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Competing or collaborating, with no symmetrical behaviour: Leadership opportunities and winning strategies under stability

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Abstract

In this paper, a new dynamic mathematical model describing leadership emergence or disappearance in agent based networks is proposed. Through a generalised Verhulst–Lotka–Volterra model, a triad of agents operates in a market where each agent detains a quota. The triad is composed of a leader, who leads communication, and two followers. Communications flows both ways from leader to followers and vice versa. Competition, collaboration and cheating are allowed. Stability solutions are investigated analytically through a fixed point analysis. Various solutions exist depending on the type of behavioural interactions. Results show that communication counts: survival of the leader is a condition for stability. All configurations with the leader out of the market are unstable. Conversely, the two followers position is highly difficult. The three agents cannot all survive unless they behave under mutual collaboration and in very special conditions. For followers, cheating the leader, especially if the leader is collaborating, can be a disaster. By the way, collaboration with the leader may not always ensure market survival. However, this can be a strategy to survive and even share the leadership, in particular when the other agent cheats (or is cheated by) the leader. Cheating is a cause of instability. In fact, only a few cases reach stability: this occurs when cheating comes from the leader and the leader always wins. The leader may be interested in cheating if she does not want to share the leadership with a follower, that is to get the monopoly of the market.

Keywords: Leadership; Cooperation; Competition; Cheating; Stability solutions

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1. Introduction

In general, considerations about leadership, its emergence (or its conservation) are of wide interest. One may ask not only how one becomes a leader, under which type of opportunity, but also how, being a leader, one maintains such a leadership [20]. We emphasise that we do not wish to extend the discussion towards aspects of leadership like career development nor towards facilitating organisational learning and collective capacity building. Our approach emphasises the final goal, i.e. to become or remain a leader, rather than the role of a leader.

Knowing that strategic alliances – constellations of bilateral agreements among firms – are increasingly necessary to support innovative activities [43], we restrict agent based modelling of leadership to an algorithmic method; as a basic model, we study a triad, where the presupposed to be the "leader" is connected to others who are pending nodes. Thus, we study how agents cooperation or competition influence leadership opportunity, under (to be better defined below) endogenous constraints.

Firstly, we propose a practical model of "coopetition" [4] in business networks letting players develop dynamical strategies. A mixed-type of interactions is considered, that is a system in which competitive and cooperative scenarios could occur simultaneously among the various interacting agents. This leads to reach realistic stability states for the leadership.

Secondly, by introducing a leadership (market) capacity in the model, a realistic constraint is inserted, i.e. the maximum level which all the agents may reach in the market, thereby enforcing a natural (endogenous) limit on agent size.

Existing economic models of leadership often consider static settings only. However, leadership requires persistence [23] which can be modelled only in a dynamic context. Gearing towards multiplex complex topologies, we analyse a possibly asymmetric bidirectionally coupled dynamics.

Within a generalised Lotka–Volterra model, we include a Verhulst limited growth constraint, to better mimic ("market capacity") reality within a struggling ("prey–predator") world [30,48].

The Verhulst–Lotka–Volterra (VLV) prey–predator model is generalised by introducing a network effect through an undirected (and still) unweighted graph. The weights binary values do suitably represent competition or cooperation strategies. Elements of the resulting adjacency matrix Γ replace the strength parameters in the interaction function of the model in [7] and [6].

We emphasise features based on analytic results on stability with differential equations rather than numerical simulations, - as in [22] or through the interesting and elaborate multiagent-based, spatially explicit, evolutionary model of Hadzibeganovic et al. [19]. Useful reviews on this topic are [50] and [35].

Our results give some quantitative insights in illustrating cooperation and competition aspects, often encountered when agents are under a social network pressure. We also show that, under certain conditions, the leader can be ousted from the market. This gives hints to the other agents on how to survive, even in apparently less favourable situations.

Let it be recalled that the Lotka–Volterra model has been used in various ways to model many types of complex systems [9,33,44,47,48]. Moreover, the well-known population growth Verhulst model has been much adapted outside demography, because the Verhulst quadratic term mimics a constraining exogenous effect found in the system limited capacity [39]. With such cornerstones, a combined VLV model has been recently formulated in order to introduce the effects of competition and growth pertinent in economy and opinion formation science [47,48]. A generalisation through the introduction of a non-linear symmetric interaction function has been used to investigate competition scenarios [7], cooperation scenarios [6], and mixed types of interactions [39]. Competition scenarios resulted in finding a self-organising clustering of agents, either chaotic or non chaotic, as a result of their dependence on their size and initial state conditions. Cooperative scenarios resulted in finding the growth of different clusters of interacting agents beyond their capacity, in contrast to the competitive case. Cooperation is affected by agents' sizes, willingness to cooperate, and by environmental complexity [3]. However, one should allow for a lack of reciprocity in the coupled agent interactions. This has to do with trustworthiness [26], that is an important aspect in strategic behaviour [14]. Indeed, it may occur that an agent wishes to compete or cooperate with another one, but this latter has not necessarily the same behavioural attitude towards the former agent. This is a quite common situation, when the struggle for survival causes behaviours that are not ethical, leaders not excluded.

We take the concept of leader from complex networks theory: a leader is a node which has a number of directed links higher than the other nodes, that is, a higher in-degree [15,31]. Besides, in our context, a leader can be both a predator and a prey [38].

Finally, recall that the most basic network unit useful for complexity needs a triplet of agents. From an organisational point of view, this is called an "uneasy triangle" in networks [36]. Here the leader takes advantage of the fact that connections between the two minor agents are weak or non-existent. An important assumption is that agents are not aware of the behaviour of others. They only know their own attitude towards their direct neighbour. This allows for asymmetrical behaviour, like competing with an agent that in fact is cooperating.

This paper is organised as follows: in Section 2, a short complementary literature review is provided encompassing related papers on leadership leaning towards modelling in a Lotka–Volterra framework. In Section 3, the generalised VLV model is introduced in detail: we outline the mathematical, sociological, or economical aspects justifying the model used in this paper. In Section 4, all cases leading to stability are analysed, showing growth and/or decay effects in various scenarios. In Section 4.1, the mathematical details are illustrated in detail. Even though our results are analytical, in Section 4.2, we accompany the results with some numerical simulations to show the paths towards equilibrium. In most cases, it is observed that the two minor agents cannot survive jointly. In Section 4.3, some considerations are derived from our analytical results. It appears that the model is suitable for in various organised networks describing complex systems. Section 5 concludes. Mathematical results on stability are reported in Appendix.

2. State of the art

In order to restrict the literature review to a reasonable size, let us focus on the specificity of the leadership types we consider below, geared within a predator–prey concept as that provided by the Lotka–Volterra model. The case discussed in this paper refers to a triad, which is the simplest complex structure to analyse in a non-linear model.

The leadership aspects which we describe pertain to the criteria on which such agents base their action in order to reinforce their position [27]. In a sociological group, an agent looks for power; in an economic system, a set of agents (firms, e.g. banks, airlines, car, soda, or electronic good makers) aims at oligopoly or even monopoly. Further examples can be found in multi-lover relationships [40] or political coalitions [1,29]. In the present case, we are open to consider how a leader loses its leadership role, how a dictator (some extreme leadership type) can be overthrown, how a superior athlete can gain a victory or lose in a race due to his/her competitor framing him/her.

