318	0.9644	3
	$S_{\mathcal{H}_3}$	312	0.9638	319	0.9645	3
	$S_{\mathcal{H}_4}$	312	0.9638	318	0.9644	3

Table 5.13: *Ternary bivariate subdivision schemes for the anisotropic Laplacian problem with* $\varepsilon = 10^{-2}, 10^{-3}$.



(a) *Isotropic Laplacian:* $\varepsilon = 1$



(b) *Strongly anisotropic Laplacian:* $\varepsilon = 10^{-2}$

Figure 5.6: *Plot of* $f_0^{(\varepsilon)} / \| f_0^{(\varepsilon)} \|_\infty$ *on the reference interval* $[0,\pi]^2$ *for different values of* $\varepsilon \in (0,1]$.

cutting strategy. To the best of our knowledge, in subdivision literature, families of subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m \in \mathbb{N}$, characterized by certain regularity properties have not been defined yet. In the next chapter, we define two families of subdivision schemes with such dilation matrix M. First, we study their regularity and generation properties. Then, we construct anisotropic grid transfer operators from their symbols capable of defining a convergent V-cycle method for isotropic and anisotropic Laplacian problems.

CHAPTER 6

Anisotropic stationary subdivision schemes and grid transfer operators

In this chapter, motivated by the numerical experiments of chapter 5, subsection 5.4.3, we explore the realm of bivariate anisotropic subdivision schemes with dilation

$$M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}, \qquad m \in \mathbb{N}, \qquad \gcd(2, m) = 1. \tag{6.1}$$

We propose two types of families of anisotropic subdivision schemes: approximating and interpolating (see sections 6.1 and 6.2, respectively). Our results and numerical experiments show that both families lead to efficient grid transfer operators. Nevertheless, the computational cost of the multigrid based on interpolatory grid transfer operators is minimal due to the fewer non-zero coefficients in the corresponding subdivision rules. Indeed, our interpolatory subdivision schemes are constructed to be optimal in terms of the size of the support versus their polynomial generation properties. Similar constructions in the case of $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ are done in [50], but they are not applicable for anisotropic multigrid. Cotronei et al. in [22] and Sauer in [66] propose a general strategy to compute d-variate interpolatory subdivision symbols with dilation $M \in \mathbb{Z}^{d\times d}$ based on the Smith factorization of M. In case of diagonal dilation M in (6.1), this strategy defines the symbols of interpolatory bivariate subdivision schemes as the tensor product of univariate subdivision symbols with dilation 2 and m. The corresponding masks have more non-zero coefficients as the ones we propose and, thus, the computational cost of the associated multigrid is higher. To study the dependence of the efficiency of multigrid on the reproduction/generation properties of subdivision, we also define a family of approximating schemes. Our construction resembles the one given in [21] for the family of bi-variate pseudo-splines with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Our goal, for compatibility of our numerical experiments with approximating and interpolating grid transfer operators, is to define approximating schemes that have the same support as the interpolating ones and matching polynomial generation properties. We do not claim to have constructed a new family of anisotropic pseudo-splines.

6.1 Anisotropic interpolatory subdivision

In subsection 6.1.1, we start by introducing the family of univariate interpolatory Dubuc-Deslauriers subdivision schemes. These will be a basis for our bivariate construction in subsection 6.1.2.

6.1.1 Univariate case

In [29], Deslauriers and Dubuc proposed a general method for constructing symmetric interpolatory subdivision schemes of dilations $m \in \mathbb{N}$, $m \geq 2$. The smoothness analysis of their schemes was conducted by Eirola et al. in [41]. Recently, Diaz Fuentes proposed in his master thesis [30] a closed formula for computing the mask of the interpolatory Dubuc-Deslauriers subdivision schemes for any dilation $m \in \mathbb{N}$, $m \geq 2$.

Definition 6.1. Let $m \in \mathbb{N}$, $m \geq 2$, and $J \in \mathbb{N}$. The univariate $(2J)$-point Dubuc-Deslauriers interpolatory subdivision scheme $S_{\mathbf{a}_{m,J}}$ of dilation m is defined by its symbol

$$a_{m,J}(z) = 1 + \sum_{\varepsilon=1}^{m-1} \sum_{\beta=-J+1}^{J} \frac{(-1)^{\beta+J}}{(2J-1)!(\varepsilon/m-\beta)} \binom{2J-1}{J-\beta} \left(-J+1-\frac{\varepsilon}{m}\right)_{2J} z^{-m\beta+\varepsilon}, \qquad z \in \mathbb{C}\setminus\{0\},$$

where for any $x \in \mathbb{R}$, $(x)_\ell$ is the Pochhammer symbol defined by

$$(x)_0 := 1, \qquad \text{and} \qquad (x)_\ell := x(x+1)\cdots(x+\ell-1), \qquad \ell \in \mathbb{N}.$$

For reader's convenience (as [30] is in Spanish), we recall the main ideas in [29] behind the construction of symmetric interpolatory subdivision schemes and repeat a few computations from [30] conducted in order to obtain the symbols $a_{m,J}$ in Definition 6.1. Without loss of generality, we focus on the case $m = 2\ell+1$, $\ell \in \mathbb{N}$. The latter choice plays a role only in the definition of the range of ε in (6.2). Indeed, if $m = 2\ell$, $\ell \in \mathbb{N}$, the range of ε is slightly different. Nevertheless, a change of variable at the end of computation leads to the same formula in Definition 6.1 for even and odd dilation m.

Let $\mathbf{c} = \{c(\gamma) \in \mathbb{R} : \gamma \in \mathbb{Z}\} \in \ell(\mathbb{Z})$ and fix an integer $\alpha \in \mathbb{Z}$. Let

$$\mathbf{c}^\alpha = \{c(\gamma) : \alpha - J + 1 \leq \gamma \leq \alpha + J\} \in \ell_0(\mathbb{Z})$$

be $2J$ consecutive elements of $\mathbf{c}$ centered in $\mathrm{c}(\alpha)$. There exists a unique polynomial $\pi \in \Pi_{2J-1}$ of degree $2J-1$ which interpolates $\mathbf{c}^\alpha$ at the integers $\{\alpha-J+1,\dots,\alpha+J\}$, namely

$$\pi(t) = \sum_{\beta=\alpha-J+1}^{\alpha+J} \mathrm{c}^\alpha(\beta)\mathcal{L}_\beta^{(\alpha-J+1,2J-1)}(t) = \sum_{\beta=-J+1}^{J} \mathrm{c}^\alpha(\beta+\alpha)\mathcal{L}_{\beta+\alpha}^{(\alpha-J+1,2J-1)}(t),$$

$$\mathcal{L}_\beta^{(\alpha-J+1,2J-1)}(t) = \prod_{\substack{\gamma=\alpha-J+1\\ \gamma\neq\beta}}^{\alpha+J} \frac{t-\gamma}{\beta-\gamma}, \qquad t\in\mathbb{R}.$$

For $\beta \in \{\alpha-J+1,\dots,\alpha+J\}$, $\mathcal{L}_\beta^{(\alpha-J+1,2J-1)}$ is the Lagrange polynomial of degree $2J-1$, centered in β, defined on the $2J$ nodes $\{\alpha-J+1,\dots,\alpha+J\}$, and satisfies

$$\mathcal{L}_\beta^{(\alpha-J+1,2J-1)}(\varepsilon) = \delta_{\beta,\varepsilon}, \qquad \varepsilon\in\{\alpha-J+1,\dots,\alpha+J\}.$$

In order to define the subdivision operator $\mathcal{S}_{\mathbf{a}_{m,J}}$, we define its action on the finite sequence $\mathbf{c}^\alpha$ by

$$\begin{aligned}
(\mathcal{S}_{\mathbf{a}_{m,J}}\mathbf{c}^\alpha)(m\alpha+\varepsilon) &:= \pi\left(\alpha+\frac{\varepsilon}{m}\right) \\
&= \sum_{\beta=-J+1}^{J} \mathcal{L}_{\beta+\alpha}^{(\alpha-J+1,2J-1)}\left(\alpha+\frac{\varepsilon}{m}\right)\mathrm{c}^\alpha(\beta+\alpha) && \text{(by definition of } \pi) \\
&= \sum_{\beta=\alpha-J+1}^{\alpha+J} \mathrm{a}_{m,J}\big(m(\alpha-\beta)+\varepsilon\big)\mathrm{c}^\alpha(\beta) && \text{(by definition of subdivision operator)} \\
&= \sum_{\beta=-J+1}^{J} \mathrm{a}_{m,J}\big(-m\beta+\varepsilon\big)\mathrm{c}^\alpha(\beta+\alpha), && \varepsilon = -\frac{m-1}{2},\dots,\frac{m-1}{2}.
\end{aligned} \tag{6.2}$$

For $\varepsilon=0$ in (6.2), $(\mathcal{S}_{\mathbf{a}_{m,J}}\mathbf{c}^\alpha)(m\alpha) = \pi(\alpha) = \mathrm{c}^\alpha(\alpha)$, thus $\mathrm{a}_{m,J}(m\beta) = \delta_{\beta,0}$.
For $\varepsilon\neq 0$ in (6.2), using simple properties of $\mathcal{L}_\beta^{(\alpha-J+1,2J-1)}$ we get

$$\begin{aligned}
\mathrm{a}_{m,J}\big(-m\beta+\varepsilon\big) &= \mathcal{L}_{\beta+\alpha}^{(\alpha-J+1,2J-1)}\left(\alpha+\frac{\varepsilon}{m}\right), \\
&= \mathcal{L}_\beta^{(-J+1,2J-1)}\left(\frac{\varepsilon}{m}\right), \\
&= \frac{(-1)^{\beta+J}}{(2J-1)!(\varepsilon/m-\beta)}\binom{2J-1}{J-\beta}\left(-J+1-\frac{\varepsilon}{m}\right)_{2J}.
\end{aligned}$$

Definition (6.1) follows from property

$$a_{m,J}(z) = \sum_{\alpha\in\mathbb{Z}} \mathrm{a}_{m,J}(\alpha)z^\alpha = \sum_{\varepsilon=0}^{m-1}\sum_{\beta=-J+1}^{J} \mathrm{a}_{m,J}(-m\beta+\varepsilon)z^{-m\beta+\varepsilon}, \qquad z\in\mathbb{C}\setminus\{0\}.$$

By construction, for any $m\in\mathbb{N}$, $m\geq 2$, and $J\in\mathbb{N}$, the univariate $(2J)$-point Dubuc-Deslauriers interpolatory subdivision schemes of dilation m generate and reproduce polynomials up to degree $2J-1$. We recall that $S_{\mathbf{a}_{m,J}}$ is the unique univariate subdivision scheme of dilation m such that

- ⋆ it is interpolatory,
- ⋆ it generates polynomials up to degree $2J-1$,
- ⋆ its mask $\mathbf{a}_{m,J}$ is symmetric and has support $\{1-mJ,\dots,mJ-1\}$.

***Example* 6.1.** Let $m=2$ and $J=1,2,3$. The masks $\mathbf{a}_{2,1}$, $\mathbf{a}_{2,2}$, $\mathbf{a}_{2,3}$ of the *binary* 2-, 4- and 6-point Dubuc-Deslauriers interpolatory subdivision schemes are defined by

$$\mathbf{a}_{2,1}=\left\{\tfrac{1}{2} \quad 1 \quad \tfrac{1}{2}\right\},$$
$$\mathbf{a}_{2,2}=\left\{-\tfrac{1}{16} \quad 0 \quad \tfrac{9}{16} \quad 1 \quad \tfrac{9}{16} \quad 0 \quad -\tfrac{1}{16}\right\},$$
$$\mathbf{a}_{2,3}=\left\{\tfrac{3}{256} \quad 0 \quad -\tfrac{25}{256} \quad 0 \quad \tfrac{75}{128} \quad 1 \quad \tfrac{75}{128} \quad 0 \quad -\tfrac{25}{256} \quad 0 \quad \tfrac{3}{256}\right\}.$$

◀

***Example* 6.2.** Let $m=3$ and $J=1,2,3$. The masks $\mathbf{a}_{3,1}$, $\mathbf{a}_{3,2}$, $\mathbf{a}_{3,3}$ of the *ternary* 2-, 4- and 6-point Dubuc-Deslauriers interpolatory subdivision schemes are defined by

$$\mathbf{a}_{3,1}=\left\{\tfrac{1}{3} \quad \tfrac{2}{3} \quad 1 \quad \tfrac{2}{3} \quad \tfrac{1}{3}\right\},$$
$$\mathbf{a}_{3,2}=\left\{-\tfrac{4}{81} \quad -\tfrac{5}{81} \quad 0 \quad \tfrac{10}{27} \quad \tfrac{20}{27} \quad 1 \quad \tfrac{20}{27} \quad \tfrac{10}{27} \quad 0 \quad -\tfrac{5}{81} \quad -\tfrac{4}{81}\right\},$$
$$\mathbf{a}_{3,3}=\left\{\tfrac{7}{729} \quad \tfrac{8}{729} \quad 0 \quad -\tfrac{56}{729} \quad -\tfrac{70}{729} \quad 0 \quad \tfrac{280}{729} \quad \tfrac{560}{729} \quad 1 \quad \tfrac{560}{729} \quad \tfrac{280}{729} \quad 0 \quad -\tfrac{70}{729} \quad -\tfrac{56}{729} \quad 0 \quad \tfrac{8}{729} \quad \tfrac{7}{729}\right\}.$$

◀

6.1.2 Bivariate case

From the family of univariate interpolatory Dubuc-Deslauriers subdivision schemes we build a family of bivariate interpolatory subdivision schemes with dilation matrix M in (6.1) using the approach from [21].

Definition 6.2. Let $J\in\mathbb{N}$. The *anisotropic interpolatory* subdvision scheme $S_{\mathbf{a}_{M,J}}$ of order J and dilation matrix M in (6.1) is defined by its symbol

$$a_{M,J}(\mathbf{z}):=\sum_{k=0}^{J-1}a_{2,J-k}(z_1)\,a_{m,k+1}(z_2)-\sum_{k=0}^{J-2}a_{2,J-k-1}(z_1)\,a_{m,k+1}(z_2),\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,$$

where $a_{2,k}$, $a_{m,k}$ are the symbols of the univariate $(2k)$-point Dubuc-Deslauriers interpolatory subdivision schemes in Definition 6.1 of dilation 2 and m, respectively.

Definition 6.2 is justified by the following result.

Proposition 6.1. *Let $J\in\mathbb{N}$ and M in* (6.1)*. The anisotropic subdvision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 is interpolatory.*

Proof. Let $J\in\mathbb{N}$. By Theorem 4.1, in order to prove Proposition 6.1, we need to show that

$$s_J(\mathbf{z}):=\sum_{j=0}^{m-1}a_{M,J}(z_1,\xi_j z_2)+\sum_{j=0}^{m-1}a_{M,J}(-z_1,\xi_j z_2)=|\det M|=2m,\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,$$

for $\xi_j = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{m} j}$, $j = 0,\dots,m-1$. Since $S_{\mathbf{a}_{2,J}}$ and $S_{\mathbf{a}_{m,J}}$ are univariate interpolatory subdivision schemes of dilation 2 and m respectively, Theorem 4.1 guarantees that their symbols satisfy

$$a_{2,J}(z_1) + a_{2,J}(-z_1) = 2 \qquad \text{and} \qquad \sum_{j=0}^{m-1} a_{m,J}(\xi_j z_2) = m.$$

By Definition 6.2, we have

$$\begin{aligned} s_J(\mathbf{z}) &= \sum_{k=0}^{J-1} a_{2,J-k}(z_1) \sum_{j=0}^{m-1} a_{m,k+1}(\xi_j z_2) - \sum_{k=0}^{J-2} a_{2,J-k-1}(z_1) \sum_{j=0}^{m-1} a_{m,k+1}(\xi_j z_2) \\ &\quad + \sum_{k=0}^{J-1} a_{2,J-k}(-z_1) \sum_{j=0}^{m-1} a_{m,k+1}(\xi_j z_2) - \sum_{k=0}^{J-2} a_{2,J-k-1}(-z_1) \sum_{j=0}^{m-1} a_{m,k+1}(\xi_j z_2) \\ &= m \sum_{k=0}^{J-1} (a_{2,J-k}(z_1) + a_{2,J-k}(-z_1)) - m \sum_{k=0}^{J-2} (a_{2,J-k-1}(z_1) + a_{2,J-k-1}(-z_1)) \\ &= 2mJ - 2m(J-1) = 2m. \end{aligned}$$

■

Further properties of the interpolatory subdivision schemes $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 are analyzed in subsections 6.1.3 (reproduction), 6.1.4 (minimality of the support) and 6.1.5 (convergence).

6.1.3 Reproduction property of $S_{\mathbf{a}_{M,J}}$

In this section, we show that the anisotropic interpolatory subdivision schemes $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 reproduce polynomials up to degree $2J-1$.

Proposition 6.2. *Let $J \in \mathbb{N}$ and M in (6.1). The anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 reproduces polynomials up to degree $2J-1$.*

Proof. By (4.24), in order to prove Proposition 6.2, we need to show that the symbol $a_{M,J}$ can be decomposed as

$$\begin{aligned} &a_{M,J}(\mathbf{z}) = 2m + \sum_{h=0}^{H} (1-z_1)^{\alpha_h} (1-z_2)^{\beta_h} c_{M,J,h}(\mathbf{z}), \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2, \\ &\alpha_h, \beta_h \in \mathbb{N}_0, \qquad \alpha_h + \beta_h \geq 2J, \qquad h = 0,\dots,H, \end{aligned} \tag{6.3}$$

for some $H \in \mathbb{N}$ and some suitable Laurent polynomials $c_{M,J,h}$, $h = 0,\dots,H$. We require that at least one pair $\alpha_h, \beta_h \in \mathbb{N}_0$ satisfies $\alpha_h + \beta_h = 2J$, otherwise the subdivision scheme $S_{\mathbf{a}_{M,J}}$ reproduces polynomials of degree strictly higher than $2J-1$. We recall that for any $k \in \mathbb{N}$, the univariate $(2k)$-point Dubuc-Deslauriers interpolatory subdivision schemes $S_{\mathbf{a}_{2,k}}$, $S_{\mathbf{a}_{m,k}}$ in Definition 6.1 with dilation 2 and m, respectively, both reproduce polynomials up to degree $2k-1$. Thus, from (4.23), their symbols $a_{2,k}$, $a_{m,k}$ can be written as

$$\begin{aligned} a_{2,k}(z) &= 2 + (1-z)^{2k} b_{2,k}(z), \\ a_{m,k}(z) &= m + (1-z)^{2k} b_{m,k}(z), \qquad z \in \mathbb{C} \setminus \{0\}, \end{aligned} \tag{6.4}$$

for suitable Laurent polynomials $b_{2,k}$, $b_{m,k}$. By Definition 6.2, using factorization (6.4), there exist Laurent polynomials $b_{2,J-k}$, $b_{m,k+1}$, $k = 0,\dots,J-1$, such that

$$\begin{aligned}
a_{M,J}(\mathbf{z}) &= \sum_{k=0}^{J-1} a_{2,J-k}(z_1)\, a_{m,k+1}(z_2) - \sum_{k=0}^{J-2} a_{2,J-k-1}(z_1)\, a_{m,k+1}(z_2)\\
&= \sum_{k=0}^{J-1}\Big(2+(1-z_1)^{2(J-k)}\, b_{2,J-k}(z_1)\Big)\Big(m+(1-z_2)^{2(k+1)}\, b_{m,k+1}(z_2)\Big)\\
&\quad - \sum_{k=0}^{J-2}\Big(2+(1-z_1)^{2(J-k-1)}\, b_{2,J-k-1}(z_1)\Big)\Big(m+(1-z_2)^{2(k+1)}\, b_{m,k+1}(z_2)\Big).
\end{aligned}$$

