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EFFECTIVE DARK MATTER AND DARK ENERGY FROM GENERAL RELATIVITY

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I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at any educational institution, except where due acknowledgment is made in the thesis. Any contribution made to the research by others with whom I have worked is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

The sections 2.2, 2.3 and 2.4 are based on my article [113], published in *European Physical Journal C*. The rest of the chapter 2 and the chapter 3 are respectively based on my articles [114] and [108], which are currently under review.

Primum Graius homo mortalis tollere contra
Est oculos ausus primusque obsistere contra;
Quem neque fama deum nec fulmina nec minitanti
Murmure compressit caelum, sed eo magis acrem
Inritat animi virtutem, effringere ut arta
Naturae primus portarum claustra cupiret.
Ergo vivida vis animi pervicit et extra
Processit longe flammantia moenia mundi
Atque omne immensum peragravit mente animoque,
Unde refert nobis victor quid possit oriri,
Quid nequeat, finita potestas denique cuique
Qua nam sit ratione atque alte terminus haerens.
Tito Lucrezio Caro, “De Rerum Natura”, Libro I

Ma guardate l'idrogeno tacere nel mare,
Guardate l'ossigeno al suo fianco dormire.
Soltanto una legge che io riesco a capire
Ha potuto sposarli senza farli scoppiare,
Soltanto la legge che io riesco a capire
Fabrizio de André, “Un chimico”

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Introduction: The Concordance Model and its open problems

The Cosmological Concordance Model (CCM) is the best description of the evolution of the universe we managed to have today. It applies the standard physical laws, assuming they were valid in the past, fixing the parameters with the actual observations.

First of all, the Concordance Model applies General Relativity, describing the whole space-time as a manifold, curved by the energy-matter following the Einstein Equations [1]

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{0.0.1}$$

The energy-momentum tensor $T_{\mu\nu}$ follows the laws of physics. The Einstein Equations are highly non-linear PDEs, impossible to solve in their most general form, so some simplifying postulates are needed.

It is usually assumed the Cosmological Principle, i.e., the homogeneity and isotropy of the space, at any instant. The anthropocentrism is outdated, so that it has been abandoned the existence of any preferential point or direction. However, notice that the Cosmological Principle can be seen as a philosophical statement, because we cannot observe the Universe from other location rather than Earth. Its empirical validity can only be checked indirectly, from the goodness of the cosmological theories built on it.

At the human scale, or even at the Solar System ones, the matter is evidently inhomogeneous, thus the Cosmological Principle could seem to be ill posed. For larger and larger scales, a set of measures supported the reaching of a better and better homogeneity[1] [2], [3], [4] and isotropy [5]. Hence, it is usually considered a good assumption for a whole-universe theory. Anyway, the existence of an homogeneity scale and its greatness is currently under debate; see e.g. [6], [7], and [8], [9], [10], [11].

0.1 The FLRW universe

A spatially homogeneous metric takes the form of the Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right] \tag{0.1.1}$$

everywhere, where t is the time coordinate, r the radial one, and $d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2$ includes the two angular coordinates on a sphere. Here $a(t)$ is the expansion parameter,

[1]Notice here the difference between an observed homogeneity and a true spatial homogeneity, due to the relativistic retard. As one observes deeper and deeper along the past light cone one departs from our present time hypersurface in a FLRW model, so even the standard model becomes observationally inhomogeneous. For this reason, observations on deep redshift are essential to support or confute the homogeneity; see the following references.

taken s.t. $a(t_0) := 1$ today, and K is the actual gaussian spatial curvature. Moreover, the Hubble parameter is usually defined as $H(t) := \dot{a}/a$.

For a perfect fluid, $T_{\mu\nu} := (\rho + p)U_\mu U_\nu - p g_{\mu\nu}$, which is comoving $U^\mu = \partial_t$ with respect to the FLRW coordinates, there survive just two independent Einstein Equations

$$\begin{cases} \frac{\dot{a}^2+k}{a^2} = \frac{8}{3}\pi G\rho \\ \frac{2\ddot{a}a+\dot{a}^2+k}{a^2} = -8\pi G p \end{cases} ; \tag{0.1.2}$$

and the Energy Conservation says

$$\dot{\rho} = -3H(\rho + p). \tag{0.1.3}$$

Let the fluid be a superposition of components $\rho := \sum_w \rho_w$ with State Equations $p_w = w\rho_w$.[2] Thus, assuming that the transformation of components into each others is negligible,

$$\rho_w(t) = \rho_{w0} a(t)^{-3(1+w)}. \tag{0.1.4}$$

Replacing this into the Einstein Equations, one finds the Friedmann Equation for the universe expansion

$$\frac{\dot{a}^2}{a^2} = \sum_w \Omega_{w0} a^{-3(1+w)} + \Omega_{K0} a^{-2}, \tag{0.1.5}$$

where are defined the cosmological parameters

$$\Omega_w(t) := \frac{8\pi G}{3H_0^2}\rho_w(t), \quad \Omega_{K0} := -\frac{K}{H_0^2} \quad s.t. \quad \sum_w \Omega_{w0} + \Omega_{K0} = 1. \tag{0.1.6}$$

With only matter and radiation as components, (0.1.5) returns a growing, decelerating solution $a(t)$, with an initial singularity $a(t_{BB}) = 0$ [12]. Such "Big Bang" model is empirically supported by the following results:

- The measure of luminosity distance of far galaxies returned the Hubble Law $\dot{d} = H_0 d$ with $H_0 > 0$, which support an homogeneous expansion of the space [13].
- It was found the Cosmic Microwave Background [14], that suggest the decoupling between matter and radiation at a redshift $z \cong 1100$ [15], when the expansion lead the temperature of the universe at low enough values [16], [17], [18].
- The abundance of deuterium, helium and lithium in the universe are coherent with their production during primordial nucleosythesis, when the average temperature has gone down enough to allow the formation of atomic nuclei [16], [17], [18].

The formation of structures, like star and galaxies, is described [19], [20], [21], [22], [23], [24] as the collapse of a certain inhomogeneity of matter inside the universe (0.1.1). In the Friedmann Model it is assumed to be small enough to not break the Cosmological Principle; however, it has consequences on the Cosmic Microwave Background inhomogeneities we see nowadays.

[2] *E.g.*, the radiation has a zero-trace energy-momentum tensor, which means $w = 1/3$. The pressure of matter is negligible, w.r.t. c^2, hence the matter is described as a "dust" with $w = 0$.
We do not specify the starting and finishing values for the sums on w, because in principle we can imagine the presence of components with any value of $w \in \mathbb{R}$; *e.g.*, we can't write "$\sum_{w=0}^{\infty}$", because it would the possibility of a cosmological constant $w = -1$. On the other hand, we will not write "$\sum_{w=-\infty}^{+\infty}$", or "$\sum_{w\in\mathbb{R}}$", because the usual models do not contain components as $w = -\sqrt{2}$ or $w = 1000$. w rarely varies outside $[-1; 1]$, although we can not exclude other possibilities. From now on, we will write just "$\sum_w$", meaning by this $\sum_{w\in\mathcal{W}}$, for a certain set $\mathcal{W} \subset \mathbb{R}$ of values, differently chosen for any model of the universe.

0.2 The dark energy

Some better measures of luminosity distances of far objects allowed the evaluation of the deceleration parameter

$$q_0 := -\frac{\ddot{a}(t_0)}{H_0^2} = \sum_w \frac{1+3w}{2}\Omega_{w0}. \tag{0.2.1}$$

It was expected to be positive, since there are only known matter and radiation as components. Surprisingly, it turned out to be negative[25],[26]; hence, the universe expansion is accelerating.
It requires, in a FLRW model, some kind of new component with $w < -1/3$. Such forms of energy are of exotic type. The most simple kind is the dark energy, or cosmological constant, with $w = -1$, which can be expressed as a covariant correction of Einstein Equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{0.2.2}$$

It seems to be some kind of “energy of vacuum”, i.e. an energy which belongs to the space itself, with a constant density $\rho_\Lambda := \frac{\Lambda}{8\pi G}$.

0.3 The dark matter

Another unexpected fact about universe components is a contribution to the matter parameter which does not correspond to any visible matter. It behaves as a type of matter[3] which does not interact through electromagnetism, avoiding to emit or absorb light - in this sense it is “dark”.

Dark matter was discovered measuring the virial of the Coma Cluster [27]: it corresponded to a gravitational mass quite bigger than the observed one. It was interpreted as the presence of an invisible mass. Afterwards, the evidences of dark matter increased, outlining a universe with far more dark matter than the observable one, at least the 80% of the total. We can divide these phenomena in two categories: global dark matter effects, which consists in unexpected values of cosmological parameters, and local dark matter effects, that arise from observations of astronomical objects.

0.3.1 Local dark matter (DM) phenomena

- In a spiral galaxy, the rotation velocity of the objects around the center is determined by the gravitational attraction of the mass inside their orbit. Since the luminous mass declines beyond the arms, the rotation curves are expected to decrease as $1/\sqrt{r}$ about. Instead, they remains flat for all the halo [28]. It seems to require there the presence of non-luminous matter.
- Many components systems interacting with newtonian gravity must obey the Virial Theorem, which links the velocity distribution to the total mass. Applying it to elliptic galaxy or galaxy clusters, the observed luminous mass does not match with the virial predictions [27], [29], [30].
- For GR, any massive objects curves the space-time, so that a light ray passing by is deviated. Such gravitational lensing allows to measure the gravitational mass of stars, galaxies and galaxy clusters. This provides an independent proof that the gravitation of extended objects is quite bigger than their observable mass [31].

[3] i.e. a zero-pressure energy

These are the most important local DM phenomena, but there are other ones: X-rays emitted by hot gas [30], etc...

0.3.2 Global dark matter phenomena

- According to the most recent measures [32],[33]

$$\begin{aligned} q_0 &= \frac{1}{2}\Omega_{M0} - \Omega_{\Lambda 0} = -0.527 \pm 0.0105 \\ \Omega_{K0} &= 1 - \Omega_{R0} - \Omega_{M0} - \Omega_{\Lambda 0} = 0.02 \pm 0.02 \\ \Omega_{R0} &= 8.24 \times 10^{-5} \pm 10^{-7}. \end{aligned} \tag{0.3.1}$$

  This leads to a fraction $\Omega_{\Lambda 0} = 0.685 \pm 0.007$ of actual dark energy, but provides also a value for the parameter $\Omega_{M0} = 0.315 \pm 0.007$ for the total actual matter (see §2.1.2 for a rigorous definition of these Ω parameters). However, the observed matter is only $\Omega_{BM0} = 0.0486 \pm 0.0010$, which means an amount of dark matter in the universe of $\Omega_{DM0} = 0.266 \pm 0.008 = (84.4 \pm 4.4)\% \times \Omega_{M0}$.

- The Fourier study of CMB inhomogeneities gives peaks which are related to cosmological parameters. The first one determines the curvature Ω_{K0} of the universe; the second one gives the density of baryonic matter, i.e. the matter which interacts with photons; while the third peak returns to the gravitational potential of matter, hence it is related also to non baryonic matter [15], [34]. It returns a similar amount of dark matter.

- The amount of observed matter is not enough to collapse on itself and form the structure we see (stars, galaxies, etc.), if the time passed is just the measured age if universe $\sim 13.8Gy$. However, the presence of dark matter would provide early some regions of gravitational attraction, where galaxies and clusters fall [35], [24]. It allows the structure formation in such a short time.

These are the most important global DM phenomena, but there are other ones: the baryon acoustic oscillations [36], [37], the Lyman-alpha forest [38], etc...

0.4 Inflation

A third patch on the Friedman model is the addition of the inflation epoch: a certain period in the early universe when the expansion was not dominated by the radiation or the matter, but increased exponentially [39], [40]. The most simple description is given by a huge amount of dark matter which filled the universe, making the metric to be a de Sitter space, for a time long enough to give a $\sim e^{60}$ expansion factor.

Such a inflation solves the "horizon problem": widely separated regions of the observable universe results to be in thermal equilibrium, although in a big bang with only the matter and radiation they have never come into causal contact. Instead, causal link is allowed in inflationary universe. Moreover, the inflation leads to the observed spatial flatness without requiring fine tuning on the parameters.

0.5 Baryogenesis

Here we cite one last open problem in cosmology: the asymmetry between matter and anti-matter. According to the Standard Model, there is a fundamental symmetry between particles and their respective anti-particles[4], such that their decay times are equal and any creation or destruction of a particle must correspond to a creation or destruction of an anti-particle. Hence, assuming that at the Big Bang instant there was an equal number of particles and anti-particles, it should be the same today. Instead, we observe astronomical structures of matter only.

The Concordance Model describes the disappearance of the anti-matter during a early epoch of the universe, named the Matter-AntiMatter Recombination. However, a small part of matter must have survived. This is usually expressed assuming that the amounts of matter and anti-matter at the very beginning were not exactly the same, and their difference is what we see today. However, this cosmological phenomenon could be a clue that the fundamental symmetry is slightly broken [41], [42], [43], [44].

[4]Here we are not talking about the C symmetry, which is broken for electroweak interactions. We refer to the conservation of baryonic number, and to similar conservation laws for any generation of leptons.

Chapter 1

Attempts to explain dark matter and dark energy

We can define the dark matter as a set of phenomena for which the apparent gravitation is quite bigger with respect to what it would be expected from the observable matter. Notice that what we “expect” is what it is returned by the theoretical models and related conventions, which are normally used to describe the system. For the phenomena localized in galaxies or clusters, such model is the newtonian one, since the relativistic corrections are usually assumed to be negligible. For the phenomena concerning the universe globally, we refer to the Concordance Model on FLRW metric.

1.1 True dark matter

The simplest explanation for the discrepancy between the model and the observable matter, would be the presence of some unobserved matter. It would not surprise that our current knowledge about the composition of the whole universe can be quite incomplete. However, we can try to classify the possible kinds of dark matter according to the known laws of physic.

First of all, we may wonder if such a invisible matter travels near to the speed of light, what is called “hot” dark matter, or has a negligible $v \ll c$, being a “cold” dark matter.

1.1.1 Hot dark matter

Only microscopic particles can reach so high speeds. There are a lot of particles proposed to constitute the hot dark matter; the most natural choice among the Standard Model is the neutrino. Indeed, we know that neutrinos have small masses[1], and they do not have electromagnetic interactions, but just gravitational or weak interactions. Usually, the neutrinos travel nearly at the speed of light and are very hard to be observed.

Anyway, it is possible to estimate the number of neutrinos. Before the CMB, it was produced a Cosmic Neutrino Background; even if we can not observe it directly, we can estimate its magnitude from other cosmic parameters. Also the neutrino production in stars can be estimated. The total amount from these contributions is not enough to justify the dark matter in the galaxies, neither the global dark matter [46].

There are other proposal for hot dark matter particles; e.g. sterile neutrinos, the neutrinos with right-handed chirality, which have neither weak interactions according to

[1]We know that because it was observed the oscillation between the three families of leptons, for free neutrinos. See e.g. [45]

the Standard Model [47]. A particle of such a kind has few relations to any cosmic parameter and is hardly detectable.

But all speculations about hot dark matter have to deal with the good concordance between local and global dark matter phenomena. Both of them reveal a fraction of dark matter which is about 85% of the total. Instead, any kind of hot dark matter would have a too high speed to belong to a gravitationally closed system [48]. Hence, it would contribute to the global phenomena but not to the local ones. The gap between them can be just a few percentage points. This leads to the conclusion that only a small part of the dark matter can be "hot", and it is necessary some other explanation.

1.1.2 Cold dark matter: WIMPs

A particle that constitutes cold dark matter would be quite massive, in order to have low velocities. A such hypothetical particle is usually imagined to have weak interactions, so that it can be produced thermally in the early Universe. The abundance of dark matter today puts some requirements on the cross section, to be obtained, which would be valid for a particle with a mass of the order of $100GeV$. All these features are resumed naming them "Weakly Interactive Massive Particles".

The Standard Model does not contain WIMPs, but similar particles appear in some beyond-SM hypotheses, especially in supersymmetric models. In the Minimal Supersymmetric Standard Model (MSSM), there exist four combinations between superpartners of the weak gauge bosons (bino and wino) and the superpartner of Higgs boson (higgsino), that have zero charge and no color. These leptons are called "neutralinos": $\tilde{N}_j^0$, for j from 1 to 4. The lightest neutralino would have a mass around $300GeV$, and would be stable if R-parity is conserved.

$\tilde{N}_1^0$ resulted to be an excellent WIMP candidate, a coincidence which is known as "WIMP miracle" [49]. No other supersymmetric particles has a such similarity to WIMPs. If the sneutrinos would exist with the amount required for the dark matter, their interaction events would be already detected, according to MSSM. Also gravitinos are proposed by SUGRA[2], but they would have a mass of $1eV$ about, hence their thermal production would not be enough to constitute a relevant fraction of the dark matter.

The WIMP miracle was a strong argument which made the SUSY cheering. However, the LHC detection beyond the $100GeV$ energy scale did not find any particle, except for the Higgs boson [50], [51], [52]. This seems to exclude the SUSY hypothesis; and even if SUSY particle would exist with some bigger masses, they would not have the characteristics required for WIMPs.

The lightest neutralino was the best candidate for a WIMP, but not the only one. *E.g.* in extra-dimensional theories there are analogous particles, called "Lightest Kaluza-Klein Particles"(LKPs).

There was a lot of detection experiments which tried to see interactions with WIMPs, i.e. weak interactions similar to those of neutrinos, except for the bigger mass. Even if there are no dark matter effects inside the Solar System (the orbits of planets follow the GR, indeed), weakly interactive particles should pass through the Earth, hence a particle detector should find them. But no relevant events was detected, up to now [53], [54], [55], [56], [57]. This seems to exclude the existence of any kind of WIMPs: neutralinos, LKPs, axions (another hypothetical elementary particle, postulated to resolve the strong CP problem in QCD) or any other.

[2]SUGRA means "SUper Gravity models", i.e. quantum theories of gravity which are supersymmetric. In quantum gravity, the graviton is the particle associated to the $g_{\mu\nu}$ field, and its supersymmetric partner is called gravitino.

If dark matter is constituted by some massive particles, they have no weak interactions. This makes their detection very hard, even if they would exist. Moreover, it is not clear how they could be produced; they should exist from the very beginning of universe, without relation to other parameters. This situation suggests to consider dark matter which is not a particle.

1.1.3 Cold dark matter: MaCHOs

Some kinds of astrophysical objects do not emit radiations: black holes, neutron stars, brown dwarfs and unassociated planets (i.e. planets that are not orbiting around a star). If they are present with a big enough number, along the halos of galaxies, they could constitute the dark matter without requiring any hypothetical particle. These are called "Massive Compact Halo Objects".

It is hard to detect MaCHOs, but it is possible with gravitational microlensing. Some research groups enumerated the microlensing which happens when a dark galactic object passes nearly in front of a star, with respect the our line-of-sight. A further statistical analysis estimated the total amount of MaCHOs, which turned up to be not more than the 8% of the required dark matter [58], [59].

The microlensing technique applies to objects between $10^{22}kg$ (less than the mass of the Moon) and $10^{32}kg$ (hundred times the mass of the Sun). This rules out the planets, the brown dwarfs and any relic of stars. However, for black holes there is another proposed origin, besides star's deaths. If sufficiently dense regions underwent gravitational collapse, soon after the Big Bang, they would produce "Primordial black holes" [60]. Some considerations restrain the amount of primordial black holes with mass below $10^{22}kg$.

- The Hawking radiation has an inverse proportion to the mass of black hole, thus black holes with mass less than $10^{11}kg$ have a lifetime shorter than the age of the universe [61].

- Black holes with a little more masses could be now observed to evaporate and explode. The searching of such explosions [62] concluded that black holes with mass up to $10^{13}kg$ contribute to dark matter to less than the one percent.

- Even if the microlensing of the star light can be detected only for objects with masses bigger than $10^{22}kg$, the gamma-ray burst provides far more powerful sources. The observation of such events [63] allowed to estimate the number of black holes with mass between $5 \times 10^{14}kg$ and $10^{17}kg$, and they cannot contribute importantly to the dark matter.

- If primordial black holes with masses between $10^{15}kg$ and $10^{22}kg$ had abundances comparable to that of dark matter, neutron stars in globular clusters should have captured some of them, leading to the rapid destruction of the star. Observing neutron stars in globular clusters, it was possible to rule out also this range of masses [64].

Taking all these results together, the only MaCHO candidate capable of explaining the majority of the dark matter remains the primordial black holes with masses between $10^{13}kg$ and $5 \times 10^{14}kg$. Assuming plausibly that they have a negligible amount, we can conclude that at most the 10% of the dark matter can be constituted by MaCHOs. This leaves the 77% of the gravitational phenomena unexplained by any kind of imagined true matter.

1.2 Alternatives to General Relativity

1.2.1 MOG theories

If our theories predict a certain amount of matter, but any attempt to find such a matter failed, maybe we should modify the theory itself. This challenging idea led to the MOdified Gravitational theories: some new terms are added to the usual gravitational law, small enough for interplanetary distances to justify the usual gravitation in those limit, but not negligible at galactic scale, so that an apparent dark matter arises.

This line of research started with the MOdified Newtonian Dynamic (MOND), which inserts a dependence by the acceleration inside the Newton law of gravitation[3]. It hence becomes

$$F_G = \mu(a)\frac{GMm}{r^2},$$

where the "interpolating function" $\mu(a)$ is just a factor 1 for high accelerations, but goes as a_0/a for accelerations smaller than a constant a_0. Since for galactic distances the gravitational acceleration is very small, the MOND law is

$$ma^2/a_0 = GMm/r^2,$$

which returns a velocity independent by the radius. The function $\mu(a)$ can by always chosen in order to interpolate the rotation curve of a galaxy, thus explaining this particular dark matter phenomenon [65]. It can be done the same for the virials of clusters.

Even if the Newton law is often used for the galactic dynamic, we know that the physics is relativistic; the MOND law cannot be the right gravitational law. Moreover, a lot of dark matter phenomena involve the GR, as the gravitational lensing, and almost all the global phenomena.

For these reasons, there were proposed relativistic generalizations of MOND. The most famous is the Tensor-vector-scalar gravity (TeVeS), obtained adding to the Einstein-Hilbert lagrangian

$$\mathcal{L}_{EH} = -\frac{1}{16\pi G}R\sqrt{-\det g_{\mu\nu}} \tag{1.2.1}$$

the lagrangians of a vector field u^μ

$$\mathcal{L}_u := -\frac{1}{32\pi G}\left[Kg^{\alpha\beta}g^{\mu\nu}B_{\alpha\mu}B_{\beta\nu} + 2\lambda(g^{\mu\nu}u_\mu u_\nu - 1)\right]\sqrt{-g},\ s.t.\ B_{\mu\nu} := \partial_\mu u_\nu - \partial_\nu u_\mu,$$

and of a scalar field ϕ

$$\mathcal{L}_\phi := -\frac{\sigma^2}{2}\left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{G}{2l^2}\sigma^2 F(kG\sigma^2)\right]\sqrt{-g},\ s.t.\ h^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu;$$

where K and k are the vector and the scalar coupling constants, l is a constant length, and F is the interpolating function [66]. This new law explains again the rotation curves of galaxies and the virial of clusters, and it is consistent also with gravitational lensing and some cosmological observations, with the suitable interpolation [67].

However, TeVeS does not fit with other kinds of gravitational phenomena, as a universal gravitational law should do. If the parameters are chosen in order to explain the dark matter, then the spherically symmetric solutions of such a theory result to be unstable, so that the stars cannot survive more than a few weeks [68]. Moreover, it was found [69]

[3] An even more shocking version of MOND inserts the same dependence inside the Second Law of Dynamics itself, so that it concerns any kind of force $F = \frac{ma}{\mu(a)}$, not just gravitation.

that TeVeS does not agree with the observed acoustic peaks of the CMB, that are another dark matter evidence.

There are other relativistic MOG, as the Scalar-tensor-vector gravity, and its extension, the Bi-scalar-tensor-vector gravity. However, all the MOG theories have to face the following fatal lacks.

- The interpolation is not consistent with the scientific methodology. If any galaxy can be adapted to a MOG law, then this law predicts nothing, and is not a scientific theory. The MOG interpolation seems similar to the epicycles system.
- Even if any galaxy curve or gravitational lens admit a suitable interpolation, the other phenomena requires different interpolating functions [70]. There is not a unique expression for gravity.
- A universal law, instead of an unknown kind of matter, must give the same contribution everywhere. However, the dark matter shows an inhomogeneous distribution [71]. Some galaxies or clusters have more dark matter contribution than other ones [72].
- This inhomogeneity of dark matter is particularly evident in some observed collisions of clusters, as the Bullet Cluster [73]. For such objects, the visible matter is nearer the center than the dark matter, as it is localized via gravitational lensing. It is coherent if the ordinary matter feels the friction, and if there is some dark matter which does not interact electromagnetically; but it is not explainable by any MOG theory.
- The GR was confirmed with respect to MOG alternatives, from large scale phenomena [74], [75] and from the gravitational waves observations [76].

1.2.2 Other alternative theories of gravity

The MOG theories do not run out the attempts to change the theory of gravitation. Here we consider briefly the principal alternatives.

Brans-Dicke theory

In 1961 it was developed [77] a theory presuming that the gravitational constant G is not constant, but can vary in space and time. $1/G$ is hence replaced by a scalar filed ϕ. The Brans-Dicke action for gravity is [78]

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi\right),$$

where ω is the dimensionless Dicke coupling constant.
Adding the matter lagrangian, the field equations are returned to be

$$\begin{cases} G_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) + \frac{\omega}{\phi^2}\left(\partial_\mu\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi\right) \\ \Box\phi = \frac{8\pi}{3+2\omega}T \end{cases}.$$

Such equations predicts the deflection of light and the precession of perihelia, as General Relativity does, but such effects depend on the parameter ω. Thus, the observations provide a bound on its value. Actually, they requires at least $\omega > 40000$, as for an higher and higher parameter the theory becomes empirically indistinguishable from General Relativity. Essentially, The Brans-Dicke theory is less falsifiable than General Relativity, because it has a tuneable parameter.

The $f(R)$ gravity

Let us consider again the Einstein-Hilbert action

$$S = \int d^4x \mathcal{L}_{EG},$$

where we remember (1.2.1). A possible generalization [79] consist of substitute some function $f(R)$ of the Ricci scalar

$$S = \int d^4x f(R)\sqrt{-g}.$$

Adding the matter lagrangian, the field equation results to be

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(R) = T_{\mu\nu}.$$

In the Taylor expansion

$$f(R) := a_0 + a_1R + a_2R^2 + \dots,$$

the term a_0 is equivalent to a cosmological constant, while the choice of a_1 essentially fixes the gravitational constant a_1. The second order term had an historical role in the development of the Starobinski inflation [39], since he chose $a_2 := \frac{1}{16\pi G}\frac{1}{6M^2}$, where M has the dimensions of a mass.
The presence of a certain a_2 can provide contributions to the dark matter [80]. However, observations [81] constrains $|a_2| < 4 \cdot 10^{-9}m^2$. Within these limits, the $f(R)$ gravity and General Relativity are empirically indistinguishable, analogously to the previous alternatives we saw.

Entropic gravity

The studies about the connection between gravity and thermodynamics leaded [82] to hypothesize that the gravitation is not a fundamental force, but an emergent phenomenon due to the entropy of bits of space-time information. For strong enough gravitational accelerations, the statistical analysis returns the classical Newton law. Under a threshold of approximately $a_0 \cong 1.2 \cdot 10^{-10}m/s^2$, the law flexes in a linear relation. This provides a theoretical justification for the MOND postulates, and gives the same explanation for the dark matter.

A MOND theory justified with the entropic gravity manages to overcome the lacks due to the large number of possible interpolations, because the gravitational law is not postulated *ex nihilo*, but uniquely derived. However, other lacks of the MOND recur, as the counterexample of the Bullet Cluster. Moreover, problems arise for the derivation of Einstein's equations from entropic gravity, for the energy-momentum conservation [83].

Minimally coupled gravity

A theory of gravitation arises from a lagrangian that includes the space-time metric and other fields. Besides the Einstein-Hilbert term (1.2.1) for $g_{\mu\nu}$ and some other part for the fields, the gravitational interaction must specify also the coupling term between the metric and the fields. It is possible to ask that such coupling terms have the minimal degree. This assumption leads to other alternatives to GR [84].

1.3 Fake dark matter from General Relativity

If our theory of gravitation is correct, and if all the kinds of invisible matter provide too small contributions, how can our model predict an amount of matter that is so bigger than the observable one? They were developed in the last years many lines of research that seem to explain unusual gravitational effects, such as dark matter and dark energy, without true dark matter and without changing the gravitational law. These have different nature and strategies, but all of them can work because *the model is not the theory*[4].

The model of galactic dynamic, or the Cosmological Concordance Model, are not identical to our theory of gravitation, the General Relativity. The Friedmann model assumes the homogeneity of the universe, but it holds just as approximation. The galaxies are described with the Newtonian dynamics, which is again an approximation of GR. Both of them are valid as long as such approximations are valid. Thus the dark matter, and maybe other open problems in cosmology, could be the clue that these approximations ignore some relevant contributions.

1.3.1 Off-diagonal contributions

The Newton law is often considered a good approximation of the Einstein Equations, for sub-relativistic velocities and low matter density, as they are inside a galaxy. It is true for the newtonian gauge, i.e. those choices of variables for which the space-time metric is diagonal, and their components mimic the newtonian potential, returning an evolution analogous to the quadratic-inverse law. (Here we must remember that the solar system, although it has low velocities and densities, is not described in a newtonian gauge, but rather with the IAU reference frame. All this paragraph is referred to the description of galaxies.) Higher order terms arise for strong gravity fields. This is the theoretical justification of the Newtonian framework in galactic studies.

However, this argument is not valid for observers which are not in the newtonian gauge. A series of works [85], [86], [87], [88] contain deep considerations about the coordinates system, used in General Relativity. These have consequences both for global effects, due to different time coordinates in use [89], [90], and for local ones, due to an off diagonal contribution. For a rotating matter source, a comoving observer would have off-diagonal components in the space-time metric. *These are not infinitesimal with respect to sub-relativistic velocities.* Hence, a rigorously general relativistic study would include non-negligible off-diagonal contributions, which are completely ignored in newtonian framework. A newtonian calculation would interpret them as the gravitation generated by an invisible source, i.e. a dark matter effect.

Theoretical studies [91] for a general galaxy indicate the explanation for at least a fraction of the galactic dark matter. A more specific analysis [92] was performed about the Milky Way, following the principles of relativistic astrometric modelling, which accounts for the weak field off-diagonal terms of the solar system metric, at work for the data processing delivered by Gaia. It is a first test of a GR weak field effect at Galactic scale, with the best stellar sample ever tracing the Galactic potential, that uses a suitable GR description of the observer and the observable adapted to the BG approximation. Then, [92] accounts for the 100% of DM to explain the Milky Way observed rotational curve by testing also the density and using the full set of the Einstein Equations.

[4] As a matter of fact, any model depends on the chosen theory. The point is that any modifications of the theory should be able to reproduce what is confirmed by the standard theory of gravity. On the other side, any GR approximations cannot represent the whole theory.

1.3.2 Backreaction

Considering now the cosmic scale, we have that the Friedmann model is yet a general relativistic one. However, the mathematical complexity of GR equations does not ensure that the small matter inhomogeneity, which is completely ignored by Friedmann, does generate an equally negligible curvature and gravity. Indeed, remember that all the dark matter effects are gravitational ones, i.e. are due to an unexpected curvature of space-time: the rotation galaxy curves, the gravitational lensig, and the universe accelerated expansion, are all due to the space-time metric. If a more precise resolution of Einstein Equations gives a suitable correction of the metric, all these phenomena would be naturally explained without the need of more matter.

The backreaction effect is an example of such deeper study of Einstein Equations. These equations are highly non-linear, hence *the spatial average of the metric generated by an inhomogeneous matter, in general, is not equal to the metric generated by the average of the same matter*, because the averaging operator is not transparent to the equations. The Friedmann model implements the second calculation: the matter is approximated as homogeneous at large scales, and from this averaged source it is calculated the homogeneous metric. However, the measure of cosmological parameters follows the first calculation: the real matter is inhomogeneous, generating an inhomogeneous metric, and our measurement processes take the average of this metric.

The difference between the two averaging methods is the backreaction term. The kinematical backreaction $\tilde{Q}_{\mathcal{D}}$ on a region D can be calculated as in [93], §2.1 and §3.2, by the spatial second fundamental form K_{ij}

$$\tilde{Q}_D = \frac{2}{3} Var_D(N\theta) - \langle N^2\sigma^2 \rangle_D, \tag{1.3.1}$$

where $\theta := -K_{ij}g^{ij}$ is the trace, $-\sigma_{ij} := K_{ij} + \frac{1}{3}\theta g_{ij}$ is the shear tensor and $\sigma^2 := \sigma^i_j \sigma^j_i$. The backreaction is a second order correction to the Friedmann metric, which showed to explain a fraction of the dark matter [94] and also of the dark energy [95].

1.3.3 Fractal cosmology

The backreaction does not depend on the exact conformation of the inhomogeneity, but just on statistical indices of the spatial curvature, as the variance of its trace and the average of its free-trace part. Other studies tried to specify the inhomogeneity distribution, substituting it inside the Einstein Equations instead of the Friedmannian homogeneity. Such distribution results to be a fractal one.

The Olbers Paradox calculates that, if the universe is static and eternal, and if stars have always the same absolute luminosity L and density n, then the total luminosity we should receive from sky is

$$I_{tot} = \int_0^\infty dr 4\pi r^2 \frac{L}{4\pi r^2} n = nL \int_0^\infty dr,$$

which diverges. The Paradox was historically solved with the Hubble Law, for which the universe is not static neither eternal, but Mandelbrot noticed [96] that it is possible a solution already in newtonian gravity, if density of the star is not homogeneous: if the stars lie on a fractal of Haussdorf dimension D, after a certain scale R_0, the average density decreases with the scale as $n(r)|_{r \geq R_0} \cong kr^{D-3}$; for $D < 2$ it returns

$$I_{tot} = \int_0^\infty dr 4\pi r^2 \frac{L}{4\pi r^2} n = n(0) L \int_0^{R_0} dr + kL \int_{R_0}^\infty r^{D-3} dr = R_0 \left(n(0) + \frac{n(R_0)}{2-D} \right) L < \infty.$$

This can be considered the foundation of hierarchical cosmology. A couple of arguments found by Fournier and Hoyle, from classical physics as well, suggested that $D = 1$.

The astronomical observations highlighted a hierarchical distribution for the matter structures, increasing the scale [97], [98]. The galaxies belong to galaxy clusters, which belong to superclusters, which belong to "filaments", and so on. Such a hierarchy is mathematically described as a fractal. At large scales, the filaments are divided by big "void bubbles", forming what is called "the cosmic fractal". The Mandelbrot's intuition is thus empirically confirmed [99].

A more modern measure of the fractal dimension is due to [100]. Here it was noticed that the average density $n(r) \cong kr^{D-3}$ is not well defined for fractals, e.g. cannot be chosen a suitable value for the coefficient k. It was considered the correlation function for the number density of galaxies, defined as

$$\Gamma(r) := \frac{\langle n(\underline{r}_0) n(\underline{r}) \rangle}{\langle n \rangle}.$$

In radial coordinate, it is the average conditional density, i.e. the density of a shell as a function of density of other shells. It is well defined for fractals, resulting to be $\Gamma(r) = c(D) r^{D-3}$. Applying this definition to the observed distribution of galaxies, it was found the value $D \cong 1.2$, consistent for different scales up to superclusters. A following work [101] used more recent observations about galaxy distribution, obtaining now

$$D \cong 2. \tag{1.3.2}$$

To solve the Einstein Equations with a fractal source would be a formidable task. A possible way to build a fractal model starts from the observation that, around any point of the fractal, it can be approximated considering the total matter inside a sphere[5]: it should go as $M(r) \cong \Phi r^D$. Such a "homogenized" source seems to violate the Copernican Principle, giving to the Earth a preferential role in the universe, as the center of a power law; but the Earth has not any particular role in this framework. Indeed, if any other point of the fractal is chosen, the "homogenization" would return the same law with a new center. It just describes the gravitation as it is summarily seen by a certain "material observer", i.e. an observer which stays where there is some matter, which is a point of the fractal. It is hence regained a "Conditional Cosmological Principle": the equivalence of all the material observers, what implies that D and Φ do not depend on the chosen point [102].

Such a power law for matter generates a Lemaitre-Tolman-Bondi metric. It distorts the measure of luminosity distances for far galaxies, which is the essential observation for estimate the universe expansion. Thus, if our universe has really a LTB metric, but we used a FLRW metric for interpreting the observations, our values for the universe acceleration contains a fake contribution. The dark energy phenomenon can be fully explained in this model, choosing a fractal dimension of $D = 2.87 \pm 0.02$ [103].

There is a discrepancy between the fractal dimension estimated by density correlation, and by the luminosity distances. A possible explanation can be given by a limit of the cosmic fractal. It was hence postulated that, at very large scales, the distribution of matter stops to be self-similar, approaching more and more to the friedmannian homogeneity [3], [104]. The loss of self-similarity has the suggestive name of "End of Greatness"[105], and e.g. in [4] its scale is valuated to be $L_{EG} \cong 100 Mpc$; however, we saw in the Introduction how the value of L_{EG}, and even its existence, is currently debated.

[5] The matter is defined as the integral of the T_{00} component, for a certain gauge. The radius r of the sphere is defined integrating the $ds = g_{ij} dx^i dy^j$ space metric.

A mixed model considers then an internal LTB metric, joining it with an external FLRW metric; it is called "Swiss cheese model", *e.g.* in [106] and [107]. Below the End of Greatness, the cosmic fractal would have the dimension $D \cong 2$ as estimated, and the contributes of the FLRW metric beyond the End of Greatness would justify the value $D \cong 2.87$, obtained from a completely fractal model.

LTB is a valid approximation just for the actual instant. An extension to ancient times was investigated [108], using a more general Lemaître solution, describing the evolution of the fractal matter and also of the radiation component. The origin of the fractal was brought back to the M-AM recombination epoch, which leaves relevant inhomogeneities where the matter was slightly more dense than anti-matter, due to quantum fluctuations. These results will be presented in §3.

However, we stress here that The LTB spacetime is not a necessary tool to build a fractal model, as that can be achieved within the FLRW cosmology. See e.g. [109] for a review.

1.3.4 Retarded potentials

Another line of research considers the retarded gravitational potential generated by the inhomogeneous, expanding distribution of matter in the universe. Since it is anisotropic, the Birkhoff Theorem [110] does not hold and the central point can be influenced by far objects.

The potentials from far objects are retarded, so they depend on the past matter density, which is greater than the actual one. The causal propagation in gravity, and so its retarded potentials, was recently confirmed by the observation of gravitational waves [111]. This provides a magnification effect on the total metric tensor, obtained as a superposition of all retarded potentials from all past times, which predictably have a singularity at the Big Bang time. However, there is also a reduction of the gravitational potential with the distance. It is necessary a precise calculation to see if it prevails the magnification or the reduction; in the first case, we would have an explanation of dark matter and/or dark energy effects, as a distortion of the tensor metric not due to a proportional presence of matter.

This idea was preliminary developed in [112], which is only partially general-relativistic. It is used a linearized gravity model on a minkowskian background, imposing the expansion of the universe; only the matter itself moves, on a fixed Minkowski background. From averaging the generated potentials, it arises an effective FLRW metric. Its expansion rate must coincide with the imposed Hubble parameter; this returns a compatibility condition, which fixes the amount of apparent matter related to the FLRW. The calculations give an about five times bigger effective matter, in (8) of [112]. It is near to the amount of dark matter it is usually believed to exist, more the 80% of the total.

A relevant role is attribute to a particular stochastic, fractal distribution of matter. Although the resultant gravitational force has zero average, it has statistical fluctuations that were numerically studied. The formulas (10) and (11) of [112] showed that the gravitational force would be consistent with the dark matter effects we observe in galaxies and clusters.

A truly GR model was proposed in [113]. Here the retarded potentials and their superposition are approximated with linear perturbation theory, assuming that at large scale the matter distribution is almost homogeneous, so the inhomogeneities can be seen as small perturbations. Hence we have a background metric (homogeneous) and a perturbed one (of which it is taken the spatial average), whose differences are interpreted with suitable amounts of fictitious dark matter and dark energy, dependent on the small amount of matter inhomogeneity. The PDEs for the retarded potentials are mathematically difficult to solve, thus in this first paper it is considered an un-physical background universe with constant Hubble parameter, which allows exact solutions; however, some arguments indicate that the real universe would return qualitatively the same phenomena.

In the subsequent paper [114] there were developed integral techniques which apply to all kinds of background expansion. It was obtained the "retarded potential model" for any combination of universe components - radiation, matter and cosmological constant - confirming the "magnification effect" for the real universe.
It was found a set of possible solutions, depending on the free parameters of the model. One of these fully explains the dark matter, but it required a larger cosmological constant than the usual; another one fully explains the cosmological constant, but it required more dark matter; and a third one explains a relevant fraction of both dark matter and dark energy. It will need some more empirical restrains to know which one of these solutions is that of the real universe, but anyway a redefinition of the cosmological parameters will be necessary. Moreover, there, was checked that the perturbations are small with respect to the background, and therefore the perturbative method is valid. All these results will be exposed in §2.

[113] and [114] investigated only global dark matter effects[6]. But the "retarded potential framework" can provide also local dark matter effects, since inhomogeneous matter generates inhomogeneous potentials. The anomalies in rotation curves and virials would be due to a statistical maximum of metric distortion around the galaxies and the clusters, which acts as an halo of effective dark matter. The correspondence of these maximums with the galaxies would not be a coincidence: during the formation of structure, the baryonic matter would fall into the gravitational wells, generated by far fields, so we would have automatically the formation of galaxies inside them.

[6]Remember that all the dark energy phenomena regards global observations.

Chapter 2

Retarded perturbations on FLRW background

In this chapter we will develop the methods of retarded perturbations, following [113] and [114]. After the basic definitions §2.1 and the derivations of the fundamental equations (2.2.13) in §2.2, we will apply them to the more reasonable gauge choices: the harmonic gauge §2.3, and then the newtonian gauge §2.5 .

The background for our perturbations will be always a FLRW universe, in this chapter, while an inhomogeneous background will be considered in the next chapter, following [108]. Our study will contain any possible FLRW background, with any partition in physical components (matter, radiation, dark energy, and so on). An exact solution of (2.2.13) will be possible just for a particular background §2.4, but some averaging theorems (in Appendix E) will allow us to reach quantitative results in §2.6 for any background. The consequences of these general studies for our real universe will be shown in §2.7, getting new evaluations for the cosmological parameters.

Notations

We will adopt the following terminology.
With “radiation” we mean any component of the universe with a pressure $p = w\rho$ with $w = \frac{1}{3}$. We call “matter” any component with $w = 0$; and “dark energy” any one with $w = -1$. A component with $w < 0$ will be called “exotic”.

Moreover, we call “total matter” the quantity of matter Ω_{M0}, which is required in the Cosmological Standard Model in order to explain the observed deceleration parameter; analogous for the “total dark energy” $\Omega_{\Lambda 0}$. We call “dark matter” Ω_{DM0} the part of total matter that is not observed, unless indirectly via gravitational phenomena. The observed part is essentially the “baryonic matter” $\Omega_{BM0} \cong \Omega_{M0} - \Omega_{DM0}$.

We adopt the most minus signature and natural units, so that $c = 1$. τ is the conformal time for the unperturbed metric, and $\underline{x}$ is the spatial coordinate.

Any quantity has associated an “unperturbed” version $\bar{Q}(\tau)$, computed using the background metric $\bar{g}_{\mu\nu}$, and a “perturbed” version $Q(\underline{x};\tau)$ computed using the real metric $g_{\mu\nu}$; this Q is the “true” version, since the background description is just an homogeneous approximation of an inhomogeneous universe. Its “perturbation” is the difference $\tilde{Q}(\underline{x};\tau) := Q(\underline{x};\tau) - \bar{Q}(\tau)$, which we consider negligible beyond the first order.

The dot will always denote derivation with respect to the conformal time: $\dot{Q} := \partial_\tau Q$. $a(\tau)$ is the unperturbed expansion parameter, so that $\bar{t} = \int a(\tau)d\tau$ is the usual (unperturbed) time. $H(\tau) := \frac{\dot{a}}{a}$ is the Hubble parameter for the unperturbed model. τ, a and

H are written without the overline, with abuse of notation, for better readability. Their perturbed versions will be $\mathbf{a}$ and $\mathbf{H}$.

τ_0 is the "present" instant for the unperturbed universe, defined s.t. $a(\tau_0) := 1$. t_0 is the "present" time for the true model, defined as $\mathbf{a}(t_0) := 1$. The "0" label means evaluation at the present time, i.e. the instant in which the observer lives, both for unperturbed and perturbed quantities. Notice that in general $t_0 \neq t(\tau_0)$, with still a little abuse of notation.

2.1 Framework

2.1.1 Perturbative method

Let us consider a universe, filled with any choice of components, each one characterised by its constant w. $\rho = \sum_w \rho_w$ is the true matter-energy density. Only the matter component $\rho_M(\underline{x};\tau)$ can be inhomogeneous. Let us call its homogeneous part [1]

$$\bar{\rho}(\tau) := \min_{\underline{x}} \rho(\underline{x};\tau). \tag{2.1.1}$$

Now consider a background universe, approximating our true universe at zero order, so assuming it filled with a perfectly homogeneous $\bar{\rho}$. We assume this universe to be spatially flat, in order to keep all calculations as simple as possible. For the same reason, we assume irrotational matter. Then, for a more realistic description of the universe, we perform a first order perturbative calculation. The perturbation of the energy-matter, which in fact consists only on matter, is

$$\tilde{\rho}(\underline{x};\tau) := \rho(\underline{x};\tau) - \bar{\rho}(\tau) = \rho_M(\underline{x};\tau) - \bar{\rho}_M(\tau). \tag{2.1.2}$$

Notice that this inhomogeneity is always non negative. In a symmetric way, we could also define

$$\bar{\rho}(\tau) := \max_{\underline{x}} \rho(\underline{x};\tau); \tag{2.1.3}$$

in such a case, $\tilde{\rho}$ would be non positive. It is a matter of convenience to fix the choice with always positive or negative $\tilde{\rho}$.

Along all this chapter, we will assume the Cosmological Principle. This means that $\tilde{\rho}$ will be considered small, with respect of $\bar{\rho}$, so that the perturbation method is acceptable.

2.1.2 Background evolution

The background metric is

$$\bar{g}_{\mu\nu}(\tau) = a(\tau)^2 (d\tau \otimes d\tau - \delta_{ij} dx_i \otimes dx_j). \tag{2.1.4}$$

$a(\tau)$ solves the Friedmann Equations

$$\begin{cases} 3H^2 = 8\pi G a^2 \bar{\rho} \\ \dot{\bar{\rho}} = -3H(\bar{\rho} + p) \end{cases}, \tag{2.1.5}$$

with $p(\tau) = \sum_w p_w(\tau) = \sum_w w\bar{\rho}_w(\tau)$. Indeed, the only inhomogeneity is on matter, which has no pressure. From the second Friedmann Equation we know

$$\bar{\rho}_w(\tau) = \bar{\rho}_{w0} a(\tau)^{-3(1+w)}.$$

[1] We can interpret it as the density of the intergalactic medium. It is very low, but it is not zero, since even between the galaxies we don't have the perfect void. However, a universe can also have $\bar{\rho} \equiv 0$; this would mean that all its matter is inhomogeneous.

We can describe the background components as

$$\bar{\Omega}_w(\tau) := a(\tau)^2 \frac{\bar{\rho}_w(\tau)}{\bar{\rho}_0} = \bar{\Omega}_{w0} a(\tau)^{-1-3w} \tag{2.1.6}$$

s.t. $\bar{\rho}_0 = \frac{3H_0^2}{8\pi G}$, so $\sum_w \bar{\Omega}_{w0} = 1$. The first Friedmann Equation says now that $H(\tau)^2 = H_0^2 \sum \bar{\Omega}_w(\tau)$, from which we get a ODE for the evolution of a

$$\left(\frac{\dot{a}}{H_0}\right)^2 = \sum_w \bar{\Omega}_{w0} a^{1-3w}. \tag{2.1.7}$$

We define $a(\tau)$ as a maximal solution of this ODE, with maximal domain $(\tau_I; \tau_F)$ from an "initial" to a "final" time, eventually unbounded. The initial condition for a is provided by the request that $\lim_{\tau \to \tau_I} a(\tau) = 0$. The radius of the visible universe results to be $R(\tau) = \tau - \tau_I$; notice that it is always infinite if $\tau_I = -\infty$.