To make the transition towards economic systems seen as evolving living systems within prey–predator aspects, we refer to Kamimura et al. [28]. Before this article, Guastello [18], stressing self-organisation as a biological concept, studied leadership emergence in animal troupes. As an organisational ecology modelling strategy describing how to use Lotka–Volterra models, Sungaila [42] is one of the pioneers. Symbolic and cultural aspects of organisational life are framed in the process of self-renewal, with which the dynamics of self-organisation must compete, in view of attaining an exclusive leadership. More recently, Castiaux [8] studied different patterns of relationships between organisations, as in a Lotka–Volterra system, proposing radical innovation in established organisations, like being a "knowledge predator".

3. The generalised Verhulst–Lotka–Volterra model

This work deals with a generalised prey–predator Verhulst–Lotka–Volterra model representing a market in which n agents simultaneously operate. Variables $s_i \geq 0$ denote the market size of agent i, $i = 1, \ldots, n$ while $\dot{s}_i$ are their time derivative. For sake of compactness, dependence on time for all variables is conveniently omitted.

Two are the forces that drive the model: a Verhulst logistic-type dynamics is coupled by a Lotka–Volterra's term. The first part distributes the overall market proportionally to the agents' sizes while the second allows the growth and the interaction among them. Here, interaction can be either of a cooperative or competitive kind and is modelled by suitable matrices that describe the network effect. Mixed-type interaction among agents is allowed so that cheating (that is when an agent is collaborating while the other is competing) is possible. The model reads

$$\dot{s}_i = \alpha_i s_i \left(\beta - \sum_{j=1}^{n} s_j \right) - \sum_{i \neq j}^{n} \gamma \left(s_i, s_j \right) s_i s_j \quad i = 1, \ldots, n. \tag{1}$$

The first term in (1) contains a Verhulst-like term [45,46] in which "market capacity" $\beta \geq 1$ stands for the final relative size, or amount of product or service, that could be reached by the totality of agents within the market [17,39,40]. Further, parameter α_i denotes the growth rate of agent i's share in the logistic model. For sake of simplicity, in the numerical part of this article it is assumed that agents have the same dynamical (growth) properties $\alpha_i = 1$.

Term $\beta - \sum_{j=1}^{n} s_j$ is the residual market, that is the portion of the market agents can obtain before reaching maximum capacity.

Interaction functions $\gamma(s_i, s_j)$, $i, j = 1, \ldots, n$ in (1) [7] stem from the Lotka–Volterra model [32,49].

Functions $\gamma(s_i, s_j)$, to be analytically defined below, can be seen in a [8] perspective, as a knowledge acquisition (or loss) motor, that is in our words a dynamical learning function.

As it will be shown below, these interaction terms might lead to the sum of market shares of all agents to end up being different than β.

The general case ($\beta \geq 1$) and the particular case ($\beta = 1$) are both studied to give further insights to our model. When $\beta = 1$, the leadership capacity is normalised and implies the same basic time scale for each agent. Furthermore, $\beta \geq 1$ may be called "overcapacity" of the market [39].

Consider the case $n = 3$; in order to emphasise leadership, the role of "leader" is given to agent A_1; she is allowed to interact directly (by means of functions $\gamma(s_1, s_2)$ and $\gamma(s_1, s_3)$) to agents A_2 and A_3 (the "followers" or "minors") and, therefore, has some market centrality. Agents A_2 and A_3 do not directly interact with each other: $\gamma(s_2, s_3) = \gamma(s_3, s_2) = 0$. They have a weak connection as they both interact (by means of functions $\gamma(s_2, s_1)$ and $\gamma(s_3, s_1)$) with A_1.

The organisational structure is described by the system of differential equations

$$\begin{cases} \dot{s}_1 = \alpha_1 s_1 (\beta - s_1 - s_2 - s_3) - \gamma(s_1, s_2) s_1 s_2 - \gamma(s_1, s_3) s_1 s_3 & \text{(a)} \\ \dot{s}_2 = \alpha_2 s_2 (\beta - s_1 - s_2 - s_3) - \gamma(s_2, s_1) s_2 s_1 & \text{(b)} \\ \dot{s}_3 = \alpha_3 s_3 (\beta - s_1 - s_2 - s_3) - \gamma(s_3, s_1) s_3 s_1 & \text{(c)} \end{cases} \tag{2}$$

The interaction function $\gamma(s_i, s_j)$ that measures how agents A_i and A_j affect each other is [7]

$$\gamma\left(s_i, s_j\right) = \gamma_{i,j} = \tilde{\gamma}_{i,j} e^{-\left(\frac{s_i - s_j}{\sigma}\right)^2} \tag{3}$$

where $\tilde{\gamma}_{i,j}$, $i, j = 1, 2, 3$ is an element of matrix $\Gamma \in \mathcal{M}(3, 3)$ while $\sigma > 0$ scales the intensity of agents sizes similarity.

Elements $\tilde{\gamma}_{i,j} \in \Gamma$ can get three values: $-1, 0, 1$. If $\tilde{\gamma}_{i,j} = -1$ $\left(\tilde{\gamma}_{i,j} = 1\right)$ agent A_i cooperates (competes) with agent A_j. In case no direct interaction occurs, $\tilde{\gamma}_{i,j} = 0$. Further, $\tilde{\gamma}_{i,i} = 0$, $i = 1, 2, 3$. Network representations associated to matrices Γ are depicted in Table 1.

A key feature in this analysis is that if agents A_i and A_j do not behave reciprocally, the one who competes $\left(\tilde{\gamma}_{i,j} = 1\right)$ is *cheating* her cooperating counterpart $\left(\tilde{\gamma}_{j,i} = -1\right)$. Interactions can be:

(i) **symmetric**: $\tilde{\gamma}_{i,j} = \tilde{\gamma}_{j,i}$, $i, j = 1, 2, 3$. If A_i cooperates (competes) with A_j then A_j cooperates (competes) with A_i. Here no agent is cheating. Matrices encompassing these cases are denoted with $\Gamma^{(+)}$ (Table 1, first column). Matrix $\Gamma_1^{(+)}$ represents a fully competitive system. Matrix $\Gamma_2^{(+)}$ describes a mixed-type of interaction system where A_1 and A_2 are unfriendly between each others but A_1 and A_3 reciprocally cooperate; otherwise said, couples (A_1; A_2) and (A_1; A_3) do not have the same strategic plans. Matrix $\Gamma_3^{(+)}$ describes full cooperation.

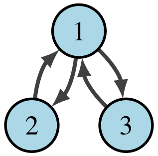
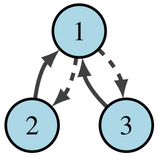
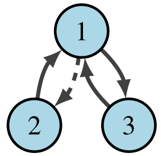
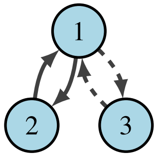
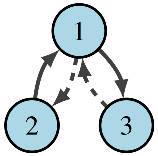
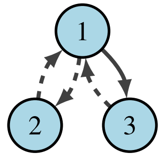
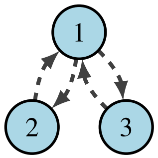
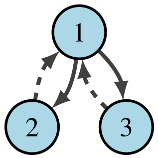
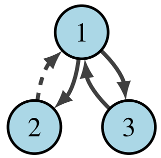
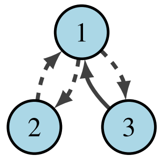
(ii) **antisymmetric**: $\tilde{\gamma}_{i,j} = -\tilde{\gamma}_{j,i}$. If A_i cooperates (competes) with A_j then A_j competes (cooperates) with A_i. Matrices encompassing these cases are denoted with $\Gamma^{(-)}$ (Table 1, second column). In this scenario, players do not act symmetrically. For each interaction between agents there is an agent cheating another one. Take for example $\Gamma_1^{(-)}$: A_1 collaborates with both A_2 and A_3, but both A_2 and A_3 compete with A_1. Due to the lack of direct connection between A_2 and A_3, this does not necessarily mean that these agents establish some sort of alliance.