Rearranging the terms in the above expression, we obtain

$$\begin{aligned}
a_{M,J}(\mathbf{z}) =&\, 2mJ + m\sum_{k=0}^{J-1}(1-z_1)^{2(J-k)}\, b_{2,J-k}(z_1) + 2\sum_{k=0}^{J-1}(1-z_2)^{2(k+1)}\, b_{m,k+1}(z_2)\\
&+ \sum_{k=0}^{J-1}(1-z_1)^{2(J-k)}(1-z_2)^{2(k+1)}\, b_{2,J-k}(z_1)\, b_{m,k+1}(z_2)\\
&- 2m(J-1) - m\sum_{k=0}^{J-2}(1-z_1)^{2(J-k-1)}\, b_{2,J-k-1}(z_1) - 2\sum_{k=0}^{J-2}(1-z_2)^{2(k+1)}\, b_{m,k+1}(z_2)\\
&- \sum_{k=0}^{J-2}(1-z_1)^{2(J-k-1)}(1-z_2)^{2(k+1)}\, b_{2,J-k-1}(z_1)\, b_{m,k+1}(z_2)\\
=&\, 2m + m(1-z_1)^{2J}\, b_{2,J}(z_1) + 2(1-z_2)^{2J}\, b_{m,J}(z_2)\\
&+ \sum_{k=0}^{J-1}(1-z_1)^{2(J-k)}(1-z_2)^{2(k+1)}\, b_{2,J-k}(z_1)\, b_{m,k+1}(z_2)\\
&- \sum_{k=0}^{J-2}(1-z_1)^{2(J-k-1)}(1-z_2)^{2(k+1)}\, b_{2,J-k-1}(z_1)\, b_{m,k+1}(z_2)\\
=&\, 2m + \sum_{h=0}^{2J}(1-z_1)^{\alpha_h}(1-z_2)^{\beta_h}\, c_{M,J,h}(\mathbf{z})
\end{aligned}$$

where

$$\alpha_h = \begin{cases} 2J & h=0,\\ 0 & h=1,\\ 2(J-h+2) & h\in\{2,\dots,J+1\},\\ 2(2J-h+1) & h\in\{J+2,\dots,2J\}, \end{cases} \qquad \beta_h = \begin{cases} 0 & h=0,\\ 2J & h=1,\\ 2(h-1) & h\in\{2,\dots,J+1\},\\ 2(h-J-1) & h\in\{J+2,\dots,2J\}, \end{cases}$$

and

$$c_{M,J,h}(\mathbf{z}) = \begin{cases} m\, b_{2,J}(z_1) & h=0,\\ 2\, b_{m,J}(z_2) & h=1,\\ b_{2,J-h+2}(z_1)\, b_{m,h-1}(z_2) & h\in\{2,\dots,J+1\},\\ -b_{2,2J-h+1}(z_1)\, b_{m,h-J-1}(z_2) & h\in\{J+2,\dots,2J\}. \end{cases}$$

Note that

$$\alpha_h+\beta_h=\begin{cases}2J & h\in\{0,1,J+2,\dots,2J\},\\ 2J+2 & h\in\{2,\dots,J+1\}.\end{cases}$$

Thus, the claim follows. ■

By Theorem 4.3, an equivalent way to check the reproduction property of the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2, $J\in\mathbb{N}$, is to compute the $\boldsymbol{\mu}$-th derivatives of the subdivision symbol $a_{M,J}$ with $\boldsymbol{\mu}\in\mathbb{N}_0^2$, $1\le|\boldsymbol{\mu}|\le 2J-1$, and to verify that they vanish at $\mathbf{1}$, namely

$$D^{\boldsymbol{\mu}}a_{M,J}(\mathbf{1})=0,\qquad \boldsymbol{\mu}\in\mathbb{N}_0^2,\qquad 1\le|\boldsymbol{\mu}|\le 2J-1.$$

Let us illustrate this fact on an example with $J=3$. The subdivision symbol $a_{M,3}$ of the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,3}}$ in Definition 6.2 is

$$\begin{aligned}a_{M,3}(\mathbf{z})&=\sum_{k=0}^{2}a_{2,3-k}(z_1)\,a_{m,k+1}(z_2)-\sum_{k=0}^{1}a_{2,3-k-1}(z_1)\,a_{m,k+1}(z_2)\\&=a_{2,3}(z_1)a_{m,1}(z_2)+a_{2,2}(z_1)a_{m,2}(z_2)+a_{2,1}(z_1)a_{m,3}(z_2)\\&\quad-a_{2,2}(z_1)a_{m,1}(z_2)-a_{2,1}(z_1)a_{m,2}(z_2),\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,\end{aligned}$$

where $a_{2,k}$, $a_{m,k}$ are the symbols of the univariate $(2k)$-point Dubuc-Deslauriers interpolatory subdivision schemes $S_{\mathbf{a}_{2,k}}$, $S_{\mathbf{a}_{m,k}}$ in Definition 6.1 with dilation 2 and m, respectively. In order to show that the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,3}}$ reproduces polynomials up to degree $2J-1=5$, we need to show that

$$D^{\boldsymbol{\mu}}a_{M,3}(\mathbf{1})=0,\qquad \boldsymbol{\mu}\in\mathbb{N}_0^2,\qquad 1\le|\boldsymbol{\mu}|\le 5.$$

Let $\boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathbb{N}_0^2$ such that $1\le|\boldsymbol{\mu}|\le 5$. Then, we have

$$\begin{aligned}D^{\boldsymbol{\mu}}a_{M,3}(\mathbf{1})&=\sum_{k=0}^{2}\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,3-k}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,k+1}(1)-\sum_{k=0}^{1}\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,3-k-1}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,k+1}(1)\\&=\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,3}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,1}(1)+\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,2}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,2}(1)+\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,1}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,3}(1)\\&\quad-\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,2}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,1}(1)-\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,1}(1)\frac{d^{\mu_2}}{dz_1^{\mu_2}}a_{m,2}(1).\end{aligned}$$

If $\mu_i\neq 0$, $i=1,2$, due to the reproduction properties of the univariate $(2k)$-point Dubuc-Deslauriers interpolatory subdivision schemes $S_{\mathbf{a}_{2,k}}$, $S_{\mathbf{a}_{m,k}}$, $k=1,2,3$, we have

$$\begin{aligned}&\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,3-k}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,k+1}(1)=0,\qquad k=0,1,2\\&\frac{d^{\mu_1}}{dz_1^{\mu_1}}a_{2,3-k-1}(1)\frac{d^{\mu_2}}{dz_2^{\mu_2}}a_{m,k+1}(1)=0,\qquad k=0,1.\end{aligned}$$

If $\mu_1 = 0$ and $\mu_2 \in \{1,\dots,5\}$, we get

$$\begin{aligned} D^{\boldsymbol{\mu}} a_{M,3}(\mathbf{1}) &= 2\sum_{k=0}^{2} \frac{d^{\mu_2}}{dz_2^{\mu_2}} a_{m,k+1}(1) - 2\sum_{k=0}^{1} \frac{d^{\mu_2}}{dz_2^{\mu_2}} a_{m,k+1}(1) \\ &= 2\frac{d^{\mu_2}}{dz_2^{\mu_2}} a_{m,3}(1) \\ &= 0, \end{aligned} \tag{6.5}$$

where the last equality holds true due to the fact the 6-point Dubuc-Deslauriers interpolatory subdivision schemes $S_{\mathbf{a}_{m,3}}$ reproduces polynomials up to degree 5. Similarly for $\mu_2 = 0$ and $\mu_1 \in \{1,\dots,5\}$. Thus, the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,3}}$ reproduces polynomials up to degree 5.

Therefore, the combination of the univariate Dubuc-Deslauriers interpolatory schemes in Definition 6.2 is chosen in a special way to guarantee the identity (6.5). This special structure is also reflected in (6.3).

6.1.4 Minimality property of $S_{\mathbf{a}_{M,J}}$

In [50], Ron and Jia constructed a family of interpolatory subdivision schemes with dilation matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and minimal support. The first aim of this subsection (see Proposition 6.3) is to generalize the result of Ron and Jia to our setting with dilation matrix M in (6.1). Then, in Theorem 6.4, using Proposition 6.3, we show that Definition 6.2 provides a closed formula for the symbols of the minimally supported interpolatory subdivision schemes. Note that the existence of such interpolatory schemes follows from the unique solvability of the corresponding interpolation problem. We are interested in providing a closed formula for their masks.

Proposition 6.3. *Let $J \in \mathbb{N}$. There exists a unique interpolatory subdivision scheme with dilation matrix M in* (6.1) *whose mask $\mathbf{c}_{M,J}$ satisfies*

(i) *$\mathbf{c}_{M,J}$ has support*

$$\{(\alpha_1,\alpha_2) \in \mathbb{Z}^2 : m|\alpha_1| + 2|\alpha_2| \le 2mJ - 2 + m\} \subset \{1-2J,\dots,2J-1\} \times \{1-mJ,\dots,mJ-1\},$$

(ii) *$\mathbf{c}_{M,J}$ is symmetric,*

(iii) *$S_{\mathbf{c}_{M,J}}$ reproduces polynomials up to degree $2J-1$.*

Before proving Proposition 6.3, we present a constructive example in order to clarify the technical steps of the proof.

***Example* 6.3.** Let $J = 3$ and $M = \mathrm{diag}(2,3)$ (i.e. $m = 3$). We construct a mask $\mathbf{c}_{M,3}$ such that

$$c_{M,3}(\alpha_1,\alpha_2) = 0, \qquad \forall (\alpha_1,\alpha_2) \notin \{(\alpha_1,\alpha_2) \in \mathbb{Z}^2 : 3|\alpha_1| + 2|\alpha_2| \le 19\} \subset \{-5,\dots,5\} \times \{-8,\dots,8\},$$

and the associated subdivision scheme $S_{\mathbf{c}_{M,3}}$ with dilation M is interpolatory, symmetric and reproduces polynomials up to degree $2J-1=5$. Notice that this size of the support is dictated by the desired polynomial reproduction property of the scheme we want to construct.

Step 1. We fix the support of the mask $\mathbf{c}_{M,3}$ (unknown entries are denoted by *)

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Step 2. We impose the interpolatory conditions $\mathrm{c}_{M,3}(0,0)=1$, $\mathrm{c}_{M,3}(2\alpha_1,3\alpha_2)=0$, $\forall \boldsymbol{\alpha} = (\alpha_1,\alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & * & * & 0 & * & * & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & * & * & 0 & * & * & 1 & * & * & 0 & * & * & 0 & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & 0 & * & * & 0 & * & * & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Step 3. We define the remaining coefficients of $\mathbf{c}_{M,3}$ symmetrically and such that they guarantee the property of polynomial reproduction of polynomials up to degree $2J-1=5$. The latter condition leads to invertible systems of equations (one interpolation problem for each submask). They yield

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} & \frac{1}{128} & \frac{3}{256} & \frac{1}{128} & \frac{1}{256} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{324} & \frac{5}{1296} & 0 & -\frac{241}{6912} & -\frac{241}{3456} & -\frac{25}{256} & -\frac{241}{3456} & -\frac{241}{6912} & 0 & \frac{5}{1296} & \frac{1}{324} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7}{1458} & \frac{4}{729} & 0 & -\frac{121}{2916} & -\frac{605}{11664} & 0 & \frac{20809}{93312} & \frac{20809}{46656} & \frac{75}{128} & \frac{20809}{46656} & \frac{20809}{93312} & 0 & -\frac{605}{11664} & -\frac{121}{2916} & 0 & \frac{4}{729} & \frac{7}{1458} \\
\frac{7}{729} & \frac{8}{729} & 0 & -\frac{56}{729} & -\frac{70}{729} & 0 & \frac{280}{729} & \frac{560}{729} & 1 & \frac{560}{729} & \frac{280}{729} & 0 & -\frac{70}{729} & -\frac{56}{729} & 0 & \frac{8}{729} & \frac{7}{729} \\
\frac{7}{1458} & \frac{4}{729} & 0 & -\frac{121}{2916} & -\frac{605}{11664} & 0 & \frac{20809}{93312} & \frac{20809}{46656} & \frac{75}{128} & \frac{20809}{46656} & \frac{20809}{93312} & 0 & -\frac{605}{11664} & -\frac{121}{2916} & 0 & \frac{4}{729} & \frac{7}{1458} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{324} & \frac{5}{1296} & 0 & -\frac{241}{6912} & -\frac{241}{3456} & -\frac{25}{256} & -\frac{241}{3456} & -\frac{241}{6912} & 0 & \frac{5}{1296} & \frac{1}{324} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} & \frac{1}{128} & \frac{3}{256} & \frac{1}{128} & \frac{1}{256} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Notice that the main column and row of $\mathbf{c}_{M,3}$ are the univariate binary $\mathbf{a}_{2,3}$ and ternary $\mathbf{a}_{3,3}$ 6-point Dubuc-Deslauriers masks defined in Examples 6.1 and 6.2, respectively. ◀

Proof. [of Proposition 6.3] Recall by (4.12) that

$$\Gamma = \{(k,j) \in \mathbb{N}_0^2 : k \in \{0,1\},\ j \in \{0,\dots,m-1\}\}$$

is a complete set of representatives of the distinct cosets of $\mathbb{Z}^2/M\mathbb{Z}^2$. Every interpolatory mask $\mathbf{a}$ reproduces polynomials up to degree $2J-1$ if and only if it satisfies the sum rules of order $2J$, i.e. by (4.17)

$$\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^2} \mathrm{a}\big(k+2\alpha_1, j+m\alpha_2\big)\big(k+2\alpha_1\big)^{\mu_1}\big(j+m\alpha_2\big)^{\mu_2} = \delta_{\boldsymbol{\mu},0},$$
$$(k,j)\in\Gamma\setminus\{(0,0)\},\qquad \boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathbb{N}_0^2 : \mu_1+\mu_2\le 2J-1,\qquad \boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\mathbb{Z}^2. \tag{6.6}$$

Notice that, for $(k,j)=(0,0)$,

$$\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^2} \mathrm{a}\big(2\alpha_1, m\alpha_2\big)\big(2\alpha_1\big)^{\mu_1}\big(m\alpha_2\big)^{\mu_2} = \delta_{\boldsymbol{\mu},0},\qquad \boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathbb{N}_0^2 : \mu_1+\mu_2\le 2J-1, \tag{6.7}$$

is automatically satisfied if we assume the interpolatory property (4.9) of $\mathbf{a}$.
The construction of the mask $\mathbf{c}_{M,J}$ satisfying *(i)* - *(iii)* is split in 3 Steps.

Step 1 (support size). We set $\mathrm{c}_{M,J}(\alpha_1,\alpha_2)=0$, $\forall\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\mathbb{Z}^2$ such that

$$|\alpha_1|>2J-1,\qquad |\alpha_2|>mJ-1,\qquad m|\alpha_1|+2|\alpha_2|>2mJ-2+m.$$

Thus, condition *(i)* is satisfied.

Step 2 (interpolation). We impose the interpolatory conditions

$$\mathrm{c}_{M,J}(0,0)=1,\qquad \mathrm{c}_{M,J}(2\alpha_1,m\alpha_2)=0,\qquad \forall\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}.$$

Thus, (6.7) is satisfied.

Step 3 (symmetry and polynomial reproduction). The system of equations in (6.6) naturally splits into $\#\Gamma - 1 = 2m-1$ separate linear systems of equations, one for each $(k,j) \in \Gamma' = \Gamma \setminus \{(0,0)\}$. Imposing the symmetry of $c_{M,J}$, the number of unknowns in the systems of equations (6.6) for $(k,j) \in \Gamma' \setminus \{(1,0)\}$ can be reduced by a factor of 2. The corresponding $J(J+1) \times J(J+1)$ system matrices are invertible, which we prove in *Step 3.a.* We treat the case $(k,j) = (1,0)$ separately, due to the special symmetry of the corresponding submask. This case is analyzed in *Step 3.b* and the corresponding $\dfrac{J(J+1)}{2} \times \dfrac{J(J+1)}{2}$ system matrix is also invertible.

Step 3.a. Let $(k,j) \in \Gamma' \setminus \{(1,0)\}$.

Symmetry in α_1: we only need to determine the coefficients $c_{M,J}(k+2\alpha_1, j+m\alpha_2)$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0 \times \mathbb{Z}$ and such that *(ii)* is satisfied, i.e.

$$0 \le k+2\alpha_1 \le 2J-1, \quad 0 \le |j+m\alpha_2| \le mJ-1, \quad 0 \le m(k+2\alpha_1)+2|j+m\alpha_2| \le 2mJ-2+m. \tag{6.8}$$

We call $\mathcal{A}$ the set of the indices in (6.8). We first determine the geometric structure and the cardinality of $\mathcal{A}$. First, we consider the inequality $0 \le k+2\alpha_1 \le 2J-1$ in (6.8). Since $k \in \{0,1\}$, we have

$$0 \le k+2\alpha_1 \le 2J-1 \iff 0 \le \alpha_1 \le J - \left\lceil \frac{1+k}{2} \right\rceil \iff 0 \le \alpha_1 \le J-1.$$

Next, we focus our attention on the inequality $0 \le |j+m\alpha_2| \le mJ-1$ in (6.8). We observe that $j+m\alpha_2 \ge 0 \iff \alpha_2 \ge 0$, thus for every $j \in \{1,\dots,m-1\}$ we have

- for $\alpha_2 \ge 0$:

$$0 \le j+m\alpha_2 \le mJ-1 \iff 0 \le \alpha_2 \le J - \left\lceil \frac{1+j}{m} \right\rceil \iff 0 \le \alpha_2 \le J-1,$$

- for $\alpha_2 < 0$:

$$0 < -j-m\alpha_2 \le mJ-1 \iff 0 < -\alpha_2 \le J + \left\lfloor \frac{j-1}{m} \right\rfloor \iff 1 \le -\alpha_2 \le J.$$

Finally, we analyze the last inequality $0 \le m(k+2\alpha_1)+2|j+m\alpha_2| \le 2mJ-2+m$ in (6.8). Let $\alpha_1 \in \{0,\dots,J-1\}$. Then, we have

- for $\alpha_2 \ge 0$:

$$\begin{aligned} 0 \le m(k+2\alpha_1)+2(j+m\alpha_2) \le 2mJ-2+m &\iff 0 \le \alpha_2 \le J-\alpha_1 - \left\lceil \frac{1+j}{m} \right\rceil + \left\lfloor \frac{1-k}{2} \right\rfloor, \\ &\iff 0 \le \alpha_2 \le J-\alpha_1-1, \end{aligned}$$

- for $\alpha_2 < 0$:

$$\begin{aligned} 0 < m(k+2\alpha_1)-2(j+m\alpha_2) \le 2mJ-2+m &\iff 0 < -\alpha_2 \le J-\alpha_1 + \left\lfloor \frac{j-1}{m} \right\rfloor + \left\lfloor \frac{1-k}{2} \right\rfloor, \\ &\iff 1 \le -\alpha_2 \le J-\alpha_1. \end{aligned}$$

Combining the above observations, we get

$$\mathcal{A} = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0 \times \mathbb{Z} : 0 \le \alpha_1 \le J-1,\ \alpha_1 - J \le \alpha_2 \le J - \alpha_1 - 1\}.$$

Thus, the cardinality of the set $\mathcal{A}$ is

$$\#\mathcal{A} = \sum_{\alpha_1=0}^{J-1} 2(J-\alpha_1) = J(J+1),$$

i.e. the number of unknowns in (6.6) for fixed $(k,j) \in \Gamma' \setminus \{(1,0)\}$ is $J(J+1)$. Moreover, due to the symmetry in α_1, (6.6) is automatically satisfied for odd μ_1. Therefore, we solve

$$\begin{cases} \sum\limits_{\substack{\boldsymbol{\alpha}\in\mathcal{A},\\ \alpha_1=0}} \mathrm{a}\big(0, j+m\alpha_2\big)\delta_{\mu_1,0}\big(j+m\alpha_2\big)^{\mu_2} + 2\sum\limits_{\substack{\boldsymbol{\alpha}\in\mathcal{A},\\ \alpha_1\neq 0}} \mathrm{a}\big(2\alpha_1, j+m\alpha_2\big)\big(2\alpha_1\big)^{2\mu_1}\big(j+m\alpha_2\big)^{\mu_2} = \delta_{\boldsymbol{\mu},\mathbf{0}} & k=0 \\ \sum\limits_{\boldsymbol{\alpha}\in\mathcal{A}} \mathrm{a}\big(k+2\alpha_1, j+m\alpha_2\big)\big(k+2\alpha_1\big)^{2\mu_1}\big(j+m\alpha_2\big)^{\mu_2} = \tfrac{1}{2}\delta_{\boldsymbol{\mu},\mathbf{0}} & k\neq 0, \end{cases} \tag{6.9}$$

with $\boldsymbol{\mu} \in \mathcal{M} = \{(\mu_1,\mu_2) \in \mathbb{N}_0^2 : 0 \le 2\mu_1 + \mu_2 \le 2J-1\}$. We notice that $\#\mathcal{M} = \#\mathcal{A} = J(J+1)$, i.e. the corresponding system matrix is indeed a square matrix.

