Potential Analysis
https://doi.org/10.1007/s11118-021-09959-4

Asymptotic Behavior of Ground States and Local Uniqueness for Fractional Schrödinger Equations with Nearly Critical Growth

Daniele Cassani[1,2] · **Youjun Wang**[2,3]

Received: 11 May 2021 / Accepted: 29 September 2021 / Published online: 22 October 2021

Abstract
We study quantitative aspects and concentration phenomena for ground states of the following nonlocal Schrödinger equation

$$(-\Delta)^s u + V(x)u = u^{2_s^*-1-\epsilon} \ \text{ in } \ \mathbb{R}^N,$$

where $\epsilon > 0$, $s \in (0,1)$, $2_s^* := \frac{2N}{N-2s}$ and $N > 4s$, as we deal with finite energy solutions. We show that the ground state u_ϵ blows up and precisely with the following rate $\|u_\epsilon\|_{L^\infty(\mathbb{R}^N)} \sim \epsilon^{-\frac{N-2s}{4s}}$, as $\epsilon \to 0^+$. We also localize the concentration points and, in the case of radial potentials V, we prove local uniqueness of sequences of ground states which exhibit a concentrating behavior.

Keywords Nonlocal equations · Fractional Laplacian · Blow-up phenomena · Ground states · Critical growth

Mathematics Subject Classification (2010) 35A15 · 35J60 · 35B40

1 Introduction

In this paper we consider the following class of nonlocal equations

$$(-\Delta)^s u + V(x)u = u^{2_s^*-1-\epsilon} \ \text{ in } \ \mathbb{R}^N, \tag{1.1}$$

✉ Daniele Cassani
daniele.cassani@uninsubria.it

Youjun Wang
scyjwang@scut.edu.cn

[1] Dip. di Scienza e Alta Tecnologia, Universitá degli Studi dell'Insubria, Varese, Italy

[2] RISM–Riemann International School of Mathematics, via G.B. Vico 46 – 21100 Varese, Italy

[3] Department of Mathematics, South China University of Technology, Guangzhou 510640, People's Republic of China

where $\epsilon \to 0^+$, $s \in (0,1)$, $2_s^* := \frac{2N}{N-2s}$, $N > 4s$, $V : \mathbb{R}^N \to \mathbb{R}$ is a potential function and

$$(-\Delta)^s u(x) = c_{N,s}\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy$$

is the fractional Laplacian. Here, $c_{N,s}$ is a normalizing constant, PV stands for the Cauchy principal value. As we are going to see, the restriction on the dimension is due to the fact that we look for finite L^2-energy solutions.

For fixed $\epsilon \in (0, 2_s^*-2)$, under suitable conditions on $V(x)$, it is known that equation (1.1) admits a positive ground state u_ϵ, see for instance [2, 21–23]. Moreover, if $V(x) = 1$, then u_ϵ is spherically symmetric, see [12, 18]. However, when $\epsilon = 0$ it follows from a Pohozaev type identity that (1.1) has no solutions in $H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ if $V(x) + \frac{1}{2s}x \cdot \nabla V(x) \geq 0$ (and $\not\equiv 0$), see Theorem 9 or [10] in the special case $V(x) \equiv 1$.

Therefore, it is natural to wonder what happens to the ground state u_ϵ as $\epsilon \to 0^+$. The main motivation of this paper is to achieve a better understanding of this phenomenon. This type of problems for semilinear equations, with the so-called nearly critical growth, were first studied in the unit ball of $\mathbb{R}^3$ by Atkinson and Peletier in [3] and then extended to spherical domains by Brezis and Peletier in [6] and non-spherical domains by Han in [25]. Indeed, they proved the solution u_ϵ blows up in the sense that $\|u_\epsilon\|_{L^\infty(\mathbb{R}^N)} \sim \epsilon^{-\frac{1}{2}}$ as $\epsilon \to 0^+$. More recently, their results were extended to nonlocal problems in bounded domains in [13]. Precisely, the authors in [13] study the following nonlocal problem

$$\begin{cases} \mathcal{A}_s u = u^{2_s^*-1-\epsilon} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{A}_s$ denotes the fractional Laplace operator $(-\Delta)^s$ in Ω defined in terms of the spectrum of the Laplacian subject to Dirichlet boundary conditions. They proved that if u_ϵ is a solution to (1.2) satisfying