(iii) **mixed reciprocal attitude**: such matrices are denoted with $\Gamma^{(0)}$ (Table 1, third column). Here one couple of agents interacts reciprocally where in the other couple there is an agent who is cheating.

In all of the three scenarios above, by permuting matrix indices accordingly, more cases can be represented. For sake of paucity, cases not considered in Table 1 are omitted. This is not limiting the analysis as, in these cases, followers' attitude switches between them.

Function (3) mimics a Gaussian-type form, is continuous and time differentiable, it avoids the dynamics to diverge, and allows a proper theoretical analysis of the system dynamics.

Table 1
Matrices Γ and network representation of interaction between agents — continuous curve: competition between agents $\left(\tilde{\gamma}_{i,j} = 1\right)$, dashed curve: collaboration between agents $\left(\tilde{\gamma}_{i,j} = -1\right)$.

 $\Gamma_1^{(+)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	 $\Gamma_1^{(-)} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	 $\Gamma_1^{(0)} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
 $\Gamma_2^{(+)} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	 $\Gamma_2^{(-)} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	 $\Gamma_2^{(0)} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
 $\Gamma_3^{(+)} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	 $\Gamma_3^{(-)} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	 $\Gamma_3^{(0)} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
		 $\Gamma_4^{(0)} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$
Symmetric (reciprocal) interactions: $\Gamma^{(+)}$	Antisymmetric (reciprocal) interactions: $\Gamma^{(-)}$	Mixed reciprocity: $\Gamma^{(0)}$

Function $\gamma_{i,j} s_i s_j$ in (2)(a)–(c), when bounded in subset $[0; +\infty) \times [0; +\infty)$, is positive and always non decreasing, if $\tilde{\gamma}_{i,j} = 1$, and negative and non increasing, if $\tilde{\gamma}_{i,j} = -1$, wrt both s_i and s_j.

In the Lotka–Volterra model interactions between species are gauged by constant parameters. In model (2)(a)–(c), as pointed out by Fernàndez et al. [12], values for the interaction function $\gamma_{i,j}$ depend on relative sizes of agents. This allows to ascertain that the relationships of closeness among agents in (3) modify the interaction (or bargaining) strength among them and, therefore, their size (payoffs) evolutions. Further, according to Caram et al. [7], such interaction coefficients are the largest (in absolute value) if agents' sizes are similar, i.e. if $s_i \sim s_j$. On the other side, $\gamma(s_i, s_j) \approx 0$, if s_i and s_j are quite different.

As it will be seen later, the asymptotic-like behaviour of $\gamma(s_i, s_j)$ is well suited to represent complex qualitative features.

4. Results on stability

The importance of stability in a market and, in general, for a whole economy as well in a price system, is crucial. Such aspect is explained in [25] and [11]. Thus, the conditions under which stability is reached, implying a stable market as well as an economy, can give useful insights on how a market should be structured.

This Section is devoted to investigation on stability for dynamical system (2)(a)–(c). In Section 4.1, the detailed analysis on stability for all cases is illustrated. The analysis indicates that a systemic stability exists, — a stability which e.g. in economic term can be referred to a possible equilibrium point of the market, for all interacting agents. In several cases, as shown, several final states may occur according to the initial quota of the agents, thus leading to metastability or instability. When stable solutions are present, for any initial condition (i.e. for any choice of initial values for s_1, s_2, and s_3) the dynamic process converges to a configuration that will never change.

Metastable solutions can also be found: here, once the initial conditions are set up, the system is stable. Finally, unstable solutions depend on various parameters.

4.1. The mathematical results

Recall (2)(a)–(c); stability, $\dot{s}_i = 0$, $i = 1, 2, 3$ returns system

$$\begin{cases} \dot{s}_1 &= s_1 \left(\beta - s_1 - \left(1 + \gamma_{1,2} \right) s_2 - \left(1 + \gamma_{1,3} \right) s_3 \right) = 0 \\ \dot{s}_2 &= s_2 \left(\beta - \left(1 + \gamma_{2,1} \right) s_1 - s_2 - s_3 \right) = 0 \\ \dot{s}_3 &= s_3 \left(\beta - \left(1 + \gamma_{3,1} \right) s_1 - s_2 - s_3 \right) = 0 \end{cases} \tag{4}$$

Analysis of Fixed Points (FPs thereafter) of dynamical system (2)(a)–(c) entails the evaluation of eigenvalues of its Jacobian matrix, computed at each corresponding FP. FPs stability of the system can be thereby determined at any time t.

All mathematical calculations are reported in the Appendix. Solutions of system (4) determine a range of FPs. A detailed list of these FPs is relegated in the Appendix. Table 5 contains FPs for $\Gamma^{(+)}$ (symmetric cases) and $\Gamma^{(-)}$ (antisymmetric cases) while Table 6 displays FPs for $\Gamma^{(0)}$ (mixed reciprocity cases).

In general terms, when the real part of each eigenvalue is negative, the system is said to be stable, that is, the equilibrium solution at long term is independent of the initial situation. If at least one eigenvalue is equal to zero and the others have their real parts negative, the system is said to be metastable (at long term, the equilibrium solution depends of the initial situation). If at least one eigenvalue has a positive real part, the system is said to be unstable.

The Jacobian matrix $J = \left[\frac{\partial}{\partial s_k} \dot{s}_i \right]$, $i, k = 1, 2, 3$ when $\alpha_i = 1$ for (2)(a)–(c) is

$$J = \begin{cases} \beta - 2s_i - \sum_{j \neq i} s_j \left(1 + \gamma_{i,j} \left(1 - \frac{2s_i (s_i - s_j)}{\sigma^2} \right) \right) & k = i \\ \\ -s_i \left(1 + \gamma_{i,k} \left(1 + \frac{2s_k (s_i - s_k)}{\sigma^2} \right) \right) & k \neq i \end{cases}.$$

An analysis of the most interesting results in terms of stability follows. Stability for FPs has been studied when $\beta = 1$ (see Tables 2, 3, and 4).

Results on stability have been analytically obtained. Besides, numerical simulations for some initial conditions have been performed (see Figs. 1). This allows to show different behaviours of agents' quotas wrt time and how these trajectories lead, in any case, to a stable scenario. In all numerical simulations $\beta = 1$.

Table 2
Stability of FPs for Symmetric cases - $\beta = 1$ - M stands for Metastable while U for Unstable solutions. For instance, if $\sigma = 1$, $s = s_2 = s_3 = 0.428...$ and $s_1 = s_2 + s_3 = 2s$.