Symmetry in α_2: it allows us to reduce the total number of linear systems in (6.9). Note that the systems in (6.9) for $j \in \{1,\dots,\frac{m-1}{2}\}$ and $m-j \in \{\frac{m+1}{2},\dots,m-1\}$ are equivalent, indeed

$$\begin{aligned} &\sum_{\boldsymbol{\alpha}\in\mathcal{A}} \mathrm{a}\big(k+2\alpha_1, m-j+m\alpha_2\big)\big(k+2\alpha_1\big)^{2\mu_1}\big(m-j+m\alpha_2\big)^{\mu_2} \\ &\quad= \sum_{\boldsymbol{\alpha}\in\mathcal{A}} \mathrm{a}\big(k+2\alpha_1, -j+m(\alpha_2+1)\big)\big(k+2\alpha_1\big)^{2\mu_1}\big(-j+m(\alpha_2+1)\big)^{\mu_2} \\ &\quad= \sum_{\boldsymbol{\beta}\in\mathcal{B}} \mathrm{a}\big(k+2\beta_1, -(j+m\beta_2)\big)\big(k+2\beta_1\big)^{2\mu_1}\big(-(j+m\beta_2)\big)^{\mu_2}, \qquad (\beta_1,\beta_2) = (\alpha_1, -\alpha_2-1), \\ &\quad= (-1)^{\mu_2}\sum_{\boldsymbol{\beta}\in\mathcal{B}} \mathrm{a}\big(k+2\beta_1, j+m\beta_2\big)\big(k+2\beta_1\big)^{2\mu_1}\big(j+m\beta_2\big)^{\mu_2}, \end{aligned}$$

where

$$\mathcal{B} = \big\{\boldsymbol{\beta} = (\beta_1,\beta_2) \in \mathbb{N}_0 \times \mathbb{Z} : 0 \le \beta_1 \le J-1,\ \beta_1 - J \le \beta_2 \le J-\beta_1-1\big\}.$$

Thus, we only need to consider the case $j \in \{1,\dots,\frac{m-1}{2}\}$. The corresponding square matrix

$$\Big((k+2\alpha_1)^{2\mu_1}(j+m\alpha_2)^{\mu_2}\Big)_{(\alpha_1,\alpha_2)\in\mathcal{A},\,(\mu_1,\mu_2)\in\mathcal{M}}$$

is non-singular [65, Theorem 3.3]. Therefore, for any $j \in \{1,\dots,\frac{m-1}{2}\}$, the linear system of equations (6.9) is uniquely solvable and its solution is

$$\Big(\mathrm{c}_{M,J}(k+2\alpha_1, j+m\alpha_2)\Big)_{(\alpha_1,\alpha_2)\in\mathcal{A}}.$$

Step 3.b. Let $(k,j)=(1,0)$. Due to the symmetry in α_1 and α_2, (6.6) reduces to the following system of equations

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}',\\ \alpha_2=0}} \mathrm{a}\big(1+2\alpha_1,0\big)\big(1+2\alpha_1\big)^{2\mu_1}\delta_{\mu_2,0}+2\sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A}',\\ \alpha_2\neq 0}} \mathrm{a}\big(1+2\alpha_1,m\alpha_2\big)\big(1+2\alpha_1\big)^{2\mu_1}\big(m\alpha_2\big)^{2\mu_2}=\frac{1}{2}\delta_{\boldsymbol{\mu},\mathbf{0}},\quad \boldsymbol{\mu}\in\mathcal{M}', \tag{6.10}$$

where

$$\mathcal{A}'=\{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2 : 0\le\alpha_1\le J-1,\, 0\le\alpha_2\le J-\alpha_1-1\},$$
$$\mathcal{M}'=\{(\mu_1,\mu_2)\in\mathbb{N}_0^2 : 0\le 2\mu_1+2\mu_2\le 2J-1\}.$$

We notice that $\#\mathcal{A}'=\#\mathcal{M}'=J(J+1)/2$. By [50, Lemma 4.1], the square matrix

$$\Big((1+2\alpha_1)^{2\mu_1}(m\alpha_2)^{2\mu_2}\Big)_{(\alpha_1,\alpha_2)\in\mathcal{A}',\,(\mu_1,\mu_2)\in\mathcal{M}'}$$

is non-singular. Therefore, the linear system of equations (6.10) is uniquely solvable and its solution is

$$\Big(\mathrm{c}_{M,J}(1+2\alpha_1,m\alpha_2)\Big)_{(\alpha_1,\alpha_2)\in\mathcal{A}'}.$$

■

We notice another special property of the masks $\mathrm{c}_{M,J}$ in Proposition 6.3.

Remark 6.1. Let $J\in\mathbb{N}$. Since the mask $\mathbf{a}_{2,J}$ of the univariate binary $(2J)$-point Dubuc-Deslauriers interpolatory subdivision scheme $S_{\mathbf{a}_{2,J}}$ in Definition 6.1 satisfies the sum rules of order $2J$, the solution of (6.10) is given by

$$\mathrm{c}_{M,J}(1+2\alpha_1,m\alpha_2)=\begin{cases}\mathrm{a}_{2,J}(1+2\alpha_1), & \alpha_2=0,\\ 0, & \alpha_2\neq 0,\end{cases}\qquad \boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\mathcal{A}'.$$

Analogously, the solution of (6.9) for $(0,j)\in\Gamma'$, $j\in\{1,\dots,m-1\}$, is given by

$$\mathrm{c}_{M,J}(2\alpha_1,j+m\alpha_2)=\begin{cases}\mathrm{a}_{m,J}(j+m\alpha_2), & \alpha_1=0,\\ 0, & \alpha_1\neq 0,\end{cases}\qquad \boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\mathcal{A},$$

where $\mathbf{a}_{m,J}$ is the mask of the univariate $(2J)$-point Dubuc-Deslauriers interpolatory subdivision scheme $S_{\mathbf{a}_{m,J}}$ with dilation m in Definition 6.1.

We now show that the masks in Definition 6.2 and the ones obtained in Proposition 6.3 actually coincide.

Theorem 6.4. *Let $J\in\mathbb{N}$ and M in* (6.1)*. The mask $\mathbf{a}_{M,J}$ of the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 satisfies*

(i) *$\mathbf{a}_{M,J}$ has support*

$$\{(\alpha_1,\alpha_2)\in\mathbb{Z}^2 : m|\alpha_1|+2|\alpha_2|\le 2mJ-2+m\}\subset\{1-2J,\dots,2J-1\}\times\{1-mJ,\dots,mJ-1\},$$

(ii) $\mathbf{a}_{M,J}$ *is symmetric,*

(iii) $S_{\mathbf{a}_{M,J}}$ *reproduces polynomials up to degree* $2J-1$.

Proof. *Step 1.* Condition *(ii)* follows directly from Definition 6.2 and from the symmetry of the univariate masks $\mathbf{a}_{2,h}$, $\mathbf{a}_{m,h}$, $h=1,\dots,J$, in Definition 6.1.
Step 2. Condition *(iii)* follows directly from Proposition 6.2.
Step 3. We focus our attention on condition *(i)*. The univariate masks $\mathbf{a}_{2,h}$, $\mathbf{a}_{m,h}$, $h\in\{1,\dots,J\}$, in Definition 6.1 have supports $\{1-2h,\dots,2h-1\}$ and $\{1-mh,\dots,mh-1\}$, respectively. Thus, for $k=0,\dots,J-1$, the masks associated to the symbol

$$a_{2,J-k}(z_1)\,a_{m,k+1}(z_2),\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,$$

in the first sum in Definition 6.2 have support

$$S_k=\{1-2(J-k),\dots,2(J-k)-1\}\times\{1-m(k+1),\dots,m(k+1)-1\}.$$

Moreover, for $k=0,\dots,J-1$, we have

$$m|\alpha_1|+2|\alpha_2|\le m\big(2(J-k)-1\big)+2\big(m(k+1)-1\big)=2mJ-2+m,\qquad(\alpha_1,\alpha_2)\in S_k.$$

Thus, the mask associated to the symbol

$$\sum_{k=0}^{J-1}a_{2,J-k}(z_1)\,a_{m,k+1}(z_2),\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,$$

in Definition 6.2 has support

$$\mathcal{A}=\{(\alpha_1,\alpha_2)\in\mathbb{Z}^2\,:\,m|\alpha_1|+2|\alpha_2|\le 2mJ-2+m\}\subset\{1-2J,\dots,2J-1\}\times\{1-mJ,\dots,mJ-1\}.$$

Using the same argument, the support of the mask associated to the symbol

$$\sum_{k=0}^{J-2}a_{2,J-k-1}(z_1)\,a_{m,k+1}(z_2),\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2,$$

in Definition 6.2 is contained in $\mathcal{A}$, so that the support of the mask $\mathbf{a}_{M,J}$ satisfies *(i)*. ■

6.1.5 Convergence of certain $S_{\mathbf{a}_{M,J}}$

In this section, we only analyze convergence of the schemes used in section 6.4. In [16], Charina and Protasov presented a detailed regularity analysis of d-variate anisotropic subdivision schemes. Especially, their results allow us to use the algorithm in [47] for the exact computation of the Hölder regularity of an anisotropic subdivision scheme.

Definition 6.3. A convergent subdivision scheme $S_{\mathbf{p}}$ with dilation M and mask $\mathbf{p}$ has Hölder regularity $\alpha_\phi\in(0,1]$ if its basic limit function ϕ has Hölder exponent α_ϕ, namely

$$\alpha_\phi=\sup\left\{\alpha\in(0,1]\,:\,\|\phi(\cdot+\mathbf{h})-\phi\|_{C(\mathbb{R}^d)}\le C\|\mathbf{h}\|^{\alpha},\,\mathbf{h}\in\mathbb{R}^d\right\}.$$

The main ingredient of the regularity analysis in [16] is the so-called *joint spectral radius* [62], which is a generalization of the classical notion of spectral radius of one square matrix to a finite (or compact) set of square matrices.

Definition 6.4. Let $\mathcal{V} = \{V_1, \dots, V_d\} \subset \mathbb{R}^{N\times N}$, $N \in \mathbb{N}$, be a finite set of square matrices. The joint spectral radius of $\mathcal{V}$ is defined by

$$\rho(\mathcal{V}) = \lim_{k\to\infty} \max\left\{ \|V_{j_1}\cdots V_{j_k}\|^{1/k} : V_{j_i} \in \mathcal{V},\, i = 1,\dots,k \right\}.$$

The limit in Definition 6.4 always exists and does not depend on the choice of the matrix norm ([62]). For practical interest (see Section 6.4), we check the continuity and compute the Hölder regularity of some elements of the family $S_{\mathbf{a}_{M,J}}$ with $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m = 3, 5$. To do so, we first define the set $\mathcal{V}$. The size of the elements of $\mathcal{V}$ depends on the support of the basic limit function and the cardinality of the set Ω in (6.12).

Let $J \in \mathbb{N}$ and $\phi_{\mathbf{a}_{M,J}}$ be the basic limit function of the anisotropic interpolatory subdvision schemes $S_{\mathbf{a}_{M,J}}$ in Definition 6.2. By [10, Proposition 2.2 and (2.7)-(2.8)], we have

$$\operatorname{supp} \phi_{\mathbf{a}_{M,J}} \subseteq K = \left\{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \sum_{j=1}^{\infty} M^{-j} \boldsymbol{\alpha}_{k_j},\, \boldsymbol{\alpha}_{k_j} \in \operatorname{supp} \mathbf{a}_{M,J} \right\} \subset \mathbb{R}^2. \tag{6.11}$$

Since $\phi_{\mathbf{a}_{M,J}}$ is compactly supported, K is a compact set. Thus, due to [10, Lemma 2.3], there exists a minimal set $\Omega \subset \mathbb{Z}^2$ such that

$$K \subset \Omega + [0,1]^2 = \bigcup_{\omega\in\Omega} (\omega + [0,1]^2). \tag{6.12}$$

The minimality of Ω reads as follows: if there exists $\tilde{\Omega} \subset \mathbb{Z}^2$ such that $K \subset \tilde{\Omega} + [0,1]^2$, then $\tilde{\Omega} \supseteq \Omega$. We refer to [10] for more details.

Let $N = \#\Omega$. By (4.12), $\Gamma = \{(k, j) \in \mathbb{N}_0^2 : k \in \{0,1\},\, j \in \{0,\dots,m-1\}\}$ is a complete set of representatives of the distinct cosets of $\mathbb{Z}^2/M\mathbb{Z}^2$. Notice that $\#\Gamma = 2m$. For every $\boldsymbol{\gamma} \in \Gamma$, we define the transition matrix

$$T_{\boldsymbol{\gamma}} = \Big(\mathrm{a}_{M,J}(M\boldsymbol{\alpha} - \boldsymbol{\beta} + \boldsymbol{\gamma})\Big)_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\Omega}.$$

We denote $\mathcal{T} = \{T_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \Gamma\}$ the set of all the transition matrices. Notice that $\#\mathcal{T} = 2m$. For every $\boldsymbol{\gamma} \in \Gamma$, the rows and columns of $T_{\boldsymbol{\gamma}}$ are enumerated by the elements from the set Ω, thus $T_{\boldsymbol{\gamma}} \in \mathbb{R}^{N\times N}$. By construction, the entries of any column of $T_{\boldsymbol{\gamma}}$ sum up to 1, thus $T_{\boldsymbol{\gamma}}$ has eigenvalue 1 (i.e. there exist $\mathbf{v}_0 \neq \mathbf{0} \in \mathbb{R}^N$ such that $T_{\boldsymbol{\gamma}}\mathbf{v}_0 = \mathbf{v}_0$). This property of $\mathcal{T}$ implies ([10, 53]) the existence of certain invariant subspaces of $\mathcal{T}$ crucial for the definition of the set $\mathcal{V}$. To determine these invariant subspaces, we define the vector-valued function

$$\begin{array}{rccl} v\colon & [0,1]^2 & \longrightarrow & \mathbb{R}^N, \\ & \mathbf{x} & \longmapsto & \Big(\phi_{\mathbf{a}_{M,J}}(\mathbf{x}+\omega)\Big)_{\omega\in\Omega}. \end{array} \tag{6.13}$$

Now we are able to define the following subspaces of $\mathbb{R}^N$

$$\begin{aligned} U &= \text{ span } \left\{ v(\mathbf{y}) - v(\mathbf{x}) : \mathbf{x},\mathbf{y} \in [0,1]^2 \right\}, \\ U_1 &= \text{ span } \left\{ v(\mathbf{y}) - v(\mathbf{x}) : \mathbf{x},\mathbf{y} \in [0,1]^2,\, \mathbf{y}-\mathbf{x} = (\alpha,0),\, \alpha \in \mathbb{R} \right\}, \\ U_2 &= \text{ span } \left\{ v(\mathbf{y}) - v(\mathbf{x}) : \mathbf{x},\mathbf{y} \in [0,1]^2,\, \mathbf{y}-\mathbf{x} = (0,\beta),\, \beta \in \mathbb{R} \right\}, \end{aligned} \tag{6.14}$$

invariant under $\mathcal{T}$. Notice that U_1, U_2 contain differences in the directions of the eigenvectors of M. Finally, we define

$$\mathcal{V} = \mathcal{T}|_U = \left\{ T_{\boldsymbol{\gamma}} \,\middle|\, _U : \boldsymbol{\gamma} \in \Gamma \right\}, \qquad \mathcal{V}_1 = \mathcal{V}|_{U_1} \qquad \text{and} \qquad \mathcal{V}_2 = \mathcal{V}|_{U_2}.$$

The following statement is a direct consequence of Theorem 1 in [16].

Theorem 6.5. *Let $J \in \mathbb{N}$. The basic limit function $\phi_{\mathbf{a}_{M,J}}$ of the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 belongs to $C(\mathbb{R}^2)$ if and only if $\rho(\mathcal{V}) < 1$. In this case, ϕ has the Hölder exponent*

$$\alpha_{\phi_{\mathbf{a}_{M,J}}} = \min\left\{ \log_{1/2} \rho(\mathcal{V}_1), \log_{1/m} \rho(\mathcal{V}_2) \right\}.$$

In order to properly end this section, we would like to answer a few questions which naturally arise after reading of the above analysis:

Q1. How to determine the spaces U, U_1 and U_2 ($\phi_{\mathbf{a}_{M,J}}$ is not known analytically)?

Q2. How to determine the sets $\mathcal{V}, \mathcal{V}_1$ and $\mathcal{V}_2$?

The questions *Q1.* and *Q2.* will be answered in the following Example.

***Example* 6.4.** Let $J = 1$ and $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The anisotropic interpolatory subdvision scheme $S_{\mathbf{a}_{M,1}}$ in Definition 6.2 has the mask

$$\mathbf{a}_{M,1} = \frac{1}{6} \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 6 & 4 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{pmatrix}.$$

Since the support of the basic limit function is a subset of $[-1,1]^2$ (see Figure 6.1 and (6.11)), we determine $\Omega = \{-1,0\}^2$, $N = \#\Omega = 4$. For

$$\Gamma = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \qquad \#\Gamma = 6,$$

the corresponding transition matrices are

$$T_{(0,0)} = \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 3 & 4 & 6 \end{pmatrix}, \qquad T_{(0,1)} = \frac{1}{6} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}, \qquad T_{(0,2)} = \frac{1}{6} \begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 6 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix},$$

$$T_{(1,0)} = \frac{1}{6} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 6 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 \end{pmatrix}, \qquad T_{(1,1)} = \frac{1}{6} \begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad T_{(1,2)} = \frac{1}{6} \begin{pmatrix} 6 & 4 & 3 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

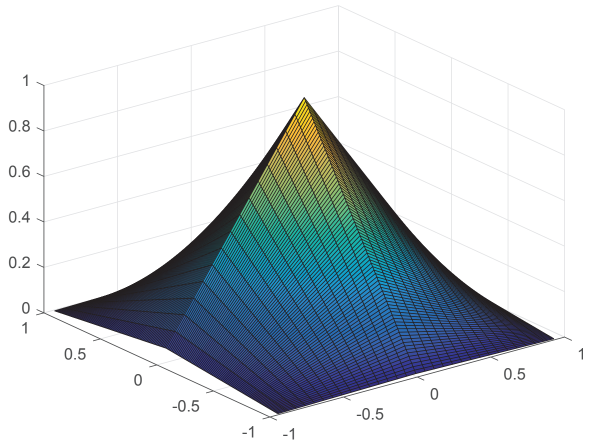


Figure 6.1: *Basic limit function of the anisotropic interpolatory subdivision schemes* $S_{\mathbf{a}_{M,1}}$, $M = \mathrm{diag}(2,3)$.

Let us compute the spaces U, U_1 and U_2.
Space U*:* the transition matrix $T_{(0,0)}$ has eigenvalue 1 with respect to the eigenvector $\mathbf{v}_0 = (0,0,0,1)^T$. To determine U, we proceed as follows using Algorithm 1 from [16].

Step 1. We define $W^{(1)} = \operatorname{span}\left\{T_{\boldsymbol{\gamma}}\mathbf{v}_0 - \mathbf{v}_0 : \boldsymbol{\gamma} \in \Gamma \setminus \{\mathbf{0}\}\right\}$.

Step 2. We compute recursively

$$W^{(k+1)} = W^{(k)} \cup \operatorname{span}\left\{T_{\boldsymbol{\gamma}}\mathbf{w}^{(k)} : \mathbf{w}^{(k)} \in W^{(k)}, \boldsymbol{\gamma} \in \Gamma\right\}, \quad 1 \le k < N-1$$

until $\dim(W^{(k)}) < \dim(W^{(k+1)})$.

Step 3. We define $U = W^{(k)}$.

Notice that the constraint $1 \le k < N-1$ makes sense since

$$U \subseteq W = \{\mathbf{w} = (w_1, \dots, w_N) \in \mathbb{R}^N : \sum_{i=1}^{N} w_i = 0\} = \{\mathbf{w} \in \mathbb{R}^N : \mathbf{w} \perp \mathbf{1}\}, \qquad \dim W = N-1.$$

In our case,

$$\begin{aligned} U &= \operatorname{span}\left\{T_{(0,1)}\mathbf{v}_0 - \mathbf{v}_0, T_{(1,0)}\mathbf{v}_0 - \mathbf{v}_0, T_{(1,1)}\mathbf{v}_0 - \mathbf{v}_0\right\} \\ &= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}, \qquad \dim U = 3. \end{aligned}$$

Space U_1*:* In order to determine U_1, we use the algorithm above with a different starting vector. By (6.14), we compute $\mathbf{v}_0 = v((1,0)) - v((0,0))$, $\|\mathbf{v}_0\|_1 = 1$. By definition (6.13) and due to

$\phi_{\mathbf{a}_{M,1}}(\boldsymbol{\alpha}) = \delta_{\boldsymbol{\alpha},\mathbf{0}}$, $\forall\, \boldsymbol{\alpha} \in \mathbb{Z}^2$, we have

$$v((1,0)) = \Big(\phi_{\mathbf{a}_{M,1}}\big((1,0)+\boldsymbol{\omega}\big)\Big)_{\boldsymbol{\omega}\in\Omega} = \begin{pmatrix} \phi_{\mathbf{a}_{M,1}}\big((1,0)+(-1,-1)\big) \\ \phi_{\mathbf{a}_{M,1}}\big((1,0)+(-1,0)\big) \\ \phi_{\mathbf{a}_{M,1}}\big((1,0)+(0,-1)\big) \\ \phi_{\mathbf{a}_{M,1}}\big((1,0)+(0,0)\big) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4,$$

$$v((0,0)) = \Big(\phi_{\mathbf{a}_{M,1}}(\boldsymbol{\omega})\Big)_{\boldsymbol{\omega}\in\Omega} = \begin{pmatrix} \phi_{\mathbf{a}_{M,1}}\big((-1,-1)\big) \\ \phi_{\mathbf{a}_{M,1}}\big((-1,0)\big) \\ \phi_{\mathbf{a}_{M,1}}\big((0,-1)\big) \\ \phi_{\mathbf{a}_{M,1}}\big((0,0)\big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4.$$

Similarly to the computation of U with $\mathbf{v}_0 = (0,1/2,0,-1/2)^T$, we obtain

$$U_1 = \operatorname{span}\left\{ T_{(0,1)}\mathbf{v}_0 - \mathbf{v}_0,\, T_{(0,2)}\mathbf{v}_0 - \mathbf{v}_0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}, \qquad \dim U_1 = 2.$$

Space U_2: In order to determine U_2, by (6.14), we compute $\mathbf{v}_0 = v((0,1)) - v((0,0))$, $\|\mathbf{v}_0\|_1 = 1$. Following the procedure described for the construction of U_1, we have $\mathbf{v}_0 = (0,0,1/2,-1/2)^T$ and

$$U_2 = \operatorname{span}\left\{ T_{(0,1)}\mathbf{v}_0 - \mathbf{v}_0,\, T_{(1,0)}\mathbf{v}_0 - \mathbf{v}_0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}, \qquad \dim U_2 = 2.$$

Note that $U = U_1 \bigcup U_2$.