2.1.3 Comparison with the Cosmological Concordance Model

Averaging it $\tilde{\rho}$ on the whole space, we get $\langle\tilde{\rho}\rangle(\tau)$. The average of a certain $\mathcal{S}$ on the whole $\underline{x} \in \mathbb{R}^3$ is defined as the average on some increasing sequence of compact domains $\mathcal{D} \subset \mathbb{R}^3$ filling the whole space in the limit

$$\langle\mathcal{S}\rangle := \lim_{\mathcal{D} \nearrow \mathbb{R}^3} \langle\mathcal{S}\rangle_{\mathcal{D}}. \tag{2.1.8}$$

The exact process of averaging is described in [115]. Since we need the average on the whole space, we perform the averaging prescription called $J(\mathcal{S}, \rho, V_0, A, B_s, \Delta B_s)$ with the thickness ΔB_s tending to infinity, and with trivial weight $\rho \equiv 1$. The choice of B and B_s are hence irrelevant, and we take $V := A := \tau$. Indeed, we do not need a lightlike gradient for V, as it is prescribed in [115] for averaging cosmological observables: a set of measures suffers from the delay of information, due to the speed of light, but the quantities ρ or $g_{\mu\nu}$ we need to average are not measures, but rather inhomogeneous fields that we want to consider as they were homogeneous, at a certain instant of the time foliation. Moreover, the relativistic delay of information is already accomplished by the study of the retarded potentials, as in §2.2. For these reasons, we choose a time section for V.
As we will see in §2.5, the space-time can be put in the form $g_{\mu\nu} = a^2[(1+2\Psi)d\tau \otimes d\tau + (-1+2\Psi)\delta_{ij} dx_i \otimes dx_j]$. Its determinant is $g = -a^8(1-4\Psi) + o(\Omega_{IM0})$, and we can rewrite the factors in $J(\mathcal{S}, 1, \tau_0, \tau, B_s, \Delta B_s \to \infty)$ as $n^\mu \nabla_\mu \Theta(V_0 - V) = \delta(V_0 - V) \frac{\partial^\mu A \partial_\mu V}{\sqrt{|\partial_\nu A \partial^\nu A|}} = \delta(V_0 - V)\sqrt{|g^{\mu\nu}\partial_\mu V \partial_\nu V|}$, where $\partial_\mu V = \delta_{\mu\tau}$ and $g^{\tau\tau} = \frac{1}{a^2(1+2\Psi)}$. Thus we have

$$\begin{aligned} &J(\mathcal{S}, \rho \equiv 1, V_0 := \bar{t}_0, A := \bar{t}, B_s, \Delta B_s \to \infty) = \\ &= \int d^4x \sqrt{-g}\mathcal{S} \overbrace{\rho}^{1} n^\mu \nabla_\mu \Theta(V_0 - V) \overbrace{\Theta(B_s + \Delta B_s - B)\Theta(B - B_s)}^{1} = \\ &= \int d^4x a^4 \sqrt{1 - 4\Psi + o(\Omega_{IM0})}\mathcal{S} \frac{\delta(\tau_0 - \tau)}{a\sqrt{1+2\Psi}} = a(\tau)^3 \int \mathcal{S}(\underline{x};\tau)[1 - 3\Psi(\underline{x};\tau)] d^3\underline{x}|_{\tau=\tau_0} + o(\Omega_{IM0}). \end{aligned}$$

The a^3 factor is simplified dividing by the volume $J(1, \rho, V_0, A, B_s, \Delta B_s)$. Thus, for a first order calculation, this cosmological average is equal to the average of $\mathcal{S} - 3\Psi\mathcal{S}$ on a usual euclidean metric. Notice that the second term vanishes whenever $\mathcal{S}$ is a first order quantity, as it will be throughout this chapter.

We can define the "inhomogeneous matter" as a part of the total matter component

$$\Omega_{IM}(\tau) := \frac{\langle\tilde{\rho}\rangle(\tau)}{\rho_0} = \frac{8\pi G}{3\mathbf{H}_0^2}\langle\tilde{\rho}\rangle(\tau). \tag{2.1.9}$$

The Cosmological Concordance Model measures the cosmic components via the observed deceleration parameter

$$\begin{cases} \sum_w \Omega_{w0} = 1 \\ \frac{1}{2}\sum_w (1+3w)\Omega_{w0} = q_0 := -\frac{\partial_t^2 \mathbf{a}_0}{\mathbf{H}_0^2} \end{cases}, \tag{2.1.10}$$

but it assumes a homogeneous ρ. Since it is not our case, $\mathbf{a}$ will be obtained by just an adaptation of the true space-time metric to a FLRW one

$$\langle g_{\mu\nu}\rangle := dt \otimes dt - \mathbf{a}(t)\delta_{ij}dx_i \otimes dx_j. \tag{2.1.11}$$

This provides a distortion of the expansion law, so that in general $\mathbf{a}(t) \neq a(\bar{t})$, $q_0 \neq -\frac{\ddot{a}}{H_0^2}$, and $\Omega_{w0} \neq \bar{\Omega}_{w0}$. Interpreting the distortion as the unexpected presence of matter and dark energy, we will see an effect of "fictitious matter and dark energy". We evaluate them as Ω_{FM0}, $\Omega_{F\Lambda 0}$.

Remark 1. Since they come from a global evaluation and we used a first order approximation, these fictitious components will result to be proportional to the total perturbation Ω_{IM0}. Thus, these global effects do not depend on the spatial distribution of the matter inhomogeneities, but only on their total amount.

The matter and the dark energy components used in the CCM have a "true" and a fictitious part

$$\Omega_{M0} := \Omega_{TM0} + \Omega_{FM0}, \quad \Omega_{\Lambda 0} := \Omega_{T\Lambda 0} + \Omega_{F\Lambda 0}. \tag{2.1.12}$$

Indeed, we should consider the possibility that it exists a certain amount of matter of unknown nature, as WIMPs or MaCHOs; or that our framework is not able to explain a part of the dark matter phenomena, which find explanation from an alternative theory of gravity, or from the backreaction, or the fractal cosmology. Similarly, we should consider the possibility that a dark energy truly exists, although its quantity is different from what is assumed by the CCM. Our goal is to correct the cosmological parameters related to dark matter and dark energy, according to the retarded potentials' mechanism.

The other components are all "true". The "true" parts must be proportional to the same components of the background universe

$$\Omega_{Tw0} := \frac{\bar{\Omega}_{w0}}{\sum_{w'} \Omega_{Tw'0}} = \frac{\bar{\Omega}_{w0}}{1 - \Omega_{FM0} - \Omega_{F\Lambda 0}}, \tag{2.1.13}$$

with the exception of matter, for which we have to add again the inhomogeneous part

$$\Omega_{TM0} = \frac{\bar{\Omega}_{M0}}{1 - \Omega_{FM0} - \Omega_{F\Lambda 0}} + \Omega_{IM0} := \Omega_{HM0} + \Omega_{IM0} := \Omega_{BM0} + \Omega_{TDM0}. \tag{2.1.14}$$

Remember that Ω_{IM0} can be considered as positive or negative. In the second case, the homogeneous approximation $\bar{\rho}$ is a rounding up, so that $\Omega_{HM0} > \Omega_{TM0}$.

Some of the true matter must be the baryonic matter we know to exists. If there is still some part left, it is "true dark matter" Ω_{TDM0}. It is some kind of matter that actually exists, which gravitational action is not just a relativistic effect, but that is not a directly observable matter, like MaCHOs, WIMPs, and so on and so forth.

To fit the two conditions (2.1.10) two more parameters are needed, *e.g.* Ω_{FM0}, $\Omega_{F\Lambda 0}$.

2.1.4 Classification of possible results

We can apply this framework to a universe filled by any choice of components $\{w\}$.

Definition 1. We will call "selfconsistent" a choice for which

- all calculations return a finite result;
- the linearized Einstein Equations give a unique solution;
- the perturbative method is justified by small enough perturbations $\tilde{g}_{\mu\nu}$ with respect to the background $\bar{g}_{\mu\nu}$.

We can write the last condition as

$$\begin{cases} |\tilde{g}_{\mu\nu}| \ll |g_{\mu\nu}| \\ |\Omega_{IM0}| \ll \Omega_{TM0} \end{cases} ,$$

where the second condition means that the Cosmological Principle holds in beyond-End of Greatness limit.

Selfconsistence means that the choice of $\{w\}$ returns a physical universe which is mathematically possible.

Definition 2. We will call "acceptable" a choice for which

$$\begin{cases} \forall w : 0 \leq \Omega_{Tw0} \leq 1 \\ \Omega_{TDM0} \geq 0 \end{cases} .$$

The second part states that all the baryonic matter we see is really existing, so it is included in the model.

Acceptability means that the choice of $\{w\}$ returns a universe that is coherent with our empirical data.

Remark 2. Notice that the fictitious components can be negative, and such a case means that the dark matter and/or the dark energy is not explained at all, but rather its quantity is more than what is predicted by the CCM.

Definition 3. We will call "good" the choices for which both the dark matter and the dark energy are explained, at least for some fraction, i.e.

$$\begin{cases} \Omega_{TDM0} < \Omega_{DM0} \\ \Omega_{T\Lambda 0} < \Omega_{\Lambda 0} \end{cases} .$$

Even better choices are whose which fully explain the dark matter and/or the dark energy, i.e. $\Omega_{TDM0} = 0$, $\Omega_{T\Lambda 0} = 0$.

2.2 Einstein Equations linearized on a FLRW background, with an irrotational perfect fluid

As usual [23], [24], we express the perturbation as

$$g_{\mu\nu}(\tau;\underline{x}) = \overline{g}_{\mu\nu}(\tau) + \tilde{g}_{\mu\nu}(\tau;\underline{x}) = a(\tau)^2 \begin{pmatrix} 1 & \vec{0} \\ \vec{0} & -\delta_{ij} \end{pmatrix} + a(\tau)^2 \begin{pmatrix} 2A & -\vec{B} \\ -\vec{B} & h_{ij} \end{pmatrix} . \quad (2.2.1)$$

As about the energy-momentum tensor, we set

$$T_{\mu\nu}(\tau;\underline{x}) = (\rho+p)U_\mu U_\nu - p g_{\mu\nu} = \overline{T}_{\mu\nu}(\tau) + \tilde{T}_{\mu\nu}(\tau;\underline{x}), \tag{2.2.2}$$

where in general the energy density is $\rho = \overline{\rho} + \tilde{\rho}$, the pressure is $p = \overline{p}$ and the four-velocity field is $U_\mu = a\delta_{\mu\tau} + \tilde{U}_\mu$.

The unperturbed Ricci tensor and the unperturbed Einstein tensor are

$$\overline{R}_{\mu\nu} = \begin{pmatrix} -3\dot{H} & \vec{0} \\ \vec{0} & (\dot{H}+2H^2)\delta_{ij} \end{pmatrix}, \ \overline{G}_{\mu\nu} = \begin{pmatrix} 3H^2 & \vec{0} \\ \vec{0} & -(2\dot{H}+H^2)\delta_{ij} \end{pmatrix}. \tag{2.2.3}$$

The unperturbed Einstein Equations are nothing but the Friedmann equations (2.1.5), which means an evolution as (2.1.7)

After performing the scalar-vector-tensor decomposition of the metric

$$\begin{gathered} \vec{B} := \vec{\nabla}B + \hat{B},\ h_{ij} := 2C\delta_{ij} + 2\left(\partial_{ij} - \tfrac{1}{3}\delta_{ij}\nabla^2\right)E + (\partial_i\hat{E}_j + \partial_j\hat{E}_i) + 2\hat{E}_{ij} \\ s.t.\ \vec{\nabla}\cdot\hat{B} = 0,\ \vec{\nabla}\cdot\hat{E} = 0,\ \textstyle\sum_j \partial_j\hat{E}_{ij} = 0,\ \sum_j \hat{E}_{jj} = 0 \end{gathered} \tag{2.2.4}$$

we can express the perturbation of the Ricci tensor as in (A.1.11), (A.1.12), (A.1.13). We can use the geometric condition $g_{\mu\nu}U^\mu U^\nu = 1 = \overline{g}_{\mu\nu}\overline{U}^\mu\overline{U}^\nu$, to get the perturbation of velocities as

$$\tilde{U}^\mu = a^{-1}\begin{pmatrix} -A \\ \vec{v} \end{pmatrix}, \ \tilde{U}_\mu = a\begin{pmatrix} A & -\vec{v}-\vec{B} \end{pmatrix} \tag{2.2.5}$$

and so, remembering $\tilde{p} = 0$, the perturbation of stress-energy tensor is

$$\tilde{T}_{\mu\nu} = a^2\begin{pmatrix} \tilde{\rho}+2\overline{\rho}A & -\vec{q}-\overline{\rho}\vec{B} \\ -\vec{q}-\overline{\rho}\vec{B} & -ph_{ij} \end{pmatrix}, \tag{2.2.6}$$

where $\vec{v}(\tau;\underline{x})$ is the field of spatial velocities, and we defined $\vec{q} := (\overline{\rho}+p)\vec{v}$. The last has nothing to do with the deceleration parameter q_0, but it is an expression for the (irrotational) velocity field.

2.2.1 Choose the harmonic gauge

We want now to deduce the equations for the retarded potentials. To this end, we fix the harmonic gauge, usually convenient for studying gravitational waves [116]. Abstracting from the background metric, the harmonic condition on the perturbation of connection is

$$\tilde{\Gamma}^{\lambda\mu}_{\ \ \mu} = 0. \tag{2.2.7}$$

We obtain a scalar and a vector condition on A, $\vec{B}$, h_{ij}

$$\begin{cases} \dot{A} + \nabla^2 B + 3\dot{C} + 4HA = 0 \\ \vec{\nabla}A + \dot{\vec{B}} - \vec{\nabla}C + \nabla^2\vec{E} + 2H\vec{B} = 0 \end{cases}. \tag{2.2.8}$$

In this gauge, the second order part of $\tilde{R}_{\mu\nu}$ is a flat d'alambertian $\Box := \eta_{\mu\nu}\partial^\mu\partial^\nu$. Indeed, we can rewrite the perturbation of Ricci as

$$\tilde{R}_{\mu\nu} = \left[\frac{1}{2}\Box - H\partial_\tau - 2(\dot{H}+H^2)\right]\begin{pmatrix} 2A & -\vec{B} \\ -\vec{B} & h_{ij} \end{pmatrix} + \dot{H}\begin{pmatrix} 0 & \vec{0} \\ \vec{0} & h_{ij} - 2A\delta_{ij} \end{pmatrix}. \tag{2.2.9}$$

See Appendix A. This is what we are looking for, because predictably the linearized Einstein Equations will have the form of wave equations.

Following again the gravitational waves formalism, we express the Einstein Field Equations as

$$R_{\mu\nu} = 8\pi G S_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T^{\lambda}_{\lambda} g_{\mu\nu} \right). \tag{2.2.10}$$

The linearized version, simplified using equations (2.1.5), is

$$\begin{cases} \Box A - 2H\dot{A} + 2(\dot{H} - 2H^2)A = 4\pi G a^2 \tilde{\rho} \\ \Box \vec{B} - 2H\dot{\vec{B}} + 2(\dot{H} - 2H^2)\vec{B} = 16\pi G a^2 \vec{q} \\ \Box h_{ij} - 2H\dot{h}_{ij} = 4(\dot{H}A + 2\pi G a^2 \tilde{\rho})\delta_{ij} \end{cases}. \tag{2.2.11}$$

These contains the linearized Conservation Laws for energy and momentum

$$\begin{cases} \dot{\tilde{\rho}} + \vec{\nabla} \cdot \vec{q} + 3H\tilde{\rho} - 3(\overline{\rho} + p)\dot{C} = 0 \\ \dot{\vec{q}} + 4H\vec{q} + (\overline{\rho} + p)(\vec{\nabla} C + \nabla^2 \hat{E}) + [(\dot{\overline{\rho}} + \dot{p}) + 2H(\overline{\rho} + p)]\vec{B} = 0 \end{cases}. \tag{2.2.12}$$

See Appendix A for explicit calculations.

2.2.2 Simplification of wave equations

First, we observe that h_{ij} has no traceless source, hence we can choose a solution with $h_{ij} = 2C\delta_{ij}$.

Moreover, let us decompose $\vec{q} := \vec{\nabla} q + \hat{q}$. The divergenceless part $\hat{B}$ has only $\hat{q}$ as source and both of them are decoupled from the rest of the system. Indeed, they survive only in the wave equation for $\vec{B}$ and in the Momentum Conservation, which becomes

$$0 = \vec{\nabla} \left[\dot{q} + 4Hq + (\overline{\rho} + p)C + [(\dot{\overline{\rho}} + \dot{p}) + 2H(\overline{\rho} + p)]B \right] + \left[\dot{\hat{q}} + 4H\hat{q} + [(\dot{\overline{\rho}} + \dot{p}) + 2H(\overline{\rho} + p)]\hat{B} \right].$$

We set both to zero, because we considered an irrotational dust as inhomogeneous matter. The system (2.2.11) becomes

$$\begin{cases} \Box A - 2H\dot{A} + 2(\dot{H} - 2H^2)A = 4\pi G a^2 \tilde{\rho} \\ \Box B - 2H\dot{B} + 2(\dot{H} - 2H^2)B = 16\pi G a^2 q \\ \Box C - 2H\dot{C} = 2\dot{H}A + 4\pi G a^2 \tilde{\rho} \end{cases}. \tag{2.2.13}$$

The Conservation Laws become

$$\begin{cases} \dot{\tilde{\rho}} + \nabla^2 q + 3H\tilde{\rho} - 3(\overline{\rho} + p)\dot{C} = 0 \\ \dot{q} + 4Hq + (\overline{\rho} + p)C + [(\dot{\overline{\rho}} + \dot{p}) + 2H(\overline{\rho} + p)]B = 0 \end{cases}. \tag{2.2.14}$$

A general solution of the PDE system (2.2.11) has also a $\tilde{g}_{\mu\nu}$ term such that

$$\Box \tilde{g}_{\mu\nu} - 2H\dot{\tilde{g}}_{\mu\nu} + 2(\dot{H} - 2H^2)\tilde{g}_{\mu\nu} = 0,$$

describing gravitational waves on an expanding space-time. We will not consider it, since we are seeking for selfconsistent choices, so we want that the linearized Einsten Equations have a unique solution.

For a given distribution of matter and velocities as source, the PDEs return the correspondent space-time metric perturbation. For a bounded distribution of matter, the solution without gravitational waves is such that the metric is asymptotically zero, and we choose this solution as gravitational potential. Similarly to the usual wave equation, the characteristic curves are light rays, and so the potentials are retarded accordingly to the speed

of light.
Near τ_I, the matter inhomogeneities cannot have yet generated the metric perturbations A, B, C. For a selfconstistent solution we ask that A, B, C are zero at τ_I, as initial conditions for (2.2.13).

Remark 3. Here we notice that any effect we will find with the retarded perturbation approach should be added to the fictitious dark energy/matter which arises in [95] or [94]. Indeed, the backreaction effect is due to the *local* inhomogeneity on a certain compact support $\mathcal{D}$, while the retarded perturbations come mostly from far and ancient inhomogeneities, external to the region $\mathcal{D}$, giving a *global* contribution that cannot be reduced to the fields in $\mathcal{D}$.

Let us consider now the total metric $g_{\mu\nu} = a^2 \begin{pmatrix} 1+2A & -\vec{\nabla}B \\ -\vec{\nabla}B & (2C-1)\delta_{ij} \end{pmatrix}$. The spatial part is flat, even for its perturbed version, so we won't have backreaction. Indeed, on any $\mathcal{D}$, the spatial second fundamental form K_{ij} will have a constant trace $\theta := -K_{ij}g^{ij}$ and an identically zero shear tensor $\sigma_{ij} := -K_{ij} - \frac{1}{3}\theta g_{ij} \equiv 0$. So the kinematical backreaction (1.3.1) will be always zero.

Moreover, the backreaction is a second-order quantity, so it never can be found in a calculation at first order, as it is the ours.

2.3 Averaged metric in harmonic gauge

All the wave equations have the PDE form

$$\Box u + \mathcal{H}(\tau)\dot{u} + \mathcal{K}(\tau)u = \mathcal{S}(\tau;\underline{x}) \tag{2.3.1}$$

on the generic field u, with time dependent coefficients $\mathcal{H}, \mathcal{K}$ and source $\mathcal{S}$.
Let $G(\tau, \underline{x}; \tau', \underline{x}')$ be its Green function[2]. It is zero for $|\underline{x}-\underline{x}'| > \tau-\tau'$, because of causality. It is also spatially homogeneous and isotropic

$$G(\tau, \underline{x}; \tau', \underline{x}') = G(\tau, \underline{x}-\underline{x}'; \tau', \underline{0}) = G(\tau, |\underline{x}-\underline{x}'|; \tau', \underline{0}).$$

Assuming separation of variables for a generic source, $\mathcal{S}(\tau;\underline{x}) = T(\tau)\mathcal{S}_0(\underline{x})$, we can express the retarded potential through convolutions

$$\begin{gathered} u(\tau;\underline{x}) = \int_{\tau_I}^{\tau} d\tau' \int d^3\underline{x}' G(\tau, \underline{x}; \tau', \underline{x}')T(\tau')\mathcal{S}_0(\underline{x}') = \int_{|\underline{x}'-\underline{x}|<\tau-\tau_I} \mathcal{S}_0(\underline{x}') f(\tau; |\underline{x}'-\underline{x}|) d^3\underline{x}' \\ s.t.\ f(\tau; |\underline{r}|) := \int_{\tau_I}^{\tau} G(\tau, \underline{r}; \tau', \underline{0}) T(\tau') d\tau'. \end{gathered} \tag{2.3.2}$$

The auxiliary quantity f represents the superposition of all the retarded potentials generated by a point of the source at all times, from the Big Bang up to now. The resultant solution u is again the superposition for all the causally linked points.

2.3.1 Average theorems

To study the global effects, we take the average of these potentials over all the space, obtaining a function depending only on time. We get

Proposition 2.3.1. *A field as in (2.3.2) has average*

$$\langle u\rangle(\tau) = 4\pi\langle\mathcal{S}_0\rangle \int_0^{R(\tau)} f(\tau; r) r^2 dr, \tag{2.3.3}$$

where $R(\tau) := \tau - \tau_I$ is the radius of observable universe.

[2] i.e. the asymptotically vanishing solution for a source $\delta(\tau-\tau')\delta^{(3)}(\underline{x}-\underline{x}')$

Proof. The spatial average of $u(\tau;\underline{x})$ is defined as (2.1.8). Let us fix the time τ. What we obtain immediately for u, from (2.1.8), is

$$
\begin{aligned}
\langle u\rangle(\tau) &:= \lim_{\mathcal{D}\to\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} u(\tau;\underline{x})d^3\underline{x} = \\
&= \lim_{\mathcal{D}\to\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} d^3\underline{x}\int_{|\underline{x}'-\underline{x}|<R(\tau)} d^3\underline{x}'\mathcal{S}_0(\underline{x}')f(\tau;|\underline{x}-\underline{x}'|) = \\
&= \lim_{\mathcal{D}\to\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} d^3\underline{x}\int_{|\underline{r}|<R(\tau)} d^3\underline{r}\mathcal{S}_0(\underline{r}+\underline{x})f(\tau;|\underline{r}|) = \\
&= \int_{|\underline{r}|<R(\tau)} f(\tau;|\underline{r}|)\left[\lim_{\mathcal{D}\to\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} d^3\underline{x}\mathcal{S}_0(\underline{r}+\underline{x})\right]d^3\underline{r} = \\
&= \int_{|\underline{r}|<R(\tau)} f(\tau;|\underline{r}|)\left[\lim_{\mathcal{D}+\underline{r}\to\mathbb{R}^3}\langle\mathcal{S}_0\rangle_{\mathcal{D}+\underline{r}}\right]d^3\underline{r} = \\
&= \int_{r<R(\tau)} f(\tau;r)\langle\mathcal{S}_0\rangle 4\pi r^2 dr,
\end{aligned}
$$

where we call $\mathcal{D}+\underline{r} := \{\underline{x}+\underline{r}|\underline{x}\in\mathcal{D}\}$ in the fourth passage, and we changed the variables to polar in the fifth passage. This proves the proposition. □

Remark 4. A and C are not decoupled, in general. The source $2\dot{H}A$ of C has not separable variables, as we assumed. Thus, the previous Proposition is not applicable to $u = C$, unless for the cases in which the separation of variables for $A(\tau;\underline{x})$ is valid.
It is for the constant Hubble parameter case, since the coupling term in $\dot{H}A$ vanishes. Moreover, we will show in §2.4.5 that a decoupling procedure is possible whenever the universe is dominated by a single component. A general universe with choice $\{w\}$ can be always approximately described sticking a succession of epochs, each one with single-component domination.

For B it is necessary a different procedure, since it is not a component of the metric, but their partial derivatives $\vec{\nabla}B$ are. Its average results to be zero.

Proposition 2.3.2. *For a field as in (2.3.2) it holds*

$$\langle\vec{\nabla}u\rangle = 0. \tag{2.3.4}$$

Proof. Similarly to u, we find now the average of $\vec{\nabla}u$.

$$
\begin{aligned}
\langle\vec{\nabla}u\rangle(\tau) &:= \lim_{\mathcal{D}\nearrow\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \vec{\nabla}u(\tau;\underline{x})d^3\underline{x} = \\
&= \lim_{\mathcal{D}\nearrow\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} d^3\underline{x}\int_{|\underline{x}'-\underline{x}|<R(\tau)} d^3\underline{x}'\mathcal{S}_0(\underline{x}')\vec{\nabla}_{\underline{x}}f(\tau;|\underline{x}-\underline{x}'|) = \\
&= \lim_{\mathcal{D}\nearrow\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} d^3\underline{x}\int_{|\underline{r}|<R(\tau)} d^3\underline{r}\mathcal{S}_0(\underline{r}+\underline{x})\vec{\nabla}_{\underline{r}}f(\tau;|\underline{r}|) = \\
&= \int_{|\underline{r}|<R(\tau)} \left(\vec{\nabla}_{\underline{r}}f(\tau;|\underline{r}|)\right)\left[\lim_{\mathcal{D}\nearrow\mathbb{R}^3}\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} d^3\underline{x}\mathcal{S}_0(\underline{r}+\underline{x})\right]d^3\underline{r} = \\
&= \int_{|\underline{r}|<R(\tau)} \frac{\underline{r}}{|\underline{r}|}f'(\tau;|\underline{r}|)\langle\mathcal{S}_0\rangle d^3\underline{r} = 0,
\end{aligned}
$$

because it is an integral of an odd function over a symmetric region. □

2.3.2 Comparison with homogeneous metric

Applying the first Proposition to A and C, with the *caveats* of Remark 4, and the other one to B, we find that the averaged metric is diagonal.

$$\langle g_{\mu\nu}\rangle = \overline{g}_{\mu\nu} + \langle\tilde{g}_{\mu\nu}\rangle = a(\tau)^2(1+2\langle A\rangle(\tau))d\tau\otimes d\tau - a(\tau)^2(1-2\langle C\rangle(\tau))\delta_{ij}dx_i\otimes dx_j, \quad (2.3.5)$$

where

$$\begin{aligned}
&\langle A\rangle(\tau) = 16\pi^2 G\langle\tilde{\rho}\rangle(\tau_0)\int_0^{R(\tau)} f_A(\tau;r)r^2dr,\\
&\langle C\rangle(\tau) = 4\pi(2\dot{H}_0\langle A\rangle(\tau_0) + 4\pi G\langle\tilde{\rho}\rangle(\tau_0))\int_0^{R(\tau)} f_C(\tau;r)r^2dr,\ s.t.\\
&f_{A,C}(\tau;|\underline{r}|) := \int_{\tau_I}^{\tau} G_{A,C}(\tau,\underline{r};\tau',\underline{0})T(\tau')d\tau',\ T(\tau) = \frac{\tilde{\rho}(\tau,\underline{x})}{\tilde{\rho}(\tau_0,\underline{x})},\\
&\left(\Box - 2H\partial_\tau + 2(\dot{H}-2H^2)\right)G_A(\tau,\underline{x};\tau',\underline{x}') = \delta(\tau-\tau')\delta^{(3)}(\underline{x}-\underline{x}'),\\
&\left(\Box - 2H\partial_\tau\right)G_C(\tau,\underline{x};\tau',\underline{x}') = \delta(\tau-\tau')\delta^{(3)}(\underline{x}-\underline{x}'). \qquad (2.3.6)
\end{aligned}$$

Since the averaged metric we found is diagonal, it is possible to compare it to the homogeneous metric (2.1.11), defining suitably the variables

$$dt := a(\tau)\sqrt{1+2\langle A\rangle(\tau)}d\tau, \quad \mathbf{a} := a(\tau)\sqrt{1-2\langle C\rangle(\tau)}. \qquad (2.3.7)$$

We can express the correspondence with suitable "perturbations"

$$\begin{aligned}
&dt := \tilde{t}d\overline{t} \quad s.t. \quad \tilde{t} := \sqrt{1+2\langle A\rangle},\\
&\mathbf{a} := a\tilde{a} \quad s.t. \quad \tilde{a} := \sqrt{1-2\langle C\rangle}. \qquad (2.3.8)
\end{aligned}$$

Remark 5. We can have singularities if $\tilde{t}$ or $\tilde{a}$ reaches zero. It could happen for times ancient enough, and our perturbation theory is no more valid for previous instants, since imaginary quantities are not allowed.

If $\langle C\rangle(t_{BB}) = \frac{1}{2}$, then $\mathbf{a}(t_{BB}) = 0$ even if $a(t_{BB}) \neq 0$. t_{BB} would be a perturbed Big Bang, and we can set $t_{BB} = 0$ w.l.o.g.

If $\langle A\rangle(t_{min}) = -\frac{1}{2}$, then the perturbation theory looses validity even if there is no Big Bang. In this case, too much early epochs remain simply not describable by the model. This provides a cut off for the time integration.

Getting an expression for $\langle A\rangle$ and $\langle C\rangle$, we will be able to replace $\partial_t^2\mathbf{a}$ inside (2.1.10), so that the two parameters $\Omega_{FM0}, \Omega_{F\Lambda0}$ will be fixed.

2.4 Constant Hubble parameter case

To obtain some explicit solution of (2.3.1), at least for a simple case, from now on we consider a universe with constant Hubble parameter, so that

$$\mathcal{H} \equiv \mathcal{H}_0 := -2H_0;\ \mathcal{K} \equiv \mathcal{K}_0 := -4H_0^2\ for\ A\ and\ B,\ \mathcal{K} \equiv 0\ for\ C. \qquad (2.4.1)$$

In this case the system is completely decupled, so the problem in Remark 4 is overcome. Physically, we have a universe dominated by only one component $\bar{\Omega}_{w0} = 1$, which is a suitable form of exotic energy such that $w = -\frac{1}{3}$. It is the same expansion assumed in

[112], since the background expansion law is $a(\tau) = e^{H_0\tau}$ and $a(\bar{t}) = H_0\bar{t}$, but now it is treated in a general relativistic context.

Following the derivation in Appendix B, we obtain the Green function for (3.2.17) with constant coefficients.

$$G(\tau;\underline{x}) = \frac{e^{\frac{1}{2}\mathcal{H}_0\tau}}{4\pi}\left[-\frac{\delta(\tau-|\underline{x}|)}{|\underline{x}|} + \sqrt{\frac{\overline{\mathcal{K}}}{\tau^2-|\underline{x}|^2}}J_0'\left(\sqrt{\overline{\mathcal{K}}(\tau^2-|\underline{x}|^2)}\right)\theta(\tau-|\underline{x}|)\right], \quad (2.4.2)$$

where we defined the *discriminant* $\overline{\mathcal{K}} := -\mathcal{K}_0 - \left(\frac{\mathcal{H}_0}{2}\right)^2$, J_0' is the first derivative of the zeroth order Bessel function, and θ is the Heaviside function.

Notice that for A and B we have $\overline{\mathcal{K}} = 3H_0^2$, whence it comes a factor $\sqrt{3}$, while for C it is $\overline{\mathcal{K}} = -H_0^2$, for which J_0 is replaced by I_0, the zeroth order modified Bessel function. We easily recognize the causality in the potential, since the first term propagates at the speed of light and the second one slower. The second term is some kind of "echo", due to the difference of the differential operator from a pure d'alembertian. For constant coefficients, the PDE is homogeneous in time, so $G(\tau,\underline{x};\tau',\underline{0}) = G(\tau-\tau',\underline{x};0,\underline{0}) = G(\tau-\tau';\underline{x})$.

2.4.1 Density contrast growing rate

To get an explicit expression for $\langle A, C\rangle$, what we need now is an evaluation of their source $4\pi Ga^2\tilde{\rho} = \mathcal{S}(\tau;\underline{x}) = T(\tau)\mathcal{S}_0(\underline{x})$. Since $\mathcal{S}_0 \propto \tilde{\rho}_0$, it is enough to compute the growing law $T(\tau)$ of the matter inhomogeneities, due to the progressive attraction of more and more material from the medium $\overline{\rho}$. Let us measure it with the density contrast of matter

$$\delta_M := \frac{\tilde{\rho}}{\overline{\rho}_M} \propto \tilde{\rho}a^3 \propto \mathcal{S}_0 aT. \quad (2.4.3)$$

This quantity is studied in perturbative cosmology, since it describes the formation of structures; see *e.g.* [24], [117] and [118]. Hence we know, for real universe, that it holds the separation of variables for the density contrast whenever it dominates a single component. When the matter dominates, it is

$$\delta_M \propto a \Rightarrow T(\tau) = cost.$$

When to dominate is dark energy, the matter structures are ripped apart with the same expansion rate of the universe

$$\delta_M = cost. \Rightarrow T(\tau) = a(\tau)^{-1}.$$

When to dominate is radiation, the density contrast is well described by

$$\delta_M \propto \ln\left(\frac{4}{y}\right), \quad s.t. \quad y := \frac{a(\tau)}{a_{RM}},$$

as [24], [117], [118] say again, and a_{RM} is the value of a for which the matter starts to dominate on the radiation; see §2.7.2 for the rigorous definition. Thus, the T function is

$$T(\tau) = a(\tau)^{-1}\ln\left(\frac{4a_{RM}}{a(\tau)}\right) = a(\tau)^{-1}[\ln(4a_{RM}) - \ln a(\tau)].$$

Our case is different from each of these, since it dominates a particular form of exotic energy. It has a w parameter between that of matter and that of cosmological constant, thus we assume an analogous growing law

$$\delta_M \propto a^n, \quad (2.4.4)$$

where the parameter n is a "growing rate", which predictably has some value between 1 and 0.[3] A plausible choice is $n \cong 2/3$, since it is proportional to the proximity of $w = -1/3$ exotic energy to the $w = 0$ matter and the $w = -1$ dark energy.
We will use

$$\tilde{\rho}(\tau; \underline{x}) = a(\tau)^{n-3}\tilde{\rho}_0(\underline{x}) \;\Rightarrow\; T(\tau) = a(\tau)^{n-1}. \tag{2.4.5}$$

We can obtain a condition on the growing rate from the linearized Einstein Equation (2.2.13) and Conservation Laws (2.2.14)

$$n \in (-\Phi; -1) \sqcup (\varphi; 1), \tag{2.4.6}$$

where $\varphi = \Phi^{-1} := \frac{\sqrt{5}-1}{2} \cong 0.618\ldots$ is the golden ratio. The derivation can be found in Appendix C.
δ_M must grow, for gravity, so the physically acceptable values are $\varphi < n < 1$. Notice that the choice $n \cong 2/3$ is compatible.

2.4.2 A formula for the effective density

Now we can evaluate the average metric with the formula (2.3.3). Substituting (2.4.2) in (2.3.6), and , remembering $R(\tau) \equiv +\infty$ in our case, one finds

$$\begin{aligned}\langle A\rangle(\tau) &= 4\pi\left(\frac{1}{3}\mathcal{N}(n/\sqrt{3}) - \frac{1}{n^2}\right)\frac{G\langle\tilde{\rho}_0\rangle}{H_0^2}a(\tau)^{n-1},\\ \langle C\rangle(\tau) &= 4\pi\left(\mathcal{M}(n) - \frac{1}{n^2}\right)\frac{G\langle\tilde{\rho}_0\rangle}{H_0^2}a(\tau)^{n-1}.\end{aligned} \tag{2.4.7}$$

The integrals

$$\begin{aligned}\mathcal{N}(n) &:= \int_0^\infty\int_0^\infty e^{-n(x+y)}\frac{J_0'(\sqrt{y(y+2x)})}{\sqrt{y(y+2x)}}x^2dydx,\\ \mathcal{M}(n) &:= \int_0^\infty\int_0^\infty e^{-n(x+y)}\frac{I_0'(\sqrt{y(y+2x)})}{\sqrt{y(y+2x)}}x^2dydx\end{aligned} \tag{2.4.8}$$

are defined in Appendix D.

Since I_0 grows exponentially, $\mathcal{M}$ is divergent for $n \leq 1$. Analogously, $\mathcal{N}$ diverges for $n \leq -1$. For the admissible ranges, we can follow the procedure of Appendix D, finding the exact results

$$\mathcal{N}(n) = \frac{1}{n^2+1} - \frac{1}{n^2}, \quad \mathcal{M}(n) = \frac{1}{n^2-1} - \frac{1}{n^2}. \tag{2.4.9}$$

Remark 6. The divergence could be interpreted as an infinite quantity of apparent dark matter, in the constant coefficient case, due to the expansion law near the first instant $\tau_I = -\infty$. However, any known physical theory fails near the Big Bang, so we should put a cut-off on integrals $\mathcal{N}, \mathcal{M}$ that makes them finite.

For example, a natural cut off (Remark 5) for our theory is τ_{min} such that

$$a(\tau_{min})^{1-n} = -\left(\frac{1}{3}\frac{1}{(n/\sqrt{3})^2+1} - \frac{1}{3}\frac{1}{(n/\sqrt{3})^2} - \frac{1}{n^2}\right)\frac{8\pi G}{H_0^2}\langle\tilde{\rho}_0\rangle = 3\left(\frac{2}{n^2} - \frac{1}{n^2+3}\right)\Omega_{IM0} + o(\Omega_{IM0}). \tag{2.4.10}$$

As we will see (Remark 8), the pole in $n = 1$ will be canceled by a factor $(n - 1)$, so we can extend the $\mathcal{N}, \mathcal{M}$ functions also for $n < 1$, which are the physical values. What we perform is essentially a renormalization via analytic continuation.

[3]This n has nothing to do with the numerical density of stars for the Olbers Paradox, in §1.3.3

The divergence and the necessity of a renormalization is a clue of the unphysicality of the constant coefficients case we are studying. Indeed, the metric perturbations (2.4.7) do not satisfy the First Selfconsistence Condition (see Corollary 2.6.1). They do not go to zero at τ_I, as we required in §2.2.2; on the contrary, they have an asymptote, since $n < 1$. These lacks will be fully overcome in §2.6, with a general study of the selfonsistence.

In the constant coefficients case, the second relation of (2.1.10) can be expressed with an "effective density"

$$\Omega_{eff} := \sum_w (1+3w)\Omega_{w0} = -2\frac{\ddot{a}_D(t_0)}{\mathbf{H}_0^2}. \tag{2.4.11}$$

Remark 7. The effective density results to be proportional to the deceleration parameter: $\Omega_{eff} = 2q_0$. This means it may be negative, for universes with some components $w < -1/3$.
Pay attention: this is not a violation of the weak energy condition. The true matter-energy density is $\overline{\rho}+\tilde{\rho}$, which is always positive. This $\rho_{eff} := \frac{3\mathbf{H}_0^2}{8\pi G}\Omega_{eff}$ is only a fictitious density, without physical existence.

To obtain an explicit expression for the effective density, we will use the following conventions:

$$\begin{aligned} K(n) &:= 8\pi\left(\frac{1}{3}\mathcal{N}(n/\sqrt{3}) - \frac{1}{n^2}\right)G = 8\pi\left(\frac{1}{n^2+3} - \frac{2}{n^2}\right)G, \\ K'(n) &:= 8\pi\left(\mathcal{M}(n) - \frac{1}{n^2}\right)G = 8\pi\left(\frac{1}{n^2-1} - \frac{2}{n^2}\right)G. \end{aligned}$$

Moreover, we use $\overline{t}$ as most suitable variable, with $\overline{t}_0 := \overline{t}(t_0)$.

Theorem 2.4.1. *If on a spatially flat metric dominated by a $w = -\frac{1}{3}$ dark matter-kind, we put an inhomogeneity of matter $\tilde{\rho}$, the present deceleration parameter of the averaged perturbed metric can be interpreted with an amount of effective density*

$$\begin{aligned} \rho_{eff}(\langle\tilde{\rho}_0\rangle; \mathbf{H}_0; n) = \frac{3(1-n)}{16\pi G} H_0^4 \overline{t}_0^2 \frac{H_0^2\overline{t}_0^2 - 1}{[(1+K/K')H_0^2\overline{t}_0^2 - K/K']^2} \times \\ \times \left[\big((3-n)K/K' + (7-n)\big) H_0^2\overline{t}_0^2 - \big((5-n)K/K' - (n+3)\big)\right], \end{aligned} \tag{2.4.12}$$

such that

$$\begin{cases} K' H_0^{n-1} \langle\tilde{\rho}_0\rangle \overline{t}_0^{n+1} + 1 = H_0^2\overline{t}_0^2 \\ \mathbf{H}_0 = \frac{H_0}{2}\frac{(1-n)H_0^2\overline{t}_0^2 + (1+n)}{\sqrt{(1+K/K')H_0^2\overline{t}_0^2 - K/K'}} \end{cases}. \tag{2.4.13}$$

Proof. Remember

$$\mathbf{a}(\overline{t}) = H_0\overline{t}\sqrt{1 - K' H_0^{n-3}\langle\tilde{\rho}_0\rangle\overline{t}^{n-1}}.$$

We can differentiate it exploiting

$$\frac{dt}{d\overline{t}} = \sqrt{1 + K H_0^{n-3}\langle\tilde{\rho}_0\rangle\overline{t}^{n-1}},$$

and substituting inside the first two equations of (2.1.10) we get (2.4.13). Remember the known parameters are $\langle\tilde{\rho}_0\rangle, \mathbf{H}_0, n$, so this system determines $\overline{t}_0, H_0$. Differentiating $\mathbf{a}$ again, we get (2.4.12). □

This gives the quantity of fictitious matter and other components an observer would need to justify the measured distortion of deceleration parameter, if there is an average inhomogeneity of matter $\langle\tilde{\rho}_0\rangle$. Substituting the values for $\langle\tilde{\rho}_0\rangle$, it is possible to evaluate the magnitude of these effects.

Remark 8. The factor $(1-n)$ in the ρ_{eff} compensates for the pole at $n=1$ inside $K'(n)$. This justifies the renormalization we performed for the $\mathcal{M},\mathcal{N}$ integrals.

2.4.3 Numerical values

Since we assumed inhomogeneities are small, so that we can use the perturbative approach, here it is possible to expand all the theorem's quantities at the first order in $\langle\tilde{\rho}_0\rangle$. From (2.4.13) we get

$$\begin{aligned}\bar{t}_0 &= \frac{1}{\mathbf{H}_0} + \frac{K+(n+1)K'}{2\mathbf{H}_0^3}\langle\tilde{\rho}_0\rangle + o(\langle\tilde{\rho}_0\rangle),\\ H_0 &= \mathbf{H}_0 + \frac{K+nK'}{2\mathbf{H}_0}\langle\tilde{\rho}_0\rangle + o(\langle\tilde{\rho}_0\rangle).\end{aligned} \tag{2.4.14}$$

The factor $(H_0^2\bar{t}_0^2-1)$ inside ρ_{eff} has no zeroth order term, so we must take only the zeroth order term for all the other factors, which simplifies the calculation of

$$\begin{aligned}\rho_{eff} =&\frac{3(1-n)}{16\pi G}H_0^2\frac{K'H_0^{n-1}\bar{t}_0^{n+1}\langle\tilde{\rho}_0\rangle}{[(1+K/K')-K/K']^2}\left[((3-n)\frac{K}{K'}+(7-n))-((5-n)\frac{K}{K'}-(n+3))\right]+o(\langle\tilde{\rho}_0\rangle)=\\ =&\frac{3}{2}(1-n)\frac{K'}{8\pi G}\left[-2\frac{K}{K'}+10\right]\langle\tilde{\rho}_0\rangle+o(\langle\tilde{\rho}_0\rangle)=3(1-n)\left[5\frac{K'}{8\pi G}-\frac{K}{8\pi G}\right]\langle\tilde{\rho}_0\rangle+o(\langle\tilde{\rho}_0\rangle)=\\ =&3(1-n)\left[5\left(\frac{1}{n^2-1}-\frac{2}{n^2}\right)-\left(\frac{1}{n^2+3}-\frac{2}{n^2}\right)\right]\langle\tilde{\rho}_0\rangle+o(\langle\tilde{\rho}_0\rangle)=\\ =&3(1-n)\left[\frac{5}{n^2-1}-\frac{1}{n^2+3}-\frac{8}{n^2}\right]\langle\tilde{\rho}_0\rangle+o(\langle\tilde{\rho}_0\rangle).\end{aligned} \tag{2.4.15}$$

We see clearly here how the pole $n=1$ vanishes. We can express our result in terms of the quantity

$$ract(n) := \frac{\rho_{eff}}{\rho_{IM0}}; \tag{2.4.16}$$

its name means "ratio". Hence we have

$$ract(n) = \frac{\Omega_{M0}-2\Omega_{\Lambda 0}}{\Omega_{IM0}} \cong 3\left(\frac{n-1}{n^2+3}+8\frac{n-1}{n^2}-\frac{5}{n+1}\right). \tag{2.4.17}$$

Except for matter and cosmological constant, which are almost fictitious, the only component is $\Omega_{E0} := \Omega_{w0}|_{w=-\frac{1}{3}}$, that particular kind of dark energy which gives a background expansion with constant Hubble parameter $H(\tau)\equiv H_0$. Remembering $\overline{\Omega}_{E0}=1$, as it was dominant in the background expansion, we have at first order

$$\Omega_{E0} = \left(\frac{H_0}{\mathbf{H}_0}\right)^2 = 1 + \frac{K+nK'}{\mathbf{H}_0^2}\langle\tilde{\rho}_0\rangle + o(\langle\tilde{\rho}_0\rangle). \tag{2.4.18}$$

Since the sum of all Ω_{w0} is always 1, we can define another quantity which describes such "sum", getting

$$sum(n) := \frac{\Omega_{M0} + \Omega_{\Lambda 0}}{\Omega_{IM0}} \cong -\frac{K + nK'}{8\pi G} = -3\left[\left(\frac{1}{n^2+3} - \frac{2}{n^2}\right) + n\left(\frac{1}{n^2-1} - \frac{2}{n^2}\right)\right] = \\ = 3\left(\frac{2}{n} + \frac{2}{n^2} - \frac{1}{n^2+3} - \frac{n}{n^2-1}\right). \tag{2.4.19}$$

This allows us to obtain $\Omega_{\Lambda,M0}$ as function of the perturbation Ω_{IM0}. As we did not put any cosmological constant in the true universe, and we put the matter only as the inhomogeneous one,

$$\begin{aligned} &\Omega_{F\Lambda 0} = \Omega_{\Lambda 0} \cong \frac{sum(n) - ract(n)}{3}\Omega_{IM0} \\ &\Omega_{M0} \cong \frac{2sum(n) + ract(n)}{3}\Omega_{IM0} \\ &\Omega_{FM0} = \Omega_{M0} - \Omega_{IM0} \cong \left(\frac{2sum(n) + ract(n)}{3} - 1\right)\Omega_{IM0} \end{aligned} \tag{2.4.20}$$

are the fictitious matter (with the exception of Ω_{IM0}, which is a real matter added to the background) and the fictitious cosmological constant.

Evaluation of parameters with $n \cong 2/3$

Remembering the condition (2.4.6), we can try to replace a numerically simple choice as $n \cong 2/3$; this gives

$$\begin{aligned} &ract(2/3) \cong -27.29; \quad sum(2/3) =\cong 25.23 \Rightarrow \\ &\frac{\Omega_{\Lambda 0}}{\Omega_{IM0}} \cong 17.51; \quad \frac{\Omega_{M0}}{\Omega_{IM0}} \cong 7.72; \quad \frac{\Omega_{FM0}}{\Omega_{IM0}} \cong 6.72\ . \end{aligned} \tag{2.4.21}$$

In particular, we obtain $\Omega_{M0} > \Omega_{IM0}$ and $\Omega_{FM0} > 0$, so the magnification effect is verified.

We can compare the formulas (2.4.21) with the most recent measures of cosmological parameters in our universe [33].

$$\Omega_{B0} = 0.043 \pm 0.004; \quad \Omega_{M0} = 0.315 \pm 0.007; \quad \Omega_{DM0} \cong 0.272 \pm 0.011 \quad \Omega_{\Lambda 0} = 0.685 \pm 0.007\ .$$

What we find is

$$\Omega_{IM0} \cong \frac{\Omega_{M0}}{7.72} \cong 0.0408 \Rightarrow \Omega_{FM0} \cong 6.72\Omega_{IM0} \cong 0.274, \tag{2.4.22}$$

which is inside the confidence interval of Ω_{DM0}; i.e. all the dark matter is explained in this model.

The real matter is just the baryonic one, divided into inhomogeneous Ω_{IM0} and homogeneous $\Omega_{HM0} \cong 0.0022$; unfortunately, the relevant fraction of inhomogeneous matter (about 95%) breaks put out outside the Definition 1 of selfconsistence universe.