FP		Symmetric cases $\Gamma^{(+)}$		
Type		$\Gamma_1^{(+)}$	$\Gamma_2^{(+)}$	$\Gamma_3^{(+)}$
	No agent in the market			
(I)	$s_i = 0;\ i = 1, 2, 3$	U	U	U
	One agent in the market			
(IIa)	$s_1 \neq 0;\ s_2, s_3 = 0$	**Stable**	U	U
(IIb)	$s_2 \neq 0;\ s_1, s_3 = 0$	M	M	U
(IIc)	$s_3 \neq 0;\ s_1, s_2 = 0$	M	U	U
	Two agents in the market			
(IIIa)	$s_1 = 0;\ s_2, s_3 \neq 0$	M	M, U[b]	U
(IIIb)	$s_2 = 0;\ s_1, s_3 \neq 0$	U	**Stable**	**Stable**
(IIIc)	$s_3 = 0;\ s_1, s_2 \neq 0$	U	U	**Stable**
	All agents in the market ($s_i \neq 0;\ i = 1, 2, 3$)			
(IVa)	$s_1 = s_2 + s_3;\ s_2 = s_3$	U	–	**Stable**[a]
(IVb)	$2s_1 = s_2 + s_3$	–	–	–

[a]There is a unique solution for each σ.
[b]M or U depending on s.

Table 3
Stability of FPs for Antisymmetric cases - $\beta = 1$ - M stands for Metastable while U for Unstable solutions.

FP		Antisymmetric cases $\Gamma^{(-)}$		
Type		$\Gamma_1^{(-)}$	$\Gamma_2^{(-)}$	$\Gamma_3^{(-)}$
	No agent in the market			
(I)	$s_i = 0;\ i = 1, 2, 3$	U	U	U
	One agent in the market			
(IIa)	$s_1 \neq 0;\ s_2, s_3 = 0$	U	U	**Stable**
(IIb)	$s_2 \neq 0;\ s_1, s_3 = 0$	M	M	U
(IIc)	$s_3 \neq 0;\ s_1, s_2 = 0$	M	U	U
	Two agents in the market			
(IIIa)	$s_1 = 0;\ s_2, s_3 \neq 0$	M	M, U[a]	U
(IIIb)	$s_2 = 0;\ s_1, s_3 \neq 0$	–	–	–
(IIIc)	$s_3 = 0;\ s_1, s_2 \neq 0$	–	–	–
	All agents in the market ($s_i \neq 0;\ i = 1, 2, 3$)			
(IVa)	$s_1 = s_2 + s_3;\ s_2 = s_3$	–	–	–
(IVb)	$2s_1 = s_2 + s_3$	–	–	–

[a]M or U depending on s.

4.2. Stability and numerical simulations

Here a discussion of stable FPs is presented (Tables 2, 3, and 4). Several alternatives in a cell indicate that in such a case stability depends on the value for β and, possibly, for other parameters. All Subfigures below are collected in Fig. 1.

Symmetric interaction: when all agents compete, matrix $\Gamma_1^{(+)}$, no stable solution with all agents surviving is found (see case IVa in Table 2, first column). This can be considered in line, however extended to three agents, with the exclusion principle [24]. Here stability is reached in case IIa, with the leader only survivor (i.e. A_2 and A_3 ousted from the market). Fig. 1a depicts how market quotas reach the equilibrium.

Table 4
Stability of FPs for Mixed Reciprocity cases - $\beta = 1$ - M stands for Metastable while U for Unstable solutions.

FP		Mixed reciprocity cases $\Gamma^{(0)}$			
Type		$\Gamma_1^{(0)}$	$\Gamma_2^{(0)}$	$\Gamma_3^{(0)}$	$\Gamma_4^{(0)}$
	No agent in the market				
(I)	$s_i = 0;\ i = 1, 2, 3$	U	U	U	U
	One agent in the market				
(IIa)	$s_1 \neq 0;\ s_2, s_3 = 0$	U	U	**Stable**	U
(IIb)	$s_2 \neq 0;\ s_1, s_3 = 0$	M	U	U	U
(IIc)	$s_3 \neq 0;\ s_1, s_2 = 0$	M	U	M	M
	Two agents in the market				
(IIIa)	$s_1 = 0;\ s_2, s_3 \neq 0$	M	U	M	M
(IIIb)	$s_2 = 0;\ s_1, s_3 \neq 0$	U	–	U	–
(IIIc)	$s_3 = 0;\ s_1, s_2 \neq 0$	–	**Stable**	–	**Stable**
	All agents in the market ($s_i \neq 0, i = 1, 2, 3$)				
(IVa)	$s_1 = s_2 + s_3,\ s_2 = s_3$	–	–	–	–
(IVb)	$2s_1 = s_2 + s_3$	–	–	U	U

Other interactions with two agents out of three out of the market are possible; they are never stable, being either metastable or unstable.

This suggest that full competition causes minor players to go out of the market, regardless of their initial quotas.

When A_1 and A_2 compete while A_1 and A_3 collaborate (interaction as in matrix $\Gamma_2^{(+)}$), the only stable configuration (that is, of course, reached for all initial conditions) is reached for A_1 and A_3 in the market and A_2 forced out of it. Further, Fig. 1b shows that equilibrium values for A_1 and A_3 are the same and equal to 1. This means that the interaction between these two agents is capable of duplicating the capacity of the market while forcing A_2 getting ousted. Otherwise said, competing with the leader poses a follower out of the market.

Under full cooperation, interaction matrix $\Gamma_3^{(+)}$, stability can be reached either for all three agents surviving (case IVa under specific conditions $s_1 = s_2 + s_3$ and $s_2 = s_3$) or for one of the “minors” getting out: in case IIIb (IIIc) agent A_2 (A_3) succumbs while A_1 and A_3 (A_2) remains in the market. Further, Fig. 1c (1d) shows that in case IIIb (IIIc) the two surviving agents duplicate market capacity.

Antisymmetric interaction: when cheating is practised by all agents, the only stable configuration arises with A_1 cheating both A_2 and A_3 (Table 3, matrix $\Gamma_3^{(-)}$, case IIa). The final configuration is with both followers ruled out of the market, as shown in Fig. 1e. All other configurations are unstable or metastable. Cheating against a collaborating leader is, once again, bad practice for followers.

One may wonder why the leader should cheat both followers. This is a way of getting both followers out of the market and remain the only player.

Mixed reciprocity: Table 4 shows that here three stable solutions occur. The first one (matrix $\Gamma_3^{(0)}$, case IIa) deals with the leader only surviving agent. In this setting agent A_1 collaborates with A_2 while competing with A_3. The numerical depiction of this case is in Fig. 1f.

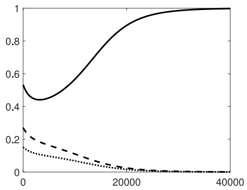
A second equilibrium (matrix $\Gamma_2^{(0)}$, case IIIc) is reached with A_3 forced out of the market while A_1 and A_2 survive (Sub Fig. 1g). Matrix $\Gamma_2^{(0)}$ explains why this occurs: A_3 is competing against A_1 while all other interactions are cooperative.

The third equilibrium (matrix $\Gamma_4^{(0)}$, case IIIc) is similar to the second one as here is agent A_2 that, while all other interactions are cooperative, competes against A_1 and is forced out of the market (Fig. 1h).

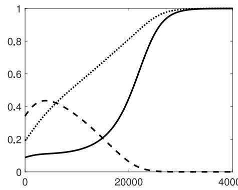
These results show that, again, followers should not compete with the leader.