$$\lim_{\epsilon\to 0} \frac{\int_\Omega |\mathcal{A}_s^{\frac{1}{2}} u_\epsilon|^2 dx}{\|u_\epsilon\|_{2_s^*}^2} = S$$

where S is the best Sobolev constant in the embedding $H^s \hookrightarrow L^{s^*}$, then

$$\lim_{\epsilon\to 0} \epsilon\|u_\epsilon\|_{L^\infty(\Omega)}^2 = b_{n,s}|\tau(x_0)|,$$

where $b_{n,s}$ is a normalizing constant and $x_0 \in \Omega$ is a critical point of the Robin function $\tau(x)$. Besides, in [27] the authors study the asymptotic behavior of solutions to the nonlocal nonlinear problem

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2-\epsilon}u & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $p_s^* = \frac{Np}{N-ps}$, $N > ps$, $p > 1$. They prove that ground state solutions concentrate at a single point in $\overline{\Omega}$ and analyze the asymptotic behavior for sequences of solutions at higher energy levels as $\epsilon \to 0$. In particular, in the semi-linear case $p = 2$, they prove that for smooth domains the concentration point cannot lie on the boundary, and identify its location in the case of annular domains. Regarding the nonlocal problem (1.3) for $p = 2$, we also refer to [29] for a profile decomposition approach and to [30] for Γ-convergence methods.

The purpose of this paper is twofold: on one side, under suitable conditions on $V(x)$, we give a complete description of the blow up behavior of the ground states of (1.1); on the other side, we identify the location of the concentration points and then we establish local uniqueness of ground states.

Before stating our main results, let us make a few assumptions on $V(x)$. Throughout this paper, we assume that $V(x)$ satisfies the following conditions:

(V_1) $V \in C^2, 0 < V_0 \le V(x) \le V_\infty := \sup_{x\in\mathbb{R}^N} V(x) = \liminf_{|x|\to+\infty} V(x) < +\infty$;

(V_2) The function $x \cdot \nabla V(x)$ stays bounded in $\mathbb{R}^N$.

We consider here the fractional Sobolev space

$$H_V^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N, V(x)\,dx) : [u]_s^2 := \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty \right\}$$

endowed with the norm

$$\|u\|_{s,V} := \left([u]_s^2 + \int_{\mathbb{R}^N} V(x)u^2 dx \right)^{\frac{1}{2}}.$$

Notice that under (V_1), $H^s(\mathbb{R}^N)$ which corresponds to the choice $V \equiv 1$ and $H^s_{V(x)}(\mathbb{R}^N)$ turn out to be equivalent in terms of norms as well as of elements. Denote by $D^s(\mathbb{R}^N)$ the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $[u]_s$. As usual $\|\cdot\|_p$ denotes $\|\cdot\|_{L^p(\mathbb{R}^N)}$ for $1 \le p \le \infty$.

Let

$$S_{2_s^*-\epsilon} := \inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_s^2}{\|u\|_{2_s^*-\epsilon}^2} = \inf_{u\in H^s(\mathbb{R}^N)} \{\|u\|_s^2 : \|u\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} = 1\}. \tag{1.4}$$

By Lions' concentration compactness, minimizers for $S_{2_s^*-\epsilon}$ always exist and one may assume they do not change sign, [18, 23]. Moreover, they are radially symmetric, see [24]. Here, we will consider only positive minimizers.

Recall also the Sobolev constant

$$S := \inf_{u\in D^s(\mathbb{R}^N)\setminus\{0\}} \frac{[u]_s^2}{\|u\|_{2_s^*}^2}. \tag{1.5}$$

For each fixed $\epsilon \in (0, 2_s^* - 2)$, let w_ϵ be a positive minimizer for

$$S^V_{2_s^*-\epsilon} = \inf_{u\in H_V^s(\mathbb{R}^N)} \{\|u\|_{s,V}^2 : \|u\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} = 1\}, \tag{1.6}$$

or equivalently,

$$S^V_{2_s^*-\epsilon} = \inf_{u\in H_V^s(\mathbb{R}^N)\setminus\{0\}} I_\epsilon(u), \tag{1.7}$$

where

$$I_\epsilon(u) = \frac{\|u\|_{s,V}^2}{\|u\|_{2_s^*-\epsilon}^2}.$$

Our main results are the following:

Theorem 1 *Assume* (V_1), (V_2), $N > 4s$ *and that* u_ϵ *is a ground state of* (1.1), *namely satisfying* (1.7), *which has a maximum point* x_ϵ *such that* $x_\epsilon \to x_0$ *as* $\epsilon \to 0^+$. *Then,*

(1) $\lim_{\epsilon\to 0^+} S^V_{2_s^*-\epsilon} = S$;

(2)

$$\lim_{\epsilon\to 0^+} \epsilon \|u_\epsilon\|_\infty^{\frac{4s}{N-2s}} = A_{N,s} \left[V(x_0) + \frac{1}{2s} x_0 \cdot \nabla V(x_0) \right]$$

where

$$A_{N,s} := \frac{2^{2(N+1)} N^2 \pi^{\frac{N}{2}} \Gamma\left(\frac{N-4s}{2}\right)}{(N-2s)^2 \Gamma(N-2s)} S^{-\frac{N}{2s}}.$$

In particular we have

Corollary 1 *Assume $N > 4s$ and that u_ϵ is a minimizer for (1.4). Then, we have:*

(1) $\lim_{\epsilon \to 0^+} S_{2_s^*-\epsilon} = S$;

(2)

$$\lim_{\epsilon \to 0^+} \epsilon \|u_\epsilon\|_\infty^{\frac{4s}{N-2s}} = A_{N,s}\ .$$

Notice that in Theorem 1, we assume that the maximum point x_ϵ does converge. However, under conditions (V_1) and (V_2), one of the main difficulties is that x_ϵ may actually escape to infinity as $\epsilon \to 0^+$. In what follows, we prove that if $N > 6s$, then the maximum point x_ϵ must be bounded, and therefore converging, up to a subsequence, to a global minimum point of $V(x)$ provided $\inf_{x \in \mathbb{R}^N} V(x) < V_\infty$. More precisely, we have the following

Theorem 2 *Assume (V_1), (V_2) with $\inf_{x\in\mathbb{R}^N} V(x) < V_\infty$, $N > 6s$ and that u_ϵ is a ground state of (1.1) (in the sense of (1.7)) which has a maximum point x_ϵ. Then, there exists a subsequence $\{x_{\epsilon_j}\}$ of $\{x_\epsilon\}$ such that:*

(1) $\lim_{j\to\infty} x_{\epsilon_j} = x_0$, *where x_0 is a global minimum point of $V(x)$;*

(2) $\lim_{j\to\infty} S^V_{2_s^*-\epsilon_j} = S$;

(3)

$$\lim_{j\to\infty} \epsilon_j \|u_{\epsilon_j}\|_\infty^{\frac{4s}{N-2s}} = A_{N,s} V(x_0).$$

What stated in Theorem 3 opens a natural question: *is there more than one blow-up ground state sequence such that the maxima concentrate at the same point?*

We do not have a full answer, however let us consider a special case. Assume $V(x)$ is radial and that there exist two radial ground state sequences $u^1_{\epsilon_j}$ and $u^2_{\epsilon_j}$ of (1.1) such that $\|u^i_j\|_\infty = u^i_j(0)$, $i = 1, 2$. Set $\mu^i_j := \|u^i_j\|_\infty^{-\frac{2_s^*-2-\epsilon_j}{2s}}$, $i = 1, 2$. We have the following local uniqueness result.

Theorem 3 *Assume (V_1), (V_2), that $V(x) = V(|x|)$ is radial, $N > 4s$ and there exist two radial ground state sequences $u^1_{\epsilon_j}$ and $u^2_{\epsilon_j}$ of (1.1) satisfying $\|u^i_j\|_\infty = u^i_j(0)$, $i = 1, 2$, (1), (2) and (3) of Theorem 3. Then, there exists $\epsilon_0 > 0$ such that for any $\epsilon_j \in (0, \epsilon_0)$, we have $u^1_{\epsilon_j} = u^2_{\epsilon_j}$, provided $\mu^1_j = \mu^2_j$. More precisely, up to rescaling, we have the following local uniqueness result*

$$u^2_{\epsilon_j}(x) = \left(\frac{\mu^1_j}{\mu^2_j}\right)^{\frac{2s}{2_s^*-2-\epsilon_j}} u^1_{\epsilon_j}\left(\frac{\mu^1_j}{\mu^2_j} x\right)$$

or equivalently

$$u_{\epsilon_j}^2(x) = \frac{\|u_{\epsilon_j}^2\|_\infty}{\|u_{\epsilon_j}^1\|_\infty} u_{\epsilon_j}^1 \left[\left(\frac{\|u_{\epsilon_j}^2\|_\infty}{\|u_{\epsilon_j}^1\|_\infty} \right)^{\frac{2_s^*-2-\epsilon_j}{2s}} x \right].$$