Let us compute the set $\mathcal{V}$.

Step 1. We extend $U \subset \mathbb{R}^4$, $\dim U = 3$, to $\mathbb{R}^4$ choosing

$$\mathbb{R}^4 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Step 2. We define the matrix S

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

By construction, the matrix S is invertible and, thus, we can compute the matrices

$$B_{\boldsymbol{\gamma}} = S^{-1} T_{\boldsymbol{\gamma}} S \in \mathbb{R}^{4\times 4}, \qquad \boldsymbol{\gamma} \in \Gamma.$$

The matrices $B_{\boldsymbol{\gamma}}$ have a block structure. More precisely, the square upper-left block of $B_{\boldsymbol{\gamma}}$ of size $\dim U \times \dim U$ is the restriction of $T_{\boldsymbol{\gamma}}$ to U, namely

$$B_{(0,0)} = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & -5 \\ 2 & 3 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}, \qquad B_{(0,1)} = \frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & -4 \\ 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}, \qquad B_{(0,2)} = \frac{1}{6}\begin{pmatrix} 3 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & 2 & 3 \\ 0 & 0 & 0 & 6 \end{pmatrix},$$

$$B_{(1,0)} = \frac{1}{6}\begin{pmatrix} 2 & 0 & 1 & -4 \\ 1 & 3 & -1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}, \quad B_{(1,1)} = \frac{1}{6}\begin{pmatrix} 3 & 1 & 1 & -2 \\ 0 & 2 & -1 & 2 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}, \quad B_{(1,2)} = \frac{1}{6}\begin{pmatrix} 4 & 2 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Finally, $\mathcal{V}$ is the set of the restrictions of $T_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \Gamma$, to U. In our case,

$$\mathcal{V} = \left\{ \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}, \right.$$
$$\left. \frac{1}{6}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 4 & 2 & 1 \\ -1 & 1 & -1 \\ -2 & -2 & 1 \end{pmatrix} \right\}.$$

Now, let us focus on the construction of $\mathcal{V}_1$, $\mathcal{V}_2$. Since U_1, U_2 are invariant subspaces under $\mathcal{V}$, we can directly compute the restrictions $\mathcal{T}|_{U_1}$ and $\mathcal{T}|_{U_2}$, respectively. Thus, we can apply the same algorithm used to determine $\mathcal{V}$ and we get

$$\mathcal{V}_1 = \left\{ \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{V}_2 = \left\{ \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Notice that $\#\mathcal{V}_1, \#\mathcal{V}_2 < 6$ since $U_1 \cap U_2 \neq \emptyset$. ◀

In Table 6.1, we check the continuity and compute the Hölder regularity of $\mathcal{S}_{\mathbf{a}_{M,J}}$ with $M \in \{\mathrm{diag}(2,3), \mathrm{diag}(2,5)\}$ and $J \in \{1,2\}$ following the procedure presented in Example 6.4. To compute the joint spectral radius, we use the software `jsr-pathcomplete` from the joint spectral radius matlab toolbox [57] based on [1].

6.2 Anisotropic approximating subdivision schemes

In this section, we consider only the dilation matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The aim of this section is to provide a family of approximating subdivision schemes as reference schemes for our multigrid examples (see Propositions 6.11 and 6.13). Especially, in Section 6.4, we show that

Dilation matrix	Subdivision scheme	$\rho(\mathcal{V})$	$\rho(\mathcal{V}_1)$	$\rho(\mathcal{V}_2)$	α_ϕ
$M=\begin{pmatrix}2&0\\0&3\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	0.50000	0.50000	0.33333	1
	$S_{\mathbf{a}_{M,2}}$	0.50000	0.50000	0.33333	1
$M=\begin{pmatrix}2&0\\0&5\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	0.50000	0.50000	0.20000	1
	$S_{\mathbf{a}_{M,2}}$	0.50000	0.50000	0.20000	1

Table 6.1: *Continuity and Hölder regularity of $S_{\mathbf{a}_{M,J}}$ (Theorem 6.5).*

for $M=\begin{pmatrix}2&0\\0&3\end{pmatrix}$ the interpolatory subdivision schemes in Definition 6.2 are computationally superior to the approximating subdivision schemes that we define in this section. We first introduce a family of symmetric four-directional box splines, see Definition 6.5, then we define a new family of symmetric four-directional approximating subdivision schemes, see Definition 6.6.

Definition 6.5. Let $J\in\mathbb{N}$. The *anisotropic symmetric four-directional box spline $S_{\mathbf{B}_J}$* of order J and dilation matrix $M=\begin{pmatrix}2&0\\0&3\end{pmatrix}$ is defined by its symbol

$$B_J(\mathbf{z})=6\left(\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{\lceil J/2\rceil}\left(\frac{(2+z_2+z_1z_2+2z_1z_2^2)(2z_1+z_2+z_1z_2+2z_2^2)}{36z_1z_2^2}\right)^{\lfloor J/2\rfloor},$$

for $\mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2$.

In order to understand the definition of the symbols B_J, $J\in\mathbb{N}$, in Definition 6.5, we need to look closely at the "basic" Laurent polynomials B_1 and B_2. For $J=1$, the symbol B_1 in Definition 6.5 becomes

$$B_1(\mathbf{z})=6\,\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2},\qquad \mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2.$$

The factors $(1+z_1)^2/(4z_1)$ and $(1+z_2+z_2^2)^2/(9z_2^2)$ are called first and second direction and they are the symbols of the univariate binary and ternary linear B-splines, respectively. Thus, the subdivision scheme $S_{\mathbf{B}_1}$ generates polynomials up to degree 1 (by a tensor product argument) and its mask $\mathbf{B}_1$ is symmetric and minimally supported,

$$\mathbf{B}_1=\frac{1}{6}\begin{pmatrix}1&2&3&2&1\\2&4&6&4&2\\1&2&3&2&1\end{pmatrix}.$$

For $J=2$, the symbol B_2 in Definition 6.5 becomes

$$B_2(\mathbf{z})=6\,\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}\frac{(2+z_2+z_1z_2+2z_1z_2^2)(2z_1+z_2+z_1z_2+2z_2^2)}{36z_1z_2^2},$$

$\mathbf{z}=(z_1,z_2)\in(\mathbb{C}\setminus\{0\})^2$.

The factor $(2+z_2+z_1z_2+2z_1z_2^2)(2z_1+z_2+z_1z_2+2z_2^2)/(36z_1z_2^2)$ represents the product of the so-called third and fourth directions. We computed such a symbol B_2 in order to guarantee that the subdivision scheme $S_{\mathbf{B}_2}$ generates polynomials up to degree 3 (Proposition 6.7) and its mask $\mathbf{B}_2$ is symmetric and minimally supported,

$$\mathbf{B}_2 = \frac{1}{216}\begin{pmatrix} 0 & 2 & 9 & 18 & 23 & 18 & 9 & 2 & 0 \\ 4 & 16 & 40 & 64 & 76 & 64 & 40 & 16 & 4 \\ 8 & 28 & 62 & 92 & 106 & 92 & 62 & 28 & 8 \\ 4 & 16 & 40 & 64 & 76 & 64 & 40 & 16 & 4 \\ 0 & 2 & 9 & 18 & 23 & 18 & 9 & 2 & 0 \end{pmatrix}.$$

Finally, the definition of such a symbol B_J, $J \in \mathbb{N}$, in Definition 6.5, guarantees that the subdivision scheme $S_{\mathbf{B}_J}$ generates polynomials up to degree $2J-1$ (Proposition 6.7) and its mask $\mathbf{B}_J$ is symmetric.

In order to prove Propositions 6.7 and 6.8, we need an auxiliary Lemma.

Lemma 6.6. *Let $J \in \mathbb{N}$. The Laurent polynomial B_J in Definition 6.5 satisfies*

$$\begin{aligned} B_J(\mathbf{z}) &= 6\sum_{j=0}^{\lfloor \frac{J}{2} \rfloor} \binom{\lfloor \frac{J}{2} \rfloor}{j} (-1)^j\, b_{J,j}^{(1)}(z_1)\, b_{J,j}^{(2)}(z_2), \\ b_{J,j}^{(1)}(z_1) &= \left(\frac{(1+z_1)^2}{4z_1}\right)^{J-j} \left(\frac{(1-z_1)^2}{4z_1}\right)^{j}, \\ b_{J,j}^{(2)}(z_2) &= \left(\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{J-j} \left(\frac{(1-z_2^2)^2}{9z_2^2}\right)^{j}, \qquad \mathbf{z} = (z_1,z_2) \in (\mathbb{C}\setminus\{0\})^2. \end{aligned}$$

Proof. We observe that the product of the third and the fourth directions can be written as

$$\frac{(2+z_2+z_1z_2+2z_1z_2^2)(2z_1+z_2+z_1z_2+2z_2^2)}{36z_1z_2^2} = \frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2} - \frac{(1-z_1)^2(1-z_2^2)^2}{36z_1z_2^2}.$$

Using this identity, for $\mathbf{z} = (z_1,z_2) \in (\mathbb{C}\setminus\{0\})^2$, we can rewrite the Laurent polynomial B_J in

Definition 6.5 as

$$
\begin{aligned}
B_J(\mathbf{z}) &= 6\left(\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{\lceil\frac{J}{2}\rceil}\left(\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}-\frac{(1-z_1)^2(1-z_2^2)^2}{36z_1z_2^2}\right)^{\lfloor\frac{J}{2}\rfloor}\\
&= 6\left(\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{\lceil\frac{J}{2}\rceil}\sum_{j=0}^{\lfloor\frac{J}{2}\rfloor}\binom{\lfloor\frac{J}{2}\rfloor}{j}(-1)^j\left(\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{\lfloor\frac{J}{2}\rfloor-j}\left(\frac{(1-z_1)^2(1-z_2^2)^2}{36z_1z_2^2}\right)^{j}\\
&= 6\sum_{j=0}^{\lfloor\frac{J}{2}\rfloor}\binom{\lfloor\frac{J}{2}\rfloor}{j}(-1)^j\left(\frac{(1+z_1)^2}{4z_1}\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{J-j}\left(\frac{(1-z_1)^2(1-z_2^2)^2}{36z_1z_2^2}\right)^{j}\\
&= 6\sum_{j=0}^{\lfloor\frac{J}{2}\rfloor}\binom{\lfloor\frac{J}{2}\rfloor}{j}(-1)^j\left(\frac{(1+z_1)^2}{4z_1}\right)^{J-j}\left(\frac{(1-z_1)^2}{4z_1}\right)^{j}\left(\frac{(1+z_2+z_2^2)^2}{9z_2^2}\right)^{J-j}\left(\frac{(1-z_2^2)^2}{9z_2^2}\right)^{j}\\
&= 6\sum_{j=0}^{\lfloor\frac{J}{2}\rfloor}\binom{\lfloor\frac{J}{2}\rfloor}{j}(-1)^j\, b^{(1)}_{J,j}(z_1)\, b^{(2)}_{J,j}(z_2).
\end{aligned}
$$

■

Proposition 6.7. *Let $J\in\mathbb{N}$. The anisotropic symmetric four-directional box spline $S_{\mathbf{B}_J}$ of order J in Definition 6.5 generates polynomials up to degree $2J-1$.*

Proof. We proceed by induction.
Step 1. For $J=1$, the base case is trivial due to a tensor product argument.
Step 2. Let us suppose that for any $J\geq 1$, $S_{\mathbf{B}_J}$ generates polynomials up to degree $2J-1$. We want to show that $S_{\mathbf{B}_{J+1}}$ generates polynomials up to degree $2(J+1)-1=2J+1$. By definition, the symbol of the anisotropic symmetric four-directional box spline $S_{\mathbf{B}_{J+1}}$ satisfies the recursive formula

$$
B_{J+1}(z_1,z_2)=\begin{cases} B_J(z_1,z_2)\cdot\frac{1}{6}B_1(z_1,z_2), & J \text{ even},\\[2ex] B_J(z_1,z_2)\cdot Q(z_1,z_2), & J\text{ odd},\end{cases}
$$

where

$$
Q(z_1,z_2)=\frac{\left(2+z_2+z_1z_2+2z_1z_2^2\right)\left(2z_1+z_2+z_1z_2+2z_2^2\right)}{36z_1z_2^2}.
$$

For J even, $S_{\mathbf{B}_J}\in\mathcal{I}_{2J-1}$ by induction and $S_{\mathbf{B}_1}\in\mathcal{I}_1$ by *Step 1*, thus, $S_{\mathbf{B}_{J+1}}\in\mathcal{I}_{2J+1}$.
For J odd, we cannot apply the same argument as before since $Q(z_1,z_2)$ does not vanish on $E_M\setminus\{(1,1)\}$. Let $\boldsymbol{\alpha}\in\mathbb{N}_0^2$, $|\boldsymbol{\alpha}|\leq 2J+1$. Applying the Leibniz formula to B_{J+1} we get

$$
D^{(\alpha_1,\alpha_2)}B_{J+1}(z_1,z_2)=\sum_{\beta_1=0}^{\alpha_1}\sum_{\beta_2=0}^{\alpha_2}\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}D^{(\beta_1,\beta_2)}B_J(z_1,z_2)\cdot D^{(\alpha_1-\beta_1,\alpha_2-\beta_2)}Q(z_1,z_2). \tag{6.15}
$$

The following analysis is split in 2 steps: $|\boldsymbol{\alpha}|=2J$ and $|\boldsymbol{\alpha}|=2J+1$.

Step 2.a. Let $|\boldsymbol{\alpha}| \in \mathbb{N}_0^2$, $|\boldsymbol{\alpha}| = 2J$. From (6.15), by induction, we get

$$D^{(\alpha_1,\alpha_2)} B_{J+1}(z_1,z_2) = D^{(\alpha_1,\alpha_2)} B_J(z_1,z_2) \cdot Q(z_1,z_2).$$

By straightforward computation, we have

$$Q(\boldsymbol{\varepsilon}) = 0, \qquad \boldsymbol{\varepsilon} \in \left\{ \left(1, \mathrm{e}^{2/3\pi\mathrm{i}}\right), \left(1, \mathrm{e}^{4/3\pi\mathrm{i}}\right), (-1,1) \right\},$$

thus

$$D^{(\alpha_1,\alpha_2)} B_{J+1}(\boldsymbol{\varepsilon}) = 0, \qquad \boldsymbol{\varepsilon} \in \left\{ \left(1, \mathrm{e}^{2/3\pi\mathrm{i}}\right), \left(1, \mathrm{e}^{4/3\pi\mathrm{i}}\right), (-1,1) \right\}.$$

Now we need to study the behavior of $D^{(\alpha_1,\alpha_2)} B_{J+1}(\boldsymbol{\varepsilon})$, i.e. the behavior of $D^{(\alpha_1,\alpha_2)} B_J(\boldsymbol{\varepsilon})$, for

$$\boldsymbol{\varepsilon} \in E_M \setminus \left\{ (1,1), \left(1, \mathrm{e}^{2/3\pi\mathrm{i}}\right), \left(1, \mathrm{e}^{4/3\pi\mathrm{i}}\right), (-1,1) \right\} = \left\{ \left(-1, \mathrm{e}^{2/3\pi\mathrm{i}}\right), \left(-1, \mathrm{e}^{4/3\pi\mathrm{i}}\right) \right\}.$$

W.l.o.g., we focus our attention on $\boldsymbol{\varepsilon} = \left(-1, \mathrm{e}^{2/3\pi\mathrm{i}}\right)$. By Lemma 6.6, for odd J, we get

$$D^{(\alpha_1,\alpha_2)} B_J(z_1,z_2) = 6 \sum_{j=0}^{\frac{J-1}{2}} \binom{\frac{J-1}{2}}{j} (-1)^j \frac{d^{\alpha_1}}{dz_1^{\alpha_1}} b_{J,j}^{(1)}(z_1) \frac{d^{\alpha_2}}{dz_2^{\alpha_2}} b_{J,j}^{(2)}(z_2), \tag{6.16}$$

thus, in order to compute $D^{(\alpha_1,\alpha_2)} B_J(\boldsymbol{\varepsilon})$, we need to study separately the behavior of

$$\frac{d^{\alpha_1}}{dz_1^{\alpha_1}} b_{J,j}^{(1)}(z_1)\Big|_{z_1=-1} = \sum_{\beta_1=0}^{\alpha_1} \binom{\alpha_1}{\beta_1} \frac{d^{\beta_1}}{dz_1^{\beta_1}} (1+z_1)^{2(J-j)}\Big|_{z_1=-1} \cdot \frac{d^{\alpha_1-\beta_1}}{dz_1^{\alpha_1-\beta_1}} \left(4^{-J} z_1^{-J} (1-z_1)^{2j}\right)\Big|_{z_1=-1}$$

and

$$\frac{d^{\alpha_2}}{dz_2^{\alpha_2}} b_{J,j}^{(2)}(z_2)\Big|_{z_2=\mathrm{e}^{2/3\pi\mathrm{i}}} = \sum_{\beta_2=0}^{\alpha_2} \binom{\alpha_2}{\beta_2} \frac{d^{\beta_2}}{dz_2^{\beta_2}} (1+z_2+z_2^2)^{2(J-j)}\Big|_{z_2=\mathrm{e}^{2/3\pi\mathrm{i}}} \cdot \frac{d^{\alpha_2-\beta_2}}{dz_2^{\alpha_2-\beta_2}} \left(9^{-J} z_2^{-2J} (1-z_2^2)^{2j}\right)\Big|_{z_2=\mathrm{e}^{2/3\pi\mathrm{i}}},$$

for $j = 0, \ldots, \frac{J-1}{2}$.
Notice that for any $j \in \{0, \ldots, \frac{J-1}{2}\}$, we have $2(J-j) \in \{J+1, \ldots, 2J\}$.
Case 1: Let $\alpha_1 \in \{0, \ldots, J\}$. For any $\beta_1 \in \{0, \ldots, \alpha_1\}$ and for any $j \in \{0, \ldots, \frac{J-1}{2}\}$, we have $\beta_1 \le \alpha_1 \le J < 2(J-j)$. Thus,

$$\frac{d^{\beta_1}}{dz_1^{\beta_1}} (1+z_1)^{2(J-j)}\Big|_{z_1=-1} = 0,$$

and we get $D^{(\alpha_1,\alpha_2)} B_J(\boldsymbol{\varepsilon}) = 0$.
Case 2: Let $\alpha_1 \in \{J+1, \ldots, 2J\}$. Then $\alpha_2 = 2J - \alpha_1 \in \{0, \ldots, J-1\}$. Using the same argument as before, for any $\beta_2 \in \{0, \ldots, \alpha_2\}$ and for any $j \in \{0, \ldots, \frac{J-1}{2}\}$, we get

$$\frac{d^{\beta_2}}{dz_2^{\beta_2}} (1+z_2+z_2^2)^{2(J-j)}\Big|_{z_2=\mathrm{e}^{2/3\pi\mathrm{i}}} = 0.$$

Thus, from (6.16), we get $D^{(\alpha_1,\alpha_2)} B_J(\boldsymbol{\varepsilon}) = 0$ for $|\boldsymbol{\alpha}| = 2J$.