4.3. Discussing cooperation, competition, leadership and cheating

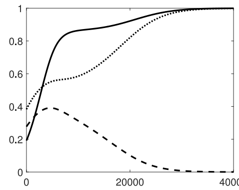
This section presents some economic comments upon stable results discussed in Section 4.2. The vocabulary mainly pertains to socio-economic-like or management cases [37], but it is easily imagined that *mutatis mutandis*



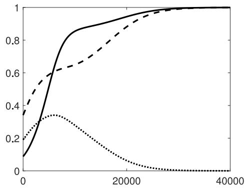
(a) $\Gamma_1^{(+)}$ - All agents are competing - A_1 only survivor - A_1 : $0.533 \to 1$, $A_2 : 0.271 \to 0$, $A_3 : 0.152 \to 0$



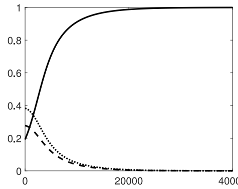
(b) $\Gamma_2^{(+)}$ - Agents A_1 and A_2 are competing, A_1 and A_3 collaborate - A_2 out of the market - $A_1 : 0.088 \to 1$, $A_2 : 0.341 \to 0$, $A_3 : 0.19 \to 1$



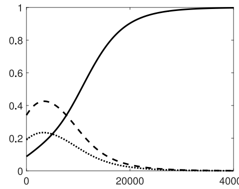
(c) $\Gamma_3^{(+)}$ - All agents are collaborating - A_2 out of the market - $A_1 : 0.192 \to 1$, $A_2 : 0.277 \to 0$, $A_3 : 0.383 \to 1$



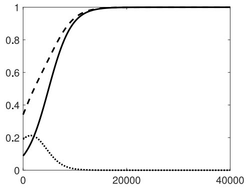
(d) $\Gamma_3^{(+)}$ - All agents are collaborating - A_3 out of the market - $A_1 : 0.088 \to 1$, $A_2 : 0.341 \to 1$, $A_3 : 0.19 \to 0$



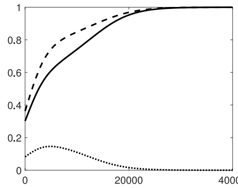
(e) $\Gamma_3^{(-)}$ - Agent A_1 is cheating both A_2 and A_3 - A_1 taking all - $A_1 : 0.192 \to 1$, $A_2 : 0.277 \to 0$, $A_3 : 0.383 \to 0$



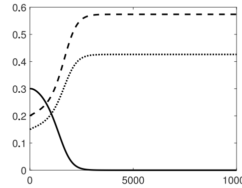
(f) $\Gamma_3^{(0)}$ - Agents A_1 and A_3 compete, A_1 is cheating A_2 - A_1 taking all - $A_1 : 0.088 \to 1$, $A_2 : 0.341 \to 0$, $A_3 : 0.19 \to 0$



(g) $\Gamma_2^{(0)}$ - Agents A_1 and A_2 collaborate, A_1 is cheating A_3 - A_3 out of the market - $A_1 : 0.088 \to 1$, $A_2 : 0.341 \to 1$, $A_3 : 0.19 \to 0$



(h) $\Gamma_4^{(0)}$ - Agents A_1 and A_2 collaborate, A_3 is cheating - A_3 out of the market - $A_1 : 0.302 \to 1$, $A_2 : 0.363 \to 1$, $A_3 : 0.082 \to 0$



(i) $\Gamma_1^{(+)}$ - Metastable equilibrium - All agents are competing - A_1 out of the market - $A_1 : 0.3 \to 0$, $A_2 : 0.2 \to 0.57143$, $A_3 : 0.15 \to 0.42857$

Fig. 1. Numerical simulations — time representation of interactions with stable equilibria ($\beta = 1, \sigma = 1$): agent A_1 (leader): continuous curve; agent A_2: dashed curve; agent A_3: dotted curve — Numerical values in each caption are the initial and the equilibrium quotas for each agent.

these comments hold for other situations, as in sport, in scientific research, in politics, in other societal processes, even in love relationships [40]. The keywords of such a discussion can be considered to be: maintenance or conservation and loss of leadership.

To illustrate these results some graphs have been depicted (see Fig. 1). The main message these plot convey is that the leader benefits of an evident as well as indisputable superiority and leaves followers to find a strategy that will not lead them of out the market.

In the considered triadic market, A_1 is in a leading position, essentially from the point of view of communication, while A_2 and A_3 share weak and negligible connections with each other. Communication counts: in fact, A_1 may be finally taking the entire market, even if she does not start with the highest initial quota of the market (see Fig. 1, Figs. 1e and 1f).

However, the competitive advantage of A_1 does not assure by itself that A_1 will definitely enjoy higher quotas or oust the other agents away or even survive into the market. Agent A_1 can be ousted from the market (cases IIb, IIc, and IIIa in Tables 2, 3, and 4) even if all these scenarios are not stable.

In case the interaction matrix is $\Gamma_1^{(+)}$, that is if all agents compete, the leader is forced out of the market whenever her initial quota is less than the sum of the initial quotas of the two followers (Fig. 1i). However, this is a metastable equilibrium. Further, lack of information on the mutual strategies does not allow a sort of alliance between minor agents. The presence of the leader is a condition for stability.

A general result is found: the three agents cannot all survive unless they behave under mutual collaboration and in very special conditions, (case IVa in matrix $\Gamma_3^{(+)}$). Thus, the need of survival triggers the behaviour (competition, cooperation and cheating) and, due to this, it may even happen that it is the leader that goes out the market. The competitive advantage is not permanent and therefore can be lost. This may be due to a lack of innovation, advancement in technology, etc., leading to a reduction in firm share in a competitive market or total exit from the market.

Conversely, there are (stable) situations in which the interaction among agents leads A_1 to the "leader takes all" type — a phenomenon which is caused by the existence of the so called Matthew effect also known as the "rich gets richer" effect [33,51].

On the other hand, the position of the two followers is highly difficult. In general, a possible good strategy for minor players is to collaborate with the leader, hoping not to be cheated. In fact, it has been discovered [2,13,16] that "Collaboration is a key driver of overall performance of companies around the world. Its impact is twice as significant as a company's aggressiveness in pursuing new market opportunities (...) and five times as significant as the external market environment (...)".

Cheating is a cause of instability. In fact, only a few cases reach stability: by the way, this occurs when cheating comes from the leader and the leader always wins.

As said above, for the minor agents, cheating the leader can end up with a disaster. Conversely, the leader may benefit from cheating, no matter the attitude of the other agents. The leader may be interested in cheating if she does not want to share the leadership with a follower, that is to get the monopoly of the market.

Even if a fair collaboration with the leader is established by one followers she, due to her size, could be forced to exit the market. This again depends on the attitude of the leader.

Collaboration with the leader may not always ensure market survival: even if all collaborate, one of the minor agents can go out. However, this can be a strategy to survive and even share the leadership, in particular when the other agent cheats (or is cheated by) the leader.

5. Conclusion

It is interesting to point out a few theoretical remarks from the literature concurring with this work's findings. Stiglitz [41], in his reflections on the state of the theory of monopolistic competition, intended to revolutionise the modelling of imperfectly competitive markets. He observed that the main distinctive features of monopolistic competition leading to economic agents' decisions are based on a few principles, sometimes (to say the least) weakly based on modelling. Hopefully, we provide some way of codifying such principles, and moreover we sustain their theoretical background. Recently, Mesak et al. [34] had also noticed that strategic changes improve performance and also leadership emergence. Thereby, by analogy with this specific reference, it is understandable that changes in our model parameters could be usefully investigated in terms of their relative influential impact on final equilibrium states.