We can evaluate also the correction on cosmological constant

$$\Omega_{F\Lambda 0} \cong 17.51\Omega_{IM0} \cong 0.714 \Rightarrow \Omega_{T\Lambda 0} \cong -0.029, \tag{2.4.23}$$

which is outside the 3σ interval of $\Omega_{\Lambda 0}$; i.e. this model explains too much cosmological constant, requiring finally a certain negative value for this parameter.

Evaluation of parameters with $\Omega_{IM0} := 1/sum$

The excessive value of $\Omega_{F\Lambda 0}$ in the last evaluation is due to the fact that, with a parameter $\Omega_{IM0} \cong 0.0408$, it returns $\Omega_{M0} + \Omega_{\Lambda 0} = sum(n)\Omega_{IM0} > 1$, hence they cannot fit with the empirical $\Omega_{M0} + \Omega_{\Lambda 0} = 1$. Imposing this physical request, we have a constraint on the model's quantities

$$\Omega_{IM0} = \frac{1}{sum(n)} \Rightarrow \Omega_{M0} = \frac{1}{3}\left(2 + \frac{ract(n)}{sum(n)}\right); \quad \Omega_{\Lambda 0} = \frac{1}{3}\left(1 - \frac{ract(n)}{sum(n)}\right). \tag{2.4.24}$$

Now, substituting the empirical Ω_{M0} (or equivalently $\Omega_{\Lambda 0}$), one fixes the value $n \cong 0.6761$ as solution of an algebraic equation. We notice that it falls inside the acceptable interval $(\varphi; 1)$, and it is very near to the conjectured $2/3$. With this growing rate,

$$\Omega_{IM0} = \frac{1}{sum(0.6761)} \cong 0.0402, \tag{2.4.25}$$

which means $\Omega_{HM0} \cong 0.0028$ as homogeneous matter.

We managed to find a really good model, which explains fully both the dark matter and dark energy, assuming a constant coefficients background expansion with growing rate $n \cong 0.6761$, and a perturbation with an amount $\Omega_{IM0} \cong 0.0402$ of inhomogeneous matter. Unfortunately, this is not a selfconsistent model. Indeed it breaks the Cosmological Principle, as $\Omega_{IM0} \cong 93.51\%\Omega_{M0}$. Moreover, we will see in §2.6 that it breaks also the First Selfconsistence Condition.

Remember that all these values are provisional. The quantitative results could change in a model with non constantly expanding background. Comparing Figure 2.1, 2.2 and 2.3, that describe the 2D Green function Γ of §B.2 for different dominant components, we can imagine that the inhomogeneity effects could be stronger under a dominance of radiation or matter, since the "echos" result to develop faster (the same shape to get which under constant expansion it needs $\tau - \tau' = 50Gy$, is reached under matter in $20Gy$ and under radiation in $13Gy$). The real universe passed a phase of radiation dominance and then of matter dominance, so we can expect higher values for $ract(n), sum(n)$ and a more complete explanation of the dark matter and the cosmological constant.

2.4.4 An inflation-like effect

If we are able to evaluate the fictitious quantity of cosmological constant, it would be interesting to obtain its variation during time and check if in the past was bigger. It would provide an explanation for the inflationary theory. So we fix a past instant t_1. We put our observer at this time and we wander how much inhomogeneity effect he sees. As in previous calculations (2.1.7), the observer considers a purely homogeneous model[4]

$$\left(\frac{\dot{a}_D}{\mathbf{H}_1}\right)^2 = \sum_w \Omega_{w1} a_D^{-3w-1}. \tag{2.4.26}$$

Since he lives in t_1, its effective expansion parameter is fixed as $a_D(t_1) = 1$. This means it is reduced by a factor $\mathbf{a}_1 := \mathbf{a}(t_1)$ with respect to $\mathbf{a}(t)$. The setting of parameters (2.1.10) becomes

$$\begin{cases} a_D(t_1) := 1 = \frac{\mathbf{a}(t_1)}{\mathbf{a}_1} \\ \dot{a}_D(t_1) = \mathbf{H}_1 := \frac{\dot{\mathbf{a}}(t_1)}{\mathbf{a}_1} \\ \ddot{a}_D(t_1) := \frac{\ddot{\mathbf{a}}(t_1)}{\mathbf{a}_1} \end{cases}. \tag{2.4.27}$$

[4]Coherently with the previous notation, we write $Q_1 := Q(t_1)$ for any quantity.

From them we have the analogous of (2.4.13) and (2.4.12)

$$\begin{aligned}
&\rho_{eff}(\langle\tilde{\rho}_0\rangle;H_{D1};n)=\frac{3(1-n)}{16\pi G}\frac{H_0^4\bar{t}_1^2}{\mathbf{a}_1^4}\frac{H_0^2\bar{t}_1^2-\mathbf{a}_1^2}{[(1+K/K')H_0^2\bar{t}_1^2-(K/K')\mathbf{a}_1^2]^2}\times\\
&\times\left[\left((3-n)K/K'+(7-n)\right)H_0^2\bar{t}_1^2-\left((5-n)K/K'-(n+3)\right)\mathbf{a}_1^2\right],\quad s.t.\\
&\begin{cases} K'H_0^{n-1}\langle\tilde{\rho}_0\rangle\bar{t}_1^{n+1}+\mathbf{a}_1^2=H_0^2\bar{t}_1^2\\ \mathbf{H}_1=\frac{H_0}{2}\frac{(1-n)\mathbf{a}_1^{-2}H_0^2\bar{t}_1^2+(1+n)}{\sqrt{(1+K/K')H_0^2\bar{t}_1^2-(K/K')\mathbf{a}_1^2}}\end{cases}. \qquad (2.4.28)
\end{aligned}$$

Remark 9. Notice that setting $t_1=t_0$, $\mathbf{a}_1=1$ and $\mathbf{H}_1=\mathbf{H}_0$, we turn back to (2.4.13) and (2.4.12).

We expand again the quantities in (2.4.28) at the first order in $\langle\tilde{\rho}_0\rangle$

$$\begin{aligned}
H_0&=\mathbf{H}_1a_1+\frac{K+nK'}{2\mathbf{H}_1}\mathbf{a}_1^{n-2}\langle\tilde{\rho}_0\rangle+o(\langle\tilde{\rho}_0\rangle);\\
\Omega_{M1}-2\Omega_{\Lambda1}&=ract(n)\mathbf{a}_1^{n-3}\Omega_{IM1}+o(\Omega_{IM1}). \qquad (2.4.29)
\end{aligned}$$

It is also

$$\Omega_{E1}=\left(\frac{H_0}{\mathbf{H}_1\mathbf{a}_1}\right)^2=1+\frac{K+nK'}{\mathbf{H}_1^2}\mathbf{a}_1^{n-3}\langle\tilde{\rho}_0\rangle+o(\langle\tilde{\rho}_0\rangle), \qquad (2.4.30)$$

from which

$$\Omega_{M1}+\Omega_{\Lambda1}=sum(n)\mathbf{a}_1^{n-3}\Omega_{IM1}+o(\Omega_{IM1}). \qquad (2.4.31)$$

We get the past fictitious matter and cosmological constant as functions of Ω_{IM1}

$$\begin{aligned}
\Omega_{F\Lambda1}=\Omega_{\Lambda1}&\cong\frac{sum(n)-ract(n)}{3}\mathbf{a}_1^{n-3}\Omega_{IM1}\\
\Omega_{M1}&\cong\frac{2sum(n)+ract(n)}{3}\mathbf{a}_1^{n-3}\Omega_{IM1}\\
\Omega_{FM1}=\Omega_{M1}-\Omega_{IM1}&\cong\left(\frac{2sum(n)+ract(n)}{3}-1\right)\mathbf{a}_1^{n-3}\Omega_{IM1}. \qquad (2.4.32)
\end{aligned}$$

However, we live in t_0 and we are not able to measure Ω_{IM1}, as an observer in t_1 does. We should convert our formulas with

$$\Omega_{IM1}=\left(\frac{\mathbf{H}_0}{\mathbf{H}_1}\right)^2\mathbf{a}_1^{n-3}\Omega_{IM0}. \qquad (2.4.33)$$

The factor Ω_{IM0} is yet at the first order of the perturbation, so we must evaluate the others at zeroth order, for which

$$\mathbf{H}_0=H_0+O(\langle\tilde{\rho}_0\rangle)=\mathbf{H}_1\mathbf{a}_1+O(\langle\tilde{\rho}_0\rangle),$$

so (2.4.32) becomes

$$\begin{aligned}
\Omega_{F\Lambda1}=\Omega_{\Lambda1}&\cong\frac{sum(n)-ract(n)}{3}\mathbf{a}_1^{2n-4}\Omega_{IM0}\\
\Omega_{M1}&\cong\frac{2sum(n)+ract(n)}{3}\mathbf{a}_1^{2n-4}\Omega_{IM0}\\
\Omega_{FM1}&\cong\left(\frac{2sum(n)+ract(n)}{3}-1\right)\mathbf{a}_1^{2n-4}\Omega_{IM0} \qquad (2.4.34)
\end{aligned}$$

Evaluating again $n \cong 0.6761$ and $\Omega_{IM0} \cong 0.0402$, the time dependence of the cosmological constant is

$$\Omega_{F\Lambda 1} \cong 0.685 \mathbf{a}_1^{-2.65}. \tag{2.4.35}$$

So Ω_Λ was pretty greater in the past, what can be interpreted as an inflationary epoch. It was even too much large, according to these equations. The cosmological constant reaches the maximum physical value $\Omega_{F\Lambda 1} = 1$ at

$$\mathbf{a}_1 = \mathbf{a}_{mmin} := (0.685)^{\frac{1}{2.65}} \cong 0.867 \ . \tag{2.4.36}$$

We labeled it with "mmin" to distinguish it from the cut off "min" of Remarks 5 and 6. Before this instant, our equations for $\Omega_{F\Lambda 1}$ are no more valid. We can heuristically state that in the previous epoch the cosmological constant is completely dominant $\Omega_{F\Lambda 1} :\cong 1$. This states until we reach the natural cut off (2.4.10). With our parameters, it is

$$\mathbf{a}_{min} = \left[3\left(\frac{2}{n^2} - \frac{1}{n^2+1}\right)\Omega_{IM0}\right]^{\frac{1}{1-n}} \cong 0.0820 \ . \tag{2.4.37}$$

We can summarize our knowledge about the cosmological constant variations as

$$\Omega_{F\Lambda}(t) \cong \begin{cases} unknown \quad for \quad \mathbf{a}(t) < 8.20 \cdot 10^{-2} \\ 1 \quad for \quad 8.20 \cdot 10^{-2} < \mathbf{a}(t) < 0.867 \\ 0.685 \cdot \mathbf{a}(t)^{-2.65} \quad for \quad \mathbf{a}(t) > 0.867 \end{cases} . \tag{2.4.38}$$

We can interpret the second epoch as inflation.

2.4.5 Is the constant coefficients case representative for the real universe dynamics?

The answer to the question may not be clear, since a constant expansion is pretty different from the our real universe. Previously, we calculated the fictitious matter in a constant coefficients universe, but even if we found some, there could be doubts about the presence of the same effect in a universe with not constant expansion. The general solution for the wave equations is quite difficult to get and it needs numerical integration, but here we show that it would lead to the same effect. This is because the Green functions have the same shape in any case, inducing similar averaged metric and similar distortion on $\mathbf{a}(t)$.

As we saw in Remark 4, the averaging procedure is not applicable as long as A, C are coupled. It is possible to decouple their PDEs whenever the universe is dominated by a single component, $\overline{\Omega}_{w0} = 1$. Indeed, we can define an auxiliary field D with equation

$$\Box D - 2H\dot{D} = 8\pi G(1-\alpha)a^2\tilde{\rho} \quad s.t. \quad \alpha := \frac{2}{3w+1} \tag{2.4.39}$$

and get C and its average as

$$C = \frac{D - A}{1 - 2\alpha}. \tag{2.4.40}$$

For a general background, with more components, it is possible to approximate the development of A, C neglecting at each instant all the components except the biggest one. The obtained law is a gluing of more single-component developments.

The general PDE (3.2.17) is isotropic, which allows us to reduce the dimension of the problem, analogously to Appendix §B.2; notice that in this case there is no more the time

symmetry, hence the potentials depends on their initial instant τ'. The one-dimensional retarded potentials $\Gamma(\tau, r; \tau')$ for A, with $\tau := \tau_0$ at the present instant, and τ' taken at a past instant, were displayed in Figure 2.1,2.2,2.3 (from [113]) for some single-component cases. Almost all of them have the same shape, which allowed us to believe that in general case a similar apparent matter will arise. The numerical value of Ω_{FM0} is different for our real universe, but we could expect the same qualitative effect.
The only difference is for the cosmological constant's dominance: in this case, the gravitational wave equation has a pure d'alembertian, thus we have no the "echo" term. However, even if in our universe there is a cosmological constant, it is not dominant until very recent times.

These cheering considerations leads us to develop more general mathematical instruments for the study of retarded potentials. The Theorem in §2.5 will be applicable to any choice of components, allowing us to find in §2.7 effective dark matter and dark energy from retarded potentials of our universe.

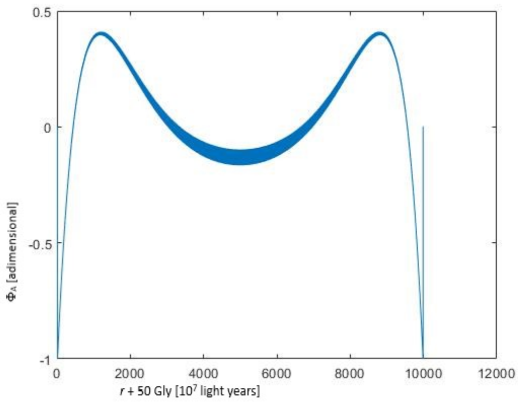

Figure 2.1: $\Gamma_A(\tau_0, r; \tau_0 - 50Gy)$ for the dominance of $w = -\frac{1}{3}$ dark energy-kind

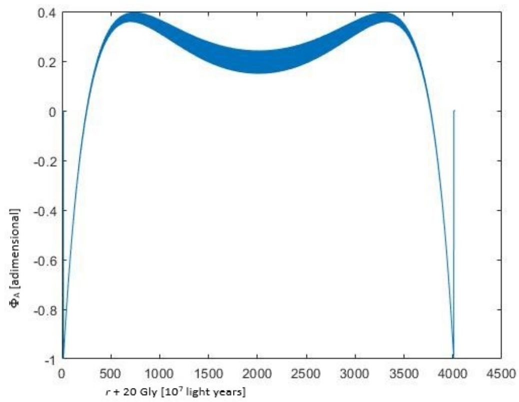

Figure 2.2: $\Gamma_A(\tau_0, r; \tau_0 - 20Gy)$ for the dominance of matter

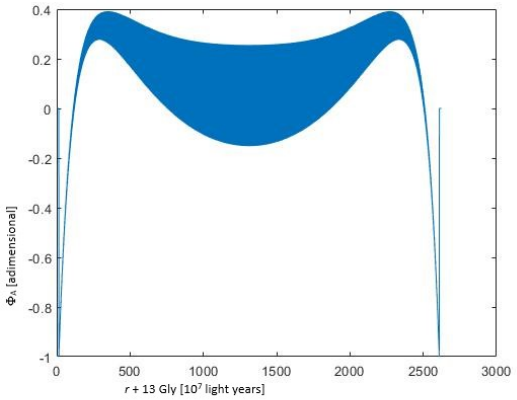

Figure 2.3: $\Gamma_A(\tau_0, r; \tau_0 - 13Gy)$ for the dominance of radiation

2.5 Averaged metric in newtonian gauge

The previous Theorems express the metric in the harmonic gauge, but what is the gauge of the metric in the comparison formula (2.1.11)? For the local effects (about galaxies, clusters...) it is used the newtonian approximation, i.e. the newtonian gauge. For the global effects, cosmologists assume a FRWL metric, which is diagonal. Anyway, it is a reasonable choice also to compare our perturbed metric to a diagonal one, and the metric is diagonalized in the newtonian gauge.

2.5.1 Gauge transformation

Lemma 2.5.1. *Via the transformation $\tau' = \tau - B(\underline{x};\tau)$, the metric $g_{\mu\nu}$ is expressed in the newtonian gauge as*

$$g_{\mu'\nu'} = a(\tau')^2 \begin{pmatrix} 2\Psi + 1 & \vec{0} \\ \vec{0} & (2\Phi - 1)\delta_{ij} \end{pmatrix}, \tag{2.5.1}$$

where the gravitational potentials are

$$\begin{aligned} \Psi &= A + \dot{B} + HB \\ \Phi &= C - HB. \end{aligned} \tag{2.5.2}$$

From now on we will use the newtonian coordinates, without writing the primes. Notice that the second gauge condition (2.2.8) guarantees

$$\Psi \equiv \Phi, \tag{2.5.3}$$

so we will call it just Ψ from now on.
The fictitious effects of matter and dark energy are not independent from the gauge, and this makes important the choice of the newtonian gauge.

Remark 10. The dependence on the gauge can be quite surprising, but it is coherent with the Lusanna's line of research, e.g. [89]. The recent measures on the Milky Way [92] arises for analogous reasons, due to the suitable definition of the observers, where the rotation of the galaxy generates a certain rotational $\vec{B}$.

2.5.2 Averaging theorems

What we will compare with the CCM is just the average of the metric, since the metric itself is not homogeneous and never allows for an exact equivalence. Such an average depends only on time

$$\langle g_{\mu\nu} \rangle = a^2 \begin{pmatrix} 2\langle\Psi\rangle + 1 & \vec{0} \\ \vec{0} & (2\langle\Psi\rangle - 1)\delta_{ij} \end{pmatrix}, \tag{2.5.4}$$

where we know from the last Lemma

$$\langle\Psi\rangle = \langle A\rangle + \langle\dot{B}\rangle + H\langle B\rangle = \langle C\rangle - H\langle B\rangle. \tag{2.5.5}$$

Now we express the averaging theorems in §2.3.1 and §2.4.5 as

Lemma 2.5.2. *Let us consider the Green functions for (2.2.13)*

$$\begin{aligned} \left(\Box - 2H\partial_\tau + 2(\dot{H} - 2H^2)\right) G_{\tau'}(\underline{x};\tau) &= \delta^{(3)}(\underline{x})\delta(\tau - \tau') \\ (\Box - 2H\partial_\tau)\, G^C_{\tau'}(\underline{x};\tau) &= \delta^{(3)}(\underline{x})\delta(\tau - \tau'), \end{aligned} \tag{2.5.6}$$

and let us assume the separation of variables for the matter inhomogeneity

$$\tilde{\rho}(\underline{x};\tau) := \tilde{\rho}_0(\underline{x})T(\tau). \tag{2.5.7}$$

Then we can express the average of metric distrortions as follows

$$\langle A\rangle(\tau) = 4\pi G\langle\tilde{\rho}_0\rangle u_A(\tau) = \frac{3}{2}\Omega_{IM0}H_0^2 u_A(\tau), \tag{2.5.8}$$

s.t.

$$u_A(\tau) := \int_{|\underline{r}|<R(\tau)}\int_{\tau_I}^{\tau} G_{\tau'}(\underline{r};\tau)a(\tau')^2T(\tau')d\tau' d^3\underline{r}. \tag{2.5.9}$$

The separation of variables does not hold exactly for A, but we can approximate

$$A(\underline{x};\tau) \propto u_A(\tau). \tag{2.5.10}$$

Then, in the same way

$$\langle C\rangle(\tau) = \frac{3}{2}\Omega_{IM0}H_0^2(2u_{AC}(\tau) + u_C(\tau)), \tag{2.5.11}$$

s.t.

$$u_{AC}(\tau) :\cong \int_{|\underline{r}|<R(\tau)}\int_{\tau_I}^{\tau} G^C_{\tau'}(\underline{r};\tau)\dot{H}(\tau')u_A(\tau')d\tau' d^3\underline{r} \tag{2.5.12}$$

and

$$u_C(\tau) := \int_{|\underline{r}|<R(\tau)}\int_{\tau_I}^{\tau} G^C_{\tau'}(\underline{r};\tau)a(\tau')^2T(\tau')d\tau' d^3\underline{r}. \tag{2.5.13}$$

Here we use the u functions to describe the time evolution of the perturbations. The separation of variables for $\tilde{\rho}$ holds when there is a single component dominating. Recall §2.4.1 for its formulas.

Since we are in the newtonian gauge, we need also the average of B, which appears in both the expressions of $\langle\Psi\rangle$. We can get it averaging the second gauge condition (2.2.8).

Lemma 2.5.3.

$$\langle B\rangle(\tau) = \frac{3}{2}\Omega_{IM0}H_0^2 u_B(\tau), \tag{2.5.14}$$

s.t.

$$u_B(\tau) := a(\tau)^{-2}\int_{\tau_I}^{\tau} a(\tau')^2(2u_{AC}(\tau') + u_C(\tau') - u_A(\tau'))d\tau'. \tag{2.5.15}$$

Proof. We know that $\langle\dot{B}\rangle + 2\frac{\dot{a}}{a}\langle B\rangle = \langle C\rangle - \langle A\rangle$. After expressing $\langle B\rangle(\tau) := a(\tau)^{-2}b(\tau)$, we get

$$a(\tau)^{-2}\dot{b}(\tau) = \langle C\rangle - \langle A\rangle = \frac{3}{2}\Omega_{IM0}H_0^2(2u_{AC}(\tau) + u_C(\tau) - u_A(\tau)) \Rightarrow$$
$$b(\tau) = \frac{3}{2}\Omega_{IM0}H_0^2\int_{\tau_I}^{\tau} a(\tau')^2(2u_{AC}(\tau') + u_C(\tau') - u_A(\tau'))d\tau',$$

which proves the assertion. □

2.5.3 Formulas for the fictitious components

The fictitious components are determined by (2.1.10). We can rewrite it using the auxiliary variables *ract* and *sum*, defined as in §2.4.3

$$\begin{cases} sum\Omega_{IM0} + o(\Omega_{IM0}) := \Omega_{FM0} + \Omega_{F\Lambda 0} = 1 - \sum_w \Omega_{Tw0} \\ ract\Omega_{IM0} + o(\Omega_{IM0}) := \Omega_{FM0} - 2\Omega_{F\Lambda 0} = 2q_0 - \sum_w (1+3w)\Omega_{Tw0} \end{cases}. \tag{2.5.16}$$

For an evaluation of these, we need to know the perturbations of q_0 and Ω_{Tw0}. The magnitude of the perturbations is determined by the comparison with the CCM metric

$$\begin{aligned} &a^2[(2\langle\Psi\rangle + 1)d\tau^2 + (2\langle\Psi\rangle - 1)\delta_{ij}dx_i dx_j] = \langle g_{\mu\nu}\rangle := dt^2 - \mathbf{a}^2\delta_{ij}dx_i dx_j \Rightarrow \\ &dt = \tilde{t}ad\tau := \sqrt{1+2\langle\Psi\rangle}ad\tau, \quad \mathbf{a} = \tilde{a}a := \sqrt{1-2\langle\Psi\rangle}a. \end{aligned} \tag{2.5.17}$$

The conditions at the present time are

$$\begin{cases} \mathbf{a}(t_0) := 1 \\ \mathbf{H}_0 = \partial_t \mathbf{a}|_{t_0} \\ q_0 = -\frac{\partial_t^2 \mathbf{a}}{\mathbf{H}}|_{t_0} \end{cases}. \tag{2.5.18}$$

From the first of these, we obtain the value of $a_0 := a(t_0) \neq a(\tau_0) = 1$, since

$$\begin{aligned} &\frac{1}{a_0} = \tilde{a}_0 = \sqrt{1 - 2\langle C\rangle_0 - 2H_0\langle B\rangle_0} \Rightarrow \\ &a_0 = 1 + \langle C\rangle_0 + H_0\langle B\rangle_0 + o(\Omega_{IM0}). \end{aligned}$$

Now, we can consider a as the time variable. By now, we denote with a prime the derivatives with respect to a. From (2.5.17)

$$dt = \tilde{t}ad\tau = \tilde{t}\frac{a}{\partial_\tau a}\partial_\tau a = \frac{\tilde{t}}{H}da,$$

so that for any given quantity Q depending on the time, we have

$$Q' := \frac{dQ}{da} = \frac{H}{\tilde{t}}\frac{dQ}{d\tau} = \frac{H}{\tilde{t}}\dot{Q}.$$

Using the relation in (2.5.18), we can find firstly the perturbations of Ω_{Tw0}. Indeed, from definition (2.1.6)

$$\Omega_{Hw} = \frac{8\pi G}{3\mathbf{H}_0^2}\mathbf{a}^2\bar{\rho}_{w0} = \left(\frac{H_0}{\mathbf{H}_0}\right)^2 \tilde{a}^2\bar{\Omega}_w. \tag{2.5.19}$$

From the second equation in (2.5.18), we can compute

$$\left(\frac{H_0}{\mathbf{H}_0}\right)^2 \tilde{a}_0^2 = 1 + 2[\langle A\rangle_0 + H_0\langle B\rangle_0 - H_0'\langle B\rangle_0 + \langle C\rangle_0'] + o(\Omega_{IM0}). \tag{2.5.20}$$

For any w it is $\Omega_{Tw0} = \Omega_{Hw0}$, with the exception of $\Omega_{TM0} = \Omega_{HM0} + \Omega_{IM0}$. Thus

$$\begin{aligned} sum\Omega_{IM0} + o(\Omega_{IM0}) &= 1 - \sum_w \Omega_{Tw0} = 1 - \Omega_{IM0} - \left(\frac{H_0}{\mathbf{H}_0}\right)^2 \tilde{a}_0^2 \sum_w \bar{\Omega}_w \Rightarrow \\ (sum+1)\Omega_{IM0} + o(\Omega_{IM0}) &= 1 - [1 + 2(\langle A\rangle_0 + H_0\langle B\rangle_0 - H_0'\langle B\rangle_0 + \langle C\rangle_0') + o(\Omega_{IM0})]1 = \\ &= -2[\langle A\rangle_0 + H_0\langle B\rangle_0 - H_0'\langle B\rangle_0 + \langle C\rangle_0'] + o(\Omega_{IM0}). \end{aligned}$$

As for the perturbation of q_0, we must remember that its zeroth order part is not zero, in general, but the background has a deceleration

$$\bar{q}_0 = \frac{1}{2}\sum_w (1+3w)\bar{\Omega}_{w0}. \tag{2.5.21}$$

It is distorted by the perturbation, then at first order we expect to have

$$q_0 := \bar{q}_0 + q_\Omega \Omega_{IM0} + o(\Omega_{IM0}) \tag{2.5.22}$$

for some coefficient q_Ω. We can compute each of these from the third of (2.5.18), obtaining

$$q_0 = -\frac{H_0'}{H_0} + \left[\langle A\rangle_0' + 2\langle C\rangle_0' + \langle C\rangle_0'' - \frac{H_0'}{H_0}(\langle C\rangle_0 + \langle C\rangle_0' - H_0'\langle B\rangle_0) - H_0''\langle B\rangle_0\right] + o(\Omega_{IM0}). \tag{2.5.23}$$

In particular, this means that

$$\frac{1}{2}\sum_w (1+3w)\bar{\Omega}_{w0} = \bar{q}_0 = -\frac{H_0'}{H_0}. \tag{2.5.24}$$

Together with (2.5.19), this gives

$$\begin{aligned}
\frac{1}{2}ract\Omega_{IM0} + o(\Omega_{IM0}) &= q_0 - \frac{1}{2}\sum_w (1+3w)\Omega_{Tw0} \\
&= [\bar{q}_0 + q_\Omega\Omega_{IM0} + o(\Omega_{IM0})] - \frac{1}{2}(1+3w)|_{w=0}\Omega_{IM0} \\
&\quad - [1 - (sum+1)\Omega_{IM0} + o(\Omega_{IM0})]\frac{1}{2}\sum_w (1+3w)\bar{\Omega}_{w0} \Rightarrow
\end{aligned}$$

$$\frac{1}{2}(ract+1)\Omega_{IM0} + o(\Omega_{IM0}) = (sum+1)\Omega_{IM0}\bar{q}_0 + q_\Omega\Omega_{IM0} + o(\Omega_{IM0}).$$

Theorem 2.5.4. *At first order, the effects of the matter inhomogeneities can be interpreted in terms of total fictitious components*

$$\begin{cases} \Omega_{FM0} = \frac{2sum+ract}{3}\Omega_{IM0} + o(\Omega_{IM0}) \\ \Omega_{F\Lambda 0} = \frac{sum-ract}{3}\Omega_{IM0} + o(\Omega_{IM0}) \end{cases}, \tag{2.5.25}$$

where the auxiliary quantities are

$$\begin{cases} \frac{1}{2}(sum+1)\Omega_{IM0} = & -\langle A\rangle_0 - H_0\langle B\rangle_0 + H_0'\langle B\rangle_0 - \langle C\rangle_0' \\ \frac{1}{2}(ract+1)\Omega_{IM0} = & \langle A\rangle_0' + 2\langle C\rangle_0' + \langle C\rangle_0'' - H_0''\langle B\rangle_0 \\ & +\frac{H_0'}{H_0}(2\langle A\rangle_0 + 2H_0\langle B\rangle_0 - H_0'\langle B\rangle_0 + \langle C\rangle_0' - \langle C\rangle_0) \end{cases}. \tag{2.5.26}$$

Now we should solve (2.2.13), replacing the resultant A, B, C inside (2.5.26). The general PDEs (2.2.13) are a formidable mathematical task. We obtained in Appendix B an exact solution for the case with constant coefficients, but it seems to be impossible an analytical solution when the coefficients depend on τ. However, here we are investigating only the global effects, which depend only on the average of A, B, C, as (2.5.26) shows. We can take inspiration from the surprising simplification of the superposition of the retarded potentials, which happens for constant coefficients case, as we can appreciate in calculations of Appendix D. We guess that, even in the spatially extended functions A, B, C can have quite complicate forms, their averages on total space can meet similar simplifications. Hence, we can study such averages without trying to solve explicitly the PDEs (2.2.13) for the non constant coefficients case.

From Lemmas 2.5.2 and 2.5.3 we have

$$\langle A\rangle = \frac{3}{2}\Omega_{IM0}H_0^2u_A(\tau), \quad \langle B\rangle = \frac{3}{2}\Omega_{IM0}H_0^2u_B(\tau), \quad \langle C\rangle \cong \frac{3}{2}\Omega_{IM0}H_0^2(2u_{AC}(\tau)+u_C(\tau));$$

where the us are studied in Appendix E. Frome those results, we get

Theorem 2.5.5. *If $\tau_I > -\infty$ and $a(\tau)^2T(\tau) \in L^1_{loc}([\tau_I;\tau_F))$, then*

$$\begin{aligned}\ddot u_A(\tau) + 2H\dot u_A(\tau) + 2(2H^2-\dot H)u_A(\tau) &= -a(\tau)^2T(\tau)\\ \ddot u_{AC}(\tau)+2H\dot u_{AC}(\tau) &= -\dot H(\tau)u_A(\tau)\\ \ddot u_C(\tau)+2H\dot u_C(\tau) &= -a(\tau)^2T(\tau).\end{aligned} \tag{2.5.27}$$

Otherwise, $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$ always diverge.

Proof. From Lemmas E.2.1, E.2.2 and E.2.3, all the us are

$$u(\tau) = \int_{-R(\tau)}^{R(\tau)} v(r;\tau)dr = \hat v(\omega;\tau)|_{\omega=0}.$$

The $\hat v$s obey equations like (E.2.2). Writing them for the us we have the assertion. □

2.6 Epochs with single component case

It is still impossible to solve analytically the evolution (2.1.7) for a and the ODEs (2.5.27) for a general choice of components $\{\bar\Omega_{w0}\}_w$. Moreover, for such a general choice it is quite difficult to determine the form of the source $\tilde\rho \propto T(\tau)$. However, we are able to solve exactly the equations when a single component $\bar\Omega_w$ dominates. We can approximate the general evolution as a succession of "epochs"; during each epoch, we consider just the dominant component

$$\forall\tau \mid \bar\Omega_w(\tau) = \max_{w'}\bar\Omega_{w'}(\tau) : \bar\Omega_{w0} \cong 1,$$

so that each epoch has a single-component evolution. The full evolution is obtained sticking the partial functions, imposing that $a(\tau)\in C^0(\tau_I;\tau_F)$, since (2.1.7) is first order, $\langle A\rangle, \langle C\rangle \in C^1(\tau_I;\tau_F)$, since (2.5.27) are second order; and $\langle B\rangle \in C^0(\tau_I;\tau_F)$, because u_B is obtained by an integral in (2.5.15).

Here we stress that the framework of this section allows to study the retarded perturbations effects on any possible universe, with any choice of components $\{w\}$. In this way, we will find mathematical inconsistencies for some choices, *e.g.* when the previous Theorem provides a divergence for the averaged metric. That is, *the retarded perturbations forbid the existence of some universes.* We will enunciate these prohibitions as Selfconsistence Conditions (SCs).

2.6.1 The First Selfconsistence Condition

Let's start solving (2.1.7) for a general epoch with $\bar\Omega_{w'0} \cong \delta_{w',w}$.

$$\left(\frac{\dot a}{H_0}\right)^2 = a^{1-3w} \Rightarrow$$

$$a(\tau) = \begin{cases}\left(\frac{1}{\alpha}H_0(\tau-c)\right)^\alpha & w\neq -\frac{1}{3}\\ e^{H_0(\tau-c)} & w=-\frac{1}{3}\end{cases} \qquad s.t. \quad \alpha(w) := \frac{2}{1+3w}; \tag{2.6.1}$$

where c is an integration constant. We get immediately the coefficients of (2.5.27)

$$H(\tau) = \begin{cases} \frac{\alpha}{\tau - c} & w \neq -\frac{1}{3} \\ H_0 & w = -\frac{1}{3} \end{cases},$$

$$2H = \begin{cases} 2\frac{\alpha}{\tau - c} & w \neq -\frac{1}{3} \\ 2H_0 & w = -\frac{1}{3} \end{cases},$$

$$2(2H^2 - \dot{H}) = \begin{cases} 2\frac{2\alpha^2+\alpha}{(\tau - c)^2} & w \neq -\frac{1}{3} \\ 4H_0^2 & w = -\frac{1}{3} \end{cases}. \tag{2.6.2}$$

Recalling (2.1.6) and that $a(\tau)$ is increasing (at least) near τ_I, we see that the epochs must be in order of decreasing w. In particular, during the first epoch it dominates $w_M := \max\{w\}$. Setting the initial condition

$$\lim_{\tau \to \tau_I} a(\tau) = 0 \Rightarrow$$
$$\tau_I \begin{cases} = -\infty & \alpha(w_M) < 0 \vee w = -\frac{1}{3} \\ \in \mathbb{R} & \alpha(w_M) > 0 \end{cases}. \tag{2.6.3}$$

By definition, it is always $\alpha \neq 0$. From the previous Theorem we get immediately

Corollary 2.6.1 (First Selfconsistence Condition)**.** *A selfconsistent choice of components* $\{w\}$ *must be such that* $w_M > -\frac{1}{3}$.

In particular, a selfconsistent universe must develop the metric perturbations as described by (2.5.27), with non constant coefficients.

Remark 11. In §2.4 we studied the constant coefficient case, filling the universe with an exotic component s.t. $w = -\frac{1}{3}$. This breaks the First Selfconsistence Condition, which explains the divergences we found in §2.4.2: it is the contribution of $rv(r;\tau)|_{r=R(\tau)} \equiv \infty$. It is possible to extract finite results even when the I SC is broken, as we did with a renormalization via analytic continuation. A general renormalization method could be to neglect always the term $rv(r;\tau)|_{r=R(\tau)} \equiv \infty$ of Lemma E.2.3, using (2.5.27) for any w_M.

As long as the I SC holds, we can fix $\tau_I := 0$ without lost of generality.

2.6.2 Decoupling

As we say in Lemma 2.5.2, for general coefficients of (2.2.13) we have just an approximated solution of $\langle C \rangle$. This is due to the coupling between C and A. Another advantage of the single component evolution is to allow the decoupling the PDEs of A and C. as we said previously.

$$\begin{cases} \Box A - 2\alpha\frac{\dot{A}}{\tau - c} - 2\alpha(2\alpha + 1)\frac{A}{(\tau - c)^2} = 4\pi G a^2 \tilde{\rho} \\ \Box C - 2\alpha\frac{\dot{C}}{\tau - c} + 2\alpha\frac{A}{(\tau - c)^2} = 4\pi G a^2 \tilde{\rho} \end{cases}.$$

Let $\alpha \neq -\frac{1}{2}$.[5] Then it is convenient to use again the auxiliary field of §2.4.5

$$D := A + (2\alpha + 1)C, \tag{2.6.4}$$

which must satisfy the PDE

$$\Box D - 2\frac{\alpha}{\tau - c}\dot{D} = 8(\alpha + 1)\pi G a^2 \tilde{\rho}. \tag{2.6.5}$$

[5] The case $\alpha = -\frac{1}{2}$ happens only for the exotic component $w = -\frac{5}{3}$.

All the results in §2.5 hold true for D, so that

$$\begin{aligned}&\langle D\rangle = 3\Omega_{IM0}H_0^2 u_D(\tau) \quad s.t.\\ &\ddot{u}_D + 2\frac{\alpha}{\tau - c}\dot{u}_D = -(\alpha+1)a(\tau)^2 T(\tau).\end{aligned} \tag{2.6.6}$$

From these we get an exact formula for $\langle C\rangle$

$$\langle C\rangle = \frac{\langle D\rangle - \langle A\rangle}{2\alpha+1} = \frac{3}{2}\Omega_{IM0}H_0^2\frac{2u_D(\tau) - u_A(\tau)}{2\alpha+1}. \tag{2.6.7}$$

Remark 12. Notice that in the dark energy epoch $\alpha = -1$ and the ODE for u_D is free of source. However, this does not imply that u_D is zero, i.e. $\langle C\rangle \neq \langle A\rangle$ in general.

2.6.3 Solving the ODEs

To solve (2.5.27) for a general w, we need the form of $T(\tau)$. We will assume

$$\delta_M \propto a(\tau)^n, \tag{2.6.8}$$

with $n(w)$ a regular function, of which we know $n(0) = 1$ and $n(-1) = 0$. This assumption does not certainly hold for the radiation epoch ($w = \frac{1}{3}$), when

$$\delta_M \propto \ln(4a_R) - \ln a(\tau) \quad s.t. \quad a_R = \max\{a(\tau)|\bar{\Omega}_R(\tau) = \max\bar{\Omega}_w(\tau)\}. \tag{2.6.9}$$

Let us start by solving for u_A. In general, it has a term u_{IA} generated by the source $-a(\tau)^2 T(\tau) = -a(\tau)^{n-1}$, and a term u_{HA} without sources. They result to be

$$\begin{aligned}u_{IA}(\tau) =& H_0^{-2}u_{A0}(H_0\tau)^{n_A} \quad s.t. \quad n_A = (n-1)\alpha + 2\\ &and \quad u_{A0} = -\frac{\alpha^{(1-n)\alpha}}{(n\alpha - \alpha + 2)(n\alpha + \alpha + 1) + 2\alpha(2\alpha+1)},\end{aligned} \tag{2.6.10}$$

$$u_{HA}(\tau) \propto (H_0\tau)^{n_H} \quad s.t. \quad n_H^2 + (2\alpha - 1)n_H + (4\alpha^2 + 2\alpha) = 0. \tag{2.6.11}$$

The exponent of u_{HA} is

$$n_H = \left(\frac{1}{2} - \alpha\right) \pm \sqrt{\frac{1}{4} - 3\alpha - 3\alpha^2}. \tag{2.6.12}$$

It has an imaginary part if and only if

$$\begin{aligned}&\alpha \in \left(-\infty; \alpha(w_+) := -\frac{1}{\sqrt{3}} - \frac{1}{2}\right) \sqcup \left(\alpha(w_-) := \frac{1}{\sqrt{3}} - \frac{1}{2}; +\infty\right) \Leftrightarrow\\ &\Leftrightarrow w \in (w_- \cong -0.9521; w_+ \cong 8.2855).\end{aligned}$$

Because of the arbitrariness of the integration constants c_1 and c_2, we can write in general

$$\begin{aligned}H_0^2 u_A(\tau) =& u_{A0}(H_0\tau)^{(n-1)\alpha+2}\\ &+ \begin{cases}\left[c_{A1}\sin(\sqrt{\xi}\ln H_0\tau) + c_{A2}\cos(\sqrt{\xi}\ln H_0\tau)\right](H_0\tau)^{\frac{1}{2}-\alpha} & w \in (w_; w_+)\\ \left[c_{A1}(H_0\tau)^{\sqrt{-\xi}} + c_{A2}(H_0\tau)^{-\sqrt{-\xi}}\right](H_0\tau)^{\frac{1}{2}-\alpha} & w \notin (w_; w_+)\end{cases},\end{aligned} \tag{2.6.13}$$

where $\xi := 3\alpha^2 + 3\alpha - \frac{1}{4}$. The solution for D is simpler.

$$\begin{aligned} H_0 u_D(\tau) &= u_{D0}(H_0\tau)^{(n-1)\alpha+2} + c_{D1}(H_0\tau)^{1-2\alpha} + c_{D2} \\ s.t. \quad u_{D0} &= -\frac{\alpha^{(1-n)\alpha}}{(n\alpha-\alpha+2)(n\alpha+\alpha+1)}. \end{aligned} \tag{2.6.14}$$

Using this, we get for C

$$\begin{aligned} H_0^2 u_C(\tau) &= u_{C0}(H_0\tau)^{(n-1)\alpha+2} + c_{D1}(H_0\tau)^{1-2\alpha} + c_{D2} \\ &+ \begin{cases} -\frac{1}{2\alpha+1}\left[c_{A1}\sin(\sqrt{\xi}\ln H_0\tau) + c_{A2}\cos(\sqrt{\xi}\ln H_0\tau)\right](H_0\tau)^{\frac{1}{2}-\alpha} & w \in (w_;w_+) \\ -\frac{1}{2\alpha+1}\left[c_{A1}(H_0\tau)^{\sqrt{-\xi}} + c_{A2}(H_0\tau)^{-\sqrt{-\xi}}\right](H_0\tau)^{\frac{1}{2}-\alpha} & w \notin (w_;w_+) \end{cases} \\ s.t. \quad u_{C0} &:= \frac{2(\alpha+1)u_{D0} - u_{A0}}{2\alpha+1}. \end{aligned} \tag{2.6.15}$$

The evolution of B is determined by Lemma 2.5.3.

$$\begin{aligned} u_B(\tau) &= H_0^{-3}\frac{u_{C0}-u_{A0}}{(n+1)\alpha+3}(H_0\tau)^{(n-1)\alpha+3} + u_{HB}(\tau) \\ s.t. \quad u_{HB}(\tau) &= a(\tau)^{-2}\int a(\tau')^2\left(\frac{2u_{HD}(\tau')-u_{HA}(\tau')}{2\alpha+1} - u_{HA}(\tau')\right)d\tau'. \end{aligned} \tag{2.6.16}$$

2.6.4 Particular components

In the following sections, we will need the single-component solutions for some particular components.
The dark energy has $w = -1 < w_-$. The perturbations evolve as

$$\begin{aligned} H_0^2 u_A(\tau) &= u_{A0}|_{w=-1}(H_0\tau - H_0c_\Lambda)^3 + c_{A1\Lambda}(H_0\tau - H_0c_\Lambda)^2 + c_{A2\Lambda}(H_0\tau - H_0c_\Lambda); \\ H_0^2 u_C(\tau) &= u_{C0}|_{w=-1}(H_0\tau - H_0c_\Lambda)^3 - c_{A1\Lambda}(H_0\tau - H_0c_\Lambda)^2 - c_{A2\Lambda}(H_0\tau - H_0c_\Lambda) \\ &\quad + c_{D1\Lambda}(H_0\tau - H_0c_\Lambda)^5 + c_{D2\Lambda}; \\ H_0^3 u_B(\tau) &= -2c_{A1\Lambda}(H_0\tau - H_0c_\Lambda)^3 - 2c_{A2\Lambda}(H_0\tau - H_0c_\Lambda)^2\ln|H_0\tau - H_0c_\Lambda| \\ &\quad + \frac{1}{4}c_{D1\Lambda}(H_0\tau - H_0c_\Lambda)^6 - c_{D2\Lambda}(H_0\tau - H_0c_\Lambda) + c_{B\Lambda}(H_0\tau - H_0c_\Lambda)^2. \end{aligned} \tag{2.6.17}$$

Indeed, $\alpha(-1) = -1 \Rightarrow \sqrt{\xi} = \frac{1}{2}$ and $n(-1) = 0$, thus

$$u_{A0}|_{w=-1} = u_{C0}|_{w=-1} = \frac{1}{2}, \Rightarrow u_{C0} - u_{A0}|_{w=-1} = 0. \tag{2.6.18}$$

The matter has $w = 0 \in (w_-; w_+)$. The perturbations evolve as

$$H_0^2 u_A(\tau) = u_{A0}|_{w=0}(H_0\tau - H_0 c_M)^2 + (H_0\tau - H_0 c_M)^{-\frac{3}{2}}[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0 c_M)\right)$$
$$+ c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0 c_M)\right)];$$
$$H_0^2 u_C(\tau) = u_{C0}|_{w=0}(H_0\tau - H_0 c_M)^2 + c_{D1M}(H_0\tau - H_0 c_M)^{-3} + c_{D2M} + \frac{1}{5}(H_0\tau - H_0 c_M)^{-\frac{3}{2}}\times$$
$$\times\left[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0 c_M)\right) + c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0 c_M)\right)\right];$$
$$H_0^3 u_B(\tau) = \frac{u_{C0} - u_{A0}}{7}|_{w=0}(H_0\tau - H_0 c_M)^3 + \frac{1}{2}c_{D1M}(H_0\tau - H_0 c_M)^{-2} + \frac{1}{5}c_{D2M}(H_0\tau - H_0 c_M)$$
$$- \frac{1}{50}(H_0\tau - H_0 c_M)^{-\frac{1}{2}}[(3c_{A1M} + \frac{\sqrt{71}}{2}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0 c_M)\right)$$
$$+ (3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0 c_M)\right)] + c_{BM}(H_0\tau - H_0 c_M)^{-4}. \tag{2.6.19}$$

Indeed, $\alpha(0) = 2 \Rightarrow \sqrt{-\xi} = \frac{\sqrt{71}}{2}$ and $n(0) = 1$, thus

$$u_{A0}|_{w=0} = -\frac{1}{30}, \quad u_{C0}|_{w=0} = -\frac{17}{150} \Rightarrow \frac{u_{C0} - u_{A0}}{7}|_{w=0} = -\frac{2}{175}. \tag{2.6.20}$$

For the peculiar evolution during the radiation epoch, we don't use $T = a^n$. The perturbations evolve as

$$H_0^2 u_A(\tau) = \frac{1}{8}(H_0\tau)\left[\ln\left(\frac{H_0\tau}{4a_R} - \frac{3}{8}\right)\right] + u_{HA}(\tau);$$
$$H_0^2 u_C(\tau) = \frac{1}{8}(H_0\tau)\left[5\ln\left(\frac{H_0\tau}{4a_R} - \frac{63}{8}\right)\right] + u_{HC}(\tau);$$
$$H_0^3 u_B(\tau) = \frac{1}{8}(H_0\tau)^2\left[\ln\left(\frac{H_0\tau}{4a_R} - \frac{17}{8}\right)\right] + u_{HB}(\tau). \tag{2.6.21}$$

2.6.5 Other Selfconsistence Conditions

Recalling our definition of a "selfconsistent" universe, the First Selfconsistence Condition ensures that there exist finite solutions for $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$. We must require also that these solutions are unique and that they describe small enough perturbations. The initial conditions for (2.2.13) were

$$\lim_{\tau\to 0}\langle A\rangle(\tau), \langle B\rangle(\tau), \langle C\rangle(\tau) = 0. \tag{2.6.22}$$

These functions are described by (2.6.13), (2.6.16) and (2.6.15) accordingly to the dominating w near $\tau = 0$, i.e. w_M. The initial conditions put some constraints on w_M and on the integration constants: we can satisfy (2.6.22) if all $u_{HA}, u_{HB}, u_{HC} \equiv 0$, and also

$$0 = \lim_{\tau\to 0}(H_0\tau)^{(n(w_M)-1)\alpha(w_M)+2} \propto \lim_{\tau\to 0} a(\tau)^{n(w_M)+3w_M} \Leftrightarrow n(w_M) + 3w_M > 0$$
$$0 = \lim_{\tau\to 0}(H_0\tau)^{(n(w_M)-1)\alpha(w_M)+3} \propto \lim_{\tau\to 0} a(\tau)^{n(w_M)+3w_M+\frac{1}{\alpha(w_M)}}$$
$$\Leftrightarrow n(w_M) + 3w_M + \frac{1}{\alpha(w_M)} > 0.$$

For the I SC one has $\alpha(w_M) > 0$, so that the first limit implies the second one.