Step 2.b. Let $|\boldsymbol{\alpha}| \in \mathbb{N}_0^2$, $|\boldsymbol{\alpha}| = 2J+1$. From (6.15) and *Step 2.a.*, we get

$$D^{(\alpha_1,\alpha_2)} B_{J+1}(z_1,z_2) = D^{(\alpha_1,\alpha_2)} B_J(z_1,z_2) \cdot Q(z_1,z_2).$$

Thus, similarly to *Step 2.a.*, we need to study the behavior of $D^{(\alpha_1,\alpha_2)} B_{J+1}(\boldsymbol{\varepsilon})$, i.e. the behavior of $D^{(\alpha_1,\alpha_2)} B_J(\boldsymbol{\varepsilon})$, for

$$\boldsymbol{\varepsilon} \in E_M \setminus \left\{ (1,1), \left(1, \mathrm{e}^{2/3\pi \mathrm{i}}\right), \left(1, \mathrm{e}^{4/3\pi \mathrm{i}}\right), (-1,1) \right\} = \left\{ \left(-1, \mathrm{e}^{2/3\pi \mathrm{i}}\right), \left(-1, \mathrm{e}^{4/3\pi \mathrm{i}}\right) \right\}.$$

We notice that

- ⋆ $\alpha_1 \in \{0,\dots,J\}$: for any $\beta_1 \in \{0,\dots,\alpha_1\}$ and for any $j \in \{0,\dots,\frac{J-1}{2}\}$, we have $\beta_1 \le \alpha_1 \le J < 2(J-j)$,
- ⋆ $\alpha_1 \in \{J+1,\dots,2J+1\}$: $\alpha_2 = 2J+1-\alpha_1 \in \{0,\dots,J\}$, thus for any $\beta_2 \in \{0,\dots,\alpha_2\}$ and for any $j \in \{0,\dots,\frac{J-1}{2}\}$, we have $\beta_2 \le \alpha_2 \le J < 2(J-j)$.

Thesis follows from the same argument of *Step 2.a.* ■

Proposition 6.8. *Let $J \in \mathbb{N}$. The anisotropic symmetric four-directional box spline $S_{\mathbf{B}_J}$ of order J in Definition 6.5 reproduces polynomials up to degree 1.*

Proof. In order to prove Proposition 6.8, by Theorem 4.3 and Proposition 6.7, we need to show that

$$\begin{array}{lll} (i) & B_J(1,1) = |\det M| = 6, & \\ (ii) & D^{\boldsymbol{\alpha}} B_J(1,1) = 0, & \forall \boldsymbol{\alpha} \in \mathbb{N}_0^2 : |\boldsymbol{\alpha}| = 1, \\ (iii) & D^{\boldsymbol{\alpha}} B_J(1,1) \neq 0 & \text{for some } \boldsymbol{\alpha} \in \mathbb{N}_0^2 : |\boldsymbol{\alpha}| = 2. \end{array}$$

(i) By Definition 6.5, $B_J(1,1) = 6$.
(ii) Let $\boldsymbol{\alpha} = (1,0)$, $|\boldsymbol{\alpha}| = 1$. Using Lemma 6.6 and noticing that $b^{(2)}_{J,j}(1) = 0$ for $j = 1,\dots,\lfloor J/2 \rfloor$, the $(1,0)$-th directional derivative of B_J evaluated at $(1,1)$ becomes

$$\begin{aligned} D^{(1,0)} B_J(1,1) &= 6 \sum_{j=0}^{\lfloor \frac{J}{2} \rfloor} \binom{\lfloor \frac{J}{2} \rfloor}{j} (-1)^j \frac{d}{dz_1} b^{(1)}_{J,j}(z_1)\Big|_{z_1=1} b^{(2)}_{J,j}(1) \\ &= 6 \frac{d}{dz_1} \left(\frac{(1+z_1)^2}{4z_1} \right)^J \Bigg|_{z_1=1} \\ &= 6J \left(\frac{(1+z_1)^2}{4z_1} \right)^{J-1} \Bigg|_{z_1=1} \underbrace{\left(\frac{1+z_1}{2z_1} - \frac{(1+z_1)^2}{4z_1^2} \right)\Bigg|_{z_1=1}}_{=0} = 0. \end{aligned}$$

Analogously for $\boldsymbol{\alpha} = (0,1)$, $|\boldsymbol{\alpha}| = 1$.
(iii) Let $\boldsymbol{\alpha} = (2,0)$, $|\boldsymbol{\alpha}| = 2$. We show that $D^{(2,0)} B_J(1,1) \neq 0$. Using the previous argument, we get

$$D^{(2,0)} B_J(1,1) = 6 \frac{d^2}{dz_1^2} \left(\frac{(1+z_1)^2}{4z_1} \right)^J \Bigg|_{z_1=1} = 3J.$$

■

Remark 6.2. Let $J \in \mathbb{N}$. The convergence of the anisotropic symmetric four-directional box spline $S_{\mathbf{B}_J}$ in Definition 6.5 follows by standard argument involving the smoothing factors.

We are now ready to define the family of anisotropic symmetric four-directional approximating schemes.

Definition 6.6. Let $J \in \mathbb{N}$, $L \in \{0, \dots, J-1\}$. The *anisotropic symmetric four-directional approximating scheme* $S_{\mathbf{B}_{J,L}}$ of order (J, L) and dilation matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is defined by its symbol

$$B_{J,L}(\mathbf{z}) = \sum_{i=0}^{L} B_{J-i}(z_1, z_2) \cdot \sum_{j=0}^{i} \mathrm{c}_J^{(i,j)} \delta_1(z_1)^{i-j} \delta_2(z_2)^j, \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2,$$

where

$$\delta_1(z_1) = -\frac{(1 - z_1^2)^2}{16 z_1^2}, \qquad \delta_2(z_2) = -\frac{(1 - z_2^3)^2}{27 z_2^3},$$

and the coefficients $\mathrm{c}_J^{(i,j)}$ are computed recursively as the solution of the system of equations

$$D^{(2(i-j),2j)} B_{J,L}(1,1) = 6\delta_{i,0}, \qquad j = 0, \dots, i, \qquad i = 0, \dots, L-1. \tag{6.17}$$

Proposition 6.9. *Let $J \in \mathbb{N}$, $L \in \{0, \dots, J-1\}$. The anisotropic symmetric four-directional approximating scheme $S_{\mathbf{B}_{J,L}}$ of order (J, L) in Definition 6.6 generates polynomials up to degree $2J-1$.*

Proof. By Proposition 6.7, for $i = 0, \dots, L$, the symmetric four-directional box spline $S_{\mathbf{B}_{J-i}} \in \mathcal{I}_{2J-2i-1}$. By definition,

$$\delta_1(z_1)^{i-j} \delta_2(z_2)^j = \left(-\frac{(1 - z_1^2)^2}{16 z_1^2} \right)^{i-j} \left(-\frac{(1 - z_2^3)^2}{27 z_2^3} \right)^j \in \mathcal{J}_{2i-1} \subset \mathcal{I}_{2i-1}.$$

Since the ideal $\mathcal{I}_k$, $k \in \mathbb{N}_0$, is closed under addition, we have

$$\sum_{j=0}^{i} \mathrm{c}_J^{(i,j)} \delta_1(z_1)^{i-j} \delta_2(z_2)^j \in \mathcal{I}_{2i-1}.$$

We recall that, if the subdivision symbols p_1, p_2 satisfy $p_1 \in \mathcal{I}_{q_1}$ and $p_2 \in \mathcal{I}_{q_2}$, $q_1, q_2 \in \mathbb{N}_0$, then $p_1 \cdot p_2 \in \mathcal{I}_{q_1+q_2+1}$. Thus,

$$B_{J-i}(\mathbf{z}) \cdot \sum_{j=0}^{i} \mathrm{c}_J^{(i,j)} \delta_1(z_1)^{i-j} \delta_2(z_2)^j \in \mathcal{I}_{2J-1},$$

which implies

$$B_{J,L}(\mathbf{z}) = \sum_{i=0}^{L} B_{J-i}\mathbf{z} \cdot \sum_{j=0}^{i} \mathrm{c}_J^{(i,j)} \delta_1(z_1)^{i-j} \delta_2(z_2)^j \quad \text{belongs to} \quad \mathcal{I}_{2J-1},$$

i.e. $S_{\mathbf{B}_{J,L}}$ generates polynomials up to degree $2J-1$. ■

Remark 6.3. Let $J \in \mathbb{N}$, $L \in \{0,\dots,J-1\}$. The convergence analysis of the anisotropic symmetric four-directional approximating scheme $S_{\mathbf{B}_{J,L}}$ in Definition 6.6 can be done similarly to subsection 6.1.5.

Finally, we conjecture that for any $J \in \mathbb{N}$, $L \in \{0,\dots,J-1\}$, the anisotropic symmetric four-directional approximating scheme $S_{\mathbf{B}_{J,L}}$ in Definition 6.6 reproduces polynomials up to degree $2L+1$, and we actually verified this fact for $J \le 10$. Notice that, if $L = J-1$, then $S_{\mathbf{B}_{J,J-1}}$ generates and reproduces polynomials up to the same degree $2J-1$. Contrary to the univariate case, in the bivariate case this property does not imply that the subdivision scheme $S_{\mathbf{B}_{J,J-1}}$ is interpolatory. Indeed, its mask $\mathbf{B}_{J,J-1}$ does not satisfy the interpolatory condition (4.9). See Examples 6.8 and 6.9, for the masks $\mathbf{B}_{2,1}$ and $\mathbf{B}_{3,2}$.

6.3 Subdivision, multigrid and examples

In this section, we provide grid transfer operators from the symbols of the anisotropic interpolatory and approximating subdivision schemes introduced in sections 6.1 and 6.2, respectively. Especially, in subsection 6.3.1, we focus on the anisotropic interpolatory subdivision schemes $S_{\mathbf{a}_{M,J}}$, $J \in \mathbb{N}$, in Definition 6.2, with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m = 3,5$. We prove that the symbols of $S_{\mathbf{a}_{M,J}}$ define appropriate grid transfer operators for the correct choice of the order J (see Proposition 6.10). In subsection 6.3.2, we focus on the anisotropic approximating subdivision schemes $S_{\mathbf{B}_J}$, $J \in \mathbb{N}$, in Definition 6.5 and $S_{\mathbf{B}_{J,L}}$, $J \in \mathbb{N}$, $L \in \{0,\dots,J-1\}$, in Definition 6.6, with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. First, we prove that the symbols of the anisotropic box splines $S_{\mathbf{B}_J}$ define appropriate grid transfer operators for the correct choice of the order J (see Proposition 6.11). Then, we focus on the approximating schemes $S_{\mathbf{B}_{J,L}}$. We explain how to determine the coefficients $\mathrm{c}_J^{(i,j)}$ in Definition 6.6. Then, for practical interest (see section 6.4), we verify that the symbols of $S_{\mathbf{B}_{J,L}}$, $J = 1,2,3$, $L \in \{0,\dots,J-1\}$, satisfy conditions *(i)* and *(ii)* of Theorem 4.7 (see Proposition 6.13).

6.3.1 Interpolatory grid transfer operators

The following result is a direct consequence of Theorem 4.5.

Proposition 6.10. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0,2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \le q-1, \quad \textit{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad q \in \mathbb{N}.$$

The grid transfer operator derived from the symbol $a_{M,J}$, $J \ge \lceil q/2 \rceil$, in Definition 6.2 satisfies the approximation property (3.10).

Proof. Let $J \ge \lceil q/2 \rceil$. By Proposition 6.2, the anisotropic interpolatory subdivision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 generates polynomials up to degree $2J-1 \ge q-1$. Thus, by Theorem 4.2, condition *(i)* of Theorem 4.5 is satisfied. Moreover, $S_{\mathbf{a}_{M,J}}$ is an interpolatory subdivision scheme and, thus, by [12, Proposition 1.3], its basic limit function is ℓ^∞-stable. Thus, *(ii)* of

Theorem 4.5 is also satisfied. ■

In Examples 6.5 and 6.6, we give several examples of masks of the anisotropic interpolatory subdvision schemes $S_{\mathbf{a}_{M,J}}$ with $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $M = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, respectively. The corresponding grid transfer operators are used in our numerical experiments.

***Example* 6.5.** We focus our attention on the case $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. For $J = 1,2,3$, the masks of the anisotropic interpolatory subdvision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 are

$$\mathbf{a}_{M,1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix},$$

$$\mathbf{a}_{M,2} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{24} & -\frac{1}{16} & -\frac{1}{24} & -\frac{1}{48} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{81} & -\frac{5}{162} & 0 & \frac{89}{432} & \frac{89}{216} & \frac{9}{16} & \frac{89}{216} & \frac{89}{432} & 0 & -\frac{5}{162} & -\frac{2}{81} \\ -\frac{4}{81} & -\frac{5}{81} & 0 & \frac{10}{27} & \frac{20}{27} & 1 & \frac{20}{27} & \frac{10}{27} & 0 & -\frac{5}{81} & -\frac{4}{81} \\ -\frac{2}{81} & -\frac{5}{162} & 0 & \frac{89}{432} & \frac{89}{216} & \frac{9}{16} & \frac{89}{216} & \frac{89}{432} & 0 & -\frac{5}{162} & -\frac{2}{81} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{24} & -\frac{1}{16} & -\frac{1}{24} & -\frac{1}{48} & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{a}_{M,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} & \frac{1}{128} & \frac{3}{256} & \frac{1}{128} & \frac{1}{256} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{324} & \frac{5}{1296} & 0 & -\frac{241}{6912} & -\frac{241}{3456} & -\frac{25}{256} & -\frac{241}{3456} & -\frac{241}{6912} & 0 & \frac{5}{1296} & \frac{1}{324} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{7}{1458} & \frac{4}{729} & 0 & -\frac{121}{2916} & -\frac{605}{11664} & 0 & \frac{20809}{93312} & \frac{20809}{46656} & \frac{75}{128} & \frac{20809}{46656} & \frac{20809}{93312} & 0 & -\frac{605}{11664} & -\frac{121}{2916} & 0 & \frac{4}{729} & \frac{7}{1458} \\ \frac{7}{729} & \frac{8}{729} & 0 & -\frac{56}{729} & -\frac{70}{729} & 0 & \frac{280}{729} & \frac{560}{729} & 1 & \frac{560}{729} & \frac{280}{729} & 0 & -\frac{70}{729} & -\frac{56}{729} & 0 & \frac{8}{729} & \frac{7}{729} \\ \frac{7}{1458} & \frac{4}{729} & 0 & -\frac{121}{2916} & -\frac{605}{11664} & 0 & \frac{20809}{93312} & \frac{20809}{46656} & \frac{75}{128} & \frac{20809}{46656} & \frac{20809}{93312} & 0 & -\frac{605}{11664} & -\frac{121}{2916} & 0 & \frac{4}{729} & \frac{7}{1458} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{324} & \frac{5}{1296} & 0 & -\frac{241}{6912} & -\frac{241}{3456} & -\frac{25}{256} & -\frac{241}{3456} & -\frac{241}{6912} & 0 & \frac{5}{1296} & \frac{1}{324} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} & \frac{1}{128} & \frac{3}{256} & \frac{1}{128} & \frac{1}{256} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

◀

***Example* 6.6.** We focus our attention on the case $M = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$. For $J = 1,2$, the masks of the anisotropic interpolatory subdvision scheme $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 are

$$\mathbf{a}_{M,1} = \begin{pmatrix} \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} & \frac{1}{2} & \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 & \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} & \frac{1}{2} & \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \end{pmatrix},$$

$$\mathbf{a}_{M,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{80} & -\frac{1}{40} & -\frac{3}{80} & -\frac{1}{20} & -\frac{1}{16} & -\frac{1}{20} & -\frac{3}{80} & -\frac{1}{40} & -\frac{1}{80} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{125} & -\frac{7}{250} & -\frac{4}{125} & -\frac{3}{125} & 0 & \frac{241}{2000} & \frac{249}{1000} & \frac{747}{2000} & \frac{241}{500} & \frac{9}{16} & \frac{241}{500} & \frac{747}{2000} & \frac{249}{1000} & \frac{241}{2000} & 0 & -\frac{3}{125} & -\frac{4}{125} & -\frac{7}{250} & -\frac{2}{125} \\ -\frac{4}{125} & -\frac{7}{125} & -\frac{8}{125} & -\frac{6}{125} & 0 & \frac{27}{125} & \frac{56}{125} & \frac{84}{125} & \frac{108}{125} & 1 & \frac{108}{125} & \frac{84}{125} & \frac{56}{125} & \frac{27}{125} & 0 & -\frac{6}{125} & -\frac{8}{125} & -\frac{7}{125} & -\frac{4}{125} \\ -\frac{2}{125} & -\frac{7}{250} & -\frac{4}{125} & -\frac{3}{125} & 0 & \frac{241}{2000} & \frac{249}{1000} & \frac{747}{2000} & \frac{241}{500} & \frac{9}{16} & \frac{241}{500} & \frac{747}{2000} & \frac{249}{1000} & \frac{241}{2000} & 0 & -\frac{3}{125} & -\frac{4}{125} & -\frac{7}{250} & -\frac{2}{125} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{80} & -\frac{1}{40} & -\frac{3}{80} & -\frac{1}{20} & -\frac{1}{16} & -\frac{1}{20} & -\frac{3}{80} & -\frac{1}{40} & -\frac{1}{80} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

6.3.2 Approximating grid transfer operators

In this section, we focus our attention on the case $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. First, we look at the anisotropic symmetric four-directional box splines $S_{\mathbf{B}_J}$, $J \in \mathbb{N}$, from Definition 6.5.

Proposition 6.11. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0, 2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \le q-1, \quad \textit{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad q \in \mathbb{N}.$$

The grid transfer operator derived from the symbol B_J, $J \ge \lceil q/2 \rceil$, in Definition 6.5 satisfies the approximation property (3.10).

Proof. Let $J \ge \lceil q/2 \rceil$. By Proposition 6.7, the anisotropic box spline $S_{\mathbf{B}_J}$ from Definition 6.5 generates polynomials up to degree $2J-1 \ge q-1$. Thus, by Theorem 4.2, condition *(i)* of Theorem 4.7 is satisfied.