Previously, Encaoua et al. [10] had stated that "In the struggle to create, maintain and expand favourable market positions, firms' actions are intended not only to affect the current conduct of rivals directly, but also to have an indirect effect by altering market structure in a way which constrains the rival's subsequent actions". These points are of great interest; they confirm the usefulness of the presently parsimonious model that could be applied, for instance, to the analysis of moral preferences, as illustrated in [5].

An interesting research question that will be tackled in the future is the following: could a follower gain a relevant role in a market if she owns a novel technology capable of granting her a stronger capability of penetrating the market itself? This requires to analyse a more general model than the one under scrutiny in this article. Innovation effect, and the consequent improved market penetration, is translated into this setting by larger values for parameters α.

Hermalin [23] suggests that "... one rationale for following is the leader has superior information about actions to be taken". Hence, the network structure applied in this article exactly matches this point.

Completely different scenarios may arise if a connection, even weak, is established between the two minor players. Coalitions are then possible and the leader might lose her predominance.

In this paper, the economic considerations are based on mathematical results and are focused on stability. This allows us drawing analytically the solutions of the differential equations system at any point in time. In this way, we can monitor the process, from the initial condition until the asymptotic fixed point. Later, it may be particularly interesting to study the metastable cases, possibly considering memory "constraints" in feedback mechanisms [21].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Determination of fixed points

This Appendix collects all Fixed Points (FPs) obtained by solving system of Eqs. (4) in the symmetric, antisymmetric (Table 5) and mixed reciprocity (Table 6) cases. Such solutions are divided in four cases, according to the number of equilibrium quotas that are strictly greater than 0.

Some FPs depend on β and on a parameter s. Bounds for s wrt to β are also provided. In some cases quotas s_i turn out being linked to a further constraint.

- **Case 1**. Stability for *Type I FPs*: all agents are forced out of the market ($s_1 = s_2 = s_3 = 0$). The Jacobian matrix is equal to an identity matrix; all its eigenvalues are therefore equal to 1, leading to an unstable equilibrium.
- **Case 2**. Stability for *Type II FPs*: only one agent survives so that the following three sub-cases, one for each agent remaining in the market, occur:
  - *Type IIa FPs* - only A_1 survives: $s_1 = \beta$, $s_2 = s_3 = 0$. The Jacobian matrix reads

$$J(\beta, 0, 0) = \begin{bmatrix} 1-2\beta & -\beta\left(1+\gamma_{1,2}\right) & -\beta\left(1+\gamma_{1,3}\right) \\ 0 & 1-\beta\left(1+\gamma_{2,1}\right) & 0 \\ 0 & 0 & 1-\beta\left(1+\gamma_{3,1}\right) \end{bmatrix}$$

    and its eigenvalues are $\lambda_1 = 1-2\beta < 0$, $\lambda_2 = 1-\beta\left(1+\gamma_{2,1}\right)$, and $\lambda_3 = 1-\beta\left(1+\gamma_{3,1}\right)$.
    Stable, metastable or unstable situations depend on β, $\gamma_{2,1}$, and $\gamma_{3,1}$. Cases $\Gamma_1^{(+)}$, $\Gamma_3^{(-)}$, $\Gamma_3^{(0)}$, according to formula (3) and recalling that $\tilde{\gamma}_{2,1} = \tilde{\gamma}_{3,1} = 1$, always lead to stability as all of the above eigenvalues are strictly negative. For the other interaction matrices stability, metastability or unstability can occur. For example, $\Gamma_2^{(+)}$ (for which $\tilde{\gamma}_{2,1} = 1$ and $\tilde{\gamma}_{3,1} = -1$) leads to a stable solution when $\beta > \frac{1}{1+\gamma_{3,1}}$, metastable when $\beta = \frac{1}{1+\gamma_{3,1}}$ or unstable, when $1 \leq \beta < \frac{1}{1+\gamma_{3,1}}$.
  - *Type IIb FPs* - only A_2 remains in the market: $s_2 = \beta$, $s_1 = s_3 = 0$. The Jacobian matrix now reads

$$J(0, \beta, 0) = \begin{bmatrix} 1-\beta\left(1+\gamma_{1,2}\right) & 0 & 0 \\ -\beta\left(1+\gamma_{2,1}\right) & 1-2\beta & -\beta \\ 0 & 0 & 1-\beta \end{bmatrix}$$

    and its eigenvalues are $\lambda_1 = 1-2\beta < 0$, $\lambda_2 = 1-\beta \leq 0$, and $\lambda_3 = 1-\beta\left(1+\gamma_{1,2}\right)$.
    Similarly to the previous case, we will have stable, metastable or unstable situations according the value of β and the sign of $\gamma_{1,2}$. Interactions $\Gamma_1^{(+)}$, $\Gamma_1^{(-)}$, $\Gamma_2^{(-)}$, $\Gamma_1^{(0)}$ are either stable or metastable; in all other cases unstability can arise.
  - *Type IIc FPs* - agent A_3 only remains in the market: $s_3 = \beta$, $s_1 = s_2 = 0$. The Jacobian matrix

$$J(0, 0, \beta) = \begin{bmatrix} 1-\beta\left(1+\gamma_{1,3}\right) & 0 & 0 \\ 0 & 1-\beta & 0 \\ \beta\left(1+\gamma_{3,1}\right) & -\beta & 1-2\beta \end{bmatrix}$$

    has eigenvalues equal to $\lambda_1 = 1-\beta \leq 0$, $\lambda_2 = 1-2\beta < 0$ and $\lambda_3 = 1-\beta\left(1+\gamma_{1,3}\right)$.
    Stable, metastable or unstable equilibria occur according to the value of β and the sign of $\gamma_{1,3}$. Interaction matrices $\Gamma_1^{(+)}$, $\Gamma_1^{(-)}$, $\Gamma_1^{(0)}$, $\Gamma_3^{(0)}$, $\Gamma_4^{(0)}$ end up in either stable or metastable equilibria, but in other cases unstability can appear.

Table 5
FPs for the Symmetric and Antisymmetric cases.