Theorem 2.6.2 (Second Selfconsistence Condition)**.** *A selfconsistent choice of components must be such that $n(w_M) + 3w_M > 0$.*

Remark 13. For a monotonically increasing $n(w)$, the II SC is equivalent to

$$w_M > w_0 \tag{2.6.23}$$

for some limit value w_0. We can estimate it with a linear interpolation $n(w) \cong 1 + w$, a generalization of the choice $n(-1/3) \cong 2/3$ we did in §4, that gives $w_0 \cong -\frac{1}{4}$.
With a bigger generality, remembering from §2.4.1 that[6] $n(-\frac{1}{3}) \in (\varphi; 1)$ and that $n(0) = 1$, we have

$$-\frac{1}{3} < w_0 < 0. \tag{2.6.24}$$

The II SC is not necessary if $w_M = \frac{1}{3}$, for which always

$$\lim_{\tau\to 0}(H_0\tau)\left[\ln\left(\frac{H_0\tau}{4a_R} - \frac{3}{8}\right)\right] = 0; \tag{2.6.25}$$

and the same for B and C.

Corollary 2.6.3. *The II SC ensures that the perturbations are small near the Big Bang.*

Proof.

$$\frac{|\langle\tilde{g}_{\mu\nu}\rangle|}{|\bar{g}_{\mu\nu}|} \propto |\langle\Psi\rangle| = |\langle C\rangle - H\langle B\rangle| \to^{\tau\to 0} 0 \ll 1; \tag{2.6.26}$$

where the second term is $H\langle B\rangle \propto \frac{1}{\tau}(H_0\tau)^{(n(w_M)-1)\alpha(w_M)+3} \propto a^{n(w_M)+3w_M} \to^{\tau\to 0} 0$ for the II SC again. □

Do the initial conditions (2.6.22) fix uniquely $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$? Not always. There are values of w for which u_{HA}, u_{HB}, u_{HC} go to zero even if the integration constants are not fixed to zero. Such cases are not selfconsistent, because the solutions are not unique. This is forbidden by

Theorem 2.6.4 (Third Selfconsistence Condition)**.** *A selfconsistent choice of components must be such that $w_- < w_M \leq 1$.*

Proof. Let us try any non zero choice for the integration constant, and check if nevertheless u_{HA} tends to zero; if it is the case, the corresponding value of w will not be selfconsistent.
First, let us consider the case $w_M \in (w_-; w_+)$. Remembering (2.6.13)

$$\lim_{\tau\to 0} u_{HA}(\tau) = 0 \Leftrightarrow \lim_{\tau\to 0}(H_0\tau)^{\frac{1}{2}-\alpha(w_M)} = 0 \Leftrightarrow \alpha(w_M) < \frac{1}{2} \Leftrightarrow w_M > 1,$$

otherwise the limit does not exixt because of oscillations. This forbids the values $w_M \in (1; w_+)$.

Considering now the case $w_M \notin (w_-; w_+)$, it means $\alpha(w_M) \in \left[-\frac{1}{\sqrt{3}} - \frac{1}{2}; \frac{1}{\sqrt{3}} - \frac{1}{2}\right]$, and in particular $\alpha < \frac{1}{2}$. Recalling (2.6.13), for a choice $c_{A1} \neq 0$, $c_{A2} = 0$

$$\begin{aligned}
&\sqrt{\frac{1}{4} - 3\alpha - 3\alpha^2} + \frac{1}{2} - \alpha|_{w_M} > \frac{1}{2} - \alpha(w_M) > 0 \\
&\Rightarrow \lim_{\tau\to 0}(H_0\tau)^{\sqrt{\frac{1}{4}-3\alpha-3\alpha^2}+\frac{1}{2}-\alpha}|_{w_M} = 0 \\
&\Rightarrow \lim_{\tau\to 0} u_{HA}(\tau) = 0,
\end{aligned}$$

[6] $\varphi \cong 0.618...$ is the golden ratio.

and this is enough to forbid all the values $w_M \notin (w_-; w_+)$.

For the allowed values $w_M \in (w_-; 1]$, the integration constants for u_C are fixed to zero as well, since (2.6.15) has the same functional form of (2.6.13). From (2.6.16) and (2.6.22) we see that also u_{HB} is fixed to zero, so that the metric perturbations are unique. □

The Three Selfconsistence Conditions we proved allow only a "selfconsistence interval" for the component dominating near the Big Bang:

$$w_M \in (w_0; 1]. \tag{2.6.27}$$

Remark 14. Our universe contains certainly radiation and matter as homogeneous components, and probably dark energy. The biggest w is that of radiation, and $w_M = \frac{1}{3}$ is included in the selfconsistence interval. This is not obvious. Some universes, as the "constant coefficient universe" studied in §4.2, break the Three Selfconsistence Conditions. The selfconsistence of our universe provides an empirical reinforcement to our model.

When I and III SC hold, the requirement of selfconsistence is reduced to asking that the perturbations are small enough to neglect orders higher than the first. This constitues a last Condition.

Lemma 2.6.5 (Fourth Selfconsistence Condition)**.** *A selfconsistent choice of components must have an inhomogeneous matter such that $\Omega_{IM0} \ll \Omega_{TM0}$, and such that $\forall t \in [0; t_0]$: $|\langle\Psi\rangle| \ll \frac{1}{2}$.*

Proof. The first requirement on Ω_{IM0} is the same we asked in Definition 1. The other requirement is evident from (2.5.4), where $2\langle\Psi\rangle = 2\langle\Phi\rangle$ are the perturbations of the metric, and must be smaller than 1. □

This is no more a Condition on w, but on Ω_{IM0}, so that the selfconsistence interval remains the same. Indeed, $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$ are proportional to Ω_{IM0}, and so are $\langle\Psi\rangle$, $\langle\Phi\rangle$: the IV SC defines a maximum value Ω^M_{IM0} for the inhomogeneity.

Remark 15. Notice that the IV SC does not imply the II SC since, in the limit case $w_M = w_0$, $\langle\Psi\rangle$ does not tend to zero, but it could be small nevertheless.

2.7 A model for the real universe

2.7.1 The 1-manifold of possible universes

Until now, our computations concerned a general choice of components $\{\bar{\Omega}_{w0}\}_w$, for which we found the Selfconsistence Conditions. Now we will apply this general method to our universe.

It contains just three components: the radiation Ω_{R0}, the matter Ω_{M0} and the dark energy $\Omega_{\Lambda 0}$. These are fixed by the measures of Ω_{R0}, of $q_0 = \Omega_{R0} + \frac{1}{2}\Omega_{M0} - \Omega_{\Lambda 0}$ and of the space flatness [32] $1 = 1 - \Omega_{k0} = \Omega_{R0} + \Omega_{M0} + \Omega_{\Lambda 0}$. The background components are as well $\bar{\Omega}_{R0}$, $\bar{\Omega}_{M0}$ and $\bar{\Omega}_{\Lambda 0}$, on which the model puts the constraints

$$\begin{cases} \left(\frac{H_0}{\mathbf{H}_0}\tilde{a}_0\right)^2 \bar{\Omega}_{R0} = \Omega_{R0} \\ \Omega_{FM0} + \left(\frac{H_0}{\mathbf{H}_0}\tilde{a}_0\right)^2 \bar{\Omega}_{M0} + \Omega_{IM0} = \Omega_{M0} \\ \Omega_{F\Lambda 0} + \left(\frac{H_0}{\mathbf{H}_0}\tilde{a}_0\right)^2 \bar{\Omega}_{\Lambda 0} = \Omega_{\Lambda 0} \end{cases} . \tag{2.7.1}$$

Notice that these are not independent, since $\bar{\Omega}_{R0}+\bar{\Omega}_{M0}+\bar{\Omega}_{\Lambda 0}:=1$. We have only two independent constraints from

$$\begin{cases} [1-(s_{um}+1)\Omega_{IM0}]\bar{\Omega}_{R0}=\Omega_{R0} \\ \frac{2s_{um}+r_{act}}{3}\Omega_{IM0}+[1-(s_{um}+1)\Omega_{IM0}]\bar{\Omega}_{M0}+\Omega_{IM0}=\Omega_{M0} \\ \frac{s_{um}-r_{act}}{3}\Omega_{IM0}+[1-(s_{um}+1)\Omega_{IM0}]\bar{\Omega}_{\Lambda 0}=\Omega_{\Lambda 0}=1-\Omega_{R0}-\Omega_{M0} \end{cases}. \tag{2.7.2}$$

However, we have three unknown parameters: the inhomogeneity Ω_{IM0} and other two among $\bar{\Omega}_{R0}$, $\bar{\Omega}_{M0}$ and $\bar{\Omega}_{\Lambda 0}$. This means that the components of our universe are not completely determined by (2.7.2), but we will find more possible solutions, when a parameter changes. We choose $\bar{\Omega}_{M0}\in[0;1]$ as parameter, with $\Omega_{\Lambda 0}(\Omega_{R0};\Omega_{M0})=1-\Omega_{R0}-\Omega_{M0}$,

$$\Omega_{IM0}(\Omega_{R0};\Omega_{M0})=\frac{1}{1+s_{um}(\Omega_{R0};\Omega_{M0})}\left(1-\frac{\Omega_{R0}}{\bar{\Omega}_{R0}}\right) \tag{2.7.3}$$

and $\bar{\Omega}_{R0}=\bar{\Omega}_{R0}(\bar{\Omega}_{M0})$ is determined by the last independent constraint of (2.7.2).

We will have to check which of these values of $\bar{\Omega}_{M0}$ gives selfconsistent (i.e., if for them hold the IV SC), acceptable and evetually good solutions.

2.7.2 Epochs of evolution

Applying (2.1.6),

$$\bar{\Omega}_R=\bar{\Omega}_{R0}a^{-4},\quad \bar{\Omega}_M=\bar{\Omega}_{R0}a^{-3},\quad \bar{\Omega}_\Lambda\equiv\bar{\Omega}_{\Lambda 0}.$$

So we can get the values of a for which the matter starts to be more than the radiation, and the same for other couples

$$\begin{aligned} \bar{\Omega}_R\geq\bar{\Omega}_M &\Leftrightarrow a\leq a_{RM}:=a(\tau_{RM})=\frac{\bar{\Omega}_{R0}}{\bar{\Omega}_{M0}} \\ \bar{\Omega}_M\geq\bar{\Omega}_\Lambda &\Leftrightarrow a\leq a_{M\Lambda}:=a(\tau_{M\Lambda})=\sqrt[3]{\frac{\bar{\Omega}_{M0}}{\bar{\Omega}_{\Lambda 0}}} \\ \bar{\Omega}_R\geq\bar{\Omega}_\Lambda &\Leftrightarrow a\leq a_{R\Lambda}:=a(\tau_{R\Lambda})=\sqrt[4]{\frac{\bar{\Omega}_{R0}}{\bar{\Omega}_{\Lambda 0}}}. \end{aligned} \tag{2.7.4}$$

The evolution of the universe until now is for $0\leq a\leq a_0\cong 1$. During this time, there may have been three or two epochs, depending on the values $\bar{\Omega}_{R0}$, $\bar{\Omega}_{M0}$ and $\bar{\Omega}_{\Lambda 0}$.

Lemma 2.7.1. *A selfconsistent background evolution can be divided in epochs in the following ways.*

- *If $a_{RM}<a_{M\Lambda}<a_0$, then there are three epochs: for radiation $[0;\tau_{RM}]$, matter $[\tau_{RM};\tau_{M\Lambda}]$, and dark energy $[\tau_{M\Lambda};\tau(t_0)]$.*
- *If the first inequality does not hold, then there are just two epochs: radiation $[0;\tau_{R\Lambda}]$ and dark energy $[\tau_{R\Lambda};\tau(t_0)]$.*
- *If the second inequality does not hold, then there are just two epochs: radiation $[0;\tau_{RM}]$ and matter $[\tau_{RM};\tau(t_0)]$.*

Proof. Since the radiation exists, we know $\bar{\Omega}_{R0} > 0$, so that $a_{RM} > 0$ and $a_{R\Lambda} > 0$ for any values of $\bar{\Omega}_{M0}$, $\bar{\Omega}_{\Lambda 0}$. Thus, we have always a radiation epoch, which is the first one after the Big Bang. The presence of other epochs depends on our parameter: the quantity of homogeneous matter $\bar{\Omega}_{M0}$.

We will not consider the case with only the radiation epoch, because it would mean that $a_{RM}, a_{R\Lambda} \geq a_0 \cong 1$, which appens for high values of $\bar{\Omega}_{R0}$; but we know from the measures [32] that the radiation is far more less than the matter. Moreover, if the homogeneous matter would be so little, it would mean that $\Omega_{IM0} \cong \Omega_{TM0}$, that is not selfconsistent. $\square$

Let us consider the two cases with a matter epoch. From (2.6.1) we get the background evolution

$$a(\tau) = \begin{cases} H_0\tau & \tau \in [0; \tau_{RM}] \\ \frac{H_0^2}{4}(\tau - c_M)^2 & \tau \in [\tau_{RM}; \tau_{M\Lambda}] \\ \frac{1}{H_0(c_\Lambda - \tau)} & \tau \in [\tau_{M\Lambda}; \tau_F] \end{cases}, \tag{2.7.5}$$

where the continuity determines

$$\begin{aligned} H_0\tau_{RM} = a_{RM} \Rightarrow H_0 c_M = a_{RM} - 2\sqrt{a_{RM}}, \\ H_0\tau_{M\Lambda} = 2\sqrt{a_{M\Lambda}} + H_0 c_M, \quad H_0 c_\Lambda = H_0\tau_{M\Lambda} + 4(H_0\tau_{M\Lambda} - H_0 c_M)^{-2}. \end{aligned} \tag{2.7.6}$$

On the other hand, in the case such that there is no matter epoch, the evolution is

$$a(\tau) = \begin{cases} H_0\tau & \tau \in [0; \tau_{R\Lambda}] \\ \frac{1}{H_0(c_R - \tau)} & \tau \in [\tau_{R\Lambda}; \tau_F] \end{cases}, \tag{2.7.7}$$

where the continuity determines

$$H_0\tau_{R\Lambda} = a_{R\Lambda} \Rightarrow H_0 c_R = a_{R\Lambda} + \frac{1}{a_{R\Lambda}}. \tag{2.7.8}$$

The explicit evolution laws can be found in Appendix F. Now we employ the most recent measures of the cosmological parameters. The space flatness is confirmed by [32]

$$\Omega_{tot} = 1 - \Omega_{k0} = 1.02 \pm 0.02. \tag{2.7.9}$$

Thus we can assume $\Omega_{R0} + \Omega_{M0} + \Omega_{\Lambda 0} = \Omega_{tot} := 1$. We have also

$$\Omega_{R0} = 8.24 \times 10^{-5} \pm 10^{-7}[32], \quad \Omega_{M0} = 0.315 \pm 0.007[33]. \tag{2.7.10}$$

These are coherent with

$$\Omega_{\Lambda 0} = 0.685 \pm 0.007, q_0 = -0.527 \pm 0.0105. \tag{2.7.11}$$

Moreover,

$$\Omega_{BM0} = 0.0486 \pm 0.0010[33] \Rightarrow \Omega_{DM0} = 0.266 \pm 0.008. \tag{2.7.12}$$

The fraction of matter unexplained by the CCM is 84.57%.

2.7.3 Searching for good solutions

For any value of the free parameter $\bar{\Omega}_{M0}$, we can get a numerical solution of $\bar{\Omega}_{R0}$. Following §2.7.2 we have, for any chosen value, the evolution of $a(\tau)$ and, from the formulas of Appendix F, the quantities *ract* and *sum*, and thus Ω_{IM0}, Ω_{FM0} and $\Omega_{F\Lambda 0}$.

Imposing (2.7.2), there could be one or more solutions for $\bar{\Omega}_{R0}$, or no one, depending on $\bar{\Omega}_{M0}$. For any solution, we have to check if it is acceptable. The selfconsistence checking will require to compute the evolution of $\langle\Psi\rangle$, since we have to find that its maximum is less than $\frac{1}{2}$. Getting a set of selfconsistent and acceptable solutions, we will seek if some of them are also good.

Applying this planwork with a numerical algorithm, we find that for a generic $\bar{\Omega}_{M0}$ there are up to two acceptable values of $\bar{\Omega}_{R0}$. E.g. we can find

$$\bar{\Omega}_{R0}|_{\bar{\Omega}_{M0}=0.5} \cong \begin{cases} 10.97 \times 10^{-5} \\ 0.2216 \end{cases} . \tag{2.7.13}$$

The set of solutions with $\bar{\Omega}_{R0} \sim 10^{-4}$ have a radiation density quite near to the value of the CCM. We can call them the "principal" solutions, and "secondary" solutions the others. Indeed, following these solutions with continuity, for $\bar{\Omega}_{M0} = 0.315 \cong \Omega_{M0}$ we find trivially

$$\Omega_{IM0} = 0 \Rightarrow \Omega_{FM0} = \Omega_{F\Lambda 0} = 0, \bar{\Omega}_{w0} \equiv \Omega_{w0}. \tag{2.7.14}$$

The secondary solutions are not selfconsistent, since all of them have $\Omega_{IM0} > 99\% \cdot \Omega_{TM0}$, so that they break the Cosmological Principle. Moreover, the secondary solutions have quite big perturbations $2\max\langle\Psi\rangle > 0.5$: they are smaller than 1 anyway, but not small enough.

On the other hand, the principal solutions are selfconsistent. $\frac{\Omega_{IM0}}{\Omega_{TM0}}$ becomes greater as $\bar{\Omega}_{M0}$ runs away from Ω_{M0}, but it is always less than 45%. It is the same for $2\max\langle\Psi\rangle$, which is always smaller than $0.28 \ll 1$.
For values $\bar{\Omega}_{M0} > 0.9997$ we would find $\Omega_{T\Lambda 0} < 0$, which is not acceptable. For values $\bar{\Omega}_{M0} < 0.0819$ we would find $\Omega_{TM0} < \Omega_{BM0}$, which is not acceptable. This mean that the acceptable principal solutions range in the interval $\bar{\Omega}_{M0} \in [0.0819; 0.9997]$.
However, most of the principal solutions are not good. We find just a little interval around $\bar{\Omega}_{M0} \cong 0.2$ for which are explained some fraction of both dark matter and dark energy. For the values $\bar{\Omega}_{M0} = 0.2, \bar{\Omega}_{R0} \cong 10.03 \times 10^{-5}, \bar{\Omega}_{\Lambda 0} \cong 0.8$ it is

$$\begin{aligned}
&\Omega_{IM0} \cong -0.0316, \quad \Omega_{TM0} \cong 0.1327 \Rightarrow |\Omega_{IM0}| \cong 23.8\% \cdot \Omega_{TM0} \ll \Omega_{TM0}, \\
&2\max\langle\Psi\rangle \cong 0.0485 \ll 1, \\
&\Omega_{FM} \cong 0.1823 \Rightarrow \Omega_{TDM0} \cong 0.0841 \cong 63.39\% \cdot \Omega_{TM0}, \\
&\Omega_{F\Lambda} \cong 0.0276 \Rightarrow \Omega_{T\Lambda 0} \cong 0.6572 \cong 95.96\% \cdot \Omega_{\Lambda 0}.
\end{aligned} \tag{2.7.15}$$

For $\bar{\Omega}_{M0} > 0.2$ we start soon to have $\Omega_{FM0} < 0$, so that the solutions are no more good. For $\bar{\Omega}_{M0} < 0.2$ it starts *vice versa* to be $\Omega_{F\Lambda 0} < 0$, and the solutions are no more good as well.

2.7.4 Searching for solutions without dark energy or dark matter

We can seek if there is a selfconsistent and acceptable solution which fully explains the dark energy as fictitious. From the last paragraph, we know it would require an high $\bar{\Omega}_{M0}$, for which $\Omega_{FM0} < 0$ and the dark matter is more than in the CCM.

The condition of nonexistence of dark energy is $\bar{\Omega}_{\Lambda 0} := 0$, to that it is automatically fixed $\bar{\Omega}_{R0} = 1 - \bar{\Omega}_{M0}$. The (2.7.2) are solved by

$$\bar{\Omega}_{R0} \cong 26.31 \times 10^{-5}, \quad \bar{\Omega}_{M0} \cong 0.9997, \quad \bar{\Omega}_{\Lambda 0} = 0. \tag{2.7.16}$$

In such a case we find

$$\begin{aligned}
&\Omega_{IM0} \cong 0.2516, \quad \Omega_{TM0} \cong 0.5647 \Rightarrow |\Omega_{IM0}| \cong 44.56\% \cdot \Omega_{TM0} \ll \Omega_{TM0},\\
&2\max\langle\Psi\rangle \cong 0.2792 \ll 1,\\
&\Omega_{FM} \cong -0.2507 \Rightarrow \Omega_{TDM0} \cong 0.5161 \cong 91.39\% \cdot \Omega_{TM0},\\
&\Omega_{F\Lambda} \cong 0.685 \Rightarrow \Omega_{T\Lambda 0} \cong 0.
\end{aligned} \tag{2.7.17}$$

On the opposite, we can seek if there is a selfconsistent and acceptable solution that fully explains the dark matter as fictitious. From the last paragraph, we know it would require a small $\bar{\Omega}_{M0}$, for which $\Omega_{F\Lambda 0} < 0$ and the dark energy is more than in the CCM. The condition of nonexistence of dark matter is $\bar{\Omega}_{TM0} := \Omega_{MB0}$. The corresponding value of $\bar{\Omega}_{R0}$ is fixed by (2.7.2), which we solve numerically

$$\bar{\Omega}_{R0} \cong 9.59 \times 10^{-5}, \quad \bar{\Omega}_{M0} \cong 0.0819, \quad \bar{\Omega}_{\Lambda 0} \cong 0.9170. \tag{2.7.18}$$

In such a case we find

$$\begin{aligned}
&\Omega_{IM0} \cong -0.0218, \quad \Omega_{TM0} = \Omega_{MB0} \cong 0.0486 \Rightarrow |\Omega_{IM0}| \cong 44.84\% \cdot \Omega_{TM0} \ll \Omega_{TM0},\\
&2\max\langle\Psi\rangle \cong 0.1608 \ll 1,\\
&\Omega_{FM} = \Omega_{DM0} \cong 0.266 \Rightarrow \Omega_{TDM0} = 0,\\
&\Omega_{F\Lambda} \cong -0.1039 \Rightarrow \Omega_{T\Lambda 0} \cong 0.7888 \cong 115.17\% \cdot \Omega_{\Lambda 0}.
\end{aligned} \tag{2.7.19}$$

2.8 Defects of the model

We approximated our calculations in many points. To overcome them would be an improvement of the framework.

- Solving numerically the evolution $a(\tau)$ it would not be necessary any sticking, but recall that this would require the form of $T(\tau)$ for the multi-component case.
- Even if we found always that $2\max\langle\Psi\rangle$ is far smaller than 1, it could not be considered fully negligible, so that an higher order calculation could provide some relevant corrections.
- Moreover, we assumed a spatially flat background metric and an irrotational matter, which is not the most general framework.

In the present chapter we considered the global dark matter effects only. Our cosmological model requires also the calculation of the local effects, to be empirically verified. This needs to overcome the averaging of $g_{\mu\nu}$, and the distribution of fictitious dark matter would depend on the spatial distribution of inhomogeneities $\tilde{\rho}_0$. A study of such distribution could start from the fractal properties of the matter structures at large scales [100], [101]. The fluctuations of the resultant potential $\Psi_0(\underline{x})$ should be compared to the dark matter halo of the galaxies.

Further improvements may also be expected to shed some light on the choice of the homogeneous density $\bar{\rho}$, which turned out to be a tricky feature within the perturbative approach based on retarded potentials. In §2.1.1 it was chosen $\bar{\rho} := \min_{\underline{x}} \rho(\underline{x})$ or $\bar{\rho} := \max_{\underline{x}} \rho(\underline{x})$, but it is not yet clear if these are physically sensible choices. If $\bar{\rho}$ is taken as the average of ρ, as seems to be more physically meaningful, it returns $\langle\tilde{\rho}\rangle = 0$, which means no effects at all from a first order calculation. We will overcome this problem with the next chapter, considering a fractal matter distribution, rather than an homogeneous one with small perturbations.

Chapter 3

The LTB background

3.1 Retarded potentials and the fractal

A tantalizing possibility is that deeper insights on cosmological matter inhomogeneities may be gained by merging the two approaches presented in §1.3.4 and §1.3.3. Along this chapter we will indeed consider a fractal matter distribution, *at least* up to the L_{EG} scale (remember that we called L_{EG} the "End of Greatness", in §1, i.e. the length scale at which the universe becomes, eventually, homogeneous). We leave undetermined such parameter. One can substitute e.g. $L_{EG} \cong 100 Mpc$, as measured in [3], or the different values in [8], or [10], and so on. It can be even expressed the hypotheses of inexistence of a homogeneity scale, as claimed e.g. in [6], making L_{EG} tend to infinity.

We will describe the resulting metric, and consider the "Swiss cheese homogenization"' as the zeroth order approximation, and we will then deal with a first order perturbative description of the real fractal $\rho(\underline{x})$. Within the choice of a LTB background, the effects due to retarded potentials will also be effectively dealt with, thus allowing for a more reliable evaluation of cosmological parameters, such as the cosmological constant and the dark matter amount.

The previous computations with retarded potentials have improved the explanation of the dark matter effects, as well as of the dark energy effects. However, from [103] it is known that a suitable LTB background can explain the appearance of dark energy. It is thus reasonable to expect that a combination of the above two approaches may result in considerable advances in the explanation, at least to some non-negligible extent, of both dark matter and dark energy.

At the end of the last chapter, we stressed that for a homogeneous background we can not manage to define the background density in a fully satisfactory and physical way. This issue does not arise at all in the fractal approach, because no such a thing as spatial averaging exists for a fractal, which is endowed with a lower and lower average as the space region under consideration widens up; eventually, the average tends to zero because of the void bubbles. The growth of void bubbles prevents the determination of a unique real fractal density Φ. Moreover, the approximation $\Phi(r)r^D$ in §1.3.3 do not hold for too large scales, since r cannot be a coordinate, but some observable distance. If one tries to define $\Phi(r) := r^{-D}M(r)$, this will result in $\Phi(r)$ oscillating indefinitely. Within a fractal, such an issue can be overcome by choosing the minimum Φ of the oscillations as the reference for the definition of the homogenization $\bar{\rho}(r)$. One can appreciate that this procedure is physically meaningful, because the perturbation $\tilde{\rho}(\underline{x})$ may have negative and positive values here and there, but its average will certainly be positive, so that first order effects will not vanish.

All in all, in this thesis we aim at consistently determining the parameters of our model: $D, \Phi, L_{EG}, \tilde{\rho}_0, \rho_{\Lambda 0}, \rho_{R0}$. They will be obtained by fitting the experimental data, such as those linked to the dark matter effects and to the luminosity distances of SNe Ia. Furthermore, local metric distortions due to retarded effects will be compared to the expected dark matter inside the single galaxy or cluster, thus discerning to what extent they can effectively be explained as relativistic effects.

3.1.1 The origin of the fractal, and the three epochs

The perturbative approach should concern the LTB approximation for ancient times, when most perturbations were generated. The validity of the LTB model over time would actually be an interesting issue *per se*, because the solution used so far is valid just around the current instant. Back in time, for ancient times, we know that radiation dominates, and the evolution of the metric gets distorted.

For what concerns the origin of the fractal distribution of matter, we put forward the conjecture that it arises out as a consequence of the matter-antimatter (M-AM) recombination process. In fact, as a tiny fraction of matter survives the annihilation, it is conceivable that it was not homogeneously distributed, but rather it is scattered only across those regions in which the matter itself turned out to have a slightly larger density. Before the recombination, the inhomogeneity of matter would be mainly due to quantum uncertainty, being very small. However, after recombination only a $\sim 10^{-9}$ fraction of the pre-existing matter survives, and thus its inhomogeneity is magnified of a factor $\sim 10^9$. For our purposes, we can suppose that matter was already distributed as a fractal in very ancient times[1]; in fact, this solves also the problem of structure formation: dark matter is not actually needed, if matter was sufficiently concentrated at the very beginning.

In our model, the evolution of the Universe is characterized in terms of three different epochs, as follows.

1. *Before M-AM recombination.* The Universe is well described by FLRW, and quantum uncertainty is the unique source of perturbations.

2. *M-AM recombination.* It generates a matter remnant with fractal distribution, exhibiting a non-negligible inhomogeneity. It generates a large amount of homogeneous radiation, as well.

3. *After M-AM recombination.* At zeroth order, it is approximated by a LTB Universe, starting with the dominance of a homogeneous radiation, progressively fading away into an epoch in which fractal matter gets dominant. The first order perturbations better approximate the actual fractal, and they give rise to retarded distortions. The superposition of these latter for all times effectively results into dark matter phenomena, both globally and locally, as the fractal geometry causes a distortion of the luminosity distances which appears as a Universe acceleration.

[1] Note that most models of fractal cosmology assume that fractality is observed through galaxy distribution. Here, we do not mean that the formation of galaxies happened so soon, at the M-AM recombination. The ancient fractal, as obtained by the M-AM recombination, is imagined as an inhomogeneous ionized gas, with matter and radiation still coupled. Only much later, certainly after the matter-radiation decoupling, the matter clumped together in order to form of galaxies.

3.1.2 The "Swiss cheese" metric

In this paper we will consider a Swiss cheese metric[2]

$$d\bar{s}^2 = \begin{cases} -dt^2 + \frac{A'^2}{f^2}dr^2 + A(r;t)^2 d\Omega^2, & L_G \leq r \leq L_{EG} \\ -dt^2 + a(t)^2[dx^2 + x^2 d\Omega^2], & x \geq L_{EG} \end{cases}, \tag{3.1.1}$$

where the coordinate arbitrariness is fixed as

$$A(r;0) \equiv A_0(r) := r, \quad a(0) \equiv a_0 := 1,$$

and we use a prime " $'$ " for r-derivative and a dot " $\dot{}$ " for t-derivative.
Today, the matter inside dominates and is homogenized as

$$\bar{M}_0(r)|_{[L_G;L_{EG}]} = \Phi r^D \Rightarrow \bar{\rho}_0(r)|_{[L_G;L_{EG}]} = \frac{D}{4\pi}\Phi r^{D-3}. \tag{3.1.2}$$

The matter outside is already homogeneous, with some value

$$\bar{\rho}_0(x)_{[L_{EG};\infty)} \equiv \bar{\rho}_{0,out}. \tag{3.1.3}$$

The fractal dimension $D \cong 2$ can be measured as in [101]. It does not deform the luminosity distances everywhere, but just until L_{EG}, which can be coherent to the different measure in [103].

For $r < L_G$, the exact metric depends on the distribution of matter in a galaxy. A first simplification is to consider the fractal of matter as made of balls, whose minimum radius is the average radius of a galaxy, $L_G \cong 30kpc$; hence, the galaxy would be approximated as a homogeneous sphere, and thus below L_G another Friedmann metric would arise. We are aware that the galaxy is a multi-structured object and that its internal metric is far more complicated than a FLRW one, but such a complexity would be an obstacle for an introductory fractal model of the whole universe, what is the goal of this Chapter, so we decide to roughly approximate the galaxies to homogeneous spheres.

We consider as $A(r;t)$ is a FLRW metric during also epoch 1. It gains an inhomogeneity during epoch 2, and the epoch 3 sees the evolution of fractal. From now on, we will try to describe such $A(r;t)$, especially during epoch 3.

3.2 The LTB metric is not suitable for describing epoch 3

3.2.1 Pure matter

A universe filled with only matter regulates the Friedmann Equation outside as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G \bar{\rho}_{0,out} a^{-3}. \tag{3.2.1}$$

There are no singularities of density, thus the metric must be almost everywhere twice derivable: $\bar{g}_{\mu\nu} \in C^1$. Such a requirement contains the Darmois junction, which defines the dependence $x(r;t)$. These have especially the consequences

$$\begin{aligned} &\bar{g}_{tr} \in C^1(L_{EG}) \Rightarrow \dot{x}(L_{EG};t) \equiv 0 \Rightarrow x(L_{EG};t) \equiv L_{EG}; \\ &\bar{g}_{\Omega\Omega} \in C^0(L_{EG}) \Rightarrow A(L_{EG};t) = a(t)x(L_{EG};t) = L_{EG}a(t), \end{aligned} \tag{3.2.2}$$

[2]The variable A has nothing to do with the perturbative quantitiy $A = O(\Omega_{IM0})$ we used in the previous chapter. In this chapter, we call with this letter a component of the background metric.

where we remember the general form of the metric $d\bar{s} = \bar{g}_{tt}dt^2 + \bar{g}_{rr}dr^2 + \bar{g}_{\Omega\Omega}d\Omega^2$, so that the last component describes both the angular coordinates.

Within the fractal assumption $\bar{M}_0(r) := \Phi r^D$, and setting $f := 1$, we get the function

$$A(r;t) = r\left[1 + \frac{3}{2}H_0(r)t\right]^{\frac{2}{3}} = r\left[1 + \frac{3}{2}\sqrt{2G\Phi}r^{\frac{D-3}{2}}t\right]^{\frac{2}{3}}. \tag{3.2.3}$$

For Darmois (3.2.2), it yields to

$$a(t) = \frac{1}{L_{EG}}A(L_{EG};t) = \left[1 + \frac{3}{2}\sqrt{2G\Phi}L_{EG}^{\frac{D-3}{2}}t\right]^{\frac{2}{3}}. \tag{3.2.4}$$

Moreover, by differentiating Darmois, one obtains

$$\begin{aligned} h_0 &:= \frac{\dot{a}_0}{a_0} = \frac{\dot{A}_0(L_{EG})}{A_0(L_{EG})} = H_0(L_{EG}) = \sqrt{2G\Phi}L_{EG}^{\frac{D-3}{2}} \Rightarrow \\ a(t) &= \left[1 + \frac{3}{2}h_0 t\right]^{\frac{2}{3}}, \quad H_0(r) = h_0\left(\frac{r}{L_{EG}}\right)^{\frac{D-3}{2}}. \end{aligned} \tag{3.2.5}$$

By imposing the Friedmann equation to hold outside, the following results are achieved

$$\begin{aligned} &\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\bar{\rho}_{0,out}a^{-3} = h_0^2 a^{-3} \Rightarrow \dot{a}^2 = h_0^2 a^{-1} \quad s.t. \quad h_0^2 = \frac{8}{3}\pi G\bar{\rho}_{0,out} \Rightarrow \\ &a(t) = \left[\frac{3}{2}h_0(t - t_I)\right]^{\frac{2}{3}} \Rightarrow \left[1 + \frac{3}{2}h_0 t\right]^{\frac{2}{3}} = \left[\frac{3}{2}h_0(t - t_I)\right]^{\frac{2}{3}} \Rightarrow \\ &t_I = -\frac{2}{3}h_0^{-1}, \end{aligned} \tag{3.2.6}$$

and

$$\begin{aligned} 2G\Phi L_{EG}^{D-3} &= H_0(L_{EG})^2 = \frac{8}{3}\pi G\bar{\rho}_{0,out} \Rightarrow \\ \bar{M}_0(L_{EG}) &= \Phi L_{EG}^D = \frac{4}{3}\pi L_{EG}^3\bar{\rho}_{0,out}. \end{aligned} \tag{3.2.7}$$

Thus, the Swiss cheese metric has a time singularity at

$$t_S = -\frac{2}{3}\left(\frac{L_G}{L_{EG}}\right)^{\frac{3-D}{2}} h_0^{-1} > t_I, \tag{3.2.8}$$

at which $A'(L_G;t_S)$ goes to infinity. Here the validity of our pure matter model reaches an end.

Remark 16. Usually, the Big Bang is set at the time singularity of the metric. However, for the pure matter model such a singularity depends on r

$$t_{BB}(r) = -\frac{2}{3}\left(\frac{r}{L_{EG}}\right)^{\frac{3-D}{2}} h_0^{-1}, \tag{3.2.9}$$

such that $t_S := \max_r t_{BB}(r)$ (3.2.8) is just the first instant without singularity.

It is not clear the physical meaning of this result, which describes different Big Bang times for different regions of the universe. Such an inhomogeneous Big Bang was theoretically conjectured, e.g. in [119]; however, let us assume from now on that the Big Bang should be the same for the whole Universe.

Therefore, the pure matter does not provide a satisfactory description, and multi-component model is needed. In particular, a component with a larger w, such as radiation, will do the job: if it dominates in the early Universe, with an initial homogeneous density, it would grant the synchronicity of Big Bang for all r. This reasoning implies that the pure matter Swiss cheese metric (3.1.1) with (3.2.3) can be a good approximation only near the current instant, but generally the evolution must concern a multi-component model.

3.2.2 The flat LTB model

A consistent description of the expansion of the Universe would involve many components - namely matter, radiation, and eventually dark energy - and their evolutions.

To this aim, we need to make $\bar{\rho}_M(r;t)$ explicit; the functional dependence on time is obtained to be

$$\bar{\rho}_M(r;t) = \frac{D}{\pi}\Phi r^{D-3}\frac{1}{[2+3H_0(r)t][2+DH_0(r)t]}. \tag{3.2.10}$$

It should be remarked that $\bar{\rho}_M(r;t)$ goes as the inverse of the volume

$$\bar{\rho}_M(r;t) = \frac{4}{[2+3H_0(r)t][2+DH_0(r)t]}\bar{\rho}_M(r;0) = \frac{r^2}{A^2\,(r;t)\,A'(r;t)}\bar{\rho}_{M0}(r), \tag{3.2.11}$$

as expected, since matter is still (see §3.3.4 for confirmation). On the other hand, the dark energy does not depend on t, so its density reads

$$\bar{\rho}_\Lambda(r;t) = \bar{\rho}_{\Lambda 0}\,(r)\,. \tag{3.2.12}$$

In case it is a cosmological constant, it should also be independent of r.
Analogously to the FLRW model, one would expect that the radiation density goes as

$$\bar{\rho}_R\,(r;t) \propto \left(\frac{r^2}{A^2\,(r;t)\,A'(r;t)}\right)^{\frac{4}{3}}, \tag{3.2.13}$$

but this should better be confirmed by a more detailed computation, cfr. (3.2.34) further below.

We will henceforth carry out a detailed treatment of the flat LTB model. The LTB metric returns a diagonal Einstein tensor, with

$$\begin{aligned} G^t_t &= -\frac{\dot{A}}{A}\left(2\frac{\dot{A}'}{A'}+\frac{\dot{A}}{A}\right); \\ G^r_r &= -2\frac{\ddot{A}}{A}-\frac{\dot{A}^2}{A^2}; \\ G^\theta_\theta &= G^\varphi_\varphi = -\frac{\ddot{A}'}{A'}-\frac{\ddot{A}}{A}-\frac{\dot{A}'\dot{A}}{A'A}. \end{aligned} \tag{3.2.14}$$

See e.g. [119] for derivation.
Hence, also $T_{\mu\nu}$ is diagonal, implying still matter. Within the assumption of mostly-plus signature and the symmetries of our system, the energy-momentum tensor of a perfect fluid reads

$$\begin{aligned} &T_{\mu\nu} := (\rho+p)U_\mu U_\nu + p g_{\mu\nu} \quad s.t. \quad U_\mu = \delta_{t\mu} \Rightarrow \\ &T^t_t = -\rho, \quad T^r_r = T^\varphi_\varphi = p. \end{aligned} \tag{3.2.15}$$

Thus, three independent Einstein equations are obtained, namely

$$\begin{cases} -\frac{\dot{A}}{A}\left(2\frac{\dot{A}'}{A'}+\frac{\dot{A}}{A}\right) = -8\pi G\rho \\ -2\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} = 8\pi G p \\ -\frac{\ddot{A}'}{A'} - \frac{\ddot{A}}{A} - \frac{\dot{A}'\dot{A}}{A'A} = 8\pi G p. \end{cases} \tag{3.2.16}$$

The Ricci equation as a Riccati equation, and its solutions

With a barotropic equation of state $\rho = \rho(p)$, one has four equations for the three unknowns A, ρ, p. This should imply some constraint on the form of A, ρ, p. Such a constraint can be obtained from the second and third Einstein equations, as follows:

$$\begin{aligned} &2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} = 8\pi G p = \frac{\ddot{A}'}{A'} + \frac{\ddot{A}}{A} + \frac{\dot{A}'\dot{A}}{A'A} \Rightarrow \\ &\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} = \frac{\ddot{A}'}{A'} + \frac{\dot{A}'\dot{A}}{A'A}. \end{aligned} \tag{3.2.17}$$

We can try to solve this non-linear PDE in A, which we will name *Ricci equation*, and search for a set of self-consistent solutions. Exploiting the definition

$$H := \frac{\dot{A}}{A}, \tag{3.2.18}$$

the identity (3.2.17) can be rewritten in a very simple way,

$$\dot{H}' + 3HH' = 0. \tag{3.2.19}$$

For a general Universe, (3.2.19) constrains the possible matter, radiation and/or dark energy content. It is easy to check that the solution found in [103] satisfies this PDE. Nevertheless, there is no uniqueness proven for the solutions of (3.2.19), so we can search for other, different solutions.

Now, (3.2.19) can be rewritten as

$$\begin{aligned} &0 = \partial_r\left(\dot{H} + \frac{3}{2}H^2\right) \Rightarrow \\ &\dot{H} + \frac{3}{2}H^2 = c(t), \end{aligned} \tag{3.2.20}$$

where the integration constant $c(t)$ does not depend on r. Eq. (3.2.20) can be recognized to be a Riccati Equation. For $c(t) \equiv 0$, we find again the solution in [103], namely

$$H(r;t) = \frac{2H_0(r)}{2 + 3H_0(r)t}. \tag{3.2.21}$$

But *e.g.* for a non-zero, constant $c(t) \equiv c$ we can find different solutions. Calling $c := \frac{3}{2}\tau^{-2}$, we get

$$H(r;t) = \frac{1}{\tau}\tanh\frac{3t}{2\tau} + H_0(r), \tag{3.2.22}$$

while for negative $c = -\frac{3}{2}\alpha^2$ one has the (quite unphysical) solution

$$H(r;t) = -\alpha\tan\frac{3\alpha t}{2} + H_0(r). \tag{3.2.23}$$

On the other hand, we observe that the second Einstein Eq. from (3.2.16) depends only on H, thus exploiting (3.2.20) we can obtain the following expression for th pressure p

$$8\pi Gp = -2\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} = -2\dot{H} - 3H^2 = -2\left(\dot{H} + \frac{3}{2}H^2\right) = -2c(t). \tag{3.2.24}$$

Therefore, the integration constant gets related to p itself: $c(t) = -4\pi Gp$, which implies that the total pressure *must be homogeneous* at any time

$$p(r;t) = p(t). \tag{3.2.25}$$

Conservation of four-momentum and separability of w's

Let us now study the conservation of the four-momentum. One can compute the conservation of energy

$$\dot{\rho} = -\left(\frac{\dot{A}'}{A'} + 2\frac{\dot{A}}{A}\right)(\rho + p), \tag{3.2.26}$$

and the conservation of momentum, which turn out to be

$$p' = 0. \tag{3.2.27}$$

This equation is just a confirmation of the result (3.2.25), expressing the homogeneity of pressure, as it was proven for a perfect fluid in a LTB flat metric[3].

Let us now consider a particular type of perfect fluid, namely a *single-component* one, defined by $p := w\rho$. The homogeneity of pressure then immediately implies

$$w\rho' = 0. \tag{3.2.28}$$

We can thus conclude that *no single-component, inhomogeneous flat LTB Universe can exist, unless such a component is matter.* It then turns out that the two solutions for the single-component case in a flat LTB Universe were actually already both studied: for $w = 0$, the pure matter flat LTB model, studied in [119], [120], [102] and [103], is retrieved; for $\rho' = 0$, one simply obtained the well-known FLRW model.

The case of a *multi-component* perfect fluid is more interesting. By setting

$$p = \sum_w p_w = \sum_w w\rho_w, \text{ s.t. } \rho = \sum_w \rho_w \tag{3.2.29}$$

as usual, the conservation of momentum (3.2.27) allows for inhomogeneities to exist for any component ρ_w, but only if the pressure inhomogeneities compensate each other

$$\sum_w w\rho_w'(r;t) = 0. \tag{3.2.30}$$

On the other hand, the conservation of energy allows one to study each component separately (i.e., by fixing the corresponding w); indeed, (3.2.26) and (3.2.29) yield

$$\sum_w \dot{\rho}_w = -\left(\frac{\dot{A}'}{A'} + 2\frac{\dot{A}}{A}\right)\sum_w (1+w)\rho_w. \tag{3.2.31}$$

[3]It is known that LTB spacetime is a dust solution, indeed we are going to prove that such a perfect fluid can be only the dust, essentially.

Within the assumption of separation of components[4], for each component w we find

$$\partial_t \ln \rho_w = \frac{\dot{\rho}_w}{\rho_w} = -(1+w)\left(\frac{\dot{A}'}{A'} + 2\frac{\dot{A}}{A}\right) = -(1+w)\partial_t(\ln A' + 2\ln A) = \partial_t \left(A^2 A'\right)^{-1-w}$$
$$\Rightarrow \rho_w(r;t) = \rho_{w0}(r)\left(\frac{A_0(r)^2 A_0'(r)}{A^2(r;t)A'(r;t)}\right)^{1+w}, \quad \forall w. \tag{3.2.32}$$

For $w = 0$ (matter), and choosing the radial coordinate s.t. $A_0(r) \equiv r$ (cfr. (3.1.2)), one retrieves (3.2.11), namely

$$\rho_M(r;t) = \rho_{M0}(r)\frac{r^2}{A^2(r;t)A'(r;t)}. \tag{3.2.33}$$

For $w = 1/3$ (radiation), Eq. (3.2.32) confirms the conjecture (3.2.13), namely

$$\rho_R(r;t) = \rho_{R0}(r)\left(\frac{r^2}{A^2\,(r;t)\,A'\,(r;t)}\right)^{4/3}. \tag{3.2.34}$$

For $w = -1$ (dark energy) the density is constant, and the previous result is confirmed, namely $\rho_\Lambda(r;t) = \rho_{\Lambda 0}$.

To recap, in a flat LTB Universe with just matter and radiation, the radiation must be homogeneous, and this holds also in presence of a cosmological constant (i.e., of homogeneous dark energy). Notice that here we deduced it from the conservation of four-momentum.

3.2.3 The approximation with epochs

No explicit, exact solutions are known for the Einstein field equations in such a general case, with many components. Thus, we will resort to the so-called "approximation with epochs" as we did in §2.7 for FLRW background.

We start by noticing that, even if radiation and dark energy are homogeneous, the matter is not; therefore, it might well be that for some r we could be in an epoch, whereas for some other r we are already in another one. We will consider the case of dominating matter further below, and we will now focus on an evolution dominated by radiation. Moreover, from now on we will not consider the dark energy component in our calculations: they would be just more complicate, without let a better understanding.

An homogeneous distribution of primordial radiation could be assumed, thus giving rise to a Friedmannian expansion during the epoch dominated by radiation.

$$\dot{a}^2 \cong h_0^2 \Omega_{R0} a^{-2} \Rightarrow a(t) = \left[2\sqrt{\Omega_{R0}}h_0(t - t_{BB})\right]^{1/2}, \quad s.t. \quad \Omega_{R0} := \frac{8\pi G}{3h_0^2}\rho_{R0}, \tag{3.2.35}$$

with the radiation evolving as $\Omega_R(t) = \Omega_{R0}a^{-4}(t)$.

Remark 17. $\rho_{R0} := \rho_R(r;t=0)|_{r \geq L_{EG}}$ is the radiation density today *beyond the End of Greatness* , at which it is still uniform. Below L_{EG}, one can reasonably assume that $\rho_R(r;0)$ is not homogeneous, since it developed through an inhomogeneous expansion. This would imply the current measurements of Ω_{R0} not to be reliable, since they would take place inside our galaxy, and thus in a point of the cosmic fractal: these would be measures of $\rho_R(L_G;0)$, which could be quite different from the average value ρ_{R0}. For instance, inside a void bubble, the density of the cosmic background would undergo a completely different development.

[4] We will see below that such an assumption would not hold during epoch 2 (cfr. §3.3).

When $\Omega_M(t) \geq \Omega_R(t)$, one would switch to the epoch dominated by matter. Ω_M must also depend on r

$$\Omega_M(r;t) := \frac{8\pi G}{3h_0^2}\bar{\rho}_M(r;t), \quad s.t. \quad \bar{\rho}_M(r;t) \propto \begin{cases} a(t)^{-3}, & t < t_{RM}(r) \\ \frac{r^2}{A^2 A'}, & t > t_{RM}(r). \end{cases} \tag{3.2.36}$$

For a fixed $r < L_{EG}$, the "soldering instant" t_{RM} is defined as

$$\Omega_M(r;t_{RM}) := \Omega_R(t_{RM}) \Leftrightarrow \frac{D\Phi r^{D-3}}{\pi[2+3H_0(r)t_{RM}][2+DH_0(r)t_{RM}]}\bar{\rho}_M(r;0) = \rho_R(r;t_{RM}). \tag{3.2.37}$$

"Swiss cheese" with two epochs

Next, we will consider again the Swiss cheese metric, in order to describe a radiation+matter Universe by soldering the corresponding two one-component solutions together.