Moreover, for $\mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2$, the symbol B_J in Definition 6.5 satisfies

$$B_J\left(\mathrm{e}^{-i\mathbf{x}}\right) = 6\, B_J^{(1)}(\mathbf{x})\, B_J^{(2)}(\mathbf{x}),$$

$$B_J^{(1)}(\mathbf{x}) = \left[\frac{1+\cos x_1}{2}\left(\frac{1+2\cos x_2}{3}\right)^2\right]^{\lceil \frac{J}{2} \rceil},$$

$$B_J^{(2)}(\mathbf{x}) = \left[\frac{1}{9}\left(\cos\left(\frac{x_1}{2}\right) + 2\cos\left(\frac{x_1}{2} - x_2\right)\right)\left(\cos\left(\frac{x_1}{2}\right) + 2\cos\left(\frac{x_1}{2} + x_2\right)\right)\right]^{\lfloor \frac{J}{2} \rfloor}.$$

First, we notice that

$$B_J^{(1)}(\mathbf{x}) \ge \left(\frac{1}{2}\left(\frac{2}{3}\right)^2\right)^{\lceil \frac{J}{2} \rceil} = \left(\frac{2}{9}\right)^{\lceil \frac{J}{2} \rceil} > 0, \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]. \tag{6.18}$$

Then, using trigonometric identities, we observe that

$$\begin{aligned} B_J^{(2)}(\mathbf{x}) &= \left(\frac{1}{18}\left(\cos x_1(4\cos x_2+5)+4\cos x_2+4\cos(2x_2)+1\right)\right)^{\left\lfloor \frac{J}{2}\right\rfloor} \\ &\geq \left(\frac{1}{18}(0+4\cdot\frac{1}{2}-4\cdot\frac{1}{2}+1)\right)^{\left\lfloor \frac{J}{2}\right\rfloor} = \left(\frac{1}{18}\right)^{\left\lfloor \frac{J}{2}\right\rfloor}, \qquad \forall\, \mathbf{x} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{3},\frac{\pi}{3}\right]. \end{aligned} \tag{6.19}$$

By (6.18) and (6.19), we get

$$B_J\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) \geq 6\left(\frac{2}{9}\right)^{\left\lceil \frac{J}{2}\right\rceil}\left(\frac{1}{18}\right)^{\left\lfloor \frac{J}{2}\right\rfloor} > 0, \qquad \forall\, \mathbf{x} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{3},\frac{\pi}{3}\right], \tag{6.20}$$

and *(ii)* of Theorem 4.7 is also satisfied. ■

***Example* 6.7.** Let $J = 1,2,3$. The masks of the anisotropic box splines $S_{\mathbf{B}_J}$ in Definition 6.5 are

$$\mathbf{B}_1 = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix},$$

$$\mathbf{B}_2 = \begin{pmatrix} 0 & \frac{1}{108} & \frac{1}{24} & \frac{1}{12} & \frac{23}{216} & \frac{1}{12} & \frac{1}{24} & \frac{1}{108} & 0 \\ \frac{1}{54} & \frac{2}{27} & \frac{5}{27} & \frac{8}{27} & \frac{19}{54} & \frac{8}{27} & \frac{5}{27} & \frac{2}{27} & \frac{1}{54} \\ \frac{1}{27} & \frac{7}{54} & \frac{31}{108} & \frac{23}{54} & \frac{53}{108} & \frac{23}{54} & \frac{31}{108} & \frac{7}{54} & \frac{1}{27} \\ \frac{1}{54} & \frac{2}{27} & \frac{5}{27} & \frac{8}{27} & \frac{19}{54} & \frac{8}{27} & \frac{5}{27} & \frac{2}{27} & \frac{1}{54} \\ 0 & \frac{1}{108} & \frac{1}{24} & \frac{1}{12} & \frac{23}{216} & \frac{1}{12} & \frac{1}{24} & \frac{1}{108} & 0 \end{pmatrix}$$

and

$$\mathbf{B}_3 = \begin{pmatrix} 0 & \frac{1}{3888} & \frac{13}{7776} & \frac{7}{1296} & \frac{5}{432} & \frac{23}{1296} & \frac{53}{2592} & \frac{23}{1296} & \frac{5}{432} & \frac{7}{1296} & \frac{13}{7776} & \frac{1}{3888} & 0 \\ \frac{1}{1944} & \frac{7}{1944} & \frac{55}{3888} & \frac{71}{1944} & \frac{5}{72} & \frac{65}{648} & \frac{49}{432} & \frac{65}{648} & \frac{5}{72} & \frac{71}{1944} & \frac{55}{3888} & \frac{7}{1944} & \frac{1}{1944} \\ \frac{1}{486} & \frac{47}{3888} & \frac{323}{7776} & \frac{379}{3888} & \frac{25}{144} & \frac{313}{1296} & \frac{233}{864} & \frac{313}{1296} & \frac{25}{144} & \frac{379}{3888} & \frac{323}{7776} & \frac{47}{3888} & \frac{1}{486} \\ \frac{1}{324} & \frac{17}{972} & \frac{113}{1944} & \frac{43}{324} & \frac{25}{108} & \frac{103}{324} & \frac{229}{648} & \frac{103}{324} & \frac{25}{108} & \frac{43}{324} & \frac{113}{1944} & \frac{17}{972} & \frac{1}{324} \\ \frac{1}{486} & \frac{47}{3888} & \frac{323}{7776} & \frac{379}{3888} & \frac{25}{144} & \frac{313}{1296} & \frac{233}{864} & \frac{313}{1296} & \frac{25}{144} & \frac{379}{3888} & \frac{323}{7776} & \frac{47}{3888} & \frac{1}{486} \\ \frac{1}{1944} & \frac{7}{1944} & \frac{55}{3888} & \frac{71}{1944} & \frac{5}{72} & \frac{65}{648} & \frac{49}{432} & \frac{65}{648} & \frac{5}{72} & \frac{71}{1944} & \frac{55}{3888} & \frac{7}{1944} & \frac{1}{1944} \\ 0 & \frac{1}{3888} & \frac{13}{7776} & \frac{7}{1296} & \frac{5}{432} & \frac{23}{1296} & \frac{53}{2592} & \frac{23}{1296} & \frac{5}{432} & \frac{7}{1296} & \frac{13}{7776} & \frac{1}{3888} & 0 \end{pmatrix}.$$

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Now, we look at the anisotropic approximating subdivision schemes $S_{\mathbf{B}_{J,L}}$, $J \in \mathbb{N}$, $L \in \{0,\dots,J-1\}$, in Definition 6.6.

Lemma 6.12. *Let $J \in \mathbb{N}$. The symbols $B_{J,L}$ of the anisotropic approximating subdivision schemes $S_{\mathbf{B}_{J,L}}$, $L \in \{0,\dots,J-1\}$, in Definition 6.6 satisfy for $\mathbf{z} = (z_1,z_2) \in (\mathbb{C}\setminus\{0\})^2$*

(i) *for* $J \geq 1$: $B_{J,0}(\mathbf{z}) = B_J(\mathbf{z})$,

(ii) *for* $J \geq 2$: $B_{J,1}(\mathbf{z}) = B_J(\mathbf{z}) + B_{J-1}(\mathbf{z})\big(J\delta_1(z_1) + 2J\delta_2(z_2)\big)$,

(iii) *for* $J \geq 3$:

$$\begin{aligned} B_{J,2}(\mathbf{z}) =& B_J(\mathbf{z}) + B_{J-1}(\mathbf{z})\big(J\delta_1(z_1) + 2J\delta_2(z_2)\big) \\ &+ B_{J-2}(\mathbf{z})\left(\frac{J(J+1)}{2}\delta_1(z_1)^2 + \left(2J(J-2) + \frac{4}{3}\left\lfloor\frac{J}{2}\right\rfloor\right)\delta_1(z_1)\delta_2(z_2) + J(2J+1)\delta_2(z_2)^2\right), \end{aligned}$$

where B_k, $k \in \mathbb{N}_0$, *is the symbol of the 2-directional box spline* $S_{\mathbf{B}_k}$ *in Definition 6.5.*

Proof. *(i)* Let $J \in \mathbb{N}$ and $L = 0$. The symbol of the anisotropic approximating subdivision scheme $S_{\mathbf{B}_{J,0}}$, in Definition 6.6 becomes

$$B_{J,0}(\mathbf{z}) = \mathrm{c}_J^{(0,0)} B_J(\mathbf{z}), \qquad \mathbf{z} \in (\mathbb{C}\setminus\{0\})^2,$$

where B_J is the symbol of the anisotropic symmetric four-directional box spline $S_{\mathbf{B}_J}$ in Definition 6.5. By (6.17), the coefficient $\mathrm{c}_J^{(0,0)} \in \mathbb{R}$ is computed as the solution of $B_{J,0}(\mathbf{1}) = 6$. By definition, $B_J(\mathbf{1}) = 6$ and, thus, we get $\mathrm{c}_J^{(0,0)} = 1$. We recall that $S_{\mathbf{B}_{J,0}} = S_{\mathbf{B}_J}$ reproduces polynomials up to degree 1.

(ii) Let $J \in \mathbb{N}$, $J \geq 2$, and $L = 1$. The symbol of the anisotropic approximating subdivision scheme $S_{\mathbf{B}_{J,1}}$, in Definition 6.6 becomes

$$\begin{aligned} B_{J,1}(\mathbf{z}) &= \mathrm{c}_J^{(0,0)} B_J(\mathbf{z}) + B_{J-1}(\mathbf{z})\big(\mathrm{c}_J^{(1,0)}\delta_1(z_1) + \mathrm{c}_J^{(1,1)}\delta_2(z_2)\big), \\ &= B_{J,0}(\mathbf{z}) + B_{J-1}(\mathbf{z})\left(-\mathrm{c}_J^{(1,0)}\frac{(1-z_1^2)^2}{16z_1^2} - \mathrm{c}_J^{(1,1)}\frac{(1-z_2^3)^2}{27z_2^3}\right), \qquad \mathbf{z} = (z_1, z_2) \in (\mathbb{C}\setminus\{0\})^2. \end{aligned}$$

Since $S_{\mathbf{B}_{J,0}}$ reproduces polynomials up to degree 1 and $\delta_1(z_1), \delta_2(z_2) \in \mathcal{J}_1$, then $S_{\mathbf{B}_{J,1}}$ reproduces at least polynomials up to degree 1. By (6.17), the coefficients $\mathrm{c}_J^{(1,0)}, \mathrm{c}_J^{(1,1)} \in \mathbb{R}$ are computed as the solution of $D^{(2,0)}B_{J,1}(\mathbf{1}) = 0$ and $D^{(0,2)}B_{J,1}(\mathbf{1}) = 0$, respectively. Due to

$$D^{(2,0)}B_{J,1}(\mathbf{1}) = -3(\mathrm{c}_J^{(1,0)} - J), \qquad \text{and} \quad D^{(0,2)}B_{J,1}(\mathbf{1}) = -4(\mathrm{c}_J^{(1,1)} - 2J),$$

we get

$$\mathrm{c}_J^{(1,0)} = J, \qquad \mathrm{c}_J^{(1,1)} = 2J.$$

By straightforward computations, $S_{\mathbf{B}_{J,1}}$ reproduces polynomials up to degree 3.

(iii) Let $J \in \mathbb{N}$, $J \geq 3$, and $L = 2$. For $\mathbf{z} = (z_1, z_2) \in (\mathbb{C}\setminus\{0\})^2$, the symbol of the anisotropic approximating subdivision scheme $S_{\mathbf{B}_{J,2}}$, in Definition 6.6 becomes

$$\begin{aligned} B_{J,2}(\mathbf{z}) &= \mathrm{c}_J^{(0,0)} B_J(\mathbf{z}) + B_{J-1}(\mathbf{z})\big(\mathrm{c}_J^{(1,0)}\delta_1(z_1) + \mathrm{c}_J^{(1,1)}\delta_2(z_2)\big) \\ &\quad + B_{J-2}(\mathbf{z})\big(\mathrm{c}_J^{(2,0)}\delta_1(z_1)^2 + \mathrm{c}_J^{(2,1)}\delta_1(z_1)\delta_2(z_2) + \mathrm{c}_J^{(2,2)}\delta_2(z_2)^2\big), \\ &= B_{J,1}(\mathbf{z}) + B_{J-2}(\mathbf{z})\left(\mathrm{c}_J^{(2,0)}\frac{(1-z_1^2)^4}{256z_1^4} + \mathrm{c}_J^{(2,1)}\frac{(1-z_1^2)^2(1-z_2^3)^2}{432z_1^2z_2^3} + \mathrm{c}_J^{(2,2)}\frac{(1-z_2^3)^4}{729z_2^6}\right). \end{aligned}$$

Since $S_{\mathbf{B}_{J,1}}$ reproduces polynomials up to degree 3 and $\delta_1(z_1)^2, \delta_1(z_1)\delta_2(z_2), \delta_2(z_2)^2 \in \mathcal{J}_3$, then $S_{\mathbf{B}_{J,2}}$ reproduces at least polynomials up to degree 3. By (6.17), the coefficients $\mathrm{c}_J^{(2,0)}, \mathrm{c}_J^{(2,1)}, \mathrm{c}_J^{(2,2)} \in \mathbb{R}$ are computed as the solution of $D^{(4,0)}B_{J,2}(\mathbf{1}) = 0$, $D^{(2,2)}B_{J,2}(\mathbf{1}) = 0$ and $D^{(0,4)}B_{J,2}(\mathbf{1}) = 0$, respectively. Due to

$$
\begin{aligned}
D^{(4,0)}B_{J,2}(\mathbf{1}) &= \frac{9}{2}\big(2\mathrm{c}_J^{(2,0)} - J(J+1)\big),\\
D^{(2,2)}B_{J,2}(\mathbf{1}) &= 2\mathrm{c}_J^{(2,1)} - 4J(J-2) - \frac{8}{3}\left\lfloor\frac{J}{2}\right\rfloor,\\
D^{(0,4)}B_{J,2}(\mathbf{1}) &= 16\big(\mathrm{c}_J^{(2,0)} - J(2J+1)\big),
\end{aligned}
$$

we get

$$
\mathrm{c}_J^{(2,0)} = \frac{J(J+1)}{2}, \qquad \mathrm{c}_J^{(2,1)} = 2J(J-2) + \frac{4}{3}\left\lfloor\frac{J}{2}\right\rfloor, \qquad \mathrm{c}_J^{(2,2)} = J(2J+1).
$$

By straightforward computations, $S_{\mathbf{B}_{J,2}}$ reproduces polynomials up to degree 3. ■

In general, for every $J \in \mathbb{N}$, $L \in \{0,\dots,N-1\}$, we use the procedure described in the proof of Lemma 6.12 to compute the coefficients $\mathrm{c}_J^{(i,j)}$ in Definition 6.6. In Examples 6.8 and 6.9, we give several examples of masks of the anisotropic symmetric four-directional approximating schemes $S_{B_{J,L}}$ in Definition 6.6 with $J = 2,3$, $L \in \{1,\dots,J-1\}$. We recall that the anisotropic symmetric four-directional approximating scheme $S_{\mathbf{B}_{J,0}}$ in Definition 6.6 is equal to the anisotropic symmetric box spline $S_{\mathbf{B}_J}$ in Definition 6.5. Thus, the masks of $S_{\mathbf{B}_{J,0}}$, $J = 1,2,3$, are equal to the masks of $S_{\mathbf{B}_J}$, $J = 1,2,3$, in Example 6.7.

***Example* 6.8.** Let $J = 2$ and $L = 1$. By *(ii)* of Lemma 6.12, for

$$
\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1}, \mathrm{e}^{-\mathrm{i}x_2}\right) \in (\mathbb{C}\setminus\{0\})^2, \qquad \mathbf{x} = (x_1,x_2) \in [0,2\pi)^2,
$$

the symbol $B_{2,1}$ in Definition 6.6 becomes a trigonometric polynomial

$$
\begin{aligned}
B_{2,1}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) &= B_2\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) + B_1\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) b_{2,1}(\mathbf{x}),\\
b_{2,1}(\mathbf{x}) &= 2\delta_1\left(\mathrm{e}^{-\mathrm{i}x_1}\right) + 4\delta_2\left(\mathrm{e}^{-\mathrm{i}x_2}\right) = \frac{1}{2}\sin^2 x_1 + \frac{16}{27}\sin^2\left(\frac{3}{2}x_2\right),
\end{aligned}
$$

that is used to define the grid transfer operator in (3.16). We notice that the symbol $B_{2,1}$ satisfies *(ii)* of Theorem 4.7. Indeed, by (6.20), we have

$$
B_{2,1}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) \geq 6\cdot\frac{2}{9}\cdot\frac{1}{18} + 6\cdot\frac{2}{9}\cdot 0 = \frac{2}{27} > 0, \qquad \forall\, \mathbf{x} = (x_1,x_2) \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \times \left[-\frac{\pi}{3},\frac{\pi}{3}\right].
$$

The corresponding mask is

$$\mathbf{B}_{2,1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{24} & -\frac{1}{16} & -\frac{1}{24} & -\frac{1}{48} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{108} & 0 & 0 & -\frac{1}{54} & 0 & 0 & \frac{1}{108} & 0 & 0 \\ -\frac{2}{81} & -\frac{5}{162} & 0 & \frac{89}{432} & \frac{89}{216} & \frac{9}{16} & \frac{89}{216} & \frac{89}{432} & 0 & -\frac{5}{162} & -\frac{2}{81} \\ -\frac{4}{81} & -\frac{5}{81} & -\frac{1}{54} & \frac{10}{27} & \frac{20}{27} & \frac{28}{27} & \frac{20}{27} & \frac{10}{27} & -\frac{1}{54} & -\frac{5}{81} & -\frac{4}{81} \\ -\frac{2}{81} & -\frac{5}{162} & 0 & \frac{89}{432} & \frac{89}{216} & \frac{9}{16} & \frac{89}{216} & \frac{89}{432} & 0 & -\frac{5}{162} & -\frac{2}{81} \\ 0 & 0 & \frac{1}{108} & 0 & 0 & -\frac{1}{54} & 0 & 0 & \frac{1}{108} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{48} & -\frac{1}{24} & -\frac{1}{16} & -\frac{1}{24} & -\frac{1}{48} & 0 & 0 & 0 \end{pmatrix}.$$

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***Example* 6.9.** Let $J = 3$ and $L = 1$. By *(ii)* of Lemma 6.12, for

$$\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1}, \mathrm{e}^{-\mathrm{i}x_2}\right) \in (\mathbb{C}\setminus\{0\})^2, \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2,$$

the symbol $B_{3,1}$ in Definition 6.6 becomes a trigonometric polynomial

$$B_{3,1}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) = B_3\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) + B_2\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) b_{3,1}(\mathbf{x}),$$
$$b_{3,1}(\mathbf{x}) = 3\delta_1\left(\mathrm{e}^{-\mathrm{i}x_1}\right) + 6\delta_2\left(\mathrm{e}^{-\mathrm{i}x_2}\right) = \frac{3}{4}\sin^2 x_1 + \frac{8}{9}\sin^2\left(\frac{3}{2}x_2\right),$$

that is used to define the grid transfer operator in (3.16). We notice that the symbol $B_{3,1}$ satisfies *(ii)* of Theorem 4.7. Indeed, by (6.20), we have

$$B_{3,1}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) \geq 6\cdot\left(\frac{2}{9}\right)^2\cdot\frac{1}{18} + 6\cdot\frac{2}{9}\cdot\frac{1}{18}\cdot 0 = \frac{4}{243} > 0, \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]. \quad (6.21)$$

Let $J = 3$ and $L = 2$. By *(iii)* of Lemma 6.12, for

$$\mathbf{z} = \mathrm{e}^{-\mathrm{i}\mathbf{x}} = \left(\mathrm{e}^{-\mathrm{i}x_1}, \mathrm{e}^{-\mathrm{i}x_2}\right) \in (\mathbb{C}\setminus\{0\})^2, \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2,$$

the symbol $B_{3,2}$ in Definition 6.6 becomes a trigonometric polynomial

$$B_{3,2}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) = B_{3,1}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) + B_1\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) b_{3,2}(\mathbf{x}),$$
$$\begin{aligned} b_{3,2}(\mathbf{x}) &= 6\delta_1^2\left(\mathrm{e}^{-\mathrm{i}x_1}\right) + \frac{22}{3}\delta_1\left(\mathrm{e}^{-\mathrm{i}x_1}\right)\delta_2\left(\mathrm{e}^{-\mathrm{i}x_2}\right) + 21\,\delta_2^2\left(\mathrm{e}^{-\mathrm{i}x_2}\right) \\ &= \frac{3}{8}\sin^4 x_1 + \frac{22}{81}\sin^2 x_1 \sin^2\left(\frac{3}{2}x_2\right) + \frac{112}{243}\sin^4\left(\frac{3}{2}x_2\right), \end{aligned}$$

that is used to define the grid transfer operator in (3.16). We notice that the symbol $B_{3,2}$ satisfies *(ii)* of Theorem 4.7. Indeed, by (6.20) and (6.21), we have

$$B_{3,2}\left(\mathrm{e}^{-\mathrm{i}\mathbf{x}}\right) \geq \frac{4}{243} + 6\cdot\frac{2}{9}\cdot 0 = \frac{4}{243} > 0, \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right].$$