FP		Symmetric cases $\Gamma^{(+)}$		
Type		$\Gamma_1^{(+)}$	$\Gamma_2^{(+)}$	$\Gamma_3^{(+)}$
	No agent in the market			
(I)	$s_i = 0;\ i = 1, 2, 3$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
	One agent in the market			
(IIa)	$s_1 \neq 0;\ s_2, s_3 = 0$	$(\beta, 0, 0)$	$(\beta, 0, 0)$	$(\beta, 0, 0)$
(IIb)	$s_2 \neq 0;\ s_1, s_3 = 0$	$(0, \beta, 0)$	$(0, \beta, 0)$	$(0, \beta, 0)$
(IIc)	$s_3 \neq 0;\ s_1, s_2 = 0$	$(0, 0, \beta)$	$(0, 0, \beta)$	$(0, 0, \beta)$
	Two agents in the market			
(IIIa)	$s_1 = 0;\ s_2, s_3 \neq 0$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$
(IIIb)	$s_2 = 0;\ s_1, s_3 \neq 0$	$(\beta/3, 0, \beta/3)$	$(\beta, 0, \beta)$	$(\beta, 0, \beta)$
(IIIc)	$s_3 = 0;\ s_1, s_2 \neq 0$	$(\beta/3, \beta/3, 0)$	$(\beta/3, \beta/3, 0)$	$(\beta, \beta, 0)$
	All agents in the market ($s_i \neq 0;\ i = 1, 2, 3$)			
(IVa)	$s_1 = s_2 + s_3;\ s_2 = s_3$	$(2s, s, s)$ $\beta/6 < s < \beta/4$ $s = \frac{\beta}{2(2+\gamma_{2,1})}$	–	$(2s, s, s)$ $\beta/4 < s < \beta/2$ $s = \frac{\beta}{2(2+\gamma_{2,1})}$
(IVb)	$2s_1 = s_2 + s_3$	–	–	–
FP		Antisymmetric cases $\Gamma^{(-)}$		
Type		$\Gamma_1^{(-)}$	$\Gamma_2^{(-)}$	$\Gamma_3^{(-)}$
	No agent in the market			
(I)	$s_i = 0;\ i = 1, 2, 3$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
	One agent in the market			
(IIa)	$s_1 \neq 0;\ s_2, s_3 = 0$	$(\beta, 0, 0)$	$(\beta, 0, 0)$	$(\beta, 0, 0)$
(IIb)	$s_2 \neq 0;\ s_1, s_3 = 0$	$(0, \beta, 0)$	$(0, \beta, 0)$	$(0, \beta, 0)$
(IIc)	$s_3 \neq 0;\ s_1, s_2 = 0$	$(0, 0, \beta)$	$(0, 0, \beta)$	$(0, 0, \beta)$
	Two agents in the market			
(IIIa)	$s_1 = 0;\ s_2, s_3 \neq 0$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$
(IIIb)	$s_2 = 0;\ s_1, s_3 \neq 0$	–	–	–
(IIIc)	$s_3 = 0;\ s_1, s_2 \neq 0$	–	–	–
	All agents in the market ($s_i \neq 0, i = 1, 2, 3$)			
(IVa)	$s_1 = s_2 + s_3,\ s_2 = s_3$	–	–	–
(IVb)	$2s_1 = s_2 + s_3$	–	–	–

- **Case 3**. Stability for *Type III FPs*: here only one agent is pushed out of the market. Again, three sub-cases occur.
  - *Type IIIa FPs* - agent A_1 is the only one that goes out the market: $s_1 = 0$, $s_2 > 0$, $s_3 > 0$. Let $s_2 = s$, so that $s_3 = \beta - s$, the Jacobian matrix

$$J(0, s, \beta - s) = \begin{bmatrix} 1 - s\left(\gamma_{1,2} - \gamma_{1,3}\right) - \beta\left(1 + \gamma_{1,3}\right) & 0 & 0 \\ -s\left(1 + \gamma_{2,1}\right) & 1 - s - \beta & -s \\ -(\beta - s)\left(1 + \gamma_{3,1}\right) & s - \beta & 1 - 2\beta + s \end{bmatrix}$$

has now eigenvalues equal to $\lambda_1 = 1 - 2\beta < 0$, $\lambda_2 = 1 - \beta \leq 0$, and

$$\lambda_3 = 1 - s\left(\gamma_{1,2} - \gamma_{1,3}\right) - \beta\left(1 + \gamma_{1,3}\right) = 1 - \beta - s\gamma_{1,2} - (\beta - s)\gamma_{1,3}.$$

Table 6
FPs for the Mixed reciprocity cases.

FP		Mixed reciprocity cases $\Gamma^{(0)}$			
Type		$\Gamma_1^{(0)}$	$\Gamma_2^{(0)}$	$\Gamma_3^{(0)}$	$\Gamma_4^{(0)}$
	No agent in the market				
(I)	$s_i = 0;\ i = 1, 2, 3$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
	One agent in the market				
(IIa)	$s_1 \neq 0;\ s_2, s_3 = 0$	$(\beta, 0, 0)$	$(\beta, 0, 0)$	$(\beta, 0, 0)$	$(\beta, 0, 0)$
(IIb)	$s_2 \neq 0;\ s_1, s_3 = 0$	$(0, \beta, 0)$	$(0, \beta, 0)$	$(0, \beta, 0)$	$(0, \beta, 0)$
(IIc)	$s_3 \neq 0;\ s_1, s_2 = 0$	$(0, 0, \beta)$	$(0, 0, \beta)$	$(0, 0, \beta)$	$(0, 0, \beta)$
	Two agents in the market				
(IIIa)	$s_1 = 0;\ s_2, s_3 \neq 0$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$	$(0, s, \beta - s)$ $0 < s < \beta$
(IIIb)	$s_2 = 0;\ s_1, s_3 \neq 0$	$(\beta/3, 0, \beta/3)$	–	$(\beta/3, 0, \beta/3)$	–
IIIc)	$s_3 = 0;\ s_1, s_2 \neq 0$	–	$(\beta, \beta, 0)$	–	$(\beta, \beta, 0)$
	All agents in the market ($s_i \neq 0, i = 1, 2, 3$)				
(IVa)	$s_1 = s_2 + s_3,\ s_2 = s_3$	–	–	–	–
(IVb)	$2s_1 = s_2 + s_3$	–	–	$(2s, s, 3s)$	$(2s, 3s, s)$
		–	–	$\beta/8 < s < \beta/6$	$\beta/6 < s < \beta/4$
		–	–	$s = \frac{\beta}{2(3+\gamma_{2,1})}$	$s = \frac{\beta}{2(3+\gamma_{2,1})}$

Again, we will have stable, metastable or unstable situations according the value of β, the value of s and the signs of $\gamma_{1,2}$ and $\gamma_{1,3}$.
In this case stability depends also on $s - \beta$, that is, how far is s from β.
Interaction matrices $\Gamma_1^{(+)}$, $\Gamma_1^{(-)}$, $\Gamma_1^{(0)}$ lead to either stable or metastable equilibria. In all the other cases unstability might arise.

- *Type IIIb FPs* - agent A_2 only goes out the market: $s_2 = 0$, $s_1 > 0$, $s_3 > 0$. All FPs verify equality $s_1 = s_3$. Letting $s_1 = s_3 = s$ it turns out that either $s = \beta/3$ or $s = \beta$. The Jacobian matrix reads

$$J(s, 0, s) = \begin{bmatrix} 1 - 2s - s(1 + \gamma_{1,3}) & -s(1 + \gamma_{1,2}) & -s(1 + \gamma_{1,3}) \\ 0 & 1 - s(1 + \gamma_{2,1}) - s & 0 \\ -s(1 + \gamma_{3,1}) & -s & 1 - 2s - s(1 + \gamma_{3,1}) \end{bmatrix}$$

and its eigenvalues are $\lambda_1 = 1 - 2s$, $\lambda_2 = 1 - s(2 + \gamma_{2,1})$, and $\lambda_3 = 1 - s(4 + \gamma_{1,3} + \gamma_{3,1})$. If $s = \beta$, interaction matrices $\Gamma_2^{(+)}$ and $\Gamma_3^{(+)}$ lead to stable situation. If, instead, $s = \beta/3$ interaction functions $\Gamma_1^{(+)}$, $\Gamma_1^{(0)}$, and $\Gamma_3^{(0)}$ will provide stable ($\beta > 3/2$), metastable ($\beta = 3/2$) or unstable ($\beta < 3/2$) equilibria.