Let us consider first the *outer* expansion, which is simpler. The Friedmann Eq. (3.2.6) at $r > L_{EG}$ reads

$$h^2 := \left(\frac{\dot{a}}{a}\right)^2 = h_0^2(\bar{\Omega}_{R0,out}a^{-4} + \bar{\Omega}_{M0,out}a^{-3}), \quad s.t. \quad \bar{\Omega}_{w0,out} := \frac{\bar{\rho}_{w0,out}}{\bar{\rho}_{0,out}}, \quad \bar{\rho}_{0,out} := \frac{3h_0^2}{8\pi G}. \tag{3.2.38}$$

The outside matter density is related to the inside matter density $\bar{\rho}_{M0}(r) = \frac{D}{4\pi}\Phi r^{D-3}$ by

$$\Phi = \frac{4}{3}\pi L_{EG}^{3-D}\bar{\rho}_{M0,out}. \tag{3.2.39}$$

If $\bar{\Omega}_{M00} > \bar{\Omega}_{R00}$, it holds that

$$a(0) := 1 > a_{RM} := \frac{\bar{\Omega}_{R0,out}}{\bar{\Omega}_{M0,out}} > a_{BB} := 0. \tag{3.2.40}$$

Thus, during both epochs, the evolution of the Universe can be approximated as if there were only one component, i.e. the dominating one.

$$a(t) = \begin{cases} (2h_0(t-t_{BB}))^{1/2}, & t_{BB} \leq t \leq t_{RM} \\ \left(\frac{3}{2}h_0 t + 1\right)^{2/3}, & t_{RM} \leq t \leq 0 \end{cases}. \tag{3.2.41}$$

Thus, one can compute the "soldering instant" t_{RM} as follows

$$\frac{3}{2}h_0 t_{RM} + 1 = a_{RM}^{3/2} \Rightarrow t_{RM} = \frac{2}{3}h_0^{-1}[a_{RM}^{3/2} - 1]. \tag{3.2.42}$$

From the continuity of $a(t)$ at t_{RM}, one obtains also the homogeneity for the Big Bang instant $t_{BB}(r) \equiv t_{BB}$.

Let us now consider the *inner* expansion; for a fixed $r < L_{EG}$, since the radiation epoch must be homogeneous, we know that the evolution is

$$A(r;t) = \begin{cases} r\,(2h_0(t-t_{BB}))^{1/2}, & t_{BB} \leq t \leq t_{RM}(r) \\ r\left(\frac{3}{2}H_0(r)t + 1\right)^{2/3}, & t_{RM}(r) \leq t \leq 0 \end{cases}, \tag{3.2.43}$$

where we recalled the result (3.2.5). When we try to compute the "soldering instant" $t_{RM}(r)$ within this regime, we can appreciate the inadequacy of the framework under consideration in order to describe a Universe with radiation and matter; indeed, we should impose the continuity of $A(r;t)$, and thus solve

$$(2h_0(t_{RM}(r) - t_{BB}))^3 = \left(\frac{3}{2}H_0(r)t_{RM}(r) + 1\right)^4 . \tag{3.2.44}$$

Surely, for $r \to L_{EG}^-$, we will retrieve the expression of t_{RM} (3.2.42) computed within the outside expansion, because $H_0(L_{EG}) = h_0$. Nevertheless, let us consider the definition of $t_{RM}(r)$ as the instant when the matter density and the radiation density are equal; by defining $x := r/L_{EG}$, one can write

$$\begin{aligned}
&\bar\rho_{R0,out}[2h_0(t - t_{BB})]^{-2} = \bar\rho_{R0,out}a^{-4} = \bar\rho_R(r;t) = \bar\rho_M(r;t)\\
&= \bar\rho_{M0}(r)\frac{r^2}{A^2A'} = \frac{D}{3}x^{D-3}\bar\rho_{M0,out}\left(1 + \frac{3}{2}x^{\frac{D-3}{2}}h_0t\right)^{-1}\left(1 + \frac{D}{2}x^{\frac{D-3}{2}}h_0t\right)^{-1} \Rightarrow\\
&a_{RM}\left(1 + \frac{3}{2}x^{\frac{D-3}{2}}h_0t\right)\left(1 + \frac{D}{2}x^{\frac{D-3}{2}}h_0t\right) = \frac{D}{3}x^{D-3}[2h_0(t - t_{BB})]^2 = \frac{D}{3}x^{D-3}\left(1 + \frac{3}{2}x^{\frac{D-3}{2}}h_0t\right)^{8/3},
\end{aligned} \tag{3.2.45}$$

where we used (3.2.44) in the last step of (3.2.45). As mentioned, for $x \to 1^-$ one should find again $t = t_{RM}$, thus obtaining

$$\begin{aligned}
&\left(\frac{3}{D}a_{RM}\right)^3\left(1 + \frac{D}{3}(a_{RM}^{3/2} - 1)\right)^3 = (1 + a_{RM}^{3/2} - 1)^5 \Rightarrow\\
&\left(\frac{3}{D}a_{RM} + a_{RM}(a_{RM}^{3/2} - 1)\right)^3 = a_{RM}^{15/2} \Leftrightarrow \left(\frac{3}{D} - 1\right)a_{RM} = 0.
\end{aligned} \tag{3.2.46}$$

Thus, we obtain that only trivial solutions are allowed for consistency, namely, the trivial FLRW solution $D = 3$, or the pure matter solution $a_{RM} = 0$.

3.2.4 Inadequacy of the *flat* LTB model

We have found that the Swiss cheese metric, with inhomogeneous matter and non-zero radiation, cannot be self-consistent when assuming a spatially flat metric and still energy-matter. In other words, a spatially flat, inhomogeneous LTB solution with still energy-matter must necessarily contain only matter, and possibly some dark energy, whose evolution $\propto Vol^0$ allows to preserve the homogeneity (however, dark energy cannot dominate near the Big Bang, which will necessarily be inhomogeneous in any such model; cfr. Remark 16 above).

By setting to zero the velocity field, the conservation of momentum implies the homogeneity of pressure ($p' = 0$) at *any* instant, so that there are no forces. Within this framework, one can appreciate that the inconsistency between inhomogeneous matter and non-zero radiation can be traced back to the homogeneity of pressure. Indeed, since the matter has vanishing pressure, the conservation of momentum yields homogeneous radiation density, at any instant. But the expansion iself is inhomogeneous, due to the matter inhomogeneity; as a consequence, even if the radiation is homogeneous at a given instant, it will evolve inhomogeneously with the expansion, thus breaking the conservation of momentum.

In a Universe undergoing a two-epochs evolution (as we are assuming in this Section), the conservation of momentum approximately holds during both epochs: as for the homogeneous expansion during the radiation-dominated epoch, so for the zero pressure expansion during the matter-dominated epoch. However, the "two-epochs approximation" fails in proximity of the "soldering instant" t_{RM}, namely when radiation and matter are about to be equal. In such an intermediate period of time, the pressure is no more negligible, but the expansion is still inhomogeneous. The inconsistency arises because the conservation of momentum prevents the determination of a well-defined "soldering instant" t_{RM}.

It is here worth remarking that this inconsistency cannot be solved by adding other components, possibly aiming at compensating the inhomogeneity of the pressure of radiation. Indeed, even if some other ρ_w allows to set $w\rho'_w + \frac{1}{3}\rho'_R = p' := 0$ for a given instant, this cannot hold for other instants, because the w component evolves as $\propto (Vol)^{-1-w}$ with $w \neq \frac{1}{3}$, whereas the radiation evolves $\propto Vol^{-4/3}$.

The above clashing of volumetric expansions implies that the consistent way to add the radiation, or any other component with $w \neq 0, -1$, to the LTB model, is *at most* two-fold, as one could consider a non-vanishing velocity field v (yielding a fourth Einstein equation, the one sourced by the component T_{tr} of energy-momentum tensor), and/or a non-vanishing spatial curvature.

3.2.5 The *non-flat* LTB model

Let us generalize the LTB metric by adding a non-vanishing spatial curvature $k := k(r)$,

$$ds^2 = -dt^2 + \frac{(A')^2}{f^2}dr^2 + A^2 d\Omega^2, \quad s.t. \quad f(r)^2 = 1 - k(r)^2. \tag{3.2.47}$$

The treatment of this metric given in [119] yields the Einstein tensor to be diagonal again; in particular, $G^t_r = 0$. In turn, this implies a diagonal energy-momentum tensor, and for a perfect fluid the conservation of momentum yields the following result:

$$0 = \partial_r T^r_r + 2\Gamma^\theta_{r\theta}(T^r_r - T^\theta_\theta) = p'. \tag{3.2.48}$$

However, the aforementioned inconsistency plaguing the flat LTB Universe is not (yet) resolved in such a non-flat Universe. In fact, a still energy-matter evolves as $\propto (Vol)^{-1-w}$, with some dependence on f in the formula of the volume Vol; consequently, the conservation of momentum still allows only matter and dark energy within an inhomogeneous Universe, still exhibiting an inhomogeneous Big Bang (cfr. Remark 16 above).

Thus, one must necessarily consider a non-vanishing velocity field ($v \neq 0$) within a non-flat LTB Universe. Since (compare with §3.4.1)

$$T^t_r = -(\rho + p)v\sqrt{1 + \left(\frac{f}{A'}v\right)^2} \neq 0, \tag{3.2.49}$$

this would imply a non-vanishing G^t_r, again forbidden by [119]. The only way out is to consider a moving energy-matter ($v \neq 0$) within a Universe with the most general type of (non-vanishing) spatial curvature (although the spherical symmetry is required nevertheless), namely $k = k\,(r;t)$, thus implying $f^2 = 1 - k(r;t)^2 = f(r;t)^2$. Indeed, the (tr)-component of the Einstein Eqs. results to be

$$\frac{A'}{A}\frac{\dot{f}}{f} = -4\pi G(\rho + p)v\sqrt{1 + \left(\frac{f}{A'}v\right)^2}. \tag{3.2.50}$$

Thus, we have four variables A, ρ, v, f for four Einstein Eqs. (namely, the three diagonal components (tt), (rr), $(\theta\theta)$, and the non-diagonal component (tr)).

3.3 Expansion during M-AM recombination

3.3.1 Inseparability of components

Let us consider again (3.2.31). In the treatment given above, we have assumed the separation of (3.2.31) into each of its w-components, and we have obtained that an inhomogeneous LTB Universe with non-vanishing radiation requires a non-zero velocity field and a spatial curvature k depending both on t and r. However, during the M-AM recombination (corresponding to the epoch 2; cfr. §3.1.1), the separation of Eq. (3.2.31) into its w-components is a sufficient *but not necessary* condition for the solution of (3.2.31) itself. In general, some mixing terms among the different components w's can occur, as a consequence of the recombination between matter and antimatter, in which a huge quantity of $w = 0$ (matter) component gets transformed into the $w = 1/3$ (radiation) component[5].

For simplicity's sake, let us consider now the case with matter ($w = 0$) *and* radiation ($w = 1/3$) only[6]. Eq.s (3.2.29) and (3.2.27) imply

$$\rho_R = \rho_R(t) = 3p(t). \tag{3.3.1}$$

Then, the equation of the conservation of the energy (3.2.31) can be written as

$$\begin{aligned}\dot\rho_M + \dot\rho_R = -\left(\frac{\dot A'}{A'} + 2\frac{\dot A}{A}\right)\left(\rho_M + \frac{4}{3}\rho_R\right) \Leftrightarrow \\ \Leftrightarrow \dot\rho_M = -\frac{\partial_t(A^2A')}{A^2A'}\rho_M - \left[3\dot p + 4\frac{\partial_t(A^2A')}{A^2A'}p\right].\end{aligned} \tag{3.3.2}$$

It can be integrated as

$$\rho_M(r;t) = \frac{\left[\mathcal{K}_M(r) + \int_0^t \dot p(\tau)A^2(r;\tau)A'(r;\tau)d\tau\right]}{A^2(r;t)A'(r;t)} - 4p(t), \tag{3.3.3}$$

where $\mathcal{K}_M(r) = r^2[\rho_{M0}(r) + 4p_0]$.

3.3.2 Einstein equations

Having obtained the explicit functional dependence of the matter density and its relation with the pressure, let us now try to solve the Einstein equations (3.2.16) within the flat LTB model[7]. By specifying only matter and radiation, and recalling (3.3.1), Einstein equations read

$$\begin{cases}\frac{\dot A^2}{A^2} + 2\frac{\dot A'\dot A}{A'A} = 8\pi G[\rho_M + 3p(t)] \\ 2\frac{\ddot A}{A} + \frac{\dot A^2}{A^2} = -8\pi G p(t) \\ \frac{\ddot A'}{A'} + \frac{\ddot A}{A} + \frac{\dot A'\dot A}{A'A} = -8\pi G p(t)\end{cases}, \tag{3.3.4}$$

where we stressed the fact that the pressure depends only on time, as expressed by (3.3.1), which in turn guarantees the conservation of momentum. From the treatment given in the previous Section, the conservation of energy is given by(3.3.2), whereas the equation of

[5] This may result into an overly simplified physical picture during M-AM recombination, but nevertheless we search for a solution within this framework.

[6] From now on, we will not put any dark energy as a component, since we want to explain the luminosity distance observations as a consequence of the fractal metric. It is possible an analogous model with a cosmological constant, but it would further complicate the equations.

[7] A detailed treatment of the Einstein equations for the *non-flat* LTB model with $k = k(r;t)$ and $v \neq 0$ will be given in §3.4.

state for matter and radiation has been taken into account by specifying $\rho = \rho_M + \rho_R = \rho_M + 3p$.

From the treatment of §3.2.2, we know that equating the second and third Einstein equations, one obtains a Riccati equation (3.2.20) for the Hubble parameter H (3.2.18)

$$\dot{H} + \frac{3}{2}H^2 + 4\pi G p(t) = 0, \tag{3.3.5}$$

where Eq. (3.2.24) has been recalled. We have discussed above the solutions for $p\,(t) = 0$ (vanishing pressure) and for $p(t) = p \neq 0$ (non-vanishing, constant pressure), respectively given by Eqs. (3.2.21) (obtained in [103]) and (3.2.22). Following the usual method to solve such a class of differential equations (cfr. *e.g.* [121]), we define the auxiliary variable $y\,(r;t)$ as follows

$$H =: \frac{2}{3}\frac{\dot{y}}{y}, \tag{3.3.6}$$

in terms of which the Riccati equation (3.3.5) becomes linear

$$\ddot{y} = -6\pi G p(t) y. \tag{3.3.7}$$

It can be appreciated that y provides an alternative description of the expansion of Universe, in place of the coefficient $A\,(r;t)$; indeed, by recalling (3.2.18) and (3.3.6), one gets $A^3 = y^2$, and $A^2 A' \propto yy'$.

Hence, one can rewrite the Eq. (3.3.2) of conservation of energy as

$$\dot{\rho}_M = -\frac{\partial_t(yy')}{yy'}\rho_M - \left[3\dot{p} + 4\frac{\partial_t(yy')}{yy'}p\right]. \tag{3.3.8}$$

Analogously, one can rewrite the other Einstein equations

$$\begin{aligned} 6\pi G[\rho_M + 3p(t)] &= \frac{\dot{y}'\dot{y}}{y'y}; \\ -6\pi G p(t) &= \frac{\ddot{y}}{y}. \end{aligned} \tag{3.3.9}$$

By construction, the third Einstein equation from (3.3.4) is equivalent to the second one via (3.3.7): both of them are (3.3.7) again. Thus, the Einstein system (3.3.4) can be rewritten in a simpler way in terms of the y function (3.3.6) as follows

$$\begin{cases} \frac{\dot{y}'\dot{y}}{y'y} = 6\pi G\left[\rho_M + 3p(t)\right] \\ \ddot{y} = -6\pi G p(t) y \end{cases}. \tag{3.3.10}$$

We observe that it is useless to substitute ρ_M from the first Einstein equation inside (3.3.2), since it gives again (3.3.7).

Thus, we end up with the system (3.3.10) composed by two independent PDE's in terms of the functions $y\,(r;t)$ (3.3.6) and $\rho_M\,(r;t)$ (3.3.3), but the 1-variable function $p(t)$ remains here undetermined. It is then evident that some other condition is needed in order to obtain a consistent evolution of the Universe; it is easy to realize that such a missing condition should be provided by the law of transformation from matter to radiation as resulting from the M-AM recombination, which we did not consider yet.

3.3.3 New variables

We observe that the linear Riccati equation (3.3.7) does not actually depend on r; thus, since it is a second order equation, its general solution $y(r;t)$ will be given by a linear combination of two purely t-dependent functions $y_1(t)$ and $y_2(t)$, with r-dependent coefficients,

$$y(r;t) := c_1(r)y_1(t) + c_2(r)y_2(t), \tag{3.3.11}$$

where

$$\ddot{y}_{1,2}(t) =: -6\pi Gp(t)y_{1,2}(t). \tag{3.3.12}$$

The conditions at $t = 0$ can be fixed *e.g.* by setting

$$\begin{cases} y_1(0) = 0 = \dot{y}_2(0) \\ \dot{y}_1(0) = 1 = y_2(0) \end{cases}, \tag{3.3.13}$$

which yields

$$y(r;t) = A_0(r)^{3/2}\left[\frac{3}{2}H_0(r)y_1(t) + y_2(t)\right]. \tag{3.3.14}$$

Next, we notice the importance of the variable

$$V := y^2 = A^3 \Rightarrow yy' = \frac{1}{2}V', \tag{3.3.15}$$

which represents the volume of the sphere centred in $\vec{0}$ with radius r. By exploiting the definition (3.3.15), the first Einstein equation of (3.3.10) can be recast in the following form (where $\rho = \rho_M + 3p$)

$$\dot{y}'\dot{y} = 3\pi G\rho V', \tag{3.3.16}$$

whereas the equation of energy conservation (3.3.8) and the formula of $\rho_M(r;t)$ (3.3.3) respectively acquire the following forms

$$\dot{\rho} = -\frac{\dot{V}'}{V'}(\rho + p), \tag{3.3.17}$$

and

$$\rho = \frac{1}{V'}\left[\mathcal{K}_M(r) - \int^t p(\tau)\dot{V}'(r;\tau)\,d\tau\right], \tag{3.3.18}$$

By inspecting Eq. (3.3.17), one can appreciate that an even better variable to be used would be the total energy inside the sphere of radius r,

$$E(r;t) := \int^r \rho(s;t)dV(s;t) = M(r;t) + 3p(t)V(r;t), \tag{3.3.19}$$

where

$$M(r;t) := \int_0^r \rho_M(s;t)V(s;t)ds. \tag{3.3.20}$$

By virtue of the fact that definition (3.3.19) implies

$$E' = \rho V', \tag{3.3.21}$$

the first Einstein equation (3.3.16) boils down to

$$\dot{y}'\dot{y} = 3\pi GE'. \tag{3.3.22}$$

It can be integrated in r, obtaining

$$\dot{y}^2 = 6\pi G E \Rightarrow \dot{E} = \frac{1}{3\pi G}\dot{y}\ddot{y} = -2y\dot{y}p = -p\dot{V}, \tag{3.3.23}$$

where the second Einstein equation (3.3.10) was used and definition (3.3.15) recalled. Finally, by integrating further in t, one gets

$$E(r;t) = E_0(r) - \int_0^t p(\tau)\dot{V}(r;\tau)d\tau, \tag{3.3.24}$$

where $E_0(r) = M_0(r) + 3p_0$.

The evaluation of (3.3.23) today (i.e. for $t = 0$) and the use of the time derivative of (3.3.14) yields the relation between $E_0(r) \equiv E(r;0)$ and $H_0(r)$ (by recalling the conditions (3.3.13)),

$$\dot{y}^2\,(r;0) = 6\pi G E_0\,(r) \Leftrightarrow H_0^2(r) = \frac{8}{3}\pi G\frac{E_0\,(r)}{A_0(r)^3} = \frac{8}{3}\pi G\overline{\rho}_0(r), \tag{3.3.25}$$

where we defined $\overline{\rho} := E/V = \bar{\rho}_M + 3p$ as the average density of energy inside the ball of radius r. The equation on the r.h.s. of (3.3.25) is a well known relation for FRW model, but in the framework under consideration it depends on r. Analogously to FRW, we can define the Ω parameters as

$$\Omega_M(r;t) := \frac{8\pi G M\bar{\rho}_M}{3H^2}, \quad \Omega_R(r;t) := \frac{8\pi G p}{H^2} \quad s.t. \quad \Omega_{M0}(r) + \Omega_{R0}(r) := 1, \tag{3.3.26}$$

where $\Omega_{M0}(r) \equiv \Omega_M(r;0)$ and $\Omega_{R0}(r) \equiv \Omega_R(r;0)$.

3.3.4 General form

By plugging the time derivative of $y(r;t)$ (3.3.14) into (3.3.23), one obtains

$$6\pi G E(r;t) = \dot{y}^2(r;t) = A_0(r)^3\left[\frac{3}{2}H_0(r)\dot{y}_1(t) + \dot{y}_2(t)\right]^2. \tag{3.3.27}$$

On the other hand, by exploiting the first Einstein equation of (3.3.10) an recalling the definition (3.3.15), one could get a similar expression for ρ, but it turns out to be non-polynomial.

Pure matter

Let us consider the pure matter case: $p \equiv 0$. From the second Einstein equation of (3.3.10), one gets

$$y_1(t) = t, \quad y_2 = 1 \tag{3.3.28}$$

yielding to

$$A(r;t) = y^{2/3}(r;t) = A_0(r)\left[\frac{3}{2}H_0(r)t + 1\right]^{2/3}. \tag{3.3.29}$$

Since $p = 0$, Eqs. (3.3.19) and (3.3.24) imply that

$$\dot{M} = \dot{E} = 0, \tag{3.3.30}$$

and the first Einstein equation of the system (3.3.10) simplifies down to

$$\rho_M\,(r;t) = \rho_{M0}\,(r)\,\frac{A_0'(r)}{\left[A_0'(r)\left(\frac{3}{2}H_0(r)t + 1\right) + A_0(r)H_0'(r)t\right]\left[\frac{3}{2}H_0(r)t + 1\right]}, \tag{3.3.31}$$

such that

$$\rho_{M0}(r) = \frac{1}{4\pi G}\frac{H_0(r)\left[\frac{3}{2}A_0'(r)H_0(r) + A_0(r)H_0'(r)\right]}{A_0'(r)}. \tag{3.3.32}$$

This is consistent with what we already know.

Pure radiation

On the other hand, in the case of pure radiation $\rho_M \equiv 0$, Eqs. (3.3.19) and (3.3.23) imply

$$E = 3pV \Rightarrow \dot{E} = 3(\dot{p}V + p\dot{V}) = -p\dot{V} \Leftrightarrow \frac{\dot{V}}{V} = -\frac{3}{4}\frac{\dot{p}}{p} \Leftrightarrow V(r;t) = A_0(r)^3 p(t)^{-3/4}, \tag{3.3.33}$$

and by recalling (3.3.15) one obtains

$$A_0(r)^3 p(t)^{-3/4} = V := y^2 = A_0(r)^3\left(\frac{3}{2}H_0(r)y_1(t) + y_2(t)\right)^2. \tag{3.3.34}$$

Therefore, in this case it holds that

$$H_0(r) \equiv H_0; \tag{3.3.35}$$

$$\frac{3}{2}H_0 y_1(t) + y_2(t) = p(t)^{-3/8}. \tag{3.3.36}$$

From

$$H_0^2 = \frac{8}{3}\pi G\frac{E_0(r)}{A_0(r)^3} = 8\pi G p_0 \Leftrightarrow p_0 = \frac{H_0^2}{8\pi G}, \tag{3.3.37}$$

one gets

$$p(t) = \frac{1}{32\pi G}\left(t + \frac{1}{2H_0}\right)^{-2}. \tag{3.3.38}$$

This result allows to explicitly solve the second Einstein equation of (3.3.10) yielding that

$$y_1(t) = \frac{(2H_0t+1)^{3/4} - (2H_0t+1)^{1/4}}{H_0}, \quad y_2(t) = \frac{3(2H_0t+1)^{1/4} - (2H_0t+1)^{3/4}}{2}, \tag{3.3.39}$$

finally leading to the following expression

$$\begin{aligned} y &= A_0(r)^{3/2}\left[\frac{3}{2}H_0\frac{1}{H_0}\left((2H_0t+1)^{3/4} - (2H_0t+1)^{1/4}\right) + \frac{1}{2}\left(3(2H_0t+1)^{1/4} - (2H_0t+1)^{3/4}\right)\right] \\ &= A_0(r)^{3/2}(2H_0t+1)^{3/4}, \end{aligned} \tag{3.3.40}$$

implying

$$A(r;t) = y(r;t)^{2/3} = A_0(r)\sqrt{2H_0t+1}, \tag{3.3.41}$$

in which we recognize a feature of the FLRW model with pure radiation.

Beyond pure models

The two functional forms (3.3.28) and (3.3.40), respectively concerning the cases of pure matter and pure radiation can be recognized to belong to a more general family of solutions[8], of the form

$$y \propto (t+\theta)^{\frac{1\pm a}{2}}, \quad s.t. \quad \begin{cases} a=1 & \text{for pure matter} \\ a=\frac{1}{2} & \text{for pure radiation} \end{cases}, \tag{3.3.42}$$

where θ is a suitable parameter with the dimension of a time.
Again, the second Einstein equation of (3.3.10) implies

$$p(t) = \frac{1-a^2}{24\pi G}(t+\theta)^{-2}. \tag{3.3.43}$$

By recalling the definition (3.3.26) of $\Omega_R(r;t)$, one then gets

$$\frac{1-a^2}{24\pi G\theta^2} = p(0) = \frac{\Omega_{R0}(r)H_0(r)^2}{8\pi G} \Leftrightarrow \theta^{-1}(r;a) = H_0(r)\sqrt{\frac{3}{1-a^2}\Omega_{R0}(r)}. \tag{3.3.44}$$

Notice that such a result implies that in general θ does depend on r (as well as on the parameter a).

We should now remember that we are considering the expansion of the Universe during M-AM recombination only in presence of matter and radiation (namely, we are disregarding the contribution of dark energy, for simplicity's sake). Thus, $\rho_M \geq 0$ and $p \geq 0$ always, which imply $|a| \leq 1$. Furthermore, it is reasonable to assume $\theta > 0$, so that the Big Bang happened in some past instant $t_{BB} = -\theta$. Within these assumptions, the expression of $y(r;t)$ for the family of solutions under consideration reads

$$\begin{aligned} y_1(t;a) &= \frac{\theta}{\alpha}\left[\left(\frac{t}{\theta}+1\right)^{\frac{1+a}{2}} - \left(\frac{t}{\theta}+1\right)^{\frac{1-a}{2}}\right]; \\ y_2(t;a) &= \frac{1}{2\alpha}\left[(a-1)\left(\frac{t}{\theta}+1\right)^{\frac{1+\alpha}{2}} + (a+1)\left(\frac{t}{\theta}+1\right)^{\frac{1-\alpha}{2}}\right], \end{aligned} \tag{3.3.45}$$

which finally allows one to explicitly write down the functional form of the a-parametrized family of solutions under consideration

$$y(r;t;a) = \frac{A_0^3(r)}{2a}\left[\left(a+\sqrt{3\frac{1-a^2}{\Omega_{R0}(r)}}-1\right)\left(\frac{t}{\theta}+1\right)^{\frac{1+a}{2}} + \left(a-\sqrt{3\frac{1-a^2}{\Omega_{R0}(r)}}+1\right)\left(\frac{t}{\theta}+1\right)^{\frac{1-a}{2}}\right], \tag{3.3.46}$$

where $\theta = \theta(r;a)$ given by (3.3.44). In turn, this implies the formula

$$\begin{aligned} A(r;t) = y^{2/3}(r;t) = \frac{A_0^2(r)}{(2a)^{2/3}} \times \\ \times \left[\left(a+\sqrt{3\frac{1-a^2}{\Omega_{R0}(r)}}-1\right)\left(\frac{t}{\theta}+1\right)^{\frac{1+a}{2}} + \left(a-\sqrt{3\frac{1-a^2}{\Omega_{R0}(r)}}+1\right)\left(\frac{t}{\theta}+1\right)^{\frac{1-a}{2}}\right]^{2/3}. \end{aligned} \tag{3.3.47}$$

[8]We should bear in mind that, physically, the "right" $p(t)$ depends on the M-AM recombination law.

3.4 The Lemaître model

Now we consider again the epoch 3, for which we saw the LTB solution is not general enough. Thus, in this section we will use its generalization, called the Lemaître model. It was described *e.g.* in [122] and [123]. We choose the coordinates which diagonalize the metric tensor, and we redefine t with respect to eq. (7) of [123] in order to get $g_{tt} := -1$, which will mean that the energy-matter has some radial velocity U_μ. Hence, in our gauge the metric results to be

$$ds^2 = -dt^2 + \left(\frac{A'}{f}\right)^2 dr^2 + A^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.4.1}$$

where the spatial curvature is $k(r;t) = \sqrt{1 - f(r;t)^2}$, as (3.2.47).

3.4.1 Einstein equations

We will now adopt the tetrad formalism, in which $ds^2 = \eta_{ab} e^a \otimes e^b$, and which allows us to compute the *Vielbein* as

$$e^0 = dt, \quad e^1 = \frac{A'}{f} dr, \quad e^2 = A d\theta, \quad e^3 = A \sin\theta d\phi. \tag{3.4.2}$$

We compute the Einstein tensor analogously to [119],

$$\begin{aligned} G^0_0 &= -2\frac{\dot{A}'\dot{A}}{A'A} - \frac{\dot{A}^2}{A^2} - \frac{k^2}{A^2} + 2\frac{f'f}{A'A} + 2\left(\frac{\dot{A}}{A} - \frac{\dot{f}}{f}\right)\frac{\dot{f}}{f}; \\ G^1_0 &= -2\frac{\dot{f}}{A}; \\ G^1_1 &= -2\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} - \frac{k^2}{A^2} + 2\frac{\dot{f}^2}{f^2}; \\ G^2_2 = G^3_3 &= -\frac{\ddot{A}}{A} - \frac{\ddot{A}'}{A'} - \frac{\dot{A}'\dot{A}}{A'A} + \frac{f'f}{A'A} + \frac{\ddot{f}}{f} + \left(\frac{2\dot{A}'}{A'} + \frac{\dot{A}}{A}\right)\frac{\dot{f}}{f}. \end{aligned} \tag{3.4.3}$$

On the other hand, in presence of a non-vanishing velocity field, the energy-momentum tensor reads

$$T^a_b = (\rho + p)U^a U_b + p\delta^a_b \quad \text{s.t.} \quad U_a = \sqrt{v^2+1}e^0 + ve^1 = \sqrt{v^2+1}dt + v\frac{A'}{f}dr, \tag{3.4.4}$$

namely

$$\begin{aligned} T^0_0 &= -(\rho+p)(v^2+1) + p = -\rho - v^2(\rho+p); \\ T^1_0 &= v\sqrt{v^2+1}(\rho+p); \\ T^1_1 &= p + v^2(\rho+p); \\ T^2_2 = T^3_3 &= p. \end{aligned} \tag{3.4.5}$$

Thus, we can finally write the Einstein equations for the Lemaître model

$$\begin{cases} 2\frac{\dot{A}'\dot{A}}{A'A} + \frac{\dot{A}^2}{A^2} + \frac{k^2}{A^2} - 2\frac{f'f}{A'A} - 2\left(\frac{\dot{A}}{A} - \frac{\dot{f}}{f}\right)\frac{\dot{f}}{f} = 8\pi G\rho + 8\pi G v^2(\rho+p) \\ \frac{\dot{f}}{A} = -4\pi G v\sqrt{v^2+1}(\rho+p) \\ 2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{k^2}{A^2} - 2\frac{\dot{f}^2}{f^2} = -8\pi G p - 8\pi G v^2(\rho+p) \\ \frac{\ddot{A}}{A} + \frac{\ddot{A}'}{A'} + \frac{\dot{A}'\dot{A}}{A'A} - \frac{f'f}{A'A} - \frac{\ddot{f}}{f} - \left(\frac{2\dot{A}'}{A'} + \frac{\dot{A}}{A}\right)\frac{\dot{f}}{f} = -8\pi G p \end{cases}. \tag{3.4.6}$$

Notice that we don't express them in terms of the M variable, defined in eq. (10) of [123]. We stress that M is not the "empirical amount of mass" we defined in §3.3.3 and we will use again in §3.4.2.

The velocity v represents the matter which falls on itself. We can assume that it will be small almost always, w.r.t. $c = 1$. Thus, we can approximate (3.4.6) up to the first order in v.

$$\begin{cases} 2\frac{\dot{A}'\dot{A}}{A'A} + \frac{\dot{A}^2}{A^2} + \frac{k^2}{A^2} - 2\frac{f'f}{A'A} - 2\frac{\dot{A}}{A}\frac{\dot{f}}{f} = 8\pi G\rho + o(v) \\ \dot{f} = -4\pi G v A(\rho + p) + o(v) \\ 2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{k^2}{A^2} = -8\pi G p + o(v) \\ \frac{\ddot{A}}{A} + \frac{\ddot{A}'}{A'} + \frac{\dot{A}'\dot{A}}{A'A} - \frac{f'f}{A'A} - \frac{\ddot{f}}{f} - \left(\frac{2\dot{A}'}{A'} + \frac{\dot{A}}{A}\right)\frac{\dot{f}}{f} = -8\pi G p \end{cases} \tag{3.4.7}$$

where we used than $\dot{f} = O(v)$ from the second equation. Moreover, the first equation can be rewritten in the more compact form

$$\partial_r[A(\dot{A}^2 + k^2)] = 8\pi G\frac{A'A^2}{f}[(f - \dot{A}v)\rho - \dot{A}vp]. \tag{3.4.8}$$

3.4.2 Conservation laws

In order to write the energy-momentum conservation, by recalling the energy-momentum tensor (3.4.4)-(3.4.5), one can approximate

$$U_\mu = \sqrt{v^2 + 1}dt + v\frac{A'}{f}dr \Rightarrow U^\mu = -\sqrt{v^2 + 1}\partial_t + v\frac{f}{A'}\partial_r, \tag{3.4.9}$$

which implies

$$\begin{aligned} T^t_t &= -\rho + o(v); \\ T^t_r &= -v\frac{A'}{f}(\rho + p) + o(v); \\ T^r_t &= v\frac{f}{A'}(\rho + p) + o(v); \\ T^r_r &= p + o(v); \\ T^\theta_\theta = T^\phi_\phi &= p. \end{aligned} \tag{3.4.10}$$

Hence, the conservation of energy reads

$$\dot{\rho} = \left[-\frac{\partial_t(A'A^2/f)}{(A'A^2/f)} + v\frac{f}{A'}\frac{\partial_r(A'A^2/f)}{(A'A^2/f)} + \partial_r\left(v\frac{f}{A'}\right)\right](\rho+p) + v\frac{f}{A'}(\rho'+p') + o(v), \tag{3.4.11}$$

whereas the conservation of momentum is

$$p' = \left[2\frac{v}{A^2}\partial_t\left(\frac{A'A^2}{f}\right) + \frac{A'A^2}{f}\partial_t\left(\frac{v}{A^2}\right)\right](\rho + p) + v\frac{A'}{f}(\dot{\rho} + \dot{p}) + o(v). \tag{3.4.12}$$

We can rewrite the conservation laws by calling $\frac{A'A^2}{f} := V'$, where $V(r;t)$ is the volume inside the sphere of radius r. The conservation of energy simplifies

$$\partial_t(V'\rho) = \partial_r\left[A^2 v(\rho + p)\right] - \dot{V}'p. \tag{3.4.13}$$

The l.h.s. of (3.4.13) is related to the total energy inside the sphere, defined by (3.3.19) as $E(r;t) := \int_0^r \rho(s;t)V'(s;t)ds$, such that (3.3.21) holds, namely $E' = V'\rho$. Within the assumption of separation of (3.4.13) into its w-components (which holds with a good approximation after the M-AM recombination, when the transformations occur only in the stars), one can write

$$\dot{E}' = \partial_r \left[A^2 v(\rho + p)\right] - \dot{V}'p \Rightarrow \dot{E}'_w = (1+w)\partial_r \left[v\frac{f}{A'}E'_w\right] - w\frac{\dot{V}'}{V'}E'_w,\ \forall w. \qquad (3.4.14)$$

Indeed, for the static case $v = 0$ we have just the volume deformation $E'_w \propto (V')^{-w}$. For the general case, the matter component has a particularly simple law,

$$\dot{M}' = \partial_r \left[v\frac{f}{A'}M'\right] \Rightarrow \dot{M} = v\frac{f}{A'}M'. \qquad (3.4.15)$$

Moreover, the conservation of momentum (3.4.12) becomes

$$V'p' = \partial_t \left[v\frac{A'}{f}V'(\rho + p)\right]. \qquad (3.4.16)$$

3.4.3 Approximated models

The PDE system to solve is given by the Einstein Equations (3.4.7), to which they are added the conservation laws.

$$\begin{cases} \partial_r[A(\dot{A}^2 + 1 - f^2)] = 8\pi G\frac{A'A^2}{f}[(f - \dot{A}v)\rho - \dot{A}vp] \\ \dot{f} = -4\pi GvA(\rho + p) + o(v) \\ 2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{k^2}{A^2} = -8\pi Gp + o(v) \\ \frac{\ddot{A}}{A} + \frac{\ddot{A}'}{A'} + \frac{\dot{A}'\dot{A}}{A'A} - \frac{f'f}{A'A} - \frac{\ddot{f}}{f} - \left(\frac{2\dot{A}'}{A'} + \frac{\dot{A}}{A}\right)\frac{\dot{f}}{f} = -8\pi Gp \\ \\ \dot{M} = v\frac{f}{A'}M' \\ 3\partial_t(V'\rho_R) = 4\partial_r(vA^2\rho_R) - \dot{V}'\rho_R \\ V'\rho'_R = \partial_t\left[v\frac{A'}{f}V'(3\rho_M + 4\rho_R)\right] \end{cases}. \qquad (3.4.17)$$

The independent variables are A, f, v, ρ_M, ρ_R, and the quantities $V := \int_0^r \frac{A'A^2}{f}$ and $M := \int_0^r V'\rho_M$ have been defined.

The resultant system is quite difficult to solve. The solution we need has some conditions[9] put on the initial time t_R, and some others[10] on the final instant. Hence, we can not use even a numerical approach, at least at the first step, because it would require a complete set of conditions at a certain instant, the initial or the final one. If we put the initial condition, it is not ensured that we will find an acceptable final state (it is very improbable, indeed), *and vice versa*.

What we need is a model, at least an approximated one, which satisfies some conditions both at the start and at the end. If it does not solve exactly the PDEs, we can nevertheless take it as a zeroth order, perturbing to the right version, even numerically.

r as a label

The crucial observation is that, for an only matter universe, the evolution law results to be $A(r;t) = r\left[1 + \frac{3}{2}H_0(r)t\right]^{2/3}$. A FLRW universe with pure matter has analogously

[9] $A = ra(t)$, $1 - f^2 = Kr^2$, $\rho_R = \rho_R(t)$

[10] $A = r$, $f = 1$, $M = \Phi r^D$

$a(t) = \left[1+\frac{3}{2}H_0t\right]^{2/3}$. Hence, the inhomogeneous universe has, at radius r, a metric $A(r;t) = ra_r(t)$ s.t. $\frac{\dot{a}_r^2}{a_r^2} = H_0(r)^2 a_r^{-3}$. If it is considered just the spherical region until r, and the total matter inside is considered as it was homogeneous, the consequent evolution law in t depends on the label r exactly as the LTB solution.

This observation works exactly just for pure matter. We see that setting $v := 0, f := 1$. The Conservations of Matter and Radiation give $\rho_M \propto (A'A^2)^{-1}$ and $\rho_R \propto (A'A^2)^{-4/3}$. Substituting inside the I EE

$$\partial_r(\dot{A}^2 A) = 8\pi G \left(\rho_{M0} + \frac{\rho_{R0}}{\sqrt[3]{A'A^2}}\right) A_0' A_0^2. \tag{3.4.18}$$

We see that it is exactly integrable for pure matter, but the radiation returns a non trivial term. Hence, from now on remember that the "radius as label" method is an approximation. It is good near M-AM, when the universe is almost FLRW; and near today, when the matter dominates; but it is worse during the intermediate period.

With this *caveat*, we try to write

$$A(r;t) := ra_r(t) \quad s.t. \quad \frac{\dot{a}_r^2}{H_0(r)^2} = \Omega_{M0}(r)a_r^{-1} + \Omega_{R0}(r)a_r^{-2}. \tag{3.4.19}$$

Since we consider here just the final components, let $\Omega_{K0} := 0$. The Omegas are defined as usual, s.t. $\Omega_{M0} + \Omega_{R0} = 1$.

We can solve it with exact integrations

$$\frac{da}{dt} = \frac{H_0}{a}\sqrt{\Omega_{M0}a + \Omega_{R0}} \Rightarrow$$

$$\int_{t_{BB}}^t H_0 dt = \int_{t_{BB}}^t \frac{ada}{\sqrt{\Omega_{M0}a+\Omega_{R0}}} = [2\Omega_{M0}a\sqrt{\Omega_{M0}a+\Omega_{R0}}]_{t_{BB}}^t - 2[\frac{2}{3}(\Omega_{M0}a+\Omega_{R0})^{3/2}]_{t_{BB}}^t$$

$$\Rightarrow \frac{3}{2}H_0(t-t_{BB}) = (\Omega_{M0}a - 2\Omega_{R0})\sqrt{\Omega_{M0}a+\Omega_{R0}} + 2\Omega_{R0}^{3/2}, \tag{3.4.20}$$

where we used the fact $a(t_{BB}) := 0$. Moreover, setting $a(0) := 1$

$$-\frac{3}{2}H_0 t_{BB} = (\Omega_{M0} - 2\Omega_{R0})\sqrt{\Omega_{M0}+\Omega_{R0}} + 2\Omega_{R0}^{3/2} = \Omega_{M0} + 2\Omega_{R0}(\Omega_{R0}^{1/2} - 1)$$

$$\Rightarrow \frac{3}{2}H_0 t = (\Omega_{M0}a - 2\Omega_{R0})\sqrt{\Omega_{M0}a+\Omega_{R0}} + (2\Omega_{R0} - \Omega_{M0}). \tag{3.4.21}$$

This is the exact evolution law $a = a(t)$, expressed implicitly. The explicit dependence can be obtained with the Cardano's Formula.

Now we set the parameters of real universe. First of all, the time singularity $t_{BB}(r)$ must be spatially homogeneous; since $t = 0$ is today, we can call T the age of the universe, so that $-t_{BB} \equiv T$. Then, we put the fractal $\rho_{M0} := \frac{D\Phi}{4\pi}r^{D-3}$. The evolution law token at $t = -T$ gives the last constraint

$$\frac{3}{2}H_0 T = \Omega_{M0} + 2\Omega_{R0}(\Omega_{R0}^{1/2} - 1) \Rightarrow 2(1-\Omega_{M0})^{3/2} + 3\Omega_{M0} - 2 = T\sqrt{\frac{3}{2}DG\Phi}r^{\frac{D-3}{2}}\Omega_{M0}^{1/2}. \tag{3.4.22}$$

$\Omega_{M0}(r)$ can be expressed as the solution of a high order algebraic equation, and is fixes also H_0, Ω_{R0} and ρ_{R0}.

It is difficult to solve exactly the algebraic equation of $\Omega_{M0}(r)$. Here we show an approximated solution, using the fact that $\Omega_{R0} \ll \Omega_{M0}$. Indeed, the evolution equation at $-T$ becomes

$$\frac{3}{2}H_0 T = \Omega_{M0} + 2\Omega_{R0}(\Omega_{R0}^{1/2} - 1) \cong \Omega_{M0} - 2\Omega_{R0} \cong \Omega_{M0}. \tag{3.4.23}$$

Substituting ρ_{M0},

$$\begin{aligned}
&\frac{3}{2}H_0T \cong \frac{2}{3}DG\Phi r^{D-3}H_0^{-2} \Rightarrow H_0(r) \cong \sqrt[3]{\frac{4DG\Phi}{9T}}r^{\frac{D}{3}-1} \\
&\Rightarrow \Omega_{M0}(r) \cong \frac{3}{2}T\sqrt[3]{\frac{4DG\Phi}{9T}}r^{\frac{D}{3}-1} = \sqrt[3]{\frac{3}{2}DG\Phi T^2}r^{\frac{D}{3}-1}, \\
&\rho_{R0}(r) = \frac{3H_0^2}{8\pi G} - \rho_{M0} \cong \frac{1}{4\pi}\left[\sqrt[3]{\frac{2D^2\Phi^2}{3GT^2}}r^{1-\frac{D}{3}} - D\Phi\right]r^{D-3}.
\end{aligned} \tag{3.4.24}$$

The numerical parameters D, Φ, T can be deduced from astronomical measures.

Even if the formulas are somehow simple, this model has a important lack: the expansion is not homogeneous near $-T$, because also the radiation is not homogeneous. It goes as

$$\dot{a}^2 \sim^{-T} H_0^2\Omega_{R0}a^{-2} \Rightarrow a_r(t) \sim \rho_{R0}(r)^{1/4}\sqrt{4\left(\frac{2}{3}\pi G\right)^{1/2}(t+T)}, \tag{3.4.25}$$

which expands faster for bigger r.

Step functions

The last model can be improved admitting an evolution of the Ωs. Indeed, we know that the matter and radiation densities do not change just because the expansion, but they move through r, as is described by the PDEs. Here is how the radiation can be homogeneous near $-T$ and inhomogeneous at $t = 0$: Ω_{R0} changes with time.

This fact can be roughly described inserting initial and final values $H_I, \Omega_{MI}, \Omega_{RI}; H_F, \Omega_{MF}, \Omega_{RF}$. In other words, $H_0 = H_0(t)$ is a step function that jumps from H_I to H_F, and the same for the others. The jumps happen in some middle instant t_m, when we consider all the changes are concentrated.

The evolution law can be written with differentials as

$$\begin{aligned}
Hdt &= \frac{ada}{\sqrt{\Omega_M a + \Omega_R}} = d\left[2\Omega_M a\sqrt{\Omega_M a\Omega_R}\right] - d\left[\frac{4}{3}(\Omega_M a + \Omega_R)^{3/2}\right] = \\
&= \frac{2}{3}d[(\Omega_M a - 2\Omega_R)\sqrt{\Omega_M a + \Omega_R}].
\end{aligned} \tag{3.4.26}$$

Setting $a(-T) := 0$ and $a(0) := 1$, it is respectively

$$\begin{cases}
\frac{3}{2}H_I(t+T) = (\Omega_{MI}a - 2\Omega_{RI})\sqrt{\Omega_{MI}a + \Omega_{RI}} + 2\Omega_{RI}^{3/2} & -T \le t \le t_m \\
\frac{3}{2}H_F t = (\Omega_{MF}a - 2\Omega_{RF})\sqrt{\Omega_{MF}a + \Omega_{RF}} - (\Omega_{MF} - 2\Omega_{RF}) & t_m \le t \le 0
\end{cases}. \tag{3.4.27}$$

For the Einstein Equations are of second order, it must be $a(t) \in C^1(t_m)$. Calling $a_m := a(t_m)$, we can write such request as

$$\begin{cases}
\frac{1}{H_I}\left[(\Omega_{MI}a_m - 2\Omega_{RI})\sqrt{\Omega_{MI}a_m + \Omega_{RI}} + 2\Omega_{RI}^{3/2}\right] = \frac{3}{2}(t_m + T) = \\
\quad = \frac{1}{H_F}\left[(\Omega_{MF}a_m - 2\Omega_{RF})\sqrt{\Omega_{MF}a_m + \Omega_{RF}} - (\Omega_{MF} - 2\Omega_{RF})\right] + \frac{3}{2}T \\
H_I^2(\Omega_{MI}a_m + \Omega_{RI}) = \dot{a}_m^2 a_m^2 = H_F^2(\Omega_{MF}a_m + \Omega_{RF})
\end{cases}. \tag{3.4.28}$$

Moreover, we can set the initial and final states as

$$\begin{cases}
\Omega_{MI}(r) + \Omega_{RI}(r) = 1 = \Omega_{MF}(r) + \Omega_{RF}(r) \\
H_I(r)^2\Omega_{RI}(r) = \frac{8}{3}\pi G\rho_{RI}; \quad H_F(r)^2\Omega_{MF}(r) = \frac{2}{3}DG\Phi r^{D-3}
\end{cases}. \tag{3.4.29}$$

These are 6 conditions; but we have 7 functions: $\Omega_{MI}(r)$, $\Omega_{RI}(r)$, $\Omega_{MF}(r)$, $\Omega_{RF}(r)$, $H_I(r)$, $H_F(r)$ and $a_m(r)$ (which is equivalent to t_m).