The corresponding masks are

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$$
\mathbf{B}_{3,1} = \begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{1}{576} & -\frac{1}{128} & -\frac{1}{64} & -\frac{23}{1152} & -\frac{1}{64} & -\frac{1}{128} & -\frac{1}{576} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3888} & -\frac{7}{3888} & -\frac{11}{1296} & -\frac{5}{216} & -\frac{49}{1296} & -\frac{59}{1296} & -\frac{49}{1296} & -\frac{5}{216} & -\frac{11}{1296} & -\frac{7}{3888} & \frac{1}{3888} & 0 & 0 \\
0 & -\frac{1}{648} & -\frac{11}{1944} & -\frac{11}{972} & -\frac{5}{1296} & \frac{1}{32} & \frac{103}{1296} & \frac{271}{2592} & \frac{103}{1296} & \frac{1}{32} & -\frac{5}{1296} & -\frac{11}{972} & -\frac{11}{1944} & -\frac{1}{648} & 0 \\
-\frac{1}{243} & -\frac{7}{486} & -\frac{113}{3888} & -\frac{49}{3888} & \frac{257}{3888} & \frac{437}{1944} & \frac{497}{1296} & \frac{595}{1296} & \frac{497}{1296} & \frac{437}{1944} & \frac{257}{3888} & -\frac{49}{3888} & -\frac{113}{3888} & -\frac{7}{486} & -\frac{1}{243} \\
-\frac{2}{243} & -\frac{25}{972} & -\frac{5}{108} & -\frac{1}{162} & \frac{983}{7776} & \frac{5543}{15552} & \frac{487}{864} & \frac{379}{576} & \frac{487}{864} & \frac{5543}{15552} & \frac{983}{7776} & -\frac{1}{162} & -\frac{5}{108} & -\frac{25}{972} & -\frac{2}{243} \\
-\frac{1}{243} & -\frac{7}{486} & -\frac{113}{3888} & -\frac{49}{3888} & \frac{257}{3888} & \frac{437}{1944} & \frac{497}{1296} & \frac{595}{1296} & \frac{497}{1296} & \frac{437}{1944} & \frac{257}{3888} & -\frac{49}{3888} & -\frac{113}{3888} & -\frac{7}{486} & -\frac{1}{243} \\
0 & -\frac{1}{648} & -\frac{11}{1944} & -\frac{11}{972} & -\frac{5}{1296} & \frac{1}{32} & \frac{103}{1296} & \frac{271}{2592} & \frac{103}{1296} & \frac{1}{32} & -\frac{5}{1296} & -\frac{11}{972} & -\frac{11}{1944} & -\frac{1}{648} & 0 \\
0 & 0 & \frac{1}{3888} & -\frac{7}{3888} & -\frac{11}{1296} & -\frac{5}{216} & -\frac{49}{1296} & -\frac{59}{1296} & -\frac{49}{1296} & -\frac{5}{216} & -\frac{11}{1296} & -\frac{7}{3888} & \frac{1}{3888} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{576} & -\frac{1}{128} & -\frac{1}{64} & -\frac{23}{1152} & -\frac{1}{64} & -\frac{1}{128} & -\frac{1}{576} & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\mathbf{B}_{3,2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} & \frac{1}{128} & \frac{3}{256} & \frac{1}{128} & \frac{1}{256} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{576} & 0 & 0 & \frac{1}{288} & 0 & 0 & -\frac{1}{576} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{324} & \frac{5}{1296} & 0 & -\frac{241}{6912} & -\frac{241}{3456} & -\frac{25}{256} & -\frac{241}{3456} & -\frac{241}{6912} & 0 & \frac{5}{1296} & \frac{1}{324} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{648} & 0 & 0 & \frac{17}{1296} & 0 & 0 & -\frac{5}{216} & 0 & 0 & \frac{17}{1296} & 0 & 0 & -\frac{1}{648} & 0 & 0 \\
\frac{7}{1458} & \frac{4}{729} & 0 & -\frac{121}{2916} & -\frac{605}{11664} & 0 & \frac{20809}{93312} & \frac{20809}{46656} & \frac{75}{128} & \frac{20809}{46656} & \frac{20809}{93312} & 0 & -\frac{605}{11664} & -\frac{121}{2916} & 0 & \frac{4}{729} & \frac{7}{1458} \\
\frac{7}{729} & \frac{8}{729} & \frac{1}{324} & -\frac{56}{729} & -\frac{70}{729} & -\frac{59}{2592} & \frac{280}{729} & \frac{560}{729} & \frac{449}{432} & \frac{560}{729} & \frac{280}{729} & -\frac{59}{2592} & -\frac{70}{729} & -\frac{56}{729} & \frac{1}{324} & \frac{8}{729} & \frac{7}{729} \\
\frac{7}{1458} & \frac{4}{729} & 0 & -\frac{121}{2916} & -\frac{605}{11664} & 0 & \frac{20809}{93312} & \frac{20809}{46656} & \frac{75}{128} & \frac{20809}{46656} & \frac{20809}{93312} & 0 & -\frac{605}{11664} & -\frac{121}{2916} & 0 & \frac{4}{729} & \frac{7}{1458} \\
0 & 0 & -\frac{1}{648} & 0 & 0 & \frac{17}{1296} & 0 & 0 & -\frac{5}{216} & 0 & 0 & \frac{17}{1296} & 0 & 0 & -\frac{1}{648} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{324} & \frac{5}{1296} & 0 & -\frac{241}{6912} & -\frac{241}{3456} & -\frac{25}{256} & -\frac{241}{3456} & -\frac{241}{6912} & 0 & \frac{5}{1296} & \frac{1}{324} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{576} & 0 & 0 & \frac{1}{288} & 0 & 0 & -\frac{1}{576} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} & \frac{1}{128} & \frac{3}{256} & \frac{1}{128} & \frac{1}{256} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Let $q \in \mathbb{N}$, $q \leq 4$. By Proposition 6.9, for $J = 2,3$, $L \in \{1,\dots,J-1\}$, the anisotropic symmetric four-directional approximating schemes $S_{B_{J,L}}$ in Examples 6.8 and 6.9 generate polynomials up to degree $2J-1 \geq q-1 = 3$. Thus, *(i)* of Theorem 4.7 is satisfied. Moreover, their symbols satisfy *(ii)* of Theorem 4.7. We proved the following result.

Proposition 6.13. *Let f be a real bivariate trigonometric polynomial such that $f(\mathbf{x}) > 0$, $\mathbf{x} \in (0,2\pi)^2$, and*

$$D^{\boldsymbol{\mu}} f(\mathbf{0}) = 0, \quad \boldsymbol{\mu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\mu}| \leq q-1, \quad \textit{and} \quad \exists \boldsymbol{\nu} \in \mathbb{N}_0^2, \quad |\boldsymbol{\nu}| = q, \quad D^{\boldsymbol{\nu}} f(\mathbf{0}) \neq 0, \quad 0 \leq q \leq 4.$$

The grid transfer operator derived from the symbols $B_{2,1}, B_{3,1}, B_{3,2}$ in Definition 6.6 satisfy the approximation property (3.10).

6.4 Numerical results

In this section, we illustrate the theoretical results of Propositions 6.10, 6.11 and 6.13 with two bivariate numerical examples of the geometric multigrid method applied to certain multilevel Toeplitz matrices. In both examples, using the notation introduced in chapter 5, section 5.4, we have

$$\mathbf{m} = (2,m) \in \mathbb{N}^2, \quad m \geq 2, \qquad \mathbf{k} = (k_1,k_2) \in \mathbb{N}^2, \qquad \ell = \min\{k_1,k_2\} - 1 \in \mathbb{N}.$$

The choice $\ell = \min\{k_1,k_2\}-1$ implies that the V-cycle has full length. For $j = 0,\dots,\ell$, we define

$$\mathbf{n}_j = (2^{k_1-j}-1, m^{k_2-j}-1), \qquad N_j = N(\mathbf{n}_j) = (2^{k_1-j}-1)(m^{k_2-j}-1),$$

and $\Omega_{\mathbf{n}_j}$ as a grid of $[0,1]^2$ of 2^{k_1-j}, m^{k_2-j} subintervals of size $(h_j)_1 = 2^{j-k_1}$, $(h_j)_2 = m^{j-k_2}$ in the coordinate directions x_1, x_2, respectively. For $j = 0,\dots,\ell$, the j-th matrix $A_{\mathbf{n}_j}$ is computed by discretizing a given continuous problem on the j-th grid $\Omega_{\mathbf{n}_j}$ using always the same discretization formula. Notice that matrices $A_{\mathbf{n}_j}$, $j = 0,\dots,\ell$, have dimension $N_j \times N_j$. The prolongation operators are defined as in (5.19) with $\mathbf{m} = (2,m)$. The restriction operators are $\dfrac{1}{2m} P^*_{\mathbf{n}_j}$, for $j = 0,\dots,\ell-1$.

Let $\mathbf{n} = \mathbf{n}_0$. To define $\mathbf{b_n} \in \mathbb{C}^N$, $N = n_1 n_2$, we choose the exact solution $X \in \mathbb{C}^{n_2 \times n_1}$ on the starting grid $\mathbf{n}$ as

$$X = \begin{pmatrix} \mathrm{x}(1,1) & \cdots & \mathrm{x}(1,n_1) \\ \vdots & \ddots & \vdots \\ \mathrm{x}(n_2,1) & \cdots & \mathrm{x}(n_2,n_1) \end{pmatrix},$$

$$\mathrm{x}(\alpha,\beta) = \sin\left(5\frac{\pi(\alpha-1)}{n_2-1}\right) + \sin\left(5\frac{\pi(\beta-1)}{n_1-1}\right), \quad \alpha = 1,\dots,n_2,\ \beta = 1,\dots,n_1,$$

we compute

$$\mathbf{x} = \begin{pmatrix} \mathrm{x}(1,1) & \cdots & \mathrm{x}(n_2,1) & \mathrm{x}(1,2) & \cdots & \mathrm{x}(n_2,2) & \cdots & \cdots & \mathrm{x}(1,n_1) & \cdots & \mathrm{x}(n_2,n_1) \end{pmatrix}^T \in \mathbb{C}^N$$

and set $\mathbf{b_n} := A_{\mathbf{n}} \mathbf{x} \in \mathbb{C}^N$.

6.4.1 Bivariate Laplacian problem

The first example we present arises from the discretization of the bivariate Laplacian problem with Dirichlet boundary conditions, namely

$$\begin{cases} -\dfrac{\partial^2}{\partial x_1^2}\psi(\mathbf{x}) - \dfrac{\partial^2}{\partial x_2^2}\psi(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} = (x_1, x_2) \in \Omega = (0,1)^2, \\ \psi|_{\partial\Omega} = 0. \end{cases} \tag{6.22}$$

Using finite difference discretization of order 2, for $j = 0,\ldots,\ell$, the system matrices $A_{\mathbf{n}_j} = T_{\mathbf{n}_j}(f_j) \in \mathbb{R}^{N_j \times N_j}$ are the bi-level Toeplitz matrices of order $\mathbf{n}_j$ generated by the bivariate trigonometric polynomials

$$f_j(\mathbf{x}) = \frac{1}{(h_j)_1^2}(2-2\cos x_1) + \frac{1}{(h_j)_2^2}(2-2\cos x_2), \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2.$$

Notice that f_j vanishes at $\mathbf{0}$ with order 2, thus by Propositions 6.10, 6.11and 6.13, the masks defined in Examples 6.5, 6.6, 6.7, 6.8 and 6.9 can be used to define the corresponding grid transfer operators. For an appropriate comparison, we use also Kobbelt's subdivision scheme $S_{\mathcal{K}}$, the Butterfly subdivision scheme $S_{\mathcal{B}}$ and our new subdivision scheme $S_{\mathcal{P}}$ introduced in chapter 5, subsection 5.3.2.
For the numerical experiments, we use as pre- and post-smoother one step of Gauss-Seidel method. The zero vector is used as the initial guess and the stopping criterion is $\|\mathbf{r}_s\|_2/\|\mathbf{r}_0\|_2 < 10^{-7}$, where $\mathbf{r}_s$ is the residual vector after s iterations and 10^{-7} is the given tolerance.
We define the starting grid $\mathbf{n}_0$ in agreement with the dilation matrix M, namely

$$\text{- for } M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}: \quad \begin{cases} \mathbf{n}_0 = (2^7-1, 2^7-1), & \text{Case 1,} \\ \mathbf{n}_0 = (2^8-1, 2^8-1), & \text{Case 2,} \end{cases}$$

$$\text{- for } M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}: \quad \begin{cases} \mathbf{n}_0 = (2^7-1, 3^4-1), & \text{Case 1,} \\ \mathbf{n}_0 = (2^8-1, 3^5-1), & \text{Case 2,} \end{cases}$$

$$\text{- for } M = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}: \quad \begin{cases} \mathbf{n}_0 = (2^7-1, 5^3-1), & \text{Case 1,} \\ \mathbf{n}_0 = (2^9-1, 5^4-1), & \text{Case 2,} \end{cases}$$

Table 6.2 shows how the number of iterations and convergence rates for the V-cycle change when the starting grid $\mathbf{n}_0$ becomes finer. The results in Table 6.2 support our theoretical analysis, as they show that subdivision schemes with different dilation matrices and appropriate degree of polynomials generation define grid transfer operators capable of guaranteeing convergence and optimality of the corresponding V-cycle method. The grid transfer operators defined from the subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ perform better than the grid transfer operators defined from the anisotropic subdivision schemes. This happens since the bivariate Laplacian problem in (6.22) is symmetric with respect to the two coordinate directions. If

Dilation matrix	Subdivision scheme	Case 1 iter	Case 1 conv. rate	Case 2 iter	Case 2 conv. rate	Generation degree
$M=\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$S_{\mathcal{K}}$	6	0.0661	7	0.0746	3
	$S_{\mathcal{B}}$	7	0.0760	7	0.0807	3
	$S_{\mathcal{P}}$	7	0.0831	7	0.0810	3
$M=\begin{pmatrix}2&0\\0&3\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	28	0.5573	23	0.4958	1
	$S_{\mathbf{a}_{M,2}}$	26	0.5297	22	0.4777	3
	$S_{\mathbf{a}_{M,3}}$	26	0.5347	23	0.4893	5
	$S_{\mathbf{B}_{2,0}}$	33	0.6082	26	0.5298	3
	$S_{\mathbf{B}_{2,1}}$	26	0.5298	22	0.4477	3
	$S_{\mathbf{B}_{3,0}}$	41	0.6718	35	0.6272	5
	$S_{\mathbf{B}_{3,1}}$	24	0.5096	22	0.4787	5
	$S_{\mathbf{B}_{3,2}}$	26	0.5347	23	0.4893	5
$M=\begin{pmatrix}2&0\\0&5\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	38	0.6529	45	0.6969	1
	$S_{\mathbf{a}_{M,2}}$	38	0.6532	40	0.6774	3

Table 6.2: *Bivariate subdivision schemes for the Laplacian problem.*

we use grid transfer operators derived from subdivision schemes with dilation $M=\begin{pmatrix}2&0\\0&2\end{pmatrix}$ or, equivalently, grid transfer operators defined from the downsampling matrix with the factor $\mathbf{m}=(2,2)$, we preserve the symmetry of the problem at each j-th step of the V-cycle, $j=0,\dots,\ell$. Moreover, in case of grid transfer operators derived from subdivision schemes with dilation $M=\begin{pmatrix}2&0\\0&m\end{pmatrix}$, at each Coarse Grid Correction step, we downsample the data with the factor $\mathbf{m}=(2,m)$ and the larger is m the more information we lose. Thus, the number of iterations required for convergence is larger for $m>2$. Finally, we notice that there is no crucial difference between polynomial generation and reproduction properties for convergence and optimality of the V-cycle method.

6.4.2 Bivariate anisotropic Laplacian problem

The second example we present arises from the discretization of the bivariate anisotropic Laplacian problem with Dirichlet boundary conditions (5.21).
Using finite difference discretization of order 2, for $j=0,\dots,\ell$, the system matrices $A_{\mathbf{n}_j}=T_{\mathbf{n}_j}(f_j)\in\mathbb{C}^{N_j\times N_j}$ are the bi-level Toeplitz matrices of order $\mathbf{n}_j$ generated by the bivariate trigonometric polynomials

$$f_j^{(\varepsilon)}(\mathbf{x})=\frac{\varepsilon}{(h_j)_1^2}(2-2\cos x_1)+\frac{1}{(h_j)_2^2}(2-2\cos x_2),\qquad \mathbf{x}=(x_1,x_2)\in[0,2\pi)^2.$$

Let $\mathbf{m}=(2,2)\in\mathbb{N}^2$, $\mathbf{k}=(k,k)\in\mathbb{N}^2$ and $\ell=k-1$. For $j=0,\dots,\ell$, the j-th grid of the V-cycle is symmetric, namely

$$\mathbf{n}_j = (2^{k-j}-1, 2^{k-j}-1), \qquad (h_j)_1 = (h_j)_2 = 2^{-(k-j)}.$$

Thus, we can rewrite the trigonometric polynomials $f_j^{(\varepsilon)}$, $j = 0, \ldots, \ell$, as

$$f_j^{(\varepsilon)}(\mathbf{x}) = 2^{2(k-j)}\Big(\varepsilon\,(2-2\cos x_1) + (2-2\cos x_2)\Big), \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2.$$

If $\varepsilon << 1$, the symbol $f_j^{(\varepsilon)}$ is numerically close to 0 on the entire line $x_2 = 0$, for all $j = 0, \ldots, \ell$ (see chapter 5, subsection 5.4.3)

Let $\mathbf{m} = (2, m) \in \mathbb{N}^2$, $m > 2$, and define $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$ such that

$$k_2 = \max\{\, k \in \mathbb{N} : m^k - 1 \leq 2^{k_1} - 1 \,\}.$$

We can rewrite the polynomials $f_j^{(\varepsilon)}$, $j = 0, \ldots, \ell$, as

$$\begin{aligned}
f_j^{(\varepsilon)}(\mathbf{x}) &= \frac{\varepsilon}{(h_j)_1^2}(2-2\cos x_1) + \frac{1}{(h_j)_2^2}(2-2\cos x_2), \\
&= \frac{1}{(h_j)_2^2}\left(\frac{\varepsilon (h_j)_2^2}{(h_j)_1^2}(2-2\cos x_1) + (2-2\cos x_2)\right), \\
&= m^{2(k_2-j)}\left(\varepsilon\,\frac{2^{2(k_1-j)}}{m^{2(k_2-j)}}(2-2\cos x_1) + (2-2\cos x_2)\right), \\
&= m^{2(k_2-j)}\Big(\varepsilon_j\,(2-2\cos x_1) + (2-2\cos x_2)\Big), \qquad \varepsilon_j = \varepsilon\,\frac{2^{2(k_1-j)}}{m^{2(k_2-j)}}, \qquad \mathbf{x} = (x_1, x_2) \in [0, 2\pi)^2.
\end{aligned}$$

The value ε_j represents the anisotropy of the discretized problem (5.21) on the j-th grid $\Omega_{\mathbf{n}_j}$ of the V-cycle, $j = 0, \ldots, \ell$. Especially, we have

$$\varepsilon_j = \varepsilon\,\frac{2^{2(k_1-j)}}{m^{2(k_2-j)}} = \frac{m^2}{4}\left(\varepsilon\,\frac{2^{2(k_1-(j-1))}}{m^{2(k_2-(j-1))}}\right) = \frac{m^2}{4}\varepsilon_{j-1} > \varepsilon_{j-1}, \qquad j = 1, \ldots, \ell. \tag{6.23}$$

This means that the matrix $A_{\mathbf{n}_j} = T_{\mathbf{n}_j}(f_j)$ at the j-th level of the V-cycle is less anisotropic than the matrix $A_{\mathbf{n}_{j-1}} = T_{\mathbf{n}_{j-1}}(f_{j-1})$ at the $(j-1)$-th level of the V-cycle, $j = 1, \ldots, \ell$. Motivated by this property and the observations related to the standard Laplacian problem in subsection 6.4.1, we propose a multigrid strategy which combines both anisotropic and symmetric cutting strategies. More precisely, we define the starting grid $\mathbf{n}_0$ by

$$\mathbf{n}_0 = (2^{k_1} - 1, m^h \cdot 2^{k_2} - 1), \qquad h \in \mathbb{N}, \tag{6.24}$$

and we choose $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$ such that

$$k_2 = \max\{\, k \in \mathbb{N} : m^h \cdot 2^k - 1 \leq 2^{k_1} - 1 \,\}.$$

We fix $\ell = \min\{k_1, h+k_2\} - 1$, in order to guarantee a V-cycle method with full length. Then, we define the j-th order of the V-cycle by

$$\mathbf{n}_j = \begin{cases} (2^{k_1-j} - 1, m^{h-j} \cdot 2^{k_2} - 1), & j = 0, \dots, h, \\ (2^{k_1-j} - 1, 2^{k_2-(j-h)} - 1), & j = h+1, \dots, \ell. \end{cases}$$

Finally, we construct the grid transfer operators $P_{\mathbf{n}_j}$ from the symbols of subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$ for $j = 0, \dots, h$, and from the symbols of subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ for $j = h+1, \dots, \ell$. Especially, for our numerical experiments, we use the bi-linear interpolation grid transfer operator defined from the symbol P_1 in Definition 5.3 for $j = h+1, \dots, \ell$. If we choose $h \in \mathbb{N}$ properly, due to (6.23), we can handle the anisotropy of the problem in h steps of the V-cycle. Thus, for $j = h+1, \dots, \ell$, a symmetric cutting strategy performs better than an anisotropic cutting strategy.

For the numerical experiments, we use as pre- and post-smoother one step of Gauss-Seidel method for $j = 1, \dots, \ell$, and 2 steps of Gauss-Seidel method for $j = 0$. The zero vector is used as the initial guess and the stopping criterion is $\|\mathbf{r}_s\|_2 / \|\mathbf{r}_0\|_2 < 10^{-5}$, where $\mathbf{r}_s$ is the residual vector after s iterations and 10^{-5} is the given tolerance.