- *Type IIIc FPs* - agent A_3 only goes out of the market: $s_3 = 0$, $s_1 > 0$, $s_2 > 0$. Similarly as above, all FPs verify $s_1 = s_2 = s$. Once again, these values can be equal to either $\beta/3$ or β. The Jacobian matrix is

$$J(s, s, 0) = \begin{bmatrix} 1 - 2s - s(1 + \gamma_{1,2}) & -s(1 + \gamma_{1,2}) & -s(1 + \gamma_{1,3}) \\ -s(1 + \gamma_{2,1}) & 1 - 2s - s(1 + \gamma_{2,1}) & -s \\ 0 & 0 & 1 - s(1 + \gamma_{3,1}) - s \end{bmatrix}$$

and its eigenvalues are $\lambda_1 = 1 - 2s$, $\lambda_2 = 1 - s(2 + \gamma_{3,1})$, and $\lambda_3 = 1 - s(4 + \gamma_{1,2} + \gamma_{2,1})$.
Stable equilibria for matrices $\Gamma_3^{(+)}$, $\Gamma_2^{(0)}$, and $\Gamma_4^{(0)}$ occur when $s = \beta$. If, instead, $s = \beta/3$ interaction matrices $\Gamma_1^{(+)}$ and $\Gamma_2^{(+)}$ will show stable, metastable or unstable situations depending on β (respectively $\beta > 3/2$, $\beta = 3/2$, $\beta < 3/2$).

- **Case 4**. Stability for *Type IVa FPs* - all agents remain in the market.

Necessary conditions for this result are $\gamma_{3,1} = \gamma_{2,1}$, and either $s_2 = s_3$ or $s_2 + s_3 = 2s_1$. Possible FPs solutions are plausible only for matrix $\Gamma_1^{(+)}$ and $\Gamma_3^{(+)}$, if $s_2 = s_3$, and for matrices $\Gamma_3^{(0)}$ and $\Gamma_4^{(+)}$ if, instead, $s_2+s_3 = 2s_1$. Suppose $s_2 = s_3$. In these two symmetric cases $\gamma_{1,2} = \gamma_{1,3} = \gamma_{2,1} = \gamma_{3,1}$. Letting these quantities be equal to γ, then $s_2 = s_3 = \frac{1}{2(2+\gamma)}$, $s_1 = s_2 + s_3 = \frac{1}{2+\gamma}$ (implicit solution). Denoting $s = s_2 = s_3$, the Jacobian matrix is

$$J(2s, s, s) = \begin{bmatrix} \frac{\gamma}{(2+\gamma)^3\sigma^2} - \frac{1}{2+\gamma} & -\frac{\gamma}{2(2+\gamma)^3\sigma^2} - \frac{\gamma+1}{2+\gamma} & -\frac{\gamma}{2(2+\gamma)^3\sigma^2} - \frac{\gamma+1}{2+\gamma} \\ -\frac{\gamma}{(2+\gamma)^3\sigma^2} - \frac{\gamma+1}{2(2+\gamma)} & \frac{\gamma}{(2+\gamma)^3\sigma^2} - \frac{1}{2(2+\gamma)} & -\frac{1}{2(2+\gamma)} \\ -\frac{\gamma}{(2+\gamma)^3\sigma^2} - \frac{\gamma+1}{2(2+\gamma)} & -\frac{1}{2(2+\gamma)} & \frac{\gamma}{(2+\gamma)^3\sigma^2} - \frac{1}{2(2+\gamma)} \end{bmatrix}$$

whose eigenvalues are

$$\lambda_1 = \frac{\gamma}{2(2+\gamma)^3\sigma^2}, \qquad \lambda_2 = -1, \qquad \lambda_3 = \frac{3\gamma}{2(2+\gamma)^3\sigma^2} + \frac{\gamma}{2+\gamma}$$

It turns out that solution is unstable for $\Gamma_1^{(0)}$, when $(\gamma > 0)$ and stable for $\Gamma_3^{(0)}$, (when $\gamma < 0$).
Further suppose $s_2 + s_3 = 2s_1$. Mixed reciprocity case $\Gamma_3^{(0)}$ is characterised by equality $\gamma_{2,1} = \gamma_{3,1} = \gamma_{1,3} = -\gamma_{1,2} > 0$. Denoting $\gamma = \gamma_{1,2} < 0$, then $s_2 = \frac{1}{2(3-\gamma)}$, $s_3 = \frac{3}{2(3-\gamma)}$, $s_1 = \frac{1}{3-\gamma}$ (implicit solution). Denoting also $s = s_2$, the Jacobian matrix is

$$J(2s, s, 3s) = \begin{bmatrix} \frac{2\gamma}{(3-\gamma)^3\sigma^2} - \frac{1}{3-\gamma} & -\frac{\gamma}{2(3-\gamma)^3\sigma^2} - \frac{\gamma+1}{3-\gamma} & -\frac{3\gamma}{2(3-\gamma)^3\sigma^2} - \frac{1-\gamma}{3-\gamma} \\ -\frac{\gamma}{2(3-\gamma)^3\sigma^2} - \frac{1-\gamma}{2(3-\gamma)} & \frac{\gamma}{2(3-\gamma)^3\sigma^2} - \frac{1}{2(3-\gamma)} & -\frac{1}{2(3-\gamma)} \\ \frac{3\gamma}{2(3-\gamma)^3\sigma^2} - \frac{3(1-\gamma)}{2(3-\gamma)} & -\frac{3}{2(3-\gamma)} & -\frac{3\gamma}{2(3-\gamma)^3\sigma^2} - \frac{3}{2(3-\gamma)} \end{bmatrix}$$

whose eigenvalues are

$$\lambda_1 = \frac{3\gamma}{2(3-\gamma)^3\sigma^2}, \qquad \lambda_2 = -\frac{\gamma}{3-\gamma}, \qquad \lambda_3 = -\frac{3\gamma}{2(3-\gamma)^3\sigma^2} - 1.$$

As $\gamma < 0$, $\lambda_2 > 0$ implies that the solution obtained is unstable.
A final comment on the mixed reciprocity case $\Gamma_4^{(0)}$ is due. Here $\gamma_{2,1} = \gamma_{3,1} = \gamma_{1,2} = -\gamma_{1,3} < 0$. Denoting $\gamma = \gamma_{1,2} < 0$, the implicit solution is $s_3 = \frac{1}{2(3+\gamma)}$, $s_2 = \frac{3}{2(3+\gamma)}$, $s_1 = \frac{1}{3+\gamma}$. Let $s = s_3$: the Jacobian matrix is

$$J(2s, 3s, s) = \begin{bmatrix} -\frac{2\gamma}{(3+\gamma)^3\sigma^2} - \frac{1}{3+\gamma} & \frac{3\gamma}{2(3+\gamma)^3\sigma^2} - \frac{\gamma+1}{3+\gamma} & \frac{\gamma}{2(3+\gamma)^3\sigma^2} - \frac{1-\gamma}{3+\gamma} \\ -\frac{3\gamma}{2(3+\gamma)^3\sigma^2} - \frac{3(\gamma+1)}{2(3+\gamma)} & -\frac{3\gamma}{2(3+\gamma)^3\sigma^2} - \frac{3}{2(3+\gamma)} & -\frac{3}{2(3+\gamma)} \\ \frac{\gamma}{2(3+\gamma)^3\sigma^2} - \frac{\gamma+1}{3+\gamma} & -\frac{1}{2(3+\gamma)} & -\frac{\gamma}{2(3+\gamma)^3\sigma^2} - \frac{1}{2(3+\gamma)} \end{bmatrix}$$

Eigenvalues for this matrix are not easy to calculate but its determinant is positive; this matrix will then have at least a real positive eigenvalue. This allows to conclude that the solution is unstable.

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