For the last condition, we remember that the change of matter and radiation densities depends both on v, according to the Conservation Laws

$$\begin{cases} \partial_t(V'\rho_M) = \partial_r(A^2 v\rho_M) \\ \partial_t\left((V')^{4/3}\rho_R\right) = \frac{4}{3}\sqrt[3]{V'}\partial_r(A^2 v\rho_R) \end{cases} . \tag{3.4.30}$$

Since here the ρs have a jump at t_m, all these derivatives have a Dirac delta peak. For this reason, we can neglect the variation $\partial_t(V')$, $\partial_r(\rho_M)$ and $\partial_r(\rho_R)$, taking them approximately constant w.r.t. the jumps. The Conservation Laws become

$$\begin{cases} V'\Delta(\rho_M) \cong \rho_M\Delta(A^2 v) \\ (V')^{4/3}\Delta\left(\rho_R\right) \cong \frac{4}{3}\sqrt[3]{V'}\rho_R\Delta(A^2 v) \end{cases} \Rightarrow \frac{4}{3}\Delta(\ln\rho_M) \cong \frac{\Delta(A^2 v)}{V'} \cong \Delta(\ln\rho_R). \tag{3.4.31}$$

The Δs on the ρs are intended as $\Delta(f) := \lim_{t\to t_m^+} f(t) - \lim_{t\to t_m^-} f(t)$. Thus, the last can be rewritten as

$$\frac{4}{3}(\ln\rho_{MF} - \ln\rho_{MI}) \cong \ln\rho_{RF} - \ln\rho_{RI} \Leftrightarrow \left(\frac{\rho_{MF}}{\rho_{MI}}\right)^{4/3} \cong \frac{\rho_{RF}}{\rho_{RI}}. \tag{3.4.32}$$

This fixes completely the functions of the model. The only parameters remained are just numbers: T, ρ_{RI}, D, Φ.

For a set of solvable algebraic equations, we exploit the approximations $\rho_{RI} \gg \rho_{MI}, \rho_{MF} \gg \rho_{RF}, a_m \gg 0$. These return

$$\begin{aligned} &\Omega_{RI} \cong 1, \quad \Omega_{MI} \cong 0, \quad \Omega_{MF} \cong 1, \quad \Omega_{RF} \cong 0 \Rightarrow \\ &H_I(r) \cong \sqrt{\frac{8}{3}\pi G\rho_{RI}}, \quad H_F(r) \cong \sqrt{\frac{8}{3}\pi G\rho_{MF}} = \sqrt{\frac{2}{3}DG\Phi r^{\frac{D-3}{2}}}, \\ &0 \cong \frac{1}{H_I}\left[(\Omega_{MI}a_m - 2\Omega_{RI})\sqrt{\Omega_{MI}a_m + \Omega_{RI}} + 2\Omega_{RI}^{3/2}\right] = \\ &= \frac{1}{H_F}\left[(\Omega_{MF}a_m - 2\Omega_{RF})\sqrt{\Omega_{MF}a_m + \Omega_{RF}} - (\Omega_{MF} - 2\Omega_{RF})\right] + \frac{3}{2}T \cong \frac{a_m^{3/2} - 1}{H_F} - \frac{3}{2}T \\ &\Rightarrow a_m(r) \cong \left[1 - \frac{3}{2}TH_F\right]^{2/3} \cong \left[1 - T\sqrt{\frac{3}{2}DG\Phi r^{\frac{D-3}{2}}}\right]^{2/3} . \end{aligned} \tag{3.4.33}$$

Since now it is known, we find from the other constrains

$$\begin{aligned} &\frac{8}{3}\pi G\rho_{RI} \cong H_I^2(\Omega_{MI}a_m + \Omega_{RI}) = H_F^2(\Omega_{MF}a_m + \Omega_{RF}) = \frac{8}{3}\pi G(\rho_{MF}a_m + \rho_{RF}) \\ &\Rightarrow \rho_{RF}(r) \cong \rho_{RI} - \rho_{MF}a_m \cong \rho_{RI} - \frac{D\Phi}{4\pi}r^{D-3}\left[1 - T\sqrt{\frac{3}{2}DG\Phi r^{\frac{D-3}{2}}}\right]^{2/3}, \\ &\left(\frac{\rho_{MF}}{\rho_{MI}}\right)^{4/3} \cong \frac{\rho_{RF}}{\rho_{RI}} \Rightarrow \rho_{MI}(r) \cong \rho_{MF}\left(\frac{\rho_{RI}}{\rho_{RF}}\right)^{3/4} \cong \rho_{MF}\left(\frac{\rho_{RI}}{\rho_{RI} - \rho_{MF}a_m}\right)^{3/4} . \end{aligned} \tag{3.4.34}$$

The four parameters of our simplified model can be empirically fixed, in order to compare quantitatively the previsions with observations. Following [100], we evaluate $D \cong 1.2$, between the scales of magnitude $L_G \cong 10^5 ly$ and $L_{EG} \cong 3 \times 10^8 ly$. The fractal density can be obtained from the amount of observed matter

$$\begin{aligned}\Phi L_{EG}^{D} &\cong M(L_{EG}) \cong \frac{4}{3}\pi L_{EG}^{3}\rho_{B0} = \frac{H_0^2}{2G}L_{EG}^{3}\Omega_{B0}\\ &\Rightarrow \Phi \cong \frac{H_0^2}{2G}L_{EG}^{3-D}\Omega_{B0} \cong 9.974\times 10^{24} kg/ly^{D},\end{aligned} \tag{3.4.35}$$

where $H_0 \cong 6.867\times 10^{-11}y^{-1}$ and $\Omega_{B0}\cong 0.044$ are the parameters of the Cosmological Concordance Model. Analogously, we can evaluate the amount of radiation

$$\begin{aligned}\frac{H_0^2}{2G}L_{EG}^{3}\Omega_{R0} \cong& \int_0^{L_{EG}} \rho_{RF}4\pi r^2 dr \cong \frac{4}{3}\pi L_G^3\rho_{RI}+\\ &+\int_{L_G}^{L_{EG}}\left[4\pi r^2\rho_{RI} - D\Phi r^{D-1}\left(1-T\sqrt{\frac{3}{2}DG\Phi}r^{\frac{D-3}{2}}\right)^{2/3}\right]dr =\\ =&\frac{4}{3}\pi L_{EG}^3\rho_{RI} - \left[\Phi r^{D}{}_2F_1\left(-\frac{2}{3},\frac{2D}{D-3};3\frac{D-1}{D-3};T\sqrt{\frac{3}{2}DG\Phi}r^{\frac{D-3}{2}}\right)\right]_{L_G}^{L_{EG}} \Rightarrow\\ \rho_{RI} \cong& \rho_{R0} + 2.5\rho_{MF}(L_{EG}){}_2F_1\left(-\frac{2}{3},-\frac{4}{3};-\frac{1}{3};T\sqrt{1.8G\Phi}L_{EG}^{-0.9}\right)+\\ &-2.5\left(\frac{L_G}{L_{EG}}\right)^3\rho_{MF}(L_G){}_2F_1\left(-\frac{2}{3},-\frac{4}{3};-\frac{1}{3};T\sqrt{1.8G\Phi}L_{G}^{-0.9}\right),\end{aligned} \tag{3.4.36}$$

where $\frac{8\pi G}{3H_0^2}\rho_{R0} := \Omega_{R0} \cong 8.24\times 10^{-5}$ are CCM parameters again, and ${}_2F_1$ is the hypergeometric function. The last parameter T can be evaluated from other cosmological observations.

3.5 Defects of the model

Of course, our calculations admit further improvements, for instance provided by a more precise solution to the evolution equations of the Lemaître model, as discussed in §3.4. After our analysis, we may reasonably wonder that a more detailed analysis would describe the fall of the matter fractal onto itself, thus providing self-consistency and stability within fractal cosmology, while the homogeneous FLRW would just be an unstable solution. Future works might also improve the description of the second epoch, e.g. implementing the transformation law of matter into radiation.

It is worth pointing out here that the whole theoretical framework dealing with LTB and Lemaître models provides a smooth approximation to the actual fractal dynamics. Indeed, a more realistic model for fractal cosmology should make use of distributional General Relativity, which is a quite formidable task, or at least of a first order perturbative approximation towards the anisotropic distribution. These latter perturbative methods, applied to an LTB or Lemaître background, should expectedly provide some amount of effective dark matter, due to retarded potentials of §2. Since the fractal approach is able to explain dark energy phenomena [103], it is conceivable that a combined framework will be able to overcome many of the drawbacks of the Cosmological Concordance Model.

Conclusions: What has been done and what remains to be done

Along this PhD thesis we developed theoretical models and solutions to improve the Cosmological Standard Model. We proposed some partial answers to its open problems, as those of the dark matter and the dark energy, and we showed also links to the problems of the inflation and of the baryogenesis. These open a lot of future perspectives, both for cosmology and theoretical physics.

Looking back

We developed in §2 the "retarded perturbations" framework, managing to apply it to a model of our universe, complete with all the components. The large number of variables leaves a free parameter, depending on which we found a one-dimensional set of possible solutions. Within this interval, more dark matter is explained less as less dark energy is, and *vice versa*. At an end of the range, dark matter is fully explained as a relativistic effect, but the same effect caused an underestimation of dark energy in the Cosmological Standard Model. At the other end of the range the numbers are analogous, with dark matter and dark energy exchanged. For a particular value, both dark matter and dark energy found a partial explanation. A new observation or a new test would determine which is the right solution, but anyway a correction of the parameters of the CCM is required.

Better measures of the density parameters will improve our estimations of Ω_{FM0} and $\Omega_{F\Lambda 0}$, but they can not fix the right parameter $\bar{\Omega}_{M0}$. The difference between $\bar{\Omega}_{R0}$ and the measured Ω_{R0}, e.g., is not matter of measure precision, but of the factor $\left(\frac{H_0}{\mathbf{H}_0}\tilde{a}_0\right)^2$, which concerns the background universe and is not measurable. Rather, a measure of the actual gravitational force $\vec{\nabla}\Psi(\underline{x};t_0)$ could put the restraint we need. Another possible measure could be the estimation of the matter inhomogeneity at large scale Ω_{IM0}, i.e. the deviation from the exact Cosmological Principle.

A fascinating concept we introduced is that of Selfconsistence Principle. They are pure mathematical statements that a universe must follow, in order to have a well defined evolution, taking account of the retarded potentials effects. We proved e.g. that any inhomogeneity generates a diverging perturbation, unless at least one component of the universe has $w > -1/3$. We can interpret that as an *a priori* requirement of "necessity" of radiation, or of some other component with high w. Such mathematical restrictions to the laws of the physics should be taken with caution, since they appear as philosophical statements. Indeed, any mathematical result is built on some hypotheses and axioms, which have a scientific and empirical nature. It is conceptually impossible that pure mathematics fixes the physics.

Other key issues about retarded potentials are the choice of the gauge and the averaging procedure. In §2, we performed firstly the calculations in the harmonic gauge, and after in the newtonian one. The final results $\Omega_{FM0}, \Omega_{F\Lambda 0}$ are not gauge-independent, what raises doubts about their physical meaning. Does the quantity of dark matter depend on the frame in which we measure it? If it is so, what we defined as "fictitious matter" is not an intrinsic quantity, and it could have no relevance. The interpretation we give in this thesis is that the gauge-dependence of $\Omega_{FM,\Lambda 0}$ just proves that they are not real components, but only apparent phenomena, arising from our choice of coordinates. In a sufficiently good frame, we would measure only the real matter and dark energy, but it is not the frame we are using today for the astronomical observations.

The exact path for exploiting calculation must be improved, not only for the gauge but also for the averaging formula, when $\langle g_{\mu\nu} \rangle$ is evaluated. E.g. in [115] are shown more possible formulas to average cosmological quantities. Our current results should be compared with other choice of averaging, wondering which is the most suitable. We need to reflect especially about the retard of information, i.e. if the averaging should be taken on a space-like sheet, or instead on a light cone.

We have started in §3 a systematic development of the framework focussed on the analysis of the consequences of fractal cosmology on the evolution of the Universe. We have proposed a genesis of the cosmic fractal, as well as a partition in epochs, both suitable to obtain quantitative results. Only the first epoch can consistently be described with the usual FLRW solution; on the other hand, the LTB solution was exploited for the second epoch, and we proved that an even more general Lemaître solution is necessary for the third epoch, because of general restrictions arising from the momentum conservation in the LTB metric.

The "necessity" of radiation stated by the Selfconsistence Principles is somehow confirmed in the fractal framework, since without it the LTB solution leads to inhomogeneous Big Bangs. However, it is the presence of such radiation that led us to abandon the LTB metric, introducing the Lemaître one for the fractal cosmological models.

Here we highlight also the role of the End of Greatness L_{EG} in our model. Letting it be an indeterminate parameter, our model is adaptable to the various empirical evaluations about the homogeneity scale; and it allows also to include the idea that the homogeneity is never reached, with the particular choice $L_{EG} \to \infty$. Moreover, a suitable L_{EG} can explain the different values of the fractal dimension D, found with different methods. Such difference can be a new way to determine L_{EG}.

Looking ahead

The study of local dark matter effects would provide corrections to the standard newtonian approximations for the dynamic of galaxies and clusters. For such calculations, we cannot assume an irrotational matter as we did here. The rotation of galaxies could provide a rotational term for the non diagonal components of the metric $\hat{B}$, which contributes to fictitious dark matter effects [91], [92]. A further publication will follow this line of research, improving the model in [91]. The total amount of the local fictitious effects could be compared to the global fictitious effects Ω_{FM0} we found here, and the equivalence between them could be the additional restraint we need to fix uniquely the parameter $\bar{\Omega}_{M0}$.

The recently observed gravitational waves have quite far sources, e.g. in [111] it was calculated a redshift $z \cong 0.09$. Hence, they originated a relevant amount of time ago, and traveled on the space while it was expanding, maybe in a non negligible way. In the next future, more and more ancient gravitational waves will be predictably detected, so that the expansion of the space during their journey will be less and less negligible. The study of the Equation (2.2.13) without source will be useful for the interpretation of such waves. We can imagine that they will exhibit a shape of a Bessel function, more than a goniometric function.

To obtain a theoretical prevision of the dark matter distribution mathematical tools able to describe the baryonic matter distribution would be necessary. Since the matter inhomogeneity seems to have a fractal shape [100], it could be useful a study of singular distributions as sources in General Relativity, and their application to cosmology. The concentrations of matter on supports with dimension less that 2 have long been believed to be banned, because pure mathematical reasons, showed in [127]. However, this no-go theorem can be bypassed, developing the suitable formalism. This is currently under investigation and will be matter for other future articles. The application of such a theory will allow far more precise fractal cosmological models, without the LTB-like homogenization.

Finally, we would like to remark that a deeper quantitative analysis of the LTB metric is of potential relevance also in other frameworks, such as the IR-completion of gravity [124], [125], or within the attempts to explain the tension of the Hubble parameter [126].

Appendices

Appendix A

Derivation of linearized Einstein Equations

A.1 Perturbed Ricci tensor

The linearized Einstein Equations on a Minkowskian background are very well known from the study of gravitational waves, *e.g.* in [116]. They are deduced from the definitions of the Ricci and Einstein tensors, neglecting the terms in $\tilde{g}, \partial\tilde{g}$ of orders higher than one, and this is what we are going to do. However, in our case the background metric is a FLRW one, which has a non zero connection

$$\bar{g}_{\mu\nu} = a^2\eta_{\mu\nu} \Rightarrow \bar{g}^{\mu\nu} = a^{-2}\eta^{\mu\nu} \Rightarrow \forall\mu : \bar{\Gamma}^{\tau}_{\mu\mu} = \bar{\Gamma}^{\mu}_{\tau\mu} = \bar{\Gamma}^{\mu}_{\mu\tau} = H; \quad others \quad \bar{\Gamma}^{\lambda}_{\mu\nu} = 0. \tag{A.1.1}$$

This will lead to additional terms respect to the usual PDE of gravitational waves.

We wrote the perturbation on metric as is usual in perturbative cosmology

$$\tilde{g}_{\mu\nu} = a^2 \begin{pmatrix} 2A & -B_i \\ -B_i & h_{ij} \end{pmatrix} := a^2 h_{\mu\nu} \Rightarrow \tilde{g}^{\mu\nu} = -\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\tilde{g}_{\alpha\beta} = a^{-2} \begin{pmatrix} -2A & -B^i \\ -B^i & -h^{ij} \end{pmatrix} = a^{-2}h^{\mu\nu} \tag{A.1.2}$$

where we used the greek indices for four-dimensional quantities, which are raised and lowered by $g_{\mu\nu}$, and the latin ones for the three-dimensional, which follow the euclidean metric δ_{ij}.

As we said, the perturbation on the connection is defined at the first order as

$$\begin{aligned} \tilde{\Gamma}^{\lambda}_{\mu\nu} &:= \Gamma^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\mu\nu} = \tilde{g}^{\lambda\rho}\bar{\Gamma}_{\rho\mu\nu} + \frac{1}{2}\bar{g}^{\lambda\rho}(\partial_\mu\tilde{g}_{\nu\rho} + \partial_\nu\tilde{g}_{\rho\mu} - \partial_\rho\tilde{g}_{\mu\nu}) = \\ &= \sum_\rho h^{\lambda\rho}\eta_{\rho\rho}\bar{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{2a^2}\eta^{\lambda\lambda}[\partial_\mu(a^2h_{\nu\lambda}) + \partial_\nu(a^2h_{\lambda\mu}) - \partial_\lambda(a^2h_{\mu\nu})]. \end{aligned} \tag{A.1.3}$$

We perform the calculations keeping the coordinates $\tau, \{i\}_1^3$, in order to find also the

Conservation of Four-Momentum afterward.

$$\tilde{\Gamma}^{\tau}_{\tau\tau} = h^{\tau\tau}\bar{\Gamma}^{\tau}_{\tau\tau} - \sum_l h^{\tau l}\bar{\Gamma}^{l}_{\tau\tau} + \frac{1}{2a^2}\partial_\tau(a^2 h_{\tau\tau}) = (-2A)H - 0 + a^{-2}\partial_\tau(a^2 A) = -2A\frac{\dot{a}}{a} + \frac{2a\dot{a}A + a^2\dot{A}}{a^2}$$
$$= \dot{A}; \tag{A.1.4}$$

$$\tilde{\Gamma}^{\tau}_{\tau j} = h^{\tau\tau}\bar{\Gamma}^{\tau}_{\tau j} - \sum_l \bar{\Gamma}^{l}_{\tau j} + \frac{1}{2a^2}\partial_j(a^2 h_{\tau\tau}) = 0 - h^{\tau j}H + \frac{1}{2}\partial_j h_{\tau\tau}$$
$$= \partial_j A + HB_j; \tag{A.1.5}$$

$$\tilde{\Gamma}^{\tau}_{ij} = h^{\tau\tau}\bar{\Gamma}^{\tau}_{ij} - \sum_l h^{\tau l}\bar{\Gamma}^{l}_{ij} + \frac{1}{2a^2}[\partial_i(a^2 h_{j\tau}) + \partial_j(a^2 h_{\tau i}) - \partial_\tau(a^2 h_{ij})] =$$
$$= (-2A)(\delta_{ij}H) - 0 + \frac{1}{2}\left[\partial_i(-B_j) + \partial_j(-B_i) - \frac{2a\dot{a}h_{ij} + a^2\dot{h}_{ij}}{a^2}\right]$$
$$= -2HA\delta_{ij} - \partial_{(i}B_{j)} - Hh_{ij} - \frac{1}{2}\dot{h}_{ij}, \tag{A.1.6}$$

where we used the symmetric notation $\partial_{(i}v_{j)} := \frac{1}{2}(\partial_i B_j + \partial_j B_i)$;

$$\tilde{\Gamma}^{k}_{\tau\tau} = h^{k\tau}\bar{\Gamma}^{\tau}_{\tau\tau} - \sum_l h^{kl}\bar{\Gamma}^{l}_{\tau\tau} - \frac{1}{2a^2}[2\partial_\tau(a^2 h_{k\tau}) - \partial_k(a^2 h_{\tau\tau})] =$$
$$= (-B_k)H - 0 - \frac{\partial_\tau(a^2(-B_k))}{a^2} + \frac{1}{2}\partial_k(2A) = -B_k H + 2HB_k + \dot{B}_k + \partial_k A$$
$$= \partial^k A + HB^k + \dot{B}^k; \tag{A.1.7}$$

$$\tilde{\Gamma}^{k}_{\tau j} = h^{k\tau}\bar{\Gamma}^{\tau}_{\tau j} - \sum_l h^{kl}\bar{\Gamma}^{l}_{\tau j} - \frac{1}{2a^2}[\partial_\tau(a^2 h_{jk}) + \partial_j(a^2 h_{k\tau}) - \partial_k(a^2 h_{\tau j})] =$$
$$= 0 - h^{kj}H - Hh_{jk} - \frac{1}{2}\dot{h}_{jk} - \frac{1}{2}\partial_j h_{k\tau} + \frac{1}{2}\partial_k h_{j\tau} = Hh_{jk} - Hh_{jk} - \frac{1}{2}\dot{h}_{jk} + \frac{1}{2}(\partial_j B_k - \partial_k B_j)$$
$$= \partial_{\{j}B^{k\}} - \frac{1}{2}\dot{h}^k_j, \tag{A.1.8}$$

where we used the antisymmetric notation $\partial_{\{i}v_{j\}} := \frac{1}{2}(\partial_i B_j - \partial_j B_i)$;

$$\tilde{\Gamma}^{k}_{ij} = h^{k\tau}\bar{\Gamma}^{\tau}_{ij} - \sum_l h^{kl}\bar{\Gamma}^{l}_{ij} - \frac{1}{2a^2}[\partial_i(a^2 h_{jk}) + \partial_j(a^2 h_{ki}) - \partial_k(a^2 h_{ij})] =$$
$$= (-B_k)(H\delta_{ij}) - 0 - \frac{1}{2}(\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij})$$
$$= -HB^k\delta_{ij} - \gamma^k_{ij}, \tag{A.1.9}$$

where we defined the purely spatial connection $\gamma^k_{ij} = \gamma_{kij} := \frac{1}{2}(\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij})$.

Analogously, the perturbed Ricci tensor is defined as

$$\tilde{R}_{\mu\nu} := R_{\mu\nu} - \bar{R}_{\mu\nu} = \partial_\sigma\tilde{\Gamma}^{\sigma}_{\mu\nu} - \partial_\nu\tilde{\Gamma}^{\sigma}_{\mu\sigma} + \tilde{\Gamma}^{\sigma}_{\sigma\rho}\bar{\Gamma}^{\rho}_{\mu\nu} + \bar{\Gamma}^{\sigma}_{\sigma\rho}\tilde{\Gamma}^{\rho}_{\mu\nu} - \tilde{\Gamma}^{\sigma}_{\mu\rho}\bar{\Gamma}^{\rho}_{\sigma\nu} - \bar{\Gamma}^{\sigma}_{\mu\rho}\tilde{\Gamma}^{\rho}_{\sigma\nu}. \tag{A.1.10}$$

To express it, we performed the scalar-vector-tensor decomposition for the metric perturbation. For a vector as $\vec{B}$, it means to exploit the exact succession of differential operators on the 3D vector fields, isolating the part on the kernel of divergence operator $\vec{\nabla}\cdot\hat{B} = 0$, and the part on the image of the gradient operator $\vec{\nabla}B$. Then, $B_i := \partial_i B + \hat{B}_i$ and

$\vec{\nabla} \cdot \vec{B} = \nabla^2 B$.
The same decomposition can be performed for each indices of the tensor h_{ij}, getting $h_{ij} = \partial_{ij} h + \partial_i \hat{h}_j + \partial_j \hat{h}_i + \hat{\hat{h}}_{ij}$; where the two vectors $\hat{h}$ are the same because symmetry and $\sum_j \partial_j \hat{h}_j = \sum_j \partial_j \hat{\hat{h}}_{ij} = 0$. We find again the variables of §2.2 defining $h := 2E, \hat{h}_j := \hat{E}_j, \hat{\hat{h}}_{ij} := 2\hat{E}_{ij} + 2\left(C - \frac{1}{3}\nabla^2 E\right)\delta_{ij}$, where it was isolated the traceless part $\sum_j \hat{E}_{jj} = 0$. In such a way, C describes the trace of h_{ij}, since $tr(h) = \sum_j h_{jj} = \sum_j \partial_j^2 h + 2\sum_j \partial_j \hat{h}_j + \sum_j \hat{\hat{h}}_{jj} = \nabla^2 E + 0 + 2\left(C - \frac{1}{3}\nabla^2 E\right) 3 = 6C$. Hence we have $\sum_j \partial_j h_{ij} = 2\partial_i C + \nabla^2 E_i$, if it is defined the vector $\vec{E} := \frac{4}{3}\vec{\nabla} E + \hat{E}$.
With these instruments we calculate

$$\begin{aligned}
\tilde{R}_{\tau\tau} =& \partial_\tau \tilde{\Gamma}^\tau_\tau + \sum_l \partial_l \tilde{\Gamma}^l_{\tau\tau} - \partial_\tau \left(\tilde{\Gamma}^\tau_{\tau\tau} + \sum_l \tilde{\Gamma}^l_{\tau l} \right) + \overbrace{\tilde{\Gamma}^\sigma_{\sigma\rho} \bar{\Gamma}^\rho_{\tau\tau}}^{\rho=\tau} + \overbrace{\bar{\Gamma}^\sigma_{\sigma\rho} \tilde{\Gamma}^\rho_{\tau\tau}}^{\rho=\tau} - \overbrace{\tilde{\Gamma}^\sigma_{\tau\rho} \bar{\Gamma}^\rho_{\tau\sigma}}^{\rho=\sigma} - \overbrace{\bar{\Gamma}^\sigma_{\tau\rho} \tilde{\Gamma}^\rho_{\tau\sigma}}^{\rho=\sigma} = \\
=& \partial_\tau(\dot{A}) + \sum_l (\partial_l A + H B_l + \dot{B}_l) - \partial_\tau \overbrace{\left(\dot{A} + \sum_l \partial_{\{l} B_{l\}} - \frac{1}{2}\dot{h}_{ll} \right)}^{\tilde{\Gamma}^\sigma_{\tau\sigma}} + \tilde{\Gamma}^\sigma_{\tau\sigma} H + 4H\dot{A} - \tilde{\Gamma}^\sigma_{\tau\sigma} H - H\tilde{\Gamma}^\sigma_{\tau\sigma} = \\
=& \nabla^2 A + H\vec{\nabla} \cdot \vec{B} + \vec{\nabla} \cdot \dot{\vec{B}} + \frac{1}{2} tr(\ddot{h}) + H[4\dot{A} - \overbrace{\left(\dot{A} - \frac{1}{2} tr(\dot{h}) \right)}^{\tilde{\Gamma}^\sigma_{\tau\sigma}}] = \\
=& \nabla^2 A + H\nabla^2 B + \nabla^2 \dot{B} + 3\ddot{C} + H(3\dot{A} + 3\dot{C}) \\
=& [\nabla^2 A + \nabla^2 \dot{B} + 3\ddot{C}] + H[3\dot{A} + \nabla^2 B + 3\dot{C}];
\end{aligned} \tag{A.1.11}$$

$$\begin{aligned}
\tilde{R}_{\tau j} =& \partial_\tau \tilde{\Gamma}^\tau_{\tau j} + \sum_l \partial_l \tilde{\Gamma}^l_{\tau j} - \partial_j \tilde{\Gamma}^\sigma_{\tau\sigma} + \overbrace{\tilde{\Gamma}^\sigma_{\sigma\rho} + \bar{\Gamma}^\rho_{\tau j}}^{\rho=j} + \overbrace{\bar{\Gamma}^\sigma_{\sigma\rho} + \sigma\Gamma^\rho_{\tau j}}^{\rho=\tau} - \overbrace{\tilde{\Gamma}^\sigma_{\tau\rho} \bar{\Gamma}^\rho_{j\sigma}}^{\{\rho;\sigma\}=\{\tau;j\}} - \overbrace{\bar{\Gamma}^\sigma_{\tau\rho} \tilde{\Gamma}^\rho_{j\sigma}}^{\rho=\sigma} = \\
=& \partial_\tau(HB_j + \partial_j A) + \sum_l \partial_l \left(\partial_{\{j} B_{l\}} - \frac{1}{2}\dot{h}_{jl} \right) - \partial_j(\dot{A} - 3\dot{C}) + \tilde{\Gamma}^\sigma_{\sigma j} H + \\
& + 4H\tilde{\Gamma}^\tau_{\tau j} - \tilde{\Gamma}^\tau_{\tau j} H - \tilde{\Gamma}^j_{\tau\tau} H - H\tilde{\Gamma}^\sigma_{j\sigma} = \dot{H} B_j + H\dot{B}_j + \frac{1}{2}\partial_j \vec{\nabla} \cdot \vec{B} - \frac{1}{2}\nabla^2 B_j + \\
& - \frac{1}{2}(2\partial_j \dot{C} + \nabla^2 \dot{E}_j) + 3\partial_j \dot{C} + 3H(HB_j + \partial_j A) - (\partial_j A + HB_j + \dot{B}_j)H = \\
=& \dot{H} B_j + \frac{1}{2}\partial_j \nabla^2 B - \frac{1}{2}\nabla^2 B_j + 2\partial_j \dot{C} - \frac{1}{2}\nabla^2 \dot{E}_j + 2H^2 B_j + 2H\partial_j A \\
=& \left[\frac{1}{2}\nabla^2(\partial_j B - B_j) + 2\partial_j \dot{C} - \frac{1}{2}\nabla^2 \dot{E}_j \right] + 2H[\partial_j A] + (\dot{H} + 2H^2)[B_j];
\end{aligned} \tag{A.1.12}$$

$$\begin{aligned}
\tilde{R}_{ij} =& \partial_\tau \tilde{\Gamma}^\tau_{ij} + \sum_l \partial_l \tilde{\Gamma}^l_{ij} - \partial_j \left(\tilde{\Gamma}^\tau_{i\tau} + \sum_l \tilde{\Gamma}^l_{il} \right) + \overbrace{\tilde{\Gamma}^\sigma_{\sigma\rho} \bar{\Gamma}^\rho_{ij}}^{\rho=\tau} + \overbrace{\bar{\Gamma}^\sigma_{\sigma\rho} \tilde{\Gamma}^\rho_{ij}}^{\rho=\tau} - \overbrace{\tilde{\Gamma}^\sigma_{i\rho} \bar{\Gamma}^\rho_{j\sigma}}^{\{\rho;\sigma\}=\{\tau;j\}} - \overbrace{\bar{\Gamma}^\sigma_{i\rho} \tilde{\Gamma}^\rho_{j\sigma}}^{\{\rho;\sigma\}=\{\tau;i\}} = \\
=& \partial_\tau \left(-2HA\delta_{ij} - \partial_{(i} B_{j)} - Hh_{ij} - \frac{1}{2}\dot{h}_{ij} \right) + \sum_l \partial_l(-HB_l \delta_{ij} - \gamma_{lij}) + \\
& - \partial_j \left[(HB_i + \partial_i A) + \sum_l (-HB_l \delta_{il} - \gamma_{ili}) \right] + \tilde{\Gamma}^\sigma_{\sigma\tau} H\delta_{ij} + 4H\tilde{\Gamma}^\tau_{ij} - \tilde{\Gamma}^\tau_{ij} H - \tilde{\Gamma}^j_{i\tau} H - H\tilde{\Gamma}^\tau_{ji} - H\tilde{\Gamma}^i_{j\tau} =
\end{aligned}$$

$$
\begin{aligned}
=&-2\dot{H}A\delta_{ij}-2H\dot{A}\delta_{ij}-\partial_{(i}\dot{B}_{j)}-\dot{H}h_{ij}-H\dot{h}_{ij}-\frac{1}{2}\ddot{h}_{ij}-H\vec{\nabla}\cdot\vec{B}\delta_{ij}+\\
&-\frac{1}{2}\sum_l\partial_l(\partial_i h_{jl}+\partial_j h_{il}-\partial_l h_{ij})-\partial_l\left(HB_i+\partial_i A-HB_i-\frac{1}{2}\sum_l\partial_i h_{ll}\right)+H[(\dot{A}-3\dot{C})\delta_{ij}+\\
&+2\left(-2HA\delta_{ij}-\partial_{(i}B_{j)}-Hh_{ij}-\frac{1}{2}\dot{h}_{ij}\right)-\left(\partial_{\{i}B_{j\}}-\frac{1}{2}\dot{h}_{ij}\right)-\left(\partial_{\{j}B_{i\}}-\frac{1}{2}\dot{h}_{ij}\right)]=\\
=&-\partial_{(i}\dot{B}_{j)}-\frac{1}{2}\ddot{h}_{ij}-\frac{1}{2}[\partial_i(2\partial_j C+\nabla^2E_j)+\partial_j(2\partial_i C+\nabla^2E_i)-\nabla^2h_{ij}]-\partial_{ij}A+3\partial_{ij}C+\\
&+H[-2\dot{A}\delta_{ij}-\dot{h}_{ij}-\nabla^2B\delta_{ij}+\dot{A}\delta_{ij}-3\dot{C}\delta_{ij}-4HA\delta_{ij}-2\partial_{(i}B_{j)}-2Hh_{ij}-\dot{h}_{ij}+\dot{h}_{ij}]+\\
&+\dot{H}(-2A\delta_{ij}-h_{ij})=-\partial_{(i}\dot{B}_{j)}+\frac{1}{2}(\nabla^2-\partial_\tau^2)h_{ij}+\partial_{ij}C-\nabla^2\partial_{(i}E_{j)}-\partial_{ij}A+\\
&+H[-\dot{A}\delta_{ij}-\dot{h}_{ij}-\nabla^2B\delta_{ij}-3\dot{C}\delta_{ij}-2\partial_{(i}B_{j)}]-2H^2(2A\delta_{ij}+h_{ij})+\dot{H}(2A\delta_{ij}+h_{ij})\\
=&\left[\frac{1}{2}\Box h_{ij}-\partial_{ij}A-\partial_{(i}\dot{B}_{j)}+\partial_{ij}C-\nabla^2\partial_{(i}E_{j)}\right]-H[\dot{A}\delta_{ij}+\nabla^2B\delta_{ij}+2\partial_{(i}B_{j)}+3\dot{C}\delta_{ij}+\dot{h}_{ij}]+\\
&-(\dot{H}+2H^2)[2A\delta_{ij}+h_{ij}],
\end{aligned}
\tag{A.1.13}
$$

where we used the flat d'alembertian $\Box := \nabla^2-\partial_\tau^2$.

A.2 Choice of harmonic gauge

The general expression for $\tilde{R}_{\mu\nu}$ is quite complicate, but we can anyway look for a suitable gauge which simplifies it. The Einstein Equations do not yield a unique solution, but there remain 4 degrees of freedom, which allow change of variables

$$x_\mu \to x_\mu + \delta x_\mu.$$

We are seeking for the Einstein Equation linearized at the first order, hence the allowed transformations must have $\delta x_\mu = O(\tilde{g}_{\mu\nu})$.
What we want to find with the present calculation is the retarded potential generated by an inhomogeneous source. Mathematically, retarded potentials are particular solutions of second order, linear PDEs, whose solution in vacuum are waves. Hence, we want to write the linearized Einstein Equation in the form of wave equations. Our choice of the suitable gauge will be then inspired by the formalism of the gravitational waves.

The studies on gravitational waves use perturbative methods analogous to the ours, with a certain perturbation $\tilde{g}_{\mu\nu}$ on the metric which carries the waves. However, it is usually assumed a minkowskian background, that we can see as the particular case

$$a\equiv 1 \Rightarrow \tilde{g}_{\mu\nu}=h_{\mu\nu},\quad H\equiv 0\Rightarrow$$

$$\Gamma^\lambda_{\mu\nu}=\tilde{\Gamma}^\lambda_{\mu\nu}+o(h)=\frac{1}{2}\eta^{\lambda\rho}(\partial_\mu h_{\nu\rho}+\partial_\nu h_{\rho\mu}-\partial_\rho h_{\mu\nu})+o(h)\Rightarrow \tag{A.2.1}$$

$$R_{\mu\nu}=\tilde{R}_{\mu\nu}+o(h)=\partial_\nu\Gamma^\lambda_{\mu\lambda}-\partial_\lambda\Gamma^\lambda_{\mu\nu}+o(h)=\frac{1}{2}(\Box h_{\mu\nu}-\partial^2_{\lambda\mu}h^\lambda_\nu-\partial^2_{\lambda\nu}h^\lambda_\mu+\partial^2_{\mu\nu}h^\lambda_\lambda)+o(h) \tag{A.2.2}$$

Following *e.g.* [116] §10.1, we see that if the 4 degrees of gauge freedom are chosen s.t. $g^{\mu\nu}\Gamma^\lambda_{\mu\nu}:=0$, i.e. the harmonic gauge. For (A.2.1), it means

$$\partial_\mu h^\mu_\nu=\frac{1}{2}\partial_\nu h^\mu_\mu; \tag{A.2.3}$$

and substituting inside (A.2.2) one finds

$$R_{\mu\nu} = \frac{1}{2}\Box h_{\mu\nu}. \tag{A.2.4}$$

The Einstein Equations in the form

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right) \tag{A.2.5}$$

are indeed a tensorial wave equation, where the field $h_{\mu\nu}$ appears as a pure d'alembertian. Its general solution in vacuum is the usual three-dimensional wave.

We perform an analogous choice, which we call again "harmonic gauge". Since on a minkowskinan background it is $\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu}$, we generalize the gauge condition as $g^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\nu} := 0$.
It results to be the right choice. We can make it explicit, for any background metric, as

$$\begin{aligned}
0 = \tilde{\Gamma}^{\lambda\mu}_{\mu} =& \tilde{g}^{\mu\nu}\bar{\Gamma}^{\lambda}_{\mu\nu} + \bar{g}^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\nu} = \tilde{g}^{\mu\nu}\bar{g}^{\lambda\rho}\bar{\Gamma}_{\rho\mu\nu} + \bar{g}^{\mu\nu}\left[\tilde{g}^{\lambda\rho}\bar{\Gamma}_{\rho\mu\nu} + \frac{1}{2}\bar{g}^{\lambda\rho}(\partial_\mu\tilde{g}_{\nu\rho} + \partial_\nu\tilde{g}_{\rho\mu} - \partial_\rho\tilde{g}_{\mu\nu})\right] = \\
=& (\tilde{g}^{\mu\nu}\bar{g}^{\lambda\rho} + \bar{g}^{\mu\nu}\tilde{g}^{\lambda\rho})\bar{\Gamma}_{\rho\mu\nu} + \frac{1}{2}\bar{g}^{\lambda\rho}(\bar{g}^{\mu\nu}\partial_\mu\tilde{g}_{\nu\rho} + \bar{g}^{\mu\nu}\partial_\mu\tilde{g}_{\rho\nu} - \bar{g}^{\mu\nu}\partial_\rho\tilde{g}_{\mu\nu}) = \\
=& (\bar{g}^{\lambda\rho}\tilde{g}^{\mu\nu} + \bar{g}^{\mu\nu}\tilde{g}^{\lambda\rho})\bar{\Gamma}_{\rho\mu\nu} + \bar{g}^{\lambda\rho}\,(\bar{g}^{\mu\nu}\partial_\mu\tilde{g}_{\nu\rho} - \bar{g}^{\mu\nu}\partial_\rho\tilde{g}_{\mu\nu}) \Rightarrow \\
0 = \bar{g}_{\lambda\sigma}\tilde{\Gamma}^{\sigma\alpha}_{\alpha} =& \tilde{g}^{\alpha\beta}\bar{\Gamma}_{\lambda\alpha\beta} + \bar{g}_{\lambda\sigma}\bar{g}^{\alpha\beta}\tilde{g}\sigma\rho\bar{\Gamma}_{\rho\alpha\beta} + \bar{g}^{\alpha\beta}\partial_\alpha\tilde{g}_{\beta\lambda} - \frac{1}{2}\bar{g}^{\alpha\beta}\partial_\lambda\tilde{g}_{\alpha\beta} = \\
=& (-\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\tilde{g}_{\mu\nu})(\bar{g}_{\lambda\sigma}\bar{\Gamma}^{\sigma}_{\alpha\beta}) + (\bar{g}_{\lambda\sigma}\bar{g}^{\alpha\beta})(-\bar{g}^{\sigma\mu}\bar{g}^{\rho\nu}\tilde{g}_{\mu\nu})\bar{\Gamma}_{\rho\alpha\beta} + \bar{g}^{\alpha\beta}\partial_\alpha\tilde{g}_{\beta\lambda} - \frac{1}{2}\bar{g}^{\alpha\beta}\partial_\lambda\tilde{g}_{\alpha\beta} = \\
=& -\tilde{g}_{\mu\nu}\bar{g}_{\lambda\sigma}\bar{\Gamma}^{\sigma\mu\nu} - \overbrace{\bar{g}_{\lambda\sigma}\bar{g}^{\sigma\mu}}^{\delta^\mu_\lambda}\tilde{g}_{\mu\nu}\bar{g}_{\alpha\beta}\bar{\Gamma}^{\nu\alpha\beta} + \bar{g}^{\alpha\beta}\partial_\alpha\tilde{g}_{\beta\lambda} - \frac{1}{2}\bar{g}^{\alpha\beta}\partial_\lambda\tilde{g}_{\alpha\beta} = \\
=& -\tilde{g}_{\alpha\beta}\bar{g}_{\lambda\sigma}\bar{\Gamma}^{\sigma\alpha\beta} - \tilde{g}_{\lambda\sigma}\bar{g}_{\alpha\beta}\bar{\Gamma}^{\sigma\alpha\beta} + \bar{g}^{\alpha\beta}\partial_\alpha\tilde{g}_{\beta\lambda} - \frac{1}{2}\bar{g}^{\alpha\beta}\partial_\lambda\tilde{g}_{\alpha\beta} \Rightarrow \\
&\forall\lambda : \bar{g}^{\alpha\beta}\partial_\alpha\tilde{g}_{\beta\lambda} = \frac{1}{2}\bar{g}^{\alpha\beta}\partial_\lambda\tilde{g}_{\alpha\beta} + \bar{\Gamma}^{\sigma\alpha\beta}(\bar{g}_{\lambda\sigma}\tilde{g}_{\alpha\beta} + \bar{g}_{\alpha\beta}\tilde{g}_{\lambda\sigma}).
\end{aligned} \tag{A.2.6}$$

If we absorb in the notation $\tilde{R}^{(I,0)}_{\mu\nu}$ any term of $\tilde{R}_{\mu\nu}$ which is of the zeroth or of the first order in $h_{\mu\nu}$, we find that the purely second order part is

$$\begin{aligned}
\tilde{R}_{\mu\nu} =& \partial_\sigma\tilde{\Gamma}^{\sigma}_{\mu\nu} - \partial_\nu\tilde{\Gamma}^{\sigma}_{\mu\sigma} + \tilde{\Gamma}^{\sigma}_{\sigma\rho}\bar{\Gamma}^{\rho}_{\mu\nu} + \bar{\Gamma}^{\sigma}_{\sigma\rho}\tilde{\Gamma}^{\rho}_{\mu\nu} - \tilde{\Gamma}^{\sigma}_{\mu\rho}\bar{\Gamma}^{\rho}_{\sigma\nu} - \bar{\Gamma}^{\sigma}_{\mu\rho}\tilde{\Gamma}^{\rho}_{\sigma\nu} = \\
=& \partial_\sigma\left[\tilde{g}^{\sigma\rho}\bar{\Gamma}_{\rho\mu\nu} + \frac{1}{2}\bar{g}^{\sigma\rho}(\partial_\mu\tilde{g}_{\nu\rho} + \partial_\nu\tilde{g}_{\rho\mu} - \partial_\rho\tilde{g}_{\mu\nu})\right] - \partial_\nu\left[\tilde{g}^{\sigma\rho}\bar{\Gamma}_{\rho\mu\sigma} + \frac{1}{2}\bar{g}^{\sigma\rho}(\partial_\mu\tilde{g}_{\sigma\rho} + \partial_\sigma\tilde{g}_{\rho\mu} - \partial_\rho\tilde{g}_{\mu\sigma})\right] + \\
&+ \tilde{R}^{(I,0)}_{\mu\nu} = \tilde{R}^{(I,0)}_{\mu\nu} + \frac{1}{2}\partial_\sigma(\bar{g}^{\sigma\rho}\partial_\mu\tilde{g}_{\nu\rho} + \bar{g}^{\sigma\rho}\partial_\nu\tilde{g}_{\rho\mu} - \bar{g}^{\sigma\rho}\partial_\rho\tilde{g}_{\mu\nu}) - \tilde{R}^{(I,0)}_{\mu\nu} + \\
&- \frac{1}{2}\partial_\nu(\bar{g}^{\sigma\rho}\partial_\mu\tilde{g}_{\sigma\rho} + \bar{g}^{\sigma\rho}\partial_\sigma\tilde{g}_{\rho\mu} - \bar{g}^{\sigma\rho}\partial_\rho\tilde{g}_{\mu\sigma}) + \tilde{R}^{(I,0)}_{\mu\nu} = \\
=& \frac{1}{2}\bar{g}^{\sigma\rho}(\partial^2_{\sigma\mu}\tilde{g}_{\nu\rho} + \partial^2_{\sigma\nu}\tilde{g}_{\rho\mu} - \partial^2_{\sigma\rho}\tilde{g}_{\mu\nu}) + \tilde{R}^{(I,0)}_{\mu\nu} + \\
&- \frac{1}{2}\partial_\nu\left[\bar{g}^{\sigma\rho}\partial_\mu\tilde{g}_{\sigma\rho} + \bar{g}^{\sigma\rho}\partial_\sigma\tilde{g}_{\rho\mu} - \left(\frac{1}{2}\bar{g}^{\rho\sigma}\partial_\mu\tilde{g}_{\rho\sigma} + \overbrace{\bar{\Gamma}^{\varphi\rho\sigma}(\bar{g}_{\mu\varphi}\tilde{g}_{\rho\sigma} + \bar{g}_{\rho\sigma}\tilde{g}_{\mu\varphi})}^{\tilde{R}^{(I,0)}_{\mu\nu}}\right)\right] + \tilde{R}^{(I,0)}_{\mu\nu} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\partial_\mu(\bar{g}^{\rho\sigma}\partial_\sigma\tilde{g}_{\nu\rho}) + \tilde{R}^{(I,0)}_{\mu\nu} + \frac{1}{2}\bar{g}^{\sigma\rho}(\partial^2_{\sigma\nu}\tilde{g}_{\rho\mu} - \partial^2_{\sigma\rho}\tilde{g}_{\mu\nu}) - \frac{1}{2}[\bar{g}^{\sigma\rho}\left(\partial^2_{\nu\mu}\tilde{g}_{\sigma\rho} + \partial^2_{\nu\sigma}\tilde{g}_{\rho\mu} - \frac{1}{2}\partial^2_{\nu\mu}\tilde{g}_{\rho\sigma}\right) + \\
&+ \tilde{R}^{(I,0)}_{\mu\nu}] + \tilde{R}^{(I,0)}_{\mu\nu} = \frac{1}{2}\partial_\mu\left[\frac{1}{2}\bar{g}^{\sigma\rho}\partial_\nu\tilde{g}_{\sigma\rho} + \overbrace{\bar{\Gamma}^{\varphi\sigma\rho}(\bar{g}_{\nu\varphi}\tilde{g}_{\sigma\rho} - \bar{g}_{\sigma\rho}\tilde{g}_{\nu\varphi})}^{\tilde{R}^{(I,0)}_{\mu\nu}}\right] + \frac{1}{2}\bar{g}^{\sigma\rho}\left(-\partial^2_{\sigma\rho}\tilde{g}_{\mu\nu} - \frac{1}{2}\partial^2_{\mu\nu}\tilde{g}_{\sigma\rho}\right) + \\
&+ \tilde{R}^{(I,0)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\sigma\rho}\left(\frac{1}{2}\partial^2_{\mu\nu}\tilde{g}_{\sigma\rho}\right) + \tilde{R}^{(I,0)}_{\mu\nu} + \frac{1}{2}\bar{g}^{\sigma\rho}\left(-\partial^2_{\sigma\rho}\tilde{g}_{\mu\nu} - \frac{1}{2}\partial^2_{\mu\nu}\tilde{g}_{\sigma\rho}\right) + \tilde{R}^{(I,0)}_{\mu\nu} = \\
&= -\frac{1}{2}\bar{g}^{\sigma\rho}\partial^2_{\sigma\rho}\tilde{g}_{\mu\nu} + \tilde{R}^{(I,0)}_{\mu\nu} \Rightarrow \\
\tilde{R}^{(II)}_{\mu\nu} &= -\frac{1}{2}(\bar{g}^{\sigma\rho}\partial^2_{\sigma\rho})\tilde{g}_{\mu\nu},
\end{aligned}
\tag{A.2.7}
$$

where it is used the previous relation (A.2.6). This formula is valid for any background, showing that with harmonic gauge the linearized Einstein Equations have always, as second order term, the background Laplace-Beltrami operator applied to the perturbation $\tilde{g}_{\mu\nu}$. Specifying this formula for the our background, we have

$$
\begin{aligned}
\tilde{R}_{\mu\nu} &= -\frac{1}{2}\bar{g}^{\sigma\rho}\partial^2_{\sigma\rho}\tilde{g}_{\mu\nu} + \tilde{R}^{(I,0)}_{\mu\nu} = -\frac{1}{2}(a^{-2}\eta^{\sigma\rho})\partial^2_{\sigma\rho}(a^2 h_{\mu\nu}) + \tilde{R}^{(I,0)}_{\mu\nu} = \\
&= -\frac{1}{2}a^{-2}\eta^{\sigma\rho}(a^2\partial^2_{\rho\sigma}h_{\mu\nu} + \tilde{R}^{(I,0)}_{\mu\nu}) + \tilde{R}^{(I,0)}_{\mu\nu} = -\frac{1}{2}\eta^{\sigma\rho}\partial^2_{\sigma\rho}h_{\mu\nu} + \tilde{R}^{(I,0)}_{\mu\nu} \Rightarrow \\
\tilde{R}^{(II)}_{\mu\nu} &= \frac{1}{2}\Box h_{\mu\nu},
\end{aligned}
\tag{A.2.8}
$$

since we use the most-minus convention $\eta_{\mu\nu} = diag(+;-;-;-)$.
This is perfectly analogous to the result (A.2.4), and we know that with this gauge choice we will have some wave equations. However, only the second order part will be a wave operator, while in our case we can find some first order and zeroth order terms, for which the usual three-dimensional wave will not be a solution.