We define the starting grid $\mathbf{n}_0$ by (6.24), namely

- for $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: $\begin{cases} \mathbf{n}_0 = (2^7 - 1, 2^7 - 1), & \text{Case 1}, \\ \mathbf{n}_0 = (2^8 - 1, 2^8 - 1), & \text{Case 2}, \end{cases}$

- for $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$: $\begin{cases} \mathbf{n}_0 = (2^7 - 1, 3^2 \cdot 2^3 - 1), & \text{Case 1}, \\ \mathbf{n}_0 = (2^8 - 1, 3^2 \cdot 2^4 - 1), & \text{Case 2}, \end{cases}$

- for $M = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$: $\begin{cases} \mathbf{n}_0 = (2^8 - 1, 5 \cdot 2^5 - 1), & \text{Case 1}, \\ \mathbf{n}_0 = (2^8 - 1, 5^2 \cdot 2^3 - 1), & \text{Case 2}, \end{cases}$

Tables 6.3 and 6.4 show how the number of iterations and convergence rates for the V-cycle change when the starting grid $\mathbf{n}_0$ becomes finer and the anisotropy ε in (5.21) decreases. The results support our theoretical analysis. Especially, the grid transfer operators defined from the anisotropic subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ perform better than all the other grid transfer operators. Indeed, after 2 steps of downsampling with the factor $\mathbf{m} = (2,3)$, the anisotropy of the problem increases by a factor $\frac{81}{16} \approx 5$. Moreover, when we downsample the data with the factor $\mathbf{m} = (2,3)$, we lose less information than when we sample the data with the factor $\mathbf{m} = (2,5)$. Among the grid transfer operators defined from the anisotropic subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, we pay special attention to the interpolatory ones. The advantage of using the anisotropic interpolatory subdivision schemes is the computational efficiency of the corresponding grid transfer operations. Indeed, the matrices $A_{\mathbf{n}_j}$, $j = 0, \dots, \ell$, are independent of the grid transfer operators and the computational cost of the restriction and

Dilation matrix	Subdivision scheme	Case 1 iter	Case 1 conv. rate	Case 2 iter	Case 2 conv. rate	Generation degree
$M=\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$S_{\mathcal{K}}$	61	0.8273	76	0.8585	3
	$S_{\mathcal{B}}$	61	0.8274	76	0.8586	3
	$S_{\mathcal{P}}$	61	0.8274	76	0.8586	3
$M=\begin{pmatrix}2&0\\0&3\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	14	0.4315	16	0.4807	1
	$S_{\mathbf{a}_{M,2}}$	14	0.4307	16	0.48	3
	$S_{\mathbf{a}_{M,3}}$	14	0.4312	16	0.4806	5
	$S_{\mathbf{B}_{2,0}}$	13	0.5145	16	0.4780	3
	$S_{\mathbf{B}_{2,1}}$	14	0.4307	16	0.48	3
	$S_{\mathbf{B}_{3,0}}$	14	0.4363	17	0.5003	5
	$S_{\mathbf{B}_{3,1}}$	13	0.4112	15	0.4633	5
	$S_{\mathbf{B}_{3,2}}$	14	0.4312	16	0.4806	5
$M=\begin{pmatrix}2&0\\0&5\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	20	0.5623	25	0.6307	1
	$S_{\mathbf{a}_{M,2}}$	21	0.5719	26	0.6385	3

Table 6.3: *Bivariate subdivision schemes for the anisotropic Laplacian problem with* $\varepsilon = 10^{-2}$.

prolongation depends only on the number of nonzero entries of the corresponding operators. Therefore, since for a fixed $J \in \mathbb{N}$, the mask $\mathbf{a}_{M,J}$ of the interpolatory subdivision schemes $S_{\mathbf{a}_{M,J}}$ in Definition 6.2 has less nonzero entries than the masks $\mathbf{B}_{J,L}$, $L = 0,\dots,J-1$, of the approximating subdivision schemes $S_{\mathbf{B}_{J,L}}$ in Definition 6.6, each iteration of the V-cycle method with the interpolatory grid transfer operator associated to $S_{\mathbf{a}_{M,J}}$ is cheaper than one V-cycle iteration with the approximating grid transfer operators associated to $S_{\mathbf{B}_{J,L}}$. Finally, we notice that there is no crucial difference between polynomial generation and reproduction properties for convergence and optimality of the V-cycle method.

Tables 6.3 and 6.4 justify the use of the dilation matrix $M = \begin{pmatrix}2&0\\0&3\end{pmatrix}$ in our analysis. Note that the schemes with $M = \begin{pmatrix}2&0\\0&5\end{pmatrix}$ have a slower convergence rate, which is influenced by the larger support sizes of their masks and by the less efficient approximation caused by inappropriate coarsening of the mesh in the y direction.

Dilation matrix	Subdivision scheme	Case 1 iter	Case 1 conv. rate	Case 2 iter	Case 2 conv. rate	Generation degree
$M=\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$\mathcal{S}_{\mathcal{K}}$	253	0.9555	251	0.9551	3
	$\mathcal{S}_{\mathcal{B}}$	253	0.9558	251	0.9551	3
	$\mathcal{S}_{\mathcal{P}}$	253	0.9556	251	0.9551	3
$M=\begin{pmatrix}2&0\\0&3\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	33	0.7051	44	0.7694	1
	$S_{\mathbf{a}_{M,2}}$	33	0.7050	44	0.7695	3
	$S_{\mathbf{a}_{M,3}}$	33	0.7050	44	0.7697	5
	$S_{\mathbf{B}_{2,0}}$	30	0.6813	42	0.7592	3
	$S_{\mathbf{B}_{2,1}}$	33	0.7050	44	0.7695	3
	$S_{\mathbf{B}_{3,0}}$	30	0.6807	41	0.7540	5
	$S_{\mathbf{B}_{3,1}}$	31	0.6893	43	0.7641	5
	$S_{\mathbf{B}_{3,2}}$	33	0.7050	44	0.7697	5
$M=\begin{pmatrix}2&0\\0&5\end{pmatrix}$	$S_{\mathbf{a}_{M,1}}$	62	0.8301	69	0.8462	1
	$S_{\mathbf{a}_{M,2}}$	62	0.8304	70	0.8479	3

Table 6.4: *Bivariate subdivision schemes for the anisotropic Laplacian problem with* $\varepsilon = 10^{-3}$.

CHAPTER 7

Conclusion

Multigrid and subdivision appeared on the mathematical horizon in the second half of the XX century and they immediately gained popularity due to their attractive features. Multigrid methods are fast iterative solvers for sparse large ill-conditioned linear systems of equations derived, for instance, via discretization of PDEs in fluid dynamics, electrostatics and continuum mechanics problems. Subdivision schemes are simple iterative algorithms for generation of smooth curves and surfaces with applications in 3D computer graphics and animation industry. Both multigrid methods and subdivision schemes are very attractive due to their efficiency to users and researchers, who analyze in depth their properties and several applications.

This thesis presents the first definition and analysis of subdivision based multigrid methods. First, we focus on algebraic multigrid methods for circulant and d-level circulant matrix algebra with general downsampling/upsampling strategy $m \in \mathbb{N}$, $m \geq 2$, in the circulant case and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, $m_i \geq 2$, $i = 1, \ldots, d$, in the d-level circulant case. In chapter 3, using the symbol approach and the formalism of trigonometric polynomials, we define new sufficient conditions for the convergence and optimality of two-grid and V-cycle methods, see Theorem 3.7 and Theorem 3.8. We highlight that Theorem 3.8 is the first result concerning the convergence of the V-cycle method for circulant and d-level circulant matrices with general downsampling/upsampling strategies.

To establish the link between multigrid and subdivision, we consider stationary primal univariate and d-variate subdivision schemes with dilation $m \in \mathbb{N}$, $m \geq 2$, in the univariate case and

$$M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_d \end{pmatrix} \in \mathbb{Z}^{d \times d}, \qquad m_i \geq 2, \qquad i = 1, \ldots, d,$$

in the d-variate case. In chapter 4, we construct grid transfer operators in the multigrid procedure from the symbols of certain subdivision schemes and we analyze the subdivision properties which guarantee the convergence and optimality of the corresponding multigrid method. We highlight that polynomial generation property plays a fundamental role in our analysis, see Theorem 4.4, together with the stability of the basic limit function, see Theorem 4.5, and zero conditions of the subdivision symbol, see Theorem 4.7. Such a catalog of the grid transfer operators based on subdivision schemes with well-known properties allows to simply choose the appropriate grid transfer operator for solving a specific problem.

The theoretical analysis carried out in chapter 4 is supported by univariate and bivariate numerical experiments in chapter 5 for both algebraic and geometric multigrid. The numerical tests show that, if the degree of polynomial generation is high enough, then the degree of polynomial reproduction does not affect the convergence of the corresponding multigrid. Indeed, if the grid transfer operators are defined from subdivision schemes with the same degree of polynomial generation but with different degrees of polynomial reproduction, the convergence rates of the corresponding V-cycle methods are the same. In the geometric multigrid tests, the grid transfer operators defined from interpolatory subdivision schemes are more competitive than the grid transfer operators defined from approximating subdivision schemes. The advantage of using interpolatory schemes is the computational efficiency of the corresponding grid transfer operations. Indeed, due to the geometric approach, the coarser matrices $A_{\mathbf{n}_j}$, $j = 1, \ldots, \ell$, are independent of the grid transfer operators and the computational cost of the restriction and prolongation depends only on the number of nonzero entries of the corresponding operators. For schemes with the same degree of polynomial generation, the mask of an approximating subdivision scheme has more nonzero entries than the mask of an interpolatory subdivision scheme. Moreover, if a subdivision scheme is interpolatory, then the corresponding basic limit function is stable and the hypotheses of Theorem 4.5 are automatically satisfied.

The numerical tests concerning the bivariate anisotropic Laplacian at the end of chapter 5 lead to the need of bivariate anisotropic subdivision schemes with dilation $M = \begin{pmatrix} 2 & 0 \\ 0 & m \end{pmatrix}$, $m > 2$. In chapter 6, using the formalism of Lauren polynomials, we construct a family of interpolatory subdivision schemes with such dilation M which are optimal in terms of the size of the support versus their polynomial generation properties, see Proposition 6.4. As reference schemes for our numerical tests, we propose two families of approximating subdivision schemes characterized by certain polynomial generation and reproduction properties, see Definition 6.5 and Definition 6.6. The numerical tests at the end of chapter 5 confirm the validity of our theoretical analysis in chapter 4.

Several research directions for future research involve dual and primal non-stationary multivariate subdivision based multigrid. The fascinating connection between such subdivision schemes and multigrid has not been explored and guarantees to enhance multigrid methods with new, efficient, subdivision based procedures.

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Bibliography

[1] A.A. Ahmadi, R.M. Jungers, P.A. Parrilo, and M. Roozbehani. Joint spectral radius and path-complete graph lyapunov functions. *SIAM J. Control Optim.*, 52(1):687–717, 2014.

[2] A. Aricò and M. Donatelli. A V-cycle Multigrid for multilevel matrix algebras: proof of optimality. *Numer. Math.*, 105(4):511–547, 2007.

[3] A. Aricò, M. Donatelli, and S. Serra-Capizzano. V-cycle optimal convergence for certain (multilevel) structured linear systems. *SIAM J. Matrix Anal. Appl.*, 26(1):186–214, 2004.

[4] F. Auricchio, L.B. Da Veiga, TJR Hughes, A. Reali, and G. Sangalli. Isogeometric collocation methods. *Mathematical Models and Methods in Applied Sciences*, 20(11):2075–2107, 2010.

[5] M. Bolten, M. Donatelli, and T. Huckle. Analysis of smoothed aggregation multigrid methods based on Toeplitz matrices. *Electron. Trans. Numer. Anal.*, 44:25–52, 2015.

[6] M. Bolten, M. Donatelli, T. Huckle, and C. Kravvaritis. Generalized grid transfer operators for multigrid methods applied on Toeplitz matrices. *BIT Numerical Mathematics*, 55(2):341–366, 2015.

[7] C. De Boor. *A practical guide to splines*, volume 27. Springer-Verlag New York, 1978.

[8] A. Brandt. Multi-level adaptive solutions to boundary-value problems. *Math. Comp.*, 31(138):333–390, 1977.

[9] A. Brandt. Guide to multigrid development. In *Multigrid methods*, pages 220–312. Springer, 1982.

[10] C.A. Cabrelli, C. Heil, and U.M. Molter. *Self-similarity and multiwavelets in higher dimensions*, volume 170. Amer. Math. Soc., 2004.

[11] A.S. Cavaretta, W. Dahmen, and C.A. Micchelli. *Stationary subdivision*, volume 453. J. Amer. Math. Soc., 1991.

[12] M. Charina and C. Conti. Polynomial reproduction of multivariate scalar subdivision schemes. *J. Comput. Appl. Math*, 240:51–61, 2013.

[13] M. Charina, C. Conti, and L. Romani. Reproduction of exponential polynomials by multivariate non-stationary subdivision schemes with a general dilation matrix. *Numer. Math.*, 127(2):223–254, 2014.

[14] M. Charina, M. Donatelli, L. Romani, and V. Turati. Anisotropic, interpolatory subdivision and multigrid. *arXiv preprint arXiv:0000.00000*.

[15] M. Charina, M. Donatelli, L. Romani, and V. Turati. Multigrid methods: grid transfer operators and subdivision schemes. *Linear Algebra Appl.*, 520:151–190, 2017.

[16] M. Charina and V.Y. Protasov. Smoothness of anisotropic wavelets, frames and subdivision schemes. *arXiv preprint arXiv:1702.00269*, 2017.

[17] C.K. Chui. *Multivariate splines*. SIAM, 1988.

[18] A. Cohen. *Ondelettes, analyses multiresolutions et traitement numerique du signal*. PhD thesis, Paris 9, 1990.

[19] C. Conti, M. Cotronei, and T. Sauer. Full rank interpolatory subdivision: A first encounter with the multivariate realm. *J. Approx. Theory*, 162(3):559–575, 2010.

[20] C. Conti and K. Hormann. Polynomial reproduction for univariate subdivision schemes of any arity. *J. Approx. Theory*, 163(4):413–437, 2011.

[21] C. Conti, k. Hormann, and C. Deng. Symmetric four-directional bivariate pseudo-splines. *arXiv preprint arXiv:1706.03056*.

[22] M. Cotronei, D. Ghisi, M. Rossini, and T. Sauer. An anisotropic directional subdivision and multiresolution scheme. *Adv. Comput. Math.*, 41(3):709–726, 2015.

[23] D. Cox, J. Little, and D. O'shea. *Ideals, varieties, and algorithms*, volume 3. Springer, 1992.

[24] I. Daubechies et al. *Ten lectures on wavelets*, volume 61. SIAM, 1992.

[25] I. Daubechies, B. Han, A. Ron, and Z. Shen. Framelets: MRA-based constructions of wavelet frames. *Applied and computational harmonic analysis*, 14(1):1–46, 2003.

[26] P.J. Davis. *Circulant Matrices*. John Wiley & Sons, New York.

[27] C. De Boor and K. Höllig. B-splines from parallelepipeds. *J. Anal. Math.*, 42(1):99–115, 1982/1983.

[28] C.l De Boor and R. DeVore. Approximation by smooth multivariate splines. *Trans. Amer. Math. Soc.*, 276(2):775–788, 1983.

[29] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. In *Constructive approximation*, pages 49–68. Springer, 1989.

[30] R. Diaz Fuentes. Perturbacion de los esquemas de Dubuc-Deslauriers para cualquier aridad. Master's thesis, University of Havana, Cuba, 2015.

[31] M. Donatelli. An algebraic generalization of local Fourier analysis for grid transfer operators in multigrid based on Toeplitz matrices. *Numer. Linear Algebra Appl.*, 17(2-3):179–197, 2010.

[32] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers. Robust and optimal multi-iterative techniques for IgA collocation linear systems. *Comput. Methods Appl. Mech. Engrg.*, 284:1120–1146, 2015.

[33] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers. Spectral analysis and spectral symbol of matrices in isogeometric collocation methods. *Math. Comp.*, 85(300):1639–1680, 2016.

[34] M. Donatelli, S. Serra-Capizzano, and D. Sesana. Multigrid methods for Toeplitz linear systems with different size reduction. *BIT Numerical Mathematics*, 52(2):305–327, 2012.

[35] B. Dong and Z. Shen. Linear independence of pseudo-splines. *Proc. Amer. Math. Soc.*, 134(9):2685–2694, 2006.

[36] B. Dong and Z. Shen. Pseudo-splines, wavelets and framelets. *Applied and Computational Harmonic Analysis*, 22(1):78–104, 2007.

[37] S. Dubuc. Interpolation through an iterative scheme. *J. Math. Anal. Appl.*, 114(1):185–204, 1986.

[38] N. Dyn, K. Hormann, M.A. Sabin, and Z. Shen. Polynomial reproduction by symmetric subdivision schemes. *J. Approx. Theory*, 155(1):28–42, 2008.

[39] N. Dyn and D. Levin. Subdivision schemes in geometric modelling. *Acta Numer.*, 11:73–144, 2002.

[40] N. Dyn, D. Levin, and J.A. Gregory. A butterfly subdivision scheme for surface interpolation with tension control. *ACM Transaction on Graphics*, 9(2):160–169, 1990.

[41] T. Eirola. Sobolev characterization of solutions of dilation equations. *SIAM J. Math. Anal.*, 23(4):1015–1030, 1992.

[42] G. Fiorentino and S. Serra-Capizzano. Multigrid methods for Toeplitz matrices. *Calcolo*, 28(3):283–305, 1991.

[43] G. Fiorentino and S. Serra-Capizzano. Multigrid methods for symmetric positive definite block Toeplitz matrices with nonnegative generating functions. *SIAM J. Sci. Comput.*, 17(5):1068–1081, 1996.

[44] R. Fischer and T. Huckle. Multigrid methods for anisotropic bttb systems. *Linear Algebra Appl.*, 417(2):314 – 334, 2006.

[45] C.F. Gauss. Werke, Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1870.

[46] G.H. Golub and C.F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, 1983.

[47] N. Guglielmi and V.Y. Protasov. Exact computation of joint spectral characteristics of linear operators. *Found. Comput. Math.*, 13(1):37–97, 2013.

[48] W. Hackbusch. *Multi-grid methods and applications*, volume 4. Springer Science & Business Media, 1985.

[49] B. Han. Classification and construction of bivariate subdivision schemes. *Curve and Surface Fitting: Saint-Malo*, pages 187–197, 2002.

[50] B. Han and R.Q. Jia. Optimal interpolatory subdivision schemes in multidimensional spaces. *SIAM J. Math. Anal.*, 36(1):105–124, 1998.

[51] C. GJ. Jacobi. Über eine neue Auflösungsart der bei der Methode der kleinsten Quadrate vorkommenden lineären Gleichungen. *Astronomische Nachrichten*, 22(20):297–306, 1845.

[52] K. Jetter and G. Plonka. A survey on L_2-approximation orders from shift-invariant spaces. In *Multivariate approximation and applications*. Cambridge University Press, 2001.

[53] R.Q. Jia. The subdivision and transition operators associated with a refinement equation. *Series in Approximations and Decompositions*, 8:139–154, 1996.

[54] R.Q. Jia. Approximation properties of multivariate wavelets. *J. Amer. Math. Soc.*, 67(222):647–665, 1998.

[55] R.Q. Jia. Interpolatory subdivision schemes induced by box splines. *Appl. Comput. Harmon. Anal.*, 8(3):286–292, 2000.

[56] R.Q. Jia and C.A. Micchelli. Using the refinement equations for the construction of pre-wavelets II: Powers of two. *Curves and surfaces*, pages 209–246, 1991.

[57] R. Jungers. The jsr toolbox. `https://it.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox`.

[58] L. Kobbelt. Interpolatory subdivision on open quadrilateral nets with arbitrary topology. In *Computer Graphics Forum*, volume 15, pages 409–420. Wiley Online Library, 1996.

[59] A. Levin. Polynomial generation and quasi-interpolation in stationary non-uniform subdivision. *Comput. Aided Geom. Design*, 20(1):41–60, 2003.

[60] G. Muntingh. Symbols and exact regularity of symmetric pseudo-splines of any arity. *arXiv preprint arXiv:1611.00618*, 2016.

[61] L.F. Richardson. The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 210:307–357, 1911.

[62] G.C. Rota and W. Strang. A note on the joint spectral radius. 1960.

[63] J.W. Ruge and K. Stüben. Algebraic multigrid. In *Multigrid methods*, pages 73–130. SIAM, Philadelphia, 1987.

[64] T. Sauer. Polynomial interpolation, ideals and approximation order of multivariate refinable functions. *Proc. Amer. Math. Soc.*, 130(11):3335–3347, 2002.

[65] T. Sauer. Lagrange interpolation on subgrids of tensor product grids. *Math. Comp.*, 73(245):181–190, 2004.

[66] T. Sauer. Shearlet multiresolution and multiple refinement. *Shearlets*, pages 199–237, 2012.

[67] S. Serra-Capizzano. Convergence analysis of two-grid methods for elliptic Toeplitz and PDEs matrix-sequences. *Numer. Math.*, 92(3):433–465, 2002.

[68] S. Serra-Capizzano and C. Tablino-Possio. Positive representation formulas for finite difference discretizations of (elliptic) second order PDEs. *Contemp. Math.*, 281:295–317, 1999.

[69] S. Serra-Capizzano and C. Tablino-Possio. Multigrid methods for multilevel circulant matrices. *SIAM J. Sci. Comput.*, 26(1):55–85, 2004.

[70] H. Sun, R.H. Chan, and Q.S. Chang. A note on the convergence of the two-grid method for Toeplitz systems. *Comput. Math. Appl.*, 34(1):11–18, 1997.

[71] U. Trottenberg, C.W. Oosterlee, and A. Schüller. *Multigrid*. Academic Press, Inc., San Diego, CA, 2001. With contributions by A. Brandt, P. Oswald and K. Stüben.

[72] E.E. Tyrtyshnikov. A unifying approach to some old and new theorems on distribution and clustering. *Linear Algebra Appl.*, 232:1 – 43, 1996.

[73] D.X. Zhou. Stability of refinable functions, multiresolution analysis, and Haar bases. *SIAM J. Math. Anal.*, 27(3):891–904, 1996.