Now we simplify our expression for $\tilde{R}_{\mu\nu}$ exploiting the gauge conditions (A.2.6). First of all, we rewrite them in terms of $A, \vec{B}, C, \hat{E}$.
For $\lambda = \tau$ we have a scalar condition.

$$
\begin{aligned}
0 &= \tilde{\Gamma}^{\tau\mu}_{\mu} = \overbrace{\bar{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\mu\nu}}^{\mu=\nu} + \overbrace{\tilde{g}^{\mu\nu}\bar{\Gamma}^{\tau}_{\mu\nu}}^{\mu=\nu} = a^{-2}\left(\tilde{\Gamma}^{\tau}_{\tau\tau} - \sum_j \tilde{\Gamma}^{\tau}_{jj}\right) + \left(\tilde{g}^{\tau\tau} + \sum_j \tilde{g}^{jj}\right)H = \\
&= a^{-2}\dot{A} - a^{-2}\sum_j\left(-2HA\delta_{jj} - \partial_{(j}B_{j)} - Hh_{jj} - \frac{1}{2}\dot{h}_{jj}\right) + H(-2a^{-2}A) + H\sum_j(-a^{-2}h_{jj}) = \\
&= a^{-2}[\dot{A} + 6HA + \vec{\nabla}\cdot\vec{B} + 6HC + 3\dot{C} - 2HA - 6HC] \Rightarrow \\
0 &= \dot{A} + \nabla^2 B + 3\dot{C} + 4HA.
\end{aligned}
\tag{A.2.9}
$$

For $\lambda = i$ we have a vector condition.

$$
\begin{aligned}
0 &= \tilde{\Gamma}^{i\mu}_{\mu} = \overbrace{\bar{g}^{\mu\nu}\tilde{\Gamma}^{i}_{\mu\nu}}^{\mu=\nu} + \overbrace{\tilde{g}^{\mu\nu}\bar{\Gamma}^{i}_{\mu\nu}}^{\{\mu;\nu\}=\{i;\tau\}} = a^{-2}\left(\tilde{\Gamma}^{i}_{\tau\tau} - \sum_j \tilde{\Gamma}^{i}_{jj}\right) + 2\tilde{g}^{i\tau}H = \\
&= a^{-2}(\partial_i A + HB_i + \dot{B}_i) - a^{-2}\sum_j(-HB_i\delta_{jj} - \gamma_{ijj}) + 2(-a^{-2}B_i)H =
\end{aligned}
$$

$$
\begin{aligned}
&= a^{-2}\left[\partial_i A + HB_i + \dot{B}_i + 3HB_i + \sum_j \left(\partial_j h_{ij} - \frac{1}{2}\partial_i h_{jj}\right) - 2HB_i\right] \\
&= a^{-2}[\partial_i A + 2HB_i + \dot{B}_i + 2\partial_i C + \nabla^2 E_i - 3\partial_i C] \Rightarrow \\
0 &= \vec{\nabla} A + \dot{\vec{B}} - \vec{\nabla} C + \nabla^2 \vec{E} + 2H\vec{B} = \\
&= \vec{\nabla}\left[A + \dot{B} - C + \frac{4}{3}\nabla^2 E + 2HB\right] + [\dot{\hat{B}} + \nabla^2 \hat{E} + 2H\hat{B}]. \qquad \text{(A.2.10)}
\end{aligned}
$$

These were written in (2.2.8).

Let us substitute them inside the component of $\tilde{R}_{\mu\nu}$.

$$
\begin{aligned}
\tilde{R}_{\tau\tau} &= \nabla^2 A + \nabla^2 \dot{B} + 3\ddot{C} + H(3\dot{A} + \nabla^2 B + 3\dot{C}) = \\
&= \nabla^2 A - \partial_\tau(\dot{A} + 4HA) + H(2\dot{A} - 4HA) = \nabla^2 A - \ddot{A} - 4\dot{H}A - 4H\dot{A} + 2H\dot{A} - 4H^2 A \\
&= \Box A - 2H\dot{A} - 4(\dot{H} + H^2)A \qquad \text{(A.2.11)} \\
\tilde{R}_{\tau j} &= -\frac{1}{2}\nabla^2 B_j + \frac{1}{2}\partial_j \nabla^2 B + 2\partial_j \dot{C} - \frac{1}{2}\nabla^2 \dot{E}_j + 2H\partial_j A + (\dot{H} + 2H^2)B_j = \\
&= -\frac{1}{2}\nabla^2 B_j - \frac{1}{2}\partial_j(\dot{A} + 3\dot{C} + 4HA) + 2\partial_j \dot{C} + \frac{1}{2}\partial_\tau(\partial_j A + \dot{B}_j - \partial_j C + 2HB_j) + 2H\partial_j A + \\
&\quad + (\dot{H} + 2H^2)B_j = -\frac{1}{2}\nabla^2 B_j + \frac{1}{2}\ddot{B}_j + \dot{H}B_j + H\dot{B}_j + (\dot{H} + 2H^2)B_j \\
&= -\frac{1}{2}\Box B_j + H\dot{B}_j + 2(\dot{H} + H^2)B_j \\
&\qquad \text{(A.2.12)} \\
\tilde{R}_{ij} &= \frac{1}{2}\Box h_{ij} - \partial_{ij}A - \partial_{(i}\dot{B}_{j)} + \partial_{ij}C - \nabla^2 \partial_{(i}E_{j)} + \\
&\quad - H(\dot{A}\delta_{ij} + \nabla^2 B\delta_{ij} + 2\partial_{(i}B_{j)} + 3\dot{C}\delta_{ij} + \dot{h}_{ij}) - (\dot{H} + 2H^2)(2A\delta_{ij} + h_{ij}) = \\
&= \frac{1}{2}\Box h_{ij} + 2H\partial_{(i}B_{j)} - H(2\partial_{(i}B_{j)} - 4HA\delta_{ij} + \dot{h}_{ij}) - (\dot{H} + 2H^2)(2A\delta_{ij} + h_{ij}) \\
&= \frac{1}{2}\Box h_{ij} - H\dot{h}_{ij} - (\dot{H} + 2H^2)h_{ij} - 2\dot{H}A\delta_{ij}. \qquad \text{(A.2.13)}
\end{aligned}
$$

These are compactly expressed in (2.2.9).

A.3 Perturbed Conservation Laws

Now we perturb the energy-momentum at the first order in $\tilde{\rho}$, which must be of the same order of $\tilde{g}_{\mu\nu}$, looking for linearized Conservation Laws and linearized Einstein Equations.

The energy-momentum tensor depends on the perturbation of four-velocity $\tilde{U}_\mu$. Its expression can be obtained remembering that a four-velocity must be always normalized

$$
\begin{aligned}
&\bar{g}_{\mu\nu}\bar{U}^\mu \bar{U}^\mu = 1 = g_{\mu\nu}U^\mu U^\nu = \bar{g}_{\mu\nu}\bar{U}^\mu \bar{U}^\mu + \tilde{g}_{\mu\nu}\bar{U}^\mu \bar{U}^\nu + 2\bar{g}_{\mu\nu}\bar{U}^\mu \tilde{U}^\nu \Rightarrow \\
&\bar{U}^\mu = a^{-1}\delta^\mu_\tau, \quad 0 = \tilde{g}_{\mu\nu}\bar{U}^\mu \bar{U}^\nu + 2\bar{g}_{\mu\nu}\bar{U}^\mu \tilde{U}^\nu = \tilde{g}_{\tau\tau}a^{-1}a^{-1} + 2a^2 a^{-1}\tilde{U}^\tau = 2(A + a\tilde{U}^\tau) \Rightarrow \\
&\tilde{U}^\mu = \frac{1}{a}\begin{pmatrix} -A \\ \vec{v} \end{pmatrix}, \qquad \text{(A.3.1)}
\end{aligned}
$$

where we defined the three-vector velocity $v^i := a\tilde{U}^i$, which is a first order variable as $\tilde{\rho}$.

The indices is lowered as

$$\begin{aligned}\tilde{U}_\mu =&\tilde{g}_{\mu\nu}\bar{U}^\nu + \bar{g}_{\mu\nu}\tilde{U}^\nu = a^2 h_{\mu\tau} a^{-1} + a^2\eta_{\mu\nu}\tilde{U}^\nu = a\left(2A \quad -\vec{B}\right) + a\left(-A \quad -\vec{v}\right)\\ =&a\left(A \quad -(\vec{v}+\vec{B})\right). \end{aligned} \tag{A.3.2}$$

In order to get the linearized Conservation Laws, we need to write the energy-momentum tensor with a raised indices

$$\tilde{T}^\mu_\nu = [(\rho+p)U^\mu U_\nu - p g^\mu_\nu] - [(\bar{\rho}+p)\bar{U}^\mu\bar{U}_\nu - p\bar{g}^\mu_\nu] = \tilde{\rho}\bar{U}^\mu\bar{U}_\nu + (\bar{\rho}+p)(\tilde{U}^\mu\bar{U}_\nu + \bar{U}^\mu\tilde{U}_\nu), \tag{A.3.3}$$

since $g^\mu_\nu = \delta^\mu_\nu = \bar{g}^\mu_\nu$. Hence,

$$\begin{aligned}\tilde{T}^\tau_\tau =&\tilde{\rho}\bar{U}^\tau\bar{U}_\tau + (\bar{\rho}+p)(\tilde{U}^\tau\bar{U}_\tau + \bar{U}^\tau\tilde{U}_\tau) = \tilde{\rho}a^{-1}a + (\bar{\rho}+p)(-a^{-1}Aa + a^{-1}aA)\\ =&\tilde{\rho}; \end{aligned} \tag{A.3.4}$$

$$\begin{aligned}\tilde{T}^i_\tau =&(\bar{\rho}+p)\tilde{U}^i\bar{U}_\tau = (\bar{\rho}+p)a^{-1}v^i a\\ =&q^i; \end{aligned} \tag{A.3.5}$$

$$\begin{aligned}\tilde{T}^\tau_j =&(\bar{\rho}+p)\tilde{U}^\tau\bar{U}_j = (\bar{\rho}+p)a^{-1}a(-v_j - B_j)\\ =&-q_j - (\bar{\rho}+p)B_j; \end{aligned} \tag{A.3.6}$$

$$\tilde{T}^i_j = 0. \tag{A.3.7}$$

Notice that these provides natural definitions for the variable $\tilde{\rho} := \tilde{T}^\tau_\tau$. Moreover, we defined the perturbed momentum $q^i := \tilde{T}^i_\tau$, which is a first order variable as well.
We write compactly

$$\tilde{T}^\mu_\nu = \begin{pmatrix}\tilde{\rho} & -q_j + (\bar{\rho}+p)B_j\\ q^i & 0\end{pmatrix}. \tag{A.3.8}$$

The Conservation of Four-Momentum is expressed in perturbation as

$$\begin{aligned}0 = \nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha}T^\alpha_\nu - \Gamma^\alpha_{\mu\nu}T^\mu_\alpha \Rightarrow\\ 0 = \partial_\mu\tilde{T}^\mu_\nu + \tilde{\Gamma}^\mu_{\mu\alpha}\bar{T}^\alpha_\nu + \bar{\Gamma}^\mu_{\mu\alpha}\tilde{T}^\alpha_\nu - \tilde{\Gamma}^\alpha_{\mu\nu}\bar{T}^\mu_\alpha - \bar{\Gamma}^\alpha_{\mu\nu}\tilde{T}^\mu_\alpha. \end{aligned} \tag{A.3.9}$$

We can substitute the perturbations we found. For $\nu = \tau$ we get the Conservation of Energy

$$\begin{aligned}0 =&\partial_\mu\tilde{T}^\mu_\tau + \overbrace{\tilde{\Gamma}^\mu_{\mu\alpha}\bar{T}^\alpha_\tau}^{\alpha=\tau} + \overbrace{\bar{\Gamma}^\mu_{\mu\alpha}\tilde{T}^\alpha_\tau}^{\alpha=\tau} - \overbrace{\tilde{\Gamma}^\alpha_{\mu\tau}\bar{T}^\mu_\alpha}^{\alpha=\mu} - \overbrace{\bar{\Gamma}^\alpha_{\mu\tau}\tilde{T}^\mu_\alpha}^{\alpha=\mu=\tau} =\\ =&\partial_\tau\tilde{T}^\tau_\tau + \sum_l \partial_l\tilde{T}^l_\tau + \left(\tilde{\Gamma}^\tau_{\tau\tau} + \sum_l \tilde{\Gamma}^l_{l\tau}\right)\bar{\rho} + 4H\tilde{T}^\tau_\tau - \tilde{\Gamma}^\tau_{\tau\tau}\bar{\rho} - \sum_l \tilde{\Gamma}^l_{l\tau}(-p) - H\tilde{T}^\tau_\tau =\\ =&\partial_\tau\tilde{\rho} + \sum_l \partial_l q^l + (\bar{\rho}+p)\sum_l\left(\partial_{\{l}B_{l\}} - \frac{1}{2}\dot{h}_{ll}\right) + 3H\tilde{\rho} \Rightarrow\\ 0 =&\dot{\tilde{\rho}} + \vec{\nabla}\cdot\vec{q} + 3H\tilde{\rho} - 3(\bar{\rho}+p)\dot{C}. \end{aligned} \tag{A.3.10}$$

For $\nu = j$ we get the Conservation of Momentum

$$
\begin{aligned}
0 =&\partial_\mu \tilde{T}^\mu_j + + \overbrace{\tilde{\Gamma}^\mu_{\mu\alpha}\bar{T}^\alpha_j}^{\alpha=j} + \overbrace{\bar{\Gamma}^\mu_{\mu\alpha}\tilde{T}^\alpha_j}^{\alpha=\tau} - \overbrace{\tilde{\Gamma}^\alpha_{\mu j}\bar{T}^\mu_\alpha}^{\alpha=\mu} - \overbrace{\bar{\Gamma}^\alpha_{\mu j}\tilde{T}^\mu_\alpha}^{\{\alpha;\mu\}=\{\tau;j\}} = \\
=&\partial_\tau \tilde{T}^\tau_j + \left(\tilde{\Gamma}^\tau_{\tau j} + \sum_l \tilde{\Gamma}^l_{lj}\right)(-p) + 4H\tilde{T}^\tau_j - \left(\bar{\rho}\tilde{\Gamma}^\tau_{\tau j} - p\sum_l \tilde{\Gamma}^l_{lj}\right) - H(\tilde{T}^\tau_j + \tilde{T}^j_\tau) = \\
=&\partial_\tau[-q_j - (\bar{\rho}+p)B_j] - (\bar{\rho}+p)(\partial_j A + HB_j) + 3H[-q_j - (\bar{\rho}+p)B_j] - Hq^j = \\
=&\dot{q}_j + 4Hq_j + (\bar{\rho}+p)(\partial_j A + \dot{B}_j) + [(\dot{\bar{\rho}}+\dot{p}) + 4H(\bar{\rho}+p)]B_j = \\
=&\dot{q}_j + 4Hq_j + (\bar{\rho}+p)(\partial_j C - 2HB_j - \nabla^2 E_j) + [(\dot{\bar{\rho}}+\dot{p}) + 4H(\bar{\rho}+p)]B_j \Rightarrow \\
0 =&\dot{\vec{q}} + 4H\vec{q} + (\bar{\rho}+p)(\vec{\nabla}C - \nabla^2\vec{E}) + [(\dot{\bar{\rho}}+\dot{p}) + 4H(\bar{\rho}+p)]\vec{B},
\end{aligned} \tag{A.3.11}
$$

where it is used the vector gauge condition.

A.4 Perturbed source

Now we write the perturbation of energy-momentum tensor, full covariant

$$
\tilde{T}_{\mu\nu} = T_{\mu\nu} - \bar{T}_{\mu\nu} = \tilde{\rho}\bar{U}_\mu\bar{U}_\nu + (\bar{\rho}+p)(\tilde{U}_\mu\bar{U}_\nu + \bar{U}_\mu\tilde{U}_\nu) - p\tilde{g}_{\mu\nu} \Rightarrow \tag{A.4.1}
$$

$$
\begin{aligned}
\tilde{T}_{\tau\tau} =&\tilde{\rho}\bar{U}_\tau\bar{U}_\tau + (\bar{\rho}+p)(\tilde{U}_\tau\bar{U}_\tau + \bar{U}_\tau\tilde{U}_\tau) - p\tilde{g}_{\tau\tau} = \tilde{\rho}aa + (\bar{\rho}+p)(aAa + aaA) - p2a^2A \\
=&a^2(\tilde{\rho} + 2\bar{\rho}A);
\end{aligned} \tag{A.4.2}
$$

$$
\begin{aligned}
\tilde{T}_{\tau j} =&(\bar{\rho}+p)\bar{U}_\tau\tilde{U}_j - p\tilde{g}_{\tau j} = (\bar{\rho}+p)aa(-v_j - B_j) - pa^2(-B_j) \\
=& - a^2(q_j + \bar{\rho}B_j);
\end{aligned} \tag{A.4.3}
$$

$$
\tilde{T}_{ij} = - p\tilde{g}_{ij} = -a^2 p h_{ij}. \tag{A.4.4}
$$

Compactly

$$
\tilde{T}_{\mu\nu} = a^2 \begin{pmatrix} \tilde{\rho} + 2\bar{\rho}A & -(\vec{q} + \bar{\rho}\vec{B}) \\ -(\vec{q} + \bar{\rho}\vec{B}) & -ph_{ij} \end{pmatrix}. \tag{A.4.5}
$$

We saw that linearized Einstein Equations are put in the form of wave equations if they are expressed with the matrix source $S_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$. Thus we calculate

$$
\begin{aligned}
\tilde{T} =&\tilde{g}^{\mu\nu}\bar{T}_{\mu\nu} + \bar{g}^{\mu\nu}\tilde{T}_{\mu\nu} = a^{-2}(-2A)a^2\bar{\rho} + \sum_j a^{-2}(-h_{jj})a^2 p + a^{-2}a^2(\tilde{\rho} + 2\bar{\rho}A) - \sum_j a^{-2}(-a^2 p h_{jj}) \\
=&\tilde{\rho},
\end{aligned} \tag{A.4.6}
$$

indeed $T = \rho + 3p$ and $\bar{T} = \bar{\rho} + 3p$. Substituting,

$$
\begin{aligned}
\tilde{S}_{\mu\nu} =&\tilde{T}_{\mu\nu} - \frac{1}{2}\bar{T}\tilde{g}_{\mu\nu} - \frac{1}{2}\tilde{T}\bar{g}_{\mu\nu} = \\
=&a^2 \begin{pmatrix} \tilde{\rho} + 2\bar{\rho}A & -(\vec{q} - \bar{\rho}\vec{B}) \\ -(\vec{q} - \bar{\rho}\vec{B}) & -ph_{ij} \end{pmatrix} - \frac{1}{2}(\bar{\rho} - 3p)a^2 \begin{pmatrix} 2A & -\vec{B} \\ -\vec{B} & h_{ij} \end{pmatrix} - \frac{1}{2}\tilde{\rho}a^2 \begin{pmatrix} 1 & \vec{0} \\ \vec{0} & -\delta_{ij} \end{pmatrix} = \\
=&\frac{a^2}{2} \begin{pmatrix} 2\tilde{\rho} - 4\bar{\rho}A - 2\bar{\rho}A + 6pA - \tilde{\rho} & -2\vec{q} - 2\bar{\rho}\vec{B} + \bar{\rho}\vec{B} - 3p\vec{B} \\ -2\vec{q} - 2\bar{\rho}\vec{B} + \bar{\rho}\vec{B} - 3p\vec{B} & 2ph_{ij} - \bar{\rho}h_{ij} + 3ph_{ij} + \tilde{\rho}\delta_{ij} \end{pmatrix} \\
=&\frac{a^2}{2} \begin{pmatrix} \tilde{\rho} + 2(\bar{\rho} + 3p)A & -2\vec{q} - (\bar{\rho} + 3p)\vec{B} \\ -2\vec{q} - (\bar{\rho} + 3p)\vec{B} & \tilde{\rho}\delta_{ij} + (p - \bar{\rho})h_{ij} \end{pmatrix}.
\end{aligned} \tag{A.4.7}
$$

Since the Einstein Equations holds for the background quantities, we can linearize them as $\tilde{R}_{\mu\nu} = 8\pi G\tilde{S}_{\mu\nu}$. Its $(\tau\tau)$ component is

$$\begin{aligned}\Box A - 2H\dot{A} - 4(\dot{H} + H^2)A =& 8\pi G\frac{a^2}{2}[\tilde{\rho} + 2(\bar{\rho} + 3p)A] = 4\pi Ga^2\tilde{\rho} + 2(-3\dot{H})A \Rightarrow \\ \Box A - 2H\dot{A} + 2(\dot{H} - 2H^2)A =& 4\pi Ga^2\tilde{\rho},\end{aligned} \tag{A.4.8}$$

where we exploited the Friedman Equations (2.1.5) as

$$4\pi Ga^2(\bar{\rho} + 3p) = \left(\frac{3}{2}H^2\right) + 3\left(-\dot{H} - \frac{1}{2}H^2\right) = -3\dot{H}.$$

We substitute the same inside the (τj) component

$$\begin{aligned}-\frac{1}{2}\Box B_j + H\dot{B}_j + 2(\dot{H} + H^2)B_j =& 8\pi G\frac{a^2}{2}[-2q_j - (\bar{\rho} + 3p)B_j] = -8\pi Ga^2 q_j - (-3\dot{H})B_j \Rightarrow \\ \Box\vec{B} - 2H\dot{\vec{B}} + 2(\dot{H} - 2H^2)\vec{B} =& 16\pi Ga^2\vec{q}.\end{aligned} \tag{A.4.9}$$

Exploiting analogously the Friedman Equations as

$$4\pi Ga^2(p - \bar{\rho}) = \left(-\dot{H} - \frac{1}{2}H^2\right) - \left(\frac{3}{2}H^2\right) = -\dot{H} - 2H^2,$$

we simplify also the (ij) component

$$\begin{aligned}\frac{1}{2}\Box h_{ij} - H\dot{h}_{ij} - (\dot{H} + 2H^2)h_{ij} - 2\dot{H}A\delta_{ij} =& 8\pi G\frac{a^2}{2}[\tilde{\rho}\delta_{ij} + (p - \bar{\rho})h_{ij}] = 4\pi Ga^2\tilde{\rho}\delta_{ij} + (-\dot{H} - 2H^2)h_{ij} \Rightarrow \\ \Box h_{ij} - 2H\dot{h}_{ij} =& 4(\dot{H}A + 2\pi Ga^2\tilde{\rho})\delta_{ij}.\end{aligned} \tag{A.4.10}$$

Appendix B

Green function for the constant coefficients case

Here we are going to derive the Green function for the PDE (2.3.1) with constant coefficients

$$\Box G + \mathcal{H}_0 \partial_\tau G + \mathcal{K}_0 G = \delta^{(3)}(\underline{x})\delta(\tau). \tag{B.0.1}$$

It is well known that solution for $\mathcal{H}_0 = \mathcal{K}_0 = 0$ is a pure shock

$$-\frac{1}{4\pi|\underline{x}|}\delta(\tau - |\underline{x}|).$$

We can expect a similar term for our Green function, but it cannot be the only one. Indeed, it is possible to set to zero the first order term with a rescaling

$$G := e^{\frac{1}{2}\mathcal{H}_0 \tau} f,$$

but the zeroth order terms remains for f unless $\mathcal{K}_0 = -\frac{1}{2}\mathcal{H}_0^2$, which would be just a particular case. The general Green function will have a shock term plus some other terms due to $\mathcal{H}_0, \mathcal{K}_0$.
We can nonetheless study the characteristic lines of the PDE (B.0.1), which come out to be light rays again, since the second order part is always a d'alembertian. Hence, any additional term in $G(\tau; \underline{x})$ can arise only at $|\underline{x}| < \tau$. It is nothing more the Causality Principle, which ensure us we are working in a relativistc frame. We can thus imagine the Green function as the usual shock term, which travels with the speed of light, and a new term due to the lower order parts, which travels slower than light, as a sort of "echo".

B.1 Reduction of dimensions

The first step to solve G is to write it as a function of only $r := |\underline{x}|; \tau$. Indeed, as the source $\mathcal{S}(\tau; \underline{x}) = \delta^{(3)}(\underline{x})\delta(\tau)$ is radially symmetric, so it must be $G(\tau; \underline{x})$.

It could be not so clear how to write the Dirac delta as function of r, so we consider it as the weak limit of mollificators

$$\mathcal{N}_\epsilon(x) := \frac{1}{\sqrt{\pi}\epsilon} e^{-x^2/\epsilon^2} \rightharpoonup^{\epsilon \to 0} \delta(x) \tag{B.1.1}$$

Then we have the mollified source

$$\mathcal{S}_\epsilon(\tau; \underline{x}) := \mathcal{N}_\epsilon(x)\mathcal{N}_\epsilon(y)\mathcal{N}_\epsilon(z)\delta(\tau) = \frac{1}{\pi^{3/2}\epsilon^3} e^{-r^2/\epsilon^2}\delta(\tau) \rightharpoonup^{\epsilon \to 0} \delta^{(3)}(\underline{x})\delta(\tau), \tag{B.1.2}$$

which generates a mollified Green function

$$G_\epsilon(\tau;\underline{x}) := u_\epsilon(\tau;|\underline{x}|) \rightharpoonup^{\epsilon\to 0} G(\tau;\underline{x}). \tag{B.1.3}$$

In terms of such $u_\epsilon(\tau;r)$, we have

$$\nabla^2 G_\epsilon = \frac{2}{r}u'_\epsilon + u''_\epsilon,$$

where the prime denotes derivation in r. We can define a further auxiliary variable so that

$$u_\epsilon := v_\epsilon/r \Rightarrow \nabla^2 G_\epsilon = v''_\epsilon/r. \tag{B.1.4}$$

Substituting inside (B.0.1), one finds

$$(\partial_r^2 - \partial_\tau^2)v_\epsilon + \mathcal{H}_0\dot{v}_\epsilon + \mathcal{K}_0 v_\epsilon = r\mathcal{S}_\epsilon = \frac{r}{\pi^{3/2}\epsilon^3}e^{-r^2/\epsilon^2}\delta(\tau) = -\frac{1}{2\pi}\mathcal{N}'_\epsilon(r)\delta(\tau). \tag{B.1.5}$$

Performing the weak limit in ϵ and rescaling of $-\frac{1}{2\pi}$, (B.0.1) is reduced to

$$G(\tau;\underline{x}) = -\frac{1}{2\pi|\underline{x}|}v(\tau;|\underline{x}|) \quad s.t. \quad \Box_{(\tau;r)}v + \mathcal{H}_0\dot{v} + \mathcal{K}_0 v = \delta'(r)\delta(\tau). \tag{B.1.6}$$

B.2 Discriminant and 2D Green function

The second step is to remove the first order term. As we said above, the suitable rescaling is

$$v(\tau;r) := e^{\frac{1}{2}\mathcal{H}_0\tau}f(\tau;r). \tag{B.2.1}$$

Indeed, the PDE in v becomes

$$\begin{aligned}\delta'(r)\delta(\tau) =& e^{\frac{1}{2}\mathcal{H}_0\tau}f'' - \left(e^{\frac{1}{2}\mathcal{H}_0\tau}\ddot{f} + 2\frac{1}{2}\mathcal{H}_0 e^{\frac{1}{2}\mathcal{H}_0\tau}\dot{f} + \frac{1}{4}\mathcal{H}_0^2 f\right) + \mathcal{H}_0\left(e^{\frac{1}{2}\mathcal{H}_0\tau}\dot{f} + \frac{1}{2}\mathcal{H}_0 e^{\frac{1}{2}\mathcal{H}_0\tau}f\right) + \\ &+ \mathcal{K}_0\left(e^{\frac{1}{2}\mathcal{H}_0\tau}\right) = e^{\frac{1}{2}\mathcal{H}_0\tau}\left[f'' - \ddot{f} + \left(\frac{1}{4}\mathcal{H}_0^2 + \mathcal{K}_0\right)f\right] \Rightarrow \\ f'' - \ddot{f} + \bar{\mathcal{K}}f =& e^{-\frac{1}{2}\mathcal{H}_0\tau}\delta'(r)\delta(\tau) = \delta'(r)\delta(\tau),\end{aligned} \tag{B.2.2}$$

where is was defined the "discriminant" $\bar{\mathcal{K}} := \frac{1}{4}\mathcal{H}_0^2 + \mathcal{K}_0$.

The third step is to express such a f with a convolution. We need the Green function of the 2D differential operator $\partial_r^2 - \partial_\tau^2 - \bar{\mathcal{K}}$, that we call $\Gamma(\tau;r)$. One has

$$f(\tau;r) = \Gamma(\tau;r) * \delta'(r) = \Gamma'(\tau;r) * \delta(r) = \Gamma'(\tau;r), \tag{B.2.3}$$

where we exploited the properties of convolution.
Substituting in (B.1.6), the problem is reduced to

$$G(\tau;\underline{x}) = -\frac{e^{\frac{1}{2}\mathcal{H}_0\tau}}{2\pi|\underline{x}|}\Gamma'(\tau;|\underline{x}|) \quad s.t. \quad \Gamma'' - \ddot{\Gamma} - \bar{\mathcal{K}}\Gamma = \delta(r)\delta(\tau). \tag{B.2.4}$$

B.3 Fourier transforms

As a fourth step, we find an expression for Γ with the Fourier transform. Following the usual method for the resolution of PDEs

$$
\begin{aligned}
&\Gamma(\tau; r) := \int \hat{\Gamma}(\omega; k) e^{i(\omega\tau - kr)} d\omega dk \Rightarrow \\
&- k^2\hat{\Gamma} - \omega^2\hat{\Gamma} - \bar{\mathcal{K}}\hat{\Gamma} = 1 \Rightarrow \hat{\Gamma} = \frac{1}{\omega^2 - k^2 - \bar{\mathcal{K}}} \Rightarrow \\
&\Gamma(\tau; r) = \int \frac{e^{i(\omega\tau - kr)}}{\omega^2 - k^2 - \bar{\mathcal{K}}} d\omega dk. \qquad \text{(B.3.1)}
\end{aligned}
$$

A possible way to solve such a double integral requires to recall the Green function for the d'alambertian in 2+1 dimensions, with variables t, x, y. Analogously to the passages above, we can express it in Fourier transform.

$$
\int \frac{e^{i(k_0 t - k_1 x - k_2 y)}}{k_0^2 - k_1^2 - k_2^2} dk_0 dk_1 dk_2 = \frac{1}{2\pi\sqrt{t^2 - x^2 - y^2}} \theta(|t| - \sqrt{x^2 + y^2}),
$$

where θ is the Heaviside function.
We can reach the integral for Γ choosing $k_0 := \omega$ and $k_1 := k$. We find

$$
\frac{1}{2\pi\sqrt{\tau^2 - r^2 - y^2}} \theta(|\tau| - \sqrt{r^2 + y^2}) = \int \frac{e^{i(\omega\tau - kr - k_2 y)}}{\omega^2 - k^2 - k_2^2} dk\omega dk dk_2 = \int \Phi(\tau; r)|_{\bar{\mathcal{K}} = k_2^2} e^{-ik_2 y} dk_2 \Rightarrow
$$

$$
\begin{aligned}
\Gamma(\tau; r) &= \int \frac{1}{2\pi\sqrt{\tau^2 - r^2 - y^2}} \theta(|\tau| - \sqrt{r^2 + y^2}) e^{ik_2 y} dy|_{\bar{\mathcal{K}} = k_2^2} = \\
&= \frac{1}{2\pi} \int \frac{1}{\sqrt{\tau^2 - r^2 - y^2}} \theta(|\tau| - |r|)\theta(\sqrt{\tau^2 - r^2} - |y|) e^{i\bar{\mathcal{K}}^{1/2} y} dy = \\
&= \frac{1}{2\pi} \theta(|\tau| - |r|) \int_{-1}^{1} \frac{1}{\sqrt{1 - \xi^2}} e^{i\bar{\mathcal{K}}^{1/2}\sqrt{\tau^2 - r^2}\xi} d\xi. \qquad \text{(B.3.2)}
\end{aligned}
$$

At the second line it was exploited the inverse of the Fourier transform; at the third line we used the equivalence $|\tau| \geq \sqrt{r^2 + y^2} \Leftrightarrow |y| \leq \sqrt{\tau^2 - r^2}$; and finally we changed the variables as $y := \sqrt{\tau^2 - r^2}\xi$.

B.4 Bessel functions

The fifth and final step requires to solve the integral from -1 to 1. We define it as a function of its parameter

$$
j(x) := \int_{-1}^{1} \frac{e^{ix\xi}}{\sqrt{1 - \xi^2}} d\xi.
$$

It is not an analytic function, but we can characterize it as it solves some particular differential equation. The right combination of its derivatives is

$$
\begin{aligned}
j'' + j'/x + j &= \int_{-1}^{1} \frac{-\xi^2 + i\xi/x + 1}{\sqrt{1 - \xi^2}} e^{ix\xi} d\xi = \int_{-1}^{1} \frac{1 - \xi^2}{\sqrt{1 - \xi^2}} e^{ix\xi} d\xi + \frac{i}{x} \int_{-1}^{1} \frac{\xi}{\sqrt{1 - \xi^2}} e^{ix\xi} d\xi = \\
&= \int_{-1}^{1} \sqrt{1 - \xi^2} e^{ix\xi} d\xi + \frac{i}{x} [-\sqrt{1 - \xi^2} e^{ix\xi}]_{-1}^{1} - \frac{i}{x} \int_{-1}^{1} -\sqrt{1 - \xi^2} ix e^{ix\xi} d\xi = \\
&= \int_{-1}^{1} \sqrt{1 - \xi^2} e^{ix\xi} d\xi - \int_{-1}^{1} \sqrt{1 - \xi^2} e^{ix\xi} d\xi = 0. \qquad \text{(B.4.1)}
\end{aligned}
$$

This is a particular case of the Bessel Equation

$$x^2 j'' + x j' + (x^2 - \alpha^2) j = 0,$$

which is solved by the Bessel functions of first $J_\alpha(x)$ and second type $Y_\alpha(x)$. The integral $j(x)$ must be some combination of J_0 and Y_0.

We can get the right combination of Bessel functions evaluating

$$j(0) = \int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} = [\sin^{-1}\xi]_{-1}^{1} = \pi. \tag{B.4.2}$$

Since Y_0 diverges at zero and $J_0(0) = 1$, we know

$$j(x) = \pi J_0(x). \tag{B.4.3}$$

Substituting in (B.3.2) we have the 2D Green function

$$\Gamma(\tau; r) = \frac{1}{2\pi}\theta(|\tau| - |r|) J_0\left(\sqrt{\bar{\mathcal{K}}(\tau^2 - r^2)}\right). \tag{B.4.4}$$

From its derivative, the shock term and the echo term emerges.

$$\begin{aligned}\Gamma'(\tau; r) &= -\frac{1}{2}\delta(|\tau| - r) J_0\left(\sqrt{\bar{\mathcal{K}}(\tau^2 - r^2)}\right) + \frac{1}{2}\theta(|\tau| - |r|)\frac{2\bar{\mathcal{K}} r}{2\sqrt{\bar{\mathcal{K}}(\tau^2 - r^2)}} J_0'\left(\sqrt{\bar{\mathcal{K}}(\tau^2 - r^2)}\right) = \\ &= \frac{1}{2}\left[-\delta(|\tau| - r) + 2\sqrt{\frac{\bar{\mathcal{K}}}{\tau^2 - r^2}} J_0'\left(\sqrt{\bar{\mathcal{K}}(\tau^2 - r^2)}\right)\theta(|\tau| - |r|)\right].\end{aligned} \tag{B.4.5}$$

Applying it to (B.2.4) we have the Green function as in (2.4.2).

Appendix C

Growing rate of density contrast for constant Hubble parameter

We consider a background expansion with constant parameter, hence dominated by a $w = -1/3$ exotic energy. We assume a density contrast with separable variable and such that

$$\tilde{\rho}a^3 \propto \tilde{\rho}/\bar{\rho}_M = \delta_M(\tau;\underline{x}) \propto a(\tau)^n.$$

Observing the linearized Einstein Equation (2.2.13) and Conservation Laws (2.2.14) we note that, for $w = -1/3$, the three variables $C, \tilde{\rho}, q$ are the only ones that appear in the equations

$$\begin{cases} \Box C - 2H_0\dot{C} = 4\pi G a^2\tilde{\rho} \\ 0 = \dot{\tilde{\rho}} + 3H_0\tilde{\rho} + \nabla^2 q - 2\bar{\rho}\dot{C} \\ 0 = \dot{q} + 4H_0 q + \frac{2}{3}\bar{\rho}C \end{cases} . \tag{C.0.1}$$

These are the wave equation for C, the Conservation of Energy and the Conservation of Momentum. Indeed, $\dot{H} = 0$ make A vanish from the C's wave equation; the factors were evaluated as $\bar{\rho} + p = \bar{\rho}_{-1/3} - \frac{1}{3}\bar{\rho}_{-1/3} = \frac{2}{3}\bar{\rho}$; and we used

$$\begin{aligned} &\bar{\rho}_w = \bar{\rho}_{w0}a^{-3(1+w)} = |_{w=-1/3}\frac{3H_0^2}{8\pi G}a^{-2} \Rightarrow \dot{\bar{\rho}} = \frac{3H_0^2}{8\pi G}(-2\dot{a})a^{-3} = -2H\bar{\rho} \Rightarrow \\ &(\dot{\bar{\rho}} + \dot{p}) + 2H(\bar{\rho} + p) = \frac{2}{3}\dot{\bar{\rho}} + 2H\left(\frac{2}{3}\bar{\rho}\right) = \frac{2}{3}(-2H\bar{\rho} + 2H\bar{\rho}) = 0, \end{aligned}$$

making B vanish from the Conservation of Momentum.

With suitable substitutions, we can reach a differential equation for the only $\tilde{\rho}$, which allows to characterize its evolution in time. Deriving the Conservation of Energy,

$$\begin{aligned} 0 =& \ddot{\tilde{\rho}} + 3H_0\dot{\tilde{\rho}} + \nabla^2\dot{q} - 2\frac{3H_0^2}{8\pi G}(-2H_0a^{-2}\dot{C} + a^{-2}\ddot{C}) = \\ =& \ddot{\tilde{\rho}} + 4H_0\dot{\tilde{\rho}} + 3H_0\dot{\tilde{\rho}} + 12H_0^2\tilde{\rho} + \nabla^2(\dot{q} + 4H_0q) + \frac{3H_0^2}{8\pi G}a^{-2}(2H_0\dot{C} - \ddot{C} - 4H_0\dot{C}) = \\ =& \ddot{\tilde{\rho}} + 7H_0\dot{\tilde{\rho}} + 12H_0^2\tilde{\rho} + \nabla^2\left(-\frac{H_0^2}{8\pi G}a^{-2}C\right) - \frac{3H_0^2}{8\pi G}a^{-2}(\ddot{C} + 2H_0\dot{C}), \end{aligned}$$

where it is used again the Conservation of Energy in the first passage, and the Conservation of Momentum in the second one.

We managed to remove q. Substituting the C's wave equation,

$$\begin{cases}\nabla^2 C + 3\ddot{C} + 6H_0\dot{C} = \frac{4\pi G}{H_0^2}a^2(\ddot{\tilde{\rho}} + 7H_0\dot{\tilde{\rho}} + 12H_0^2\tilde{\rho}) \\ \nabla^2 C - \ddot{C} - 2H_0\dot{C} = 4\pi G a^2\tilde{\rho}\end{cases} \Rightarrow$$

$$4(\ddot{C} + 2H_0\dot{C}) = 4\frac{\pi G}{H_0^2}a^2(\ddot{\tilde{\rho}} + 7H_0\dot{\tilde{\rho}} + 11H_0^2\tilde{\rho}),$$

so we remove also $\nabla^2 C$, and only time derivatives remain.
Now we need to remove finally C, and we do that applying to C both the differential operators of the last equation and of the C's wave equation.

$$H_0^2(\partial_\tau^2 + 2H_0\partial_\tau)4\pi G\tilde{\rho} = H_0^2(\partial_\tau^2 + 2H_0\partial_\tau)(\nabla^2 - \partial_\tau^2 - 2H_0\partial_\tau)C =$$
$$= (\nabla^2 - \partial_\tau^2 - 2H_0\partial_\tau)\pi G a^2(\partial_\tau^2 + 7H_0\partial_\tau + 11H_0^2)\tilde{\rho} \Rightarrow$$
$$4H_0^2 a^2(4H_0^2\tilde{\rho} + 4H_0\dot{\tilde{\rho}} + \ddot{\tilde{\rho}} + 2H_0 2H_0\tilde{\rho} + 2H_0\dot{\tilde{\rho}}) =$$
$$= a^2(\nabla^2 - \partial_\tau^2 - 4H_0\partial_\tau - 4H_0^2 - 2H_0\partial_\tau - 4H_0^2)(\partial_\tau^2 + 7H_0\partial_\tau + 11H_0^2)\tilde{\rho} \Rightarrow$$
$$4H_0^2\ddot{\tilde{\rho}} + 24H_0^3\dot{\tilde{\rho}} + 32H_0^4\tilde{\rho} = \nabla^2(\ddot{\tilde{\rho}} + 7H_0\dot{\tilde{\rho}} + 11H_0^2\tilde{\rho}) - (\ddddot{\tilde{\rho}} + 13H_0\dddot{\tilde{\rho}} + 61H_0^2\ddot{\tilde{\rho}} + 122H_0^3\dot{\tilde{\rho}} + 88H_0^4\tilde{\rho}) \Rightarrow$$
$$\nabla^2 H_0^2 P_2(H_0^{-1}\partial_\tau)\tilde{\rho} = H_0^4 P_4(H_0^{-1}\partial_\tau)\tilde{\rho}, \tag{C.0.2}$$

where are defined the suitable polynomials

$$P_2(x) := x^2 + 7x + 11 \tag{C.0.3}$$
$$P_4(x) := x^4 + 13x^3 + 65x^2 + 146x + 120 = (x+2)(x+4)(x^2 + 7x + 15) \tag{C.0.4}$$

to express compactly the fourth order differential operators.

According to our assumptions on separation of variables and growing rate, we have to substitute inside (C.0.2) a source

$$\tilde{\rho}(\tau; \underline{x}) = X(H_0\underline{x})T(H_0\tau), \tag{C.0.5}$$

with an unknown spatial factor X which can be fixed by the actual matter inhomogeneity, and the time factor is

$$T(H_0\tau) = a^n a^{-3} = a(\tau)^{n-3} = e^{(n-3)H_0\tau}. \tag{C.0.6}$$

The substitution returns

$$(\nabla^2 X)P_2(n-3)T = XP_4(n-3)T \Rightarrow \nabla^2 X = \frac{p_4(n)}{p_2(n)}X, \tag{C.0.7}$$

where

$$p_2(x) := P_2(x-3) = x^2 + x - 1, \tag{C.0.8}$$
$$p_4(x) := P_4(x-3) = (x-1)(x+1)(x^2 + x + 3). \tag{C.0.9}$$

In other words, X is a eigenfunction for the laplacian, with eigenvalue $p_4(n)/p_2(n)$.

We recall that form of X is unknown, so we can not know what is its eigenvalue. However, it must be a negative numbers, for the properties of the laplacian. This provides an inequality for n, easy to solve

$$\frac{(n-1)(n+1)(n^2 + n + 3)}{n^2 + n - 1} < 0 \Rightarrow -\Phi < n < -1 \vee \phi < n < 1, \tag{C.0.10}$$

where $-\Phi = \frac{-\sqrt{5}-1}{2}, \phi = \frac{\sqrt{5}-1}{2} = \Phi^{-1}$ are the roots of the polynomial $p_2(x) = x^2 + x - 1$. $\phi \cong 0.61803398875\dots$ is the "golden ratio", a very important number for theoretical and applied mathematics.

Appendix D

About $\mathcal{N}$ and $\mathcal{M}$ integrals

D.1 Derivation of the integrals

The integrals introduced in §2.4.2 comes from the averaging formulas of §2.3.1, applied to a source as in §2.4.1 and with the Green function 2.4.2.
The separation of variables for the source $\mathcal{S}(\tau;\underline{x}) = \mathcal{S}_0(\underline{x})T(\tau)$ is guaranteed by (2.4.4), for which

$$\begin{aligned}&\mathcal{S}(\tau;\underline{x}) = 4\pi G a(\tau)^2\tilde{\rho}(\tau;\underline{x}) = 4\pi G\tilde{\rho}_0(\underline{x})a(\tau)^{n-1} \Rightarrow\\ &\mathcal{S}_0 := \tilde{\rho}_0, \quad T := 4\pi G a^{n-1}.\end{aligned} \tag{D.1.1}$$

This is the source for both A and C.

Now we apply Proposition 2.3.1 to A and C. The auxiliary variable f, as is defined in (2.3.2), is obtained for A substituting (2.4.2) with $\bar{\mathcal{K}} = 3H_0^2$.

$$\begin{aligned}
f_A(\tau;r) =& \int_{\tau_I}^{\tau} G_A(\tau-\tau_0;r)T(\tau_0)d\tau_0 =\\
=& \int_{-\infty}^{\tau} \frac{e^{-H_0(\tau-\tau_0)}}{4\pi}\left[-\frac{\delta(\tau-\tau_0-r)}{r} + \frac{\sqrt{3}H_0}{\sqrt{(\tau-\tau_0)^2-r^2}}J_0'(\sqrt{3}H_0\sqrt{(\tau-\tau_0)^2-r^2})\theta(\tau-\tau_0-r)\right]\times\\
&\times 4\pi G e^{(n-1)H_0\tau_0}d\tau_0 =\\
=& G e^{-H_0\tau}\left[-\frac{e^{nH_0(\tau-r)}}{r} + \sqrt{3}H_0\int_{-\infty}^{\tau-r} e^{nH_0\tau_0}\frac{J_0'(\sqrt{3}H_0\sqrt{(\tau-\tau_0)^2-r^2})}{\sqrt{(\tau-\tau_0)^2-r^2}}d\tau_0\right] =\\
=& G e^{-H_0\tau}\left[-\frac{e^{nH_0(\tau-r)}}{r} + \sqrt{3}H_0\int_0^{\infty} e^{nH_0(\tau-\sigma-r)}\frac{J_0'(\sqrt{3}H_0\sqrt{(\sigma+r)^2-r^2})}{\sqrt{(\sigma+r)^2-r^2}}d\sigma\right]\\
=& G a(\tau)^{n-1}\left[\sqrt{3}H_0\int_0^{\infty} e^{-nH_0(\sigma+r)}\frac{J_0'(\sqrt{3}H_0\sqrt{\sigma(\sigma+2r)})}{\sqrt{\sigma(\sigma+2r)}}d\sigma - \frac{e^{-nH_0r}}{r}\right].
\end{aligned} \tag{D.1.2}$$

We performed the change of variable $\tau_0 := \tau - \sigma - r$, and remember that $R(\tau) \equiv -\tau_I = \infty$ for the solution with $w = -1/3$.

Thus, for Proposition 2.3.1 we have

$$\begin{aligned}
\langle A\rangle(\tau) =&4\pi\langle\mathcal{S}_0\rangle\int_0^\infty f_A(\tau;r)r^2dr =\\
=&4\pi\langle\tilde\rho_0\rangle\int_0^\infty Ga^{n-1}\left[\sqrt{3}H_0\int_0^\infty e^{-nH_0(\sigma+r)}\frac{J_0'(\sqrt{3}H_0\sqrt{\sigma(\sigma+2r)})}{\sqrt{\sigma(\sigma+2r)}}d\sigma-\frac{e^{-nH_0r}}{r}\right]r^2dr =\\
=&4\pi G\langle\tilde\rho_0\rangle a^{n-1}\left[\sqrt{3}H_0\int_0^\infty\int_0^\infty e^{-nH_0(\sigma+r)}\frac{J_0'(\sqrt{3}H_0\sqrt{\sigma(\sigma+2r)})}{\sqrt{\sigma(\sigma+2r)}}r^2d\sigma dr-\int_0^\infty e^{-nH_0r}rdr\right] =\\
=&4\pi G\langle\tilde\rho_0\rangle a^{n-1}\left[\sqrt{3}H_0\int_0^\infty\int_0^\infty e^{-n/\sqrt{3}(x+y)}\frac{J_0'(\sqrt{y(y+2x)})}{(\sqrt{3}H_0)^{-1}\sqrt{y(y+2x)}}(\sqrt{3}H_0)^{-4}x^2dydx-\frac{1}{n^2H_0^2}\right]\\
=&4\pi\left(\frac{1}{3}\mathcal{N}(n/\sqrt{3})-\frac{1}{n^2}\right)\frac{G\langle\tilde\rho_0\rangle}{H_0^2}a(\tau)^{n-1},
\end{aligned}\tag{D.1.3}$$

where we see it is defined the integral

$$\mathcal{N}(n):=\int_0^\infty\int_0^\infty e^{-n(x+y)}\frac{J_0'(\sqrt{y(y+2x)})}{\sqrt{y(y+2x)}}x^2dydx.\tag{D.1.4}$$

The two integration variables were both rescaled of $\sqrt{3}H_0$, in order to get an adimensional quantity.
With a perfectly similar calculation one find also

$$\langle C\rangle(\tau)=4\pi\left(\mathcal{M}(n)-\frac{1}{n^2}\right)\frac{G\langle\tilde\rho_0\rangle}{H_0^2}a(\tau)^{n-1},\quad s.t.\tag{D.1.5}$$

$$\mathcal{M}(n):=\int_0^\infty\int_0^\infty e^{-n(x+y)}\frac{I_0'(\sqrt{y(y+2x)})}{\sqrt{y(y+2x)}}x^2dydx.\tag{D.1.6}$$

D.2 Calculation of the integrals

We observe that the $\mathcal{M}$ integral has the same form of $\mathcal{N}$, with the exception of the modified Bessel function I_0 instead of J_0. From now on, all the manipulations on $\mathcal{N}$ can be reproduced on $\mathcal{M}$ with this little variation.

To solve (D.1.4), the first step will be the change of variables $(x;y)\to(x;s:=x+y)$.

$$\begin{aligned}
\mathcal{N}(n)=&\int_0^\infty\int_0^\infty e^{-n(x+y)}\frac{J_0'(\sqrt{y(y+2x)})}{\sqrt{y(y+2x)}}x^2dydx=\int_0^\infty x^2\left[\int_x^\infty\frac{J_0'(\sqrt{s^2-x^2})}{\sqrt{s^2-x^2}}e^{-ns}ds\right]dx=\\
=&\int_0^\infty e^{-ns}\left[\int_0^s\frac{J_0'(\sqrt{s^2-x^2})}{\sqrt{s^2-x^2}}x^2dx\right]ds;
\end{aligned}\tag{D.2.1}$$

the two-dimensional integration dominion is $0<x<s<\infty$.

The internal integral can be evaluated as

$$\frac{J_0'(\sqrt{s^2-x^2})}{\sqrt{s^2-x^2}}x^2 = \frac{dJ_0(\sqrt{s^2-x^2})}{d\sqrt{s^2-x^2}}\frac{x^2}{\sqrt{s^2-x^2}} = \frac{dJ_0(\sqrt{s^2-x^2})}{dx}\left(\frac{d\sqrt{s^2-x^2}}{dx}\right)^{-1}\frac{x^2}{\sqrt{s^2-x^2}} =$$
$$= \partial_x J_0(\sqrt{s^2-x^2})\left(\frac{-x}{\sqrt{s^2-x^2}}\right)^{-1}\frac{x^2}{\sqrt{s^2-x^2}} = -\partial_x J_0(\sqrt{s^2-x^2})x \Rightarrow$$
$$\int_0^s \frac{J_0'(\sqrt{s^2-x^2})}{\sqrt{s^2-x^2}}x^2 dx = -\int_0^s x\partial_x J_0(\sqrt{s^2-x^2})dx = -[xJ_0(\sqrt{s^2-x^2})]_0^s + \int_0^s J_0(\sqrt{s^2-x^2})dx =$$
$$= \int_0^s J_0(\sqrt{s^2-x^2})dx - s.$$

With this second step, the integral becomes

$$\mathcal{N}(n) = \int_0^\infty e^{-ns}\left[\int_0^s J_0(\sqrt{s^2-x^2})dx - s\right]ds = \int_0^\infty\int_0^s J_0(\sqrt{s^2-x^2})e^{-ns}dxds - \int_0^\infty se^{-ns}ds =$$
$$:= -\frac{1}{n^2} + \int_0^\infty \mathcal{N}(n;s)e^{-ns}ds. \tag{D.2.2}$$

Following the same passages, one finds

$$\mathcal{M}(n) = -\frac{1}{n^2} + \int_0^\infty \mathcal{M}(n;s)e^{-ns}ds \quad s.t. \quad \mathcal{M}(n;s) := \int_0^s I_0(\sqrt{s^2-x^2})dx. \tag{D.2.3}$$

The evaluation of such $\mathcal{N}(n;s)$ is matter for the third passage, where we exploit the Taylor expansions

$$J_0(x) = \sum_{j=0}^\infty \frac{(-1)^j}{(j!)^2}(x/2)^{2j} = \sum_{j=0}^\infty (-)^j \frac{1}{4^j (j!)^2}(x^2)^j, \quad I_0(x) = \sum_{j=0}^\infty \frac{1}{4^j (j!)^2}(x^2)^j$$
$$(s^2-x^2)^j = \sum_{k=0}^j \binom{k}{j}(s^2)^{j-k}(-x^2)^k = \sum_{k=0}^j (-)^k \frac{j!}{k!(j-k)!}x^{2k}s^{2(j-k)}.$$

Since the difference between $\mathcal{N}$ and $\mathcal{N}$ is only for a sign, we will call from now on

$$\mathcal{N}_+ := \mathcal{M}, \quad \mathcal{N}_- := \mathcal{N}. \tag{D.2.4}$$

Thus we can write at once

$$\mathcal{N}_\pm(n;s) = \int_0^s \left[\sum_{j=0}^\infty (\pm)^j \frac{1}{4^j (j!)^2}(s^2-x^2)^j\right]dx = \int_0^s \left[\sum_{j=0}^\infty (\pm)^j \frac{1}{4^j (j!)^2}\sum_{k=0}^j (-)^k \frac{j!}{k!(j-k)!}x^{2k}s^{2(j-k)}\right]dx =$$
$$= \sum_{j=0}^\infty (\pm)^j \frac{1}{4^j j!}\sum_{k=0}^j \frac{(-)^k}{k!(j-k)!}\left(\int_0^s x^{2k}dx\right)s^{2(j-k)} = \sum_{j=0}^\infty (\pm)^j \frac{1}{4^j j!}\left[\sum_{k=0}^j \frac{(-)^k}{k!(j-k)!(2k+1)}\right]s^{2j+1}. \tag{D.2.5}$$

It is known that

$$\sum_{k=0}^j \frac{(-)^k}{k!(j-k)!(2k+1)} = \sqrt{\pi}/2\frac{1}{(j+1/2)!},$$

and it holds for the properties of Gamma function

$$(j+1/2)! = (j+1/2)(j-1/2)! = \frac{(2j+1)!!}{2^j}(j-j+1/2)! = \sqrt{\pi}/2\frac{(2j+1)!!}{2^j};$$

Thus the internal sum is just $\frac{2^j}{(2j+1)!!}$ and

$$\mathcal{N}_{\pm}(n;s) = \sum_{j=0}^{\infty} \frac{(\pm)^j}{2^j j!(2j+1)!!} s^{2j+1}. \tag{D.2.6}$$

We reach the fourth and last step of the calculation, when we integrate (D.2.6) against the exponential, as is prescribed by (D.2.2) and (D.2.3).

$$\begin{aligned}
\mathcal{N}_{\pm}(n) + \frac{1}{n^2} &= \int_0^{\infty} \mathcal{N}_{\pm}(n;s) e^{-ns} ds = \sum_{j=0}^{\infty} \frac{(\pm)^j}{2^j j!(2j+1)!!} \left(\int_0^{\infty} s^{2j+1} e^{-ns} ds \right) = \\
&= \sum_{j=0}^{\infty} \frac{(\pm)^j}{2^j j!(2j+1)!!} (2j+1)! n^{-2j-2} = \frac{1}{n^2} \sum_{j=0}^{\infty} (\pm)^j \frac{(2j)!!}{2^j j!} n^{-2j} = \frac{1}{n^2} \sum_{j=0}^{\infty} (\pm n^{-2})^j \Rightarrow \\
\mathcal{N}_{\pm}(n) &= \frac{1}{n^2} \frac{1}{1-(\pm n^2)} - \frac{1}{n^2} = \frac{1}{n^2 \mp 1} - \frac{1}{n^2}. \qquad \text{(D.2.7)}
\end{aligned}$$

These are the formulas we used in §2.4.2.

Appendix E

ODEs for the averaged metric components

Here we study the quantities

$$
\begin{aligned}
u_A(\tau) &= \int_{\tau_I}^{\tau} \left[\int_{|\underline{r}<R(\tau)|} G_{\tau'}(\underline{r};\tau) d^3\underline{r} \right] a(\tau')^2 T(\tau') d\tau', \\
u_{AC}(\tau) &= \int_{\tau_I}^{\tau} \left[\int_{|\underline{r}<R(\tau)|} G^C_{\tau'}(\underline{r};\tau) d^3\underline{r} \right] \dot{H}(\tau') u_A(\tau') d\tau', \\
u_C(\tau) &= \int_{\tau_I}^{\tau} \left[\int_{|\underline{r}<R(\tau)|} G^C_{\tau'}(\underline{r};\tau) d^3\underline{r} \right] a(\tau')^2 T(\tau') d\tau', \\
u_B(\tau) &= a(\tau)^{-2} \int_{\tau_I}^{\tau} a(\tau')^2 \left(2u_{AC}(\tau') + u_C(\tau') - u_A(\tau') \right) d\tau';
\end{aligned}
$$

expressed with the Green functions for non constant parameters

$$
\begin{aligned}
\left(\Box - 2H\partial_\tau + 2(\dot{H} - 2H^2) \right) G_{\tau'}(\underline{x};\tau) &= \delta^{(3)}(\underline{x})\delta(\tau - \tau'), \\
\left(\Box - 2H\partial_\tau \right) G^C_{\tau'}(\underline{x};\tau) &= \delta^{(3)}(\underline{x})\delta(\tau - \tau').
\end{aligned}
$$

We seek for simple ODEs which regulate such u functions.

E.1 Reduction of the dimensions

Since the Green functions are symmetric under spatial rotation, we can reduce the spatial dimensions to one, as we did already in the constant coefficients case (cfr. §B.1)

$$
\begin{aligned}
&G_{\tau'}(\underline{r};\tau) = -\frac{1}{2\pi|\underline{r}|}\partial_r \Gamma_{\tau'}(|\underline{r}|;\tau) \quad s.t. \\
&\left(\partial_r^2 - \partial_\tau^2 - 2H\partial_\tau + 2(\dot{H} - 2H^2) \right) \Gamma_{\tau'}(r;\tau) = \delta(r)\delta(\tau - \tau'); \qquad \text{(E.1.1)}
\end{aligned}
$$

and the same for $G^C_{\tau'}(\underline{r};\tau)$. This allow us to express in another way the terms as

$$\begin{aligned}\int_{|\underline{r}<R(\tau)|} G_{\tau'}(\underline{r};\tau)d^3\underline{r} &= \int_0^{R(\tau)} \left[-\frac{1}{2\pi r}\partial_r\Gamma_{\tau'}(r;\tau)\right] 4\pi r^2 dr \\ &= -2\int_0^{R(\tau)} r\partial_r\Gamma_{\tau'}(r;\tau)dr \\ &= 2\left(\int_0^{R(\tau)} \Gamma_{\tau'}(r;\tau)dr - [r\Gamma_{\tau'}(r;\tau)]_{r=0}^{R(\tau)}\right).\end{aligned}$$

Now let us define the auxiliary field

$$v_A(r;\tau) := \int_{\tau_I}^{\tau} \Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')d\tau'; \tag{E.1.2}$$

and similar for v_{AC} and v_C. Then, we can prove

Lemma E.1.1. *The metric perturbations evolve as*

$$\begin{aligned}u_A(\tau) &= \int_{-R(\tau)}^{R(\tau)} v_A(r;\tau)dr - 2[rv_A(r;\tau)]_{r=0}^{R(\tau)}, \\ u_{AC}(\tau) &= \int_{-R(\tau)}^{R(\tau)} v_{AC}(r;\tau)dr - 2[rv_{AC}(r;\tau)]_{r=0}^{R(\tau)}, \\ u_C(\tau) &= \int_{-R(\tau)}^{R(\tau)} v_C(r;\tau)dr - 2[rv_C(r;\tau)]_{r=0}^{R(\tau)};\end{aligned} \tag{E.1.3}$$

where the v fields solve the 2D PDEs

$$\begin{aligned}\left(\partial_r^2 - \partial_\tau^2 - 2H\partial_\tau + 2(\dot{H} - 2H^2)\right) v_A(r;\tau) &= \delta(r)a(\tau)^2T(\tau), \\ \left(\partial_r^2 - \partial_\tau^2 - 2H\partial_\tau\right) v_{AC}(r;\tau) &= \delta(r)\dot{H}(\tau)u_A(\tau), \\ \left(\partial_r^2 - \partial_\tau^2 - 2H\partial_\tau\right) v_C(r;\tau) &= \delta(r)a(\tau)^2T(\tau).\end{aligned} \tag{E.1.4}$$

Proof. Let us start from the time derivatives of v_A.

$$\begin{aligned}\dot{v}_A(r;\tau) &= [\Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')]|_{\tau'=\tau} + \int_{\tau'}^{\tau} \partial_\tau\Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')d\tau', \\ \ddot{v}_A(r;\tau) &= \partial_\tau[\Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')]|_{\tau'=\tau} + [\partial_\tau\Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')]|_{\tau'=\tau} \\ &\quad + \int_{\tau'}^{\tau} \partial_\tau^2\Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')d\tau'.\end{aligned}$$

$\Gamma_{\tau'}(r;\tau)$ satisfies a wave equation, so it holds a causality principle

$$\forall|r| > \tau - \tau' : \Gamma_{\tau'}(r;\tau) \equiv 0.$$

Setting $\tau' = \tau$ it becomes

$$\forall|r| > 0 : \Gamma_\tau(r;\tau) \equiv 0.$$

Continuity in $r = 0$ requires

$$[\Gamma_{\tau'}(r;\tau)a(\tau')^2T(\tau')]|_{\tau'=\tau} = \Gamma_\tau(r;\tau)a(\tau)^2T(\tau) \equiv 0;$$

so that the boundary terms of $\dot{v}_A$ and $\ddot{v}_A$ must vanish. Now, we can check by substitution

$$\begin{aligned}
&\partial_r^2 v_A - \ddot{v}_A - 2H\dot{v}_A + 2(\dot{H} - 2H^2)v_A \\
&\qquad = \int_{\tau_I}^{\tau} [\partial_r^2 \Gamma_{\tau'} - \partial_\tau^2 \Gamma_{\tau'} - 2H\partial_\tau \Gamma_{\tau'} + 2(\dot{H} - 2H^2)\Gamma_{\tau'}] a(\tau')^2 T(\tau') d\tau' \\
&\qquad = \int_{\tau_I}^{\tau} \delta(r)\delta(\tau - \tau') a(\tau')^2 T(\tau') d\tau' = \delta(r) a(\tau)^2 T(\tau).
\end{aligned}$$

Notice that this PDE is symmetric under $r \to -r$, so that $v_A(r;\tau) = v_A(-r;\tau)$. This allows us to write

$$\begin{aligned}
u_A(\tau) &= \int_{\tau_I}^{\tau} \left[\int_{|\underline{r} < R(\tau)|} G_{\tau'}(\underline{r};\tau) d^3\underline{r} \right] a(\tau')^2 T(\tau') d\tau' \\
&= 2\int_{\tau_I}^{\tau} \left(\int_0^{R(\tau)} \Gamma_{\tau'}(r;\tau) dr - [r\Gamma_{\tau'}(r;\tau)]_{r=0}^{R(\tau)} \right) a(\tau')^2 T(\tau') d\tau' \\
&= 2\left(\int_0^{R(\tau)} v_A(r;\tau) dr - [r v_A(r;\tau)]_{r=0}^{R(\tau)} \right) \\
&= \int_{-R(\tau)}^{R(\tau)} v_A(r;\tau) dr - 2[r v_A(r;\tau)]_{r=0}^{R(\tau)}.
\end{aligned}$$

The proof is analogous for v_{AC} and v_C. □

We will not try to write explicitly these $\Gamma_{\tau'}(r;\tau)$ or $v(r;\tau)$. Instead, we will exploit their properties to get simple characterizations for their averages.

E.2 Fourier transform

Inspired again by the constant coefficients case, we can eliminate the derivatives w.r.t. the spatial variable r by writing (E.1.4) for the vs in Fourier transform, analogously to §B.3 . If we define

$$v_A(r;\tau) := \frac{1}{2\pi} \int \hat{v}_A(\omega;\tau) e^{-ir\omega} d\omega; \tag{E.2.1}$$

then, the corresponding PDE becomes

$$\left(-\omega^2 - \partial_\tau^2 - 2H\partial_\tau + 2(\dot{H} - 2H^2) \right) \hat{v}_A(\omega;\tau) = a(\tau)^2 T(\tau). \tag{E.2.2}$$

The analogous holds for the other vs. Now we manipulate the term in the us.

Lemma E.2.1.

$$\begin{aligned}
\int_{-R(\tau)}^{R(\tau)} v_A(r;\tau) dr &= \hat{v}_A(\omega;\tau)|_{\omega=0}, \\
\int_{-R(\tau)}^{R(\tau)} v_{AC}(r;\tau) dr &= \hat{v}_{AC}(\omega;\tau)|_{\omega=0}, \\
\int_{-R(\tau)}^{R(\tau)} v_C(r;\tau) dr &= \hat{v}_C(\omega;\tau)|_{\omega=0}.
\end{aligned} \tag{E.2.3}$$

Proof. First, consider that v_A satisfies the wave equation (E.1.4), so that we can impose the causality condition

$$\forall |r| > \tau - \tau_I = R(\tau) : v_A(r;\tau) \equiv 0.$$

Therefore

$$\int_{-R(\tau)}^{R(\tau)} v_A(r;\tau)dr = \int_{-\infty}^{+\infty} v_A(r;\tau)dr.$$

After applying the Fourier transform and switching the integrals we find

$$\begin{aligned}\int v_A(r;\tau)dr &= \int \left[\frac{1}{2\pi}\int \hat{v}_A(\omega;\tau)e^{-ir\omega}d\omega\right]dr \\ &= \int \left[\int \frac{1}{2\pi}e^{-ir\omega}dr\right]\hat{v}_A(\omega;\tau)d\omega = \int \delta(\omega)\hat{v}_A(\omega;\tau)d\omega;\end{aligned}$$

which proves the assertion. For the others vs the proof is analogous. □

Now, we need to evaluate the boundary terms $[rv(r;\tau)]_{r=0}^{R(\tau)}$.

Lemma E.2.2. *The term $rv(r;\tau)|_{r=0}$ always vanishes.*

Proof. For Fourier properties

$$rv_A(r;\tau)|_{r=0} = -i\int \partial_\omega \hat{v}_A(\omega;\tau)e^{-ir\omega}d\omega|_{r=0} = -i\int \partial_\omega \hat{v}_A(\omega;\tau)d\omega = -i[\hat{v}_A(\omega;\tau)]_{-\infty}^{+\infty}.$$

For an evaluation of $\hat{v}_A(\omega;\tau)$ when ω goes to infinity, we can manipulate the corresponding ODE

$$\begin{aligned}\hat{v}_A(\omega;\tau) &= \frac{1}{\omega^2}[(-\partial_\tau^2 - 2H\partial_\tau + 2(\dot{H} - 2H^2))\hat{v}_A(\omega;\tau) - a(\tau)^2T(\tau)] \\ &\sim^{\omega\to\pm\infty} \frac{1}{\omega^2}(-\partial_\tau^2 - 2H\partial_\tau + 2(\dot{H} - 2H^2))\hat{v}_A(\omega;\tau).\end{aligned}$$

A solution is

$$\hat{v}_A(\omega;\tau) \sim^{\omega\to\pm\infty} 0;$$

which proves the assertion. The proof is analogous for the others vs. □

For the other term, we don't need the Fourier transform.

Lemma E.2.3. *The term $rv(r;\tau)|_{r=R(\tau)}$ vanishes if and only if $\tau_I > -\infty$ and $a(\tau)^2T(\tau) \in L^1_{loc}([\tau_I;\tau_F))$. Otherwise, it it divergent.*

Proof. We know that v_A satisfies a wave equation (E.1.4), whose principal symbol is the same as for a 2D d'alembertian. As in §2.4, near the wave boundary $r \to R(\tau)$ the solution depends on the principal symbol only, and we can neglect the terms $-2H\dot{v}_A + 2(\dot{H} - 2H^2)$ in that asymptotic region:

$$v_A(r;\tau) \sim^{r\to R(\tau)} \bar{v}_A(r;\tau) \quad s.t. \quad (\partial_r^2 - \partial_\tau^2)\bar{v}_A(r;\tau) = \delta(r)a(\tau)^2T(\tau).$$

It is easily solved by

$$\bar{v}_A(r;\tau) = S(\tau - |r|) \quad s.t. \quad S(x) := \int_{\tau_I}^{x} a(\tau)^2T(\tau)d\tau.$$

Notice that S diverges if $a(\tau)^2T(\tau) \notin L^1_{loc}([\tau_I;\tau_F))$. After replacing it

$$\begin{aligned} rv_A(r;\tau)|_{r=R(\tau)} &= \lim_{r\to R(\tau)} r\bar{v}_A(r;\tau) = \lim_{r\to\tau-\tau_I} rS(\tau-|r|) \\ &= \lim_{x\to\tau_I}(\tau-x)S(x) = \lim_{x\to\tau_I}(\tau-x)\int_{\tau_I}^{x} a(\tau)^2T(\tau)d\tau. \end{aligned}$$

Let now consider the case $\tau_I > -\infty$, so that the requirement on S becomes $a(\tau)^2T(\tau) \in L^1_{loc}([0;\tau_F))$. Then, the integral $\int_{\tau_I}^{x} a(\tau)^2T(\tau)d\tau$ goes to zero and $rv_A(r;\tau)|_{r=R(\tau)} \equiv 0$. On the other hand, in the case $\tau_I = -\infty$ we see that $\tau - x \to +\infty$. Since $a(\tau)^2T(\tau)$ is always positive, we get the divergence $rv_A(r;\tau)|_{r=R(\tau)} \equiv +\infty$.

The proof is analogous for v_C. For v_{AC} we obtain

$$rv_{AC}(r;\tau)|_{r=R(\tau)} = \lim_{x\to\tau_I}(\tau-x)\int_{\tau_I}^{x}\dot{H}(\tau)u_A(\tau)d\tau.$$

As before, it diverges if $\tau_I = -\infty$. If $\tau_I > -\infty$ but $a(\tau)^2T(\tau) \notin L^1_{loc}([\tau_I;\tau_F))$, then $u_A(\tau) = \int_{-R(\tau)}^{R(\tau)} v_A(r;\tau)dr - 2rv_A(r;\tau)|_{r=R(\tau)} \equiv -\infty$ as we saw, and also v_{AC} diverges. If we have $a(\tau)^2T(\tau) \in L^1_{loc}([\tau_I;\tau_F))$, then $u_A(\tau) = \hat{v}_A(0;\tau)$ for Lemmas E.2.1 and E.2.2; it converges, and the first of (2.2.13) assures that $\dot{H}(\tau)u_A(\tau) \in L^1_{loc}([\tau_I;\tau_F))$. $\square$

Putting the last Lemmas all together, we obtain the us as solutions of the ODEs of Theorem 2.5.5 .

Appendix F

Explicit evolution and fictitious components

From the results of §2.7, now we can get the formulas for the evaluation of fictitious dark matter and dark energy. They are different for the three cases described in Lemma 2.7.1. Let us start with the case where we have all the three epochs.

F.1 Three epochs

Applying (2.6.21), (2.6.19) and (2.6.17), we get the evolution of $\langle A \rangle$

$$H_0^2 u_A(\tau) = \begin{cases} \frac{1}{8}(H_0\tau)\left[\ln\left(\frac{H_0\tau}{4a_{RM}}\right) - \frac{3}{8}\right] & \tau \in [0;\tau_{RM}] \\ -\frac{1}{30}(H_0\tau - H_0c_M)^2 & \\ \quad +(H_0\tau - H_0c_M)^{-\frac{3}{2}}[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0c_M)\right) & \\ \quad +c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0c_M)\right)] & \tau \in [\tau_{RM};\tau_{M\Lambda}] \\ \frac{1}{2}(H_0\tau - H_0c_\Lambda)^3 + c_{A1\Lambda}(H_0\tau - H_0c_\Lambda)^2 & \\ \quad +c_{A2\Lambda}(H_0\tau - H_0c_\Lambda) & \tau \in [\tau_{M\Lambda};\tau(t_0)] \end{cases} \tag{F.1.1}$$

where the C^1 regularity fixes the integration constants c_{A1M}, c_{A2M} s.t.

$$\begin{cases} -\frac{1}{8}\left(\frac{3}{8} + \ln 4\right)(H_0\tau_{RM}) = -\frac{1}{30}(H_0\tau_{RM} - H_0c_M)^2 \\ \quad +(H_0\tau_{RM} - H_0c_M)^{-\frac{3}{2}}[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right) \\ \quad +c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right)] \\ \frac{1}{8}\left(\frac{5}{8} - \ln 4\right) = -\frac{1}{15}(H_0\tau_{RM} - H_0c_M) + (H_0\tau_{RM} - H_0c_M)^{-\frac{5}{2}} \\ \quad \times[c_{A1M}\left[-\frac{3}{2}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right) + \frac{\sqrt{71}}{2}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right)\right] \\ \quad +c_{A2M}\left[-\frac{3}{2}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right) - \frac{\sqrt{71}}{2}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right)\right]] \end{cases} \tag{F.1.2}$$

(the left hand terms have been simplified using $a_{RM} = H_0\tau_{RM}$), and $c_{A1\Lambda}$, $c_{A2\Lambda}$ s.t.

$$\begin{cases} -\frac{1}{30}(H_0\tau_{M\Lambda} - H_0c_M)^2 + (H_0\tau_{M\Lambda} - H_0c_M)^{-\frac{3}{2}} \\ \quad\times\left[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right) + c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right)\right] \\ \quad= \frac{1}{2}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^3 + c_{A1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^2 + c_{A2\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda) \\ -\frac{1}{15}(H_0\tau_{M\Lambda} - H_0c_M) - \frac{1}{2}(H_0\tau_{M\Lambda} - H_0c_M)^{-\frac{5}{2}} \\ \quad\times[(3c_{A1M} + \sqrt{71}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right) \\ \quad+(3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right)] \\ \quad= \frac{3}{2}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^2 + 2c_{A1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda) + c_{A2\Lambda} \end{cases} \tag{F.1.3}$$

the evolution of $\langle C\rangle$

$$H_0^2 u_C(\tau) = \begin{cases} \frac{1}{8}(H_0\tau)\left[5\ln\left(\frac{H_0\tau}{4a_{RM}}\right) - \frac{63}{8}\right] & \tau\in[0;\tau_{RM}] \\ -\frac{17}{150}(H_0\tau - H_0c_M)^2 + c_{D1M}(H_0\tau - H_0c_M)^{-3} + c_{D2M} & \\ \quad\frac{1}{5}(H_0\tau - H_0c_M)^{-\frac{3}{2}}[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0c_M)\right) & \\ \quad+c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0c_M)\right)] & \tau\in[\tau_{RM};\tau_{M\Lambda}] \\ \frac{1}{2}(H_0\tau - H_0c_\Lambda)^3 - c_{A1\Lambda}(H_0\tau - H_0c_\Lambda)^2 & \\ \quad-c_{A2\Lambda}(H_0\tau - H_0c_\Lambda) + c_{D1\Lambda}(H_0\tau - H_0c_\Lambda)^5 + c_{D2\Lambda} & \tau\in[\tau_{M\Lambda};\tau(t_0)] \end{cases} \tag{F.1.4}$$

where the C^1 regularity fixes the integration constants c_{D1M}, c_{D2M} s.t.

$$\begin{cases} -\frac{1}{8}\left(\frac{63}{8} + 5\ln 4\right)(H_0\tau_{RM}) = -\frac{17}{150}(H_0\tau_{RM} - H_0c_M)^2 \\ \quad+c_{D1M}(H_0\tau_{RM} - H_0c_M)^{-3} + c_{D2M} + \frac{1}{5}(H_0\tau_{RM} - H_0c_M)^{-\frac{3}{2}} \\ \quad\times\left[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right) + c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right)\right] \\ -\frac{1}{8}\left(\frac{23}{8} + 5\ln 4\right) = -\frac{17}{75}(H_0\tau_{RM} - H_0c_M) - 3c_{D1M}(H_0\tau_{RM} - H_0c_M)^{-4} \\ \quad-\frac{1}{10}(H_0\tau_{RM} - H_0c_M)^{-\frac{5}{2}}[(3c_{A1M} + \sqrt{71}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right) \\ \quad+(3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right)] \end{cases} \tag{F.1.5}$$

and $c_{D1\Lambda}$, $c_{D2\Lambda}$ s.t.

$$\begin{cases} -\frac{17}{150}(H_0\tau_{M\Lambda} - H_0c_M)^2 + \frac{1}{5}(H_0\tau_{M\Lambda} - H_0c_M)^{-\frac{3}{2}}[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right) \\ \quad+c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right)] + c_{D1M}(H_0\tau_{M\Lambda} - H_0c_M)^{-3} + c_{D2M} \\ \quad= \frac{1}{2}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^3 - c_{A1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^2 - c_{A2\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda) \\ \quad+c_{D1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^5 + c_{D2\Lambda} \\ -\frac{17}{75}(H_0\tau_{M\Lambda} - H_0c_M) - 3c_{D1M}(H_0\tau_{M\Lambda} - H_0c_M)^{-4} - \frac{1}{10}(H_0\tau_{M\Lambda} - H_0c_M)^{-\frac{5}{2}} \\ \quad\times[(3c_{A1M} + \sqrt{71}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right) \\ \quad+(3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right)] = \frac{3}{2}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^2 \\ \quad-2c_{A1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda) - c_{A2\Lambda} + 5c_{D1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^4 \end{cases} \tag{F.1.6}$$

and the evolution of $\langle B\rangle$

$$H_0^3 u_B(\tau) = \begin{cases} \frac{1}{8}(H_0\tau)^2\left[\ln\left(\frac{H_0\tau}{4a_{RM}}\right) - \frac{17}{8}\right] & \tau\in[0;\tau_{RM}] \\ -\frac{2}{175}(H_0\tau - H_0c_M)^3 + \frac{1}{2}c_{D1M}(H_0\tau - H_0c_M)^{-2} & \\ \quad +\frac{1}{5}c_{D2M}(H_0\tau - H_0c_M) - \frac{1}{50}(H_0\tau - H_0c_M)^{-\frac{1}{2}} & \\ \quad \times[(3c_{A1M} + \frac{\sqrt{71}}{2}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0c_M)\right) & \\ \quad +(3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau - H_0c_M)\right)] & \\ \quad +c_{BM}(H_0\tau - H_0c_M)^{-4} & \tau\in[\tau_{RM};\tau_{M\Lambda}] \\ -2c_{A1\Lambda}(H_0\tau - H_0c_\Lambda)^3 - 2c_{A2\Lambda}(H_0\tau - H_0c_\Lambda)^2\ln|H_0\tau - H_0c_\Lambda| & \\ \quad +\frac{1}{4}c_{D1\Lambda}(H_0\tau - H_0c_\Lambda)^6 - c_{D2\Lambda}(H_0\tau - H_0c_\Lambda) + c_{B\Lambda}(H_0\tau - H_0c_\Lambda)^2 & \tau\in[\tau_{M\Lambda};\tau(t_0)] \end{cases}. \tag{F.1.7}$$

where the continuity fixes the integration constants c_{BM}, $c_{B\Lambda}$ s.t.

$$\begin{cases} -\frac{1}{8}\left(\frac{17}{8} + \ln 4\right)(H_0\tau_{RM})^2 = -\frac{2}{175}(H_0\tau_{RM} - H_0c_M)^3 + \frac{1}{2}c_{D1M}(H_0\tau_{RM} - H_0c_M)^{-2} \\ \quad +\frac{1}{5}c_{D2M}(H_0\tau_{RM} - H_0c_M) - \frac{1}{50}(H_0\tau_{RM} - H_0c_M)^{-\frac{1}{2}} \\ \quad \times[(3c_{A1M} + \sqrt{71}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right) \\ \quad +(3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{RM} - H_0c_M)\right)] + c_{BM}(H_0\tau_{RM} - H_0c_M)^{-4} \\ -\frac{2}{175}(H_0\tau_{M\Lambda} - H_0c_M)^3 + \frac{1}{2}c_{D1M}(H_0\tau_{M\Lambda} - H_0c_M)^{-2} + \frac{1}{5}c_{D2M}(H_0\tau_{M\Lambda} - H_0c_M) \\ \quad -\frac{1}{50}(H_0\tau_{M\Lambda} - H_0c_M)^{-\frac{1}{2}}[(3c_{A1M} + \sqrt{71}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right) \\ \quad +(3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(H_0\tau_{M\Lambda} - H_0c_M)\right)] + c_{BM}(H_0\tau_{M\Lambda} - H_0c_M)^{-4} \\ \quad = -2c_{A1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^3 - 2c_{A2\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^2\ln|H_0\tau_{M\Lambda} - H_0c_\Lambda| \\ \quad +\frac{1}{4}c_{D1\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^6 - c_{D2\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda) + c_{B\Lambda}(H_0\tau_{M\Lambda} - H_0c_\Lambda)^2 \end{cases} \tag{F.1.8}$$

Now, in order to get the fictitious components $\Omega_{FM0}, \Omega_{F\Lambda 0}$ we need only to apply the formulas (2.5.26). Recalling

$$\frac{H_0'}{H_0} = -\bar{q}_0 = -\frac{1}{2}\sum_w (1+3w)\bar{\Omega}_{w0} = -\bar{\Omega}_{R0} - \frac{1}{2}\bar{\Omega}_{M0} + \bar{\Omega}_{\Lambda 0} = 1 - 2\bar{\Omega}_{R0} - \frac{3}{2}\bar{\Omega}_{M0}. \tag{F.1.9}$$

In a similar way, we can evaluate the term $\frac{(H_0')^2}{H_0^2} + \frac{H_0''}{H_0}$ from (2.1.7)

$$H(a)^2 = H_0^2\sum_w \bar{\Omega}_{w0}a^{-1-3w} \Rightarrow 2HH'' + 2(H')^2 = H_0^2\sum_w(-1-3w)(-2-3w)a^{-3-3w} \Rightarrow$$

$$\frac{(H_0')^2}{H_0^2} + \frac{H_0''}{H_0} = \frac{1}{2}\sum_w(1+3w)(2+3w)\bar{\Omega}_{w0} = 3\bar{\Omega}_{R0} + \bar{\Omega}_{M0} + \bar{\Omega}_{\Lambda 0} = 1 + 2\bar{\Omega}_{R0}. \tag{F.1.10}$$

Thus, (2.5.26) become

$$\begin{cases} \frac{1}{2}(sum+1)\Omega_{IM0} = -\langle A\rangle_0 - (2\bar{\Omega}_{R0} + \frac{3}{2}\bar{\Omega}_{M0})H_0\langle B\rangle_0 - \langle C\rangle_0' \\ \frac{1}{2}(ract+1)\Omega_{IM0} = \langle A\rangle_0' + 2\langle C\rangle_0' + \langle C\rangle_0'' + (1 - 6\bar{\Omega}_{R0} - 3\bar{\Omega}_{M0})H_0\langle B\rangle_0 \\ \quad +(1 - 2\bar{\Omega}_{R0} - \frac{3}{2}\bar{\Omega}_{M0})(2\langle A\rangle_0 - \langle C\rangle_0 + \langle C\rangle_0') \end{cases}. \tag{F.1.11}$$

Here, all perturbations are evaluated today, when the dark energy dominates

$$a(\tau) = \frac{1}{H_0(c_\Lambda - \tau)} \Rightarrow H_0\tau - H_0c_\Lambda = -a^{-1}$$

$$
\begin{aligned}
\langle A\rangle &= \frac{3}{2}\Omega_{IM0}[\frac{1}{2}(-a^{-1})^3 + c_{A1\Lambda}(-a^{-1})^2 + c_{A2\Lambda}(-a^{-1})] \Rightarrow \\
\langle A\rangle_0 &= \frac{3}{2}\Omega_{IM0}[-\frac{1}{2} + c_{A1\Lambda} - c_{A2\Lambda}], \quad \langle A\rangle_0' = \frac{3}{2}\Omega_{IM0}[\frac{3}{2} - 2c_{A1\Lambda} + c_{A2\Lambda}];
\end{aligned} \tag{F.1.12}
$$

$$
\begin{aligned}
\langle B\rangle &= \frac{3}{2}\frac{\Omega_{IM0}}{H_0}[-2c_{A1\Lambda}(-a^{-1})^3 - 2c_{A2\Lambda}(-a^{-1})^2 \ln|-a^{-1}| \\
&\quad + \frac{1}{4}c_{D1\Lambda}(-a^{-1})^6 - c_{D2\Lambda}(-a^{-1}) + c_{B\Lambda}(-a^{-1})^2] \Rightarrow \\
H_0\langle B\rangle_0 &= \frac{3}{2}\Omega_{IM0}[2c_{A1\Lambda} + \frac{1}{4}c_{D1\Lambda} + c_{D2\Lambda} + c_{B\Lambda}];
\end{aligned} \tag{F.1.13}
$$

$$
\begin{aligned}
\langle C\rangle &= \frac{3}{2}\Omega_{IM0}[\frac{1}{2}(-a^{-1})^3 - c_{A1\Lambda}(-a^{-1})^2 - c_{A2\Lambda}(-a^{-1}) + c_{D1\Lambda}(-a^{-1})^5 + c_{D2\Lambda}] \Rightarrow \\
\langle C\rangle_0 &= \frac{3}{2}\Omega_{IM0}[-\frac{1}{2} - c_{A1\Lambda} + c_{A2\Lambda} - c_{D1\Lambda} + c_{D2\Lambda}], \\
\langle C\rangle_0' &= \frac{3}{2}\Omega_{IM0}[\frac{3}{2} + 2c_{A1\Lambda} - c_{A2\Lambda} + 5c_{D1\Lambda}], \\
\langle C\rangle_0'' &= \frac{3}{2}\Omega_{IM0}[-6 - 6c_{A1\Lambda} + 2c_{A2\Lambda} - 30c_{D1\Lambda}].
\end{aligned} \tag{F.1.14}
$$

Replacing (F.1.12), (F.1.13) and (F.1.14) inside (F.1.11), we obtain *ract* and *sum*.

F.2 No matter epoch

For different values of $\bar{\Omega}_{R0}, \bar{\Omega}_{M0}$, we would have only the radiation and dark energy epochs. Applying (2.6.21), (2.6.19) and (2.6.17), we get the evolution of $\langle A\rangle$

$$
H_0^2 u_A(\tau) = \begin{cases} \frac{1}{8}(H_0\tau)\left[\ln\left(\frac{H_0\tau}{4a_{RM}}\right) - \frac{3}{8}\right] & \tau \in [0;\tau_{R\Lambda}] \\ \frac{1}{2}(H_0\tau - H_0c_R)^3 + c_{A1\Lambda}(H_0\tau - H_0c_R)^2 & \\ \quad + c_{A2\Lambda}(H_0\tau - H_0c_R) & \tau \in [\tau_{R\Lambda};\tau(t_0)] \end{cases} \tag{F.2.1}
$$

where the C^1 regularity fixes the integration constants $c_{A1\Lambda}$, $c_{A2\Lambda}$ s.t.

$$
\begin{cases} -\frac{1}{8}\left(\frac{3}{8} + \ln 4\right)(H_0\tau_{R\Lambda}) = \frac{1}{2}(H_0\tau_{R\Lambda} - H_0c_R)^3 \\ \quad + c_{A1\Lambda}(H_0\tau_{R\Lambda} - H_0c_R)^2 + c_{A2\Lambda}(H_0\tau_{R\Lambda} - H_0c_R) \\ \frac{1}{8}\left(\frac{5}{8} - \ln 4\right) = \frac{3}{2}(H_0\tau_{R\Lambda} - H_0c_R)^2 + 2c_{A1\Lambda}(H_0\tau_{R\Lambda} - H_0c_R) + c_{A2\Lambda} \end{cases} \tag{F.2.2}
$$

For the evolution of $\langle C\rangle$ we get

$$
H_0^2 u_C(\tau) = \begin{cases} \frac{1}{8}(H_0\tau)\left[5\ln\left(\frac{H_0\tau}{4a_{RM}}\right) - \frac{63}{8}\right] & \tau \in [0;\tau_{RM\Lambda}] \\ \frac{1}{2}(H_0\tau - H_0c_R)^3 - c_{A1\Lambda}(H_0\tau - H_0c_R)^2 & \\ \quad - c_{A2\Lambda}(H_0\tau - H_0c_R) + c_{D1\Lambda}(H_0\tau - H_0c_R)^5 + c_{D2\Lambda} & \tau \in [\tau_{R\Lambda};\tau(t_0)] \end{cases} \tag{F.2.3}
$$

where the C^1 regularity fixes the integration constants $c_{D1\Lambda}$, $c_{D2\Lambda}$ s.t.

$$\begin{cases} -\frac{1}{8}\left(\frac{63}{8}+5\ln 4\right)(H_0\tau_{R\Lambda}) = \frac{1}{2}(H_0\tau_{R\Lambda}-H_0c_R)^3 - c_{A1\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^2 \\ \quad -c_{A2\Lambda}(H_0\tau_{R\Lambda}-H_0c_R) + c_{D1\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^5 + c_{D2\Lambda} \\ -\frac{1}{8}\left(\frac{23}{8}+5\ln 4\right) = \frac{3}{2}(H_0\tau_{R\Lambda}-H_0c_R)^2 - 2c_{A1\Lambda}(H_0\tau_{R\Lambda}-H_0c_R) \\ \quad -c_{A2\Lambda} + 5c_{D1\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^4 \end{cases} \tag{F.2.4}$$

For the evolution of $\langle B\rangle$ we have

$$H_0^3 u_B(\tau) = \begin{cases} \frac{1}{8}(H_0\tau)^2\left[\ln\left(\frac{H_0\tau}{4a_{RM}}\right) - \frac{17}{8}\right] & \tau\in[0;\tau_{R\Lambda}] \\ -2c_{A1\Lambda}(H_0\tau - H_0c_R)^3 & \\ \quad -2c_{A2\Lambda}(H_0\tau-H_0c_R)^2\ln|H_0\tau-H_0c_R| & \\ \quad +\frac{1}{4}c_{D1\Lambda}(H_0\tau-H_0c_R)^6 - c_{D2\Lambda}(H_0\tau-H_0c_R) & \\ \quad +c_{B\Lambda}(H_0\tau-H_0c_R)^2 & \tau\in[\tau_{R\Lambda};\tau(t_0)] \end{cases} \tag{F.2.5}$$

where the continuity fixes the integration constant $c_{B\Lambda}$ s.t.

$$\begin{aligned} -\frac{1}{8}\left(\frac{17}{8}+\ln 4\right)(H_0\tau_{R\Lambda})^2 = {} & -2c_{A1\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^3 \\ & - 2c_{A2\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^2\ln|H_0\tau_{R\Lambda}-H_0c_R| \\ & + \frac{1}{4}c_{D1\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^6 - c_{D2\Lambda}(H_0\tau_{R\Lambda}-H_0c_R) \\ & + c_{B\Lambda}(H_0\tau_{R\Lambda}-H_0c_R)^2 \end{aligned} \tag{F.2.6}$$

All perturbations are evaluated today, when it dominates the dark energy, as in §7.2. Substituting (F.1.12), (F.1.13) and (F.1.14) inside (F.1.11), we obtain *ract* and *sum* as well.

F.3 No dark energy epoch

A last possibility is that there are only the radiation and matter epochs. Applying (2.6.21), (2.6.19) and (2.6.17), we get the same evolutions of $\langle A\rangle$, $\langle B\rangle$ and $\langle C\rangle$ as in §7.3, just without the last parts. All perturbations are evaluated today, when the matter dominates.

$$\begin{aligned} \langle A\rangle &= \frac{3}{2}\Omega_{IM0}[-\frac{1}{30}(2a^{1/2})^2 + (2a^{1/2})^{-\frac{3}{2}} \\ &\quad \times\left[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(2a^{1/2})\right) + c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(2a^{1/2})\right)\right]] \Rightarrow \\ \langle A\rangle_0 &= \frac{3}{2}\Omega_{IM0}[-\frac{2}{15} + \frac{c_{A1M}}{2\sqrt{2}}\sin\left(\frac{\sqrt{71}}{2}\ln 2\right) + \frac{c_{A2M}}{2\sqrt{2}}\cos\left(\frac{\sqrt{71}}{2}\ln 2\right)], \\ \langle A\rangle_0' &= \frac{3}{2}\Omega_{IM0}[-\frac{2}{15} - \frac{3c_{A1M}+\sqrt{71}}{8\sqrt{2}}\sin\left(\frac{\sqrt{71}}{2}\ln 2\right) \\ &\quad - \frac{3c_{A2M}-\sqrt{71}}{8\sqrt{2}}\cos\left(\frac{\sqrt{71}}{2}\ln 2\right)]; \end{aligned} \tag{F.3.1}$$

$$\langle B\rangle = \frac{3}{2}\frac{\Omega_{IM0}}{H_0}[-\frac{2}{175}(2a^{1/2})^3 + \frac{1}{2}c_{D1M}(2a^{1/2})^{-2}$$
$$+ \frac{1}{5}c_{D2M}(2a^{1/2}) - \frac{1}{50}(2a^{1/2})^{-\frac{1}{2}}[(3c_{A1M} + \sqrt{71}c_{A2M})\sin\left(\frac{\sqrt{71}}{2}\ln(2a^{1/2})\right)$$
$$+ (3c_{A2M} - \sqrt{71}c_{A1M})\cos\left(\frac{\sqrt{71}}{2}\ln(2a^{1/2})\right)] + c_{BM}(2a^{1/2})^{-4}] \Rightarrow$$
$$H_0\langle B\rangle_0 = \frac{3}{2}\Omega_{IM0}[-\frac{16}{175} + \frac{1}{8}c_{D1M} + \frac{2}{5}c_{D2M} - \frac{3c_{A1M} + \sqrt{71}c_{A2M}}{50\sqrt{2}}\sin\left(\frac{\sqrt{71}}{2}\ln 2\right)$$
$$- \frac{3c_{A2M} - \sqrt{71}c_{A1M}}{50\sqrt{2}}\cos\left(\frac{\sqrt{71}}{2}\ln 2\right) + \frac{1}{16}c_{BM}]; \tag{F.3.2}$$

$$\langle C\rangle = \frac{3}{2}\Omega_{IM0}[-\frac{17}{150}(2a^{1/2})^2 + c_{D1M}(2a^{1/2})^{-3} + c_{D2M}$$
$$+ \frac{1}{5}(2a^{1/2})^{-\frac{3}{2}}\left[c_{A1M}\sin\left(\frac{\sqrt{71}}{2}\ln(2a^{1/2})\right) + c_{A2M}\cos\left(\frac{\sqrt{71}}{2}\ln(2a^{1/2})\right)\right]] \Rightarrow$$
$$\langle C\rangle_0 = \frac{3}{2}\Omega_{IM0}[-\frac{34}{75} + \frac{1}{8}c_{D1M} + c_{D2M} + \frac{c_{A1M}}{10\sqrt{2}}\sin\left(\frac{\sqrt{71}}{2}\ln 2\right)$$
$$+ \frac{c_{A2M}}{10\sqrt{2}}\cos\left(\frac{\sqrt{71}}{2}\ln 2\right)],$$
$$\langle C\rangle_0' = \frac{3}{2}\Omega_{IM0}[-\frac{34}{75} - \frac{3}{16}c_{D1M} - \frac{3c_{A1M} + \sqrt{71}}{40\sqrt{2}}\sin\left(\frac{\sqrt{71}}{2}\ln 2\right)$$
$$- \frac{3c_{A2M} - \sqrt{71}}{40\sqrt{2}}\cos\left(\frac{\sqrt{71}}{2}\ln 2\right)],$$

$$\langle C\rangle_0'' = \frac{3}{2}\Omega_{IM0}[\frac{15}{32}c_{D1M} + \frac{21c_{A1M} + 12c_{A2M} + 3\sqrt{71}}{160\sqrt{2}}\sin\left(\frac{\sqrt{71}}{2}\ln 2\right)$$
$$+ \frac{21c_{A2M} - 12c_{A1M} - 5\sqrt{71}}{160\sqrt{2}}\cos\left(\frac{\sqrt{71}}{2}\ln 2\right)]. \tag{F.3.3}$$

Substituting (F.3.1), (F.3.2) and (F.3.3) inside (F.1.11), we obtain *ract* and *sum*.

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