Remark 1 In Theorems 4 we assume that $\|u_j^i\|_\infty = u_j^i(0)$, $i = 1, 2$. Indeed, the results are still true if $\|u_j^i\|_\infty = u_j^i(x_j)$ for some $x_j \in \mathbb{R}^N$ and $\lim_{j\to\infty} x_j = x_0$. Let $w_j^i(x) = u_j^i(x + x_j)$, then w_j^i satisfies

$$(-\Delta)^s w_j^i + V(x + x_j) w_j^i = (w_j^i)^{2_s^*-1-\epsilon_j} \text{ in } \mathbb{R}^N.$$

Define

$$v_j^i(x) = \mu_j^{\frac{2s}{2_s^*-2-\epsilon_j}} w_j^i(\mu_j^i x).$$

Then $0 < v_j^i(x) \le 1$, $v_j^i(0) = 1$, and satisfies

$$(-\Delta)^s v_j^i + (\mu_j^i)^{2s} V(x_j + \mu_j^i x) v_j^i = (v_j^i)^{2_s^*-1-\epsilon_j} \text{ in } \mathbb{R}^N.$$

1.1 Overview

The asymptotic behavior of ground states to nonlocal problems has attracted remarkable attention in recent years. In [22], the authors studied the singularly perturbed fractional Schrödinger equation

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = u^p \text{ in } \mathbb{R}^N, \tag{1.8}$$

where $1 < p < 2_s^* - 1$. They proved that concentration points turn out to be critical points for V. Moreover, they proved that if the potential V is coercive and has a unique global minimum, then ground states concentrate at that minimum point as $\epsilon \to 0$. In [16], by means of a Lyapunov-Schmidt reduction method, the authors proved the existence of various type of concentrating solutions, such as multiple spikes and clusters, such that each of the local maxima converge to a critical point of V as $\epsilon \to 0$, see also [1, 20]. In [5], the authors considered the nonlocal scalar field equation

$$(-\Delta)^s u + \epsilon u = |u|^{p-2}u - |u|^{q-2}u \text{ in } \mathbb{R}^N, \tag{1.9}$$

where $2 < p < q$. For ϵ small, they proved the existence and qualitative properties of positive solutions when p is subcritical, supercritical or critical Sobolev exponent. For the existence of positive solutions of nonlocal equations with a small parameter see also [7, 19].

Loosely speaking, all the results mentioned above were concerned with the characterization of concentration of ground states. The purpose of this paper is quite different as we focus on quantitative aspects of concentrating solutions . Let us emphasize that Theorem 3 can be seen as a nonlocal analog of the results in [31, 40]. In [31], the authors studied the behavior of the ground states of equation

$$-\Delta u + K(x)u = u^{2^*-1-\epsilon} \text{ in } \mathbb{R}^N. \tag{1.10}$$

Under some geometric assumptions on $K(x)$, they proved the existence of ground states u_ϵ. Moreover, the maximum point x_ϵ of u_ε is bounded and $\|u\|_{L^\infty(\mathbb{R}^N)} \sim \epsilon^{-\frac{N-2}{4}}$ as $\epsilon \to 0^+$. In [40], the author further identified the location of the blow-up point. In the present paper, though conditions (V_1) and (V_2) guarantee the existence of the ground state solution u_ϵ, it is not true in general that the maximum point x_ϵ of u_ϵ stays bounded as $\epsilon \to 0^+$ and this yields a major difficulty.

The paper is organized as follows: existence of minimizers, local boundedness estimates of solutions and a Pohozaev type identity are established in the preliminary Section 2. In Section 3, we study the asymptotic behavior of ground states, including a uniform bound up to rescaling. Section 4 is devoted to identify the location of blow-up points, whence in Section 5 we prove the local uniqueness of ground states.

Throughout this paper, C will denote a positive constant which may vary from line to line.

2 Preliminaries

Here for the convenience of the reader we prove some auxiliary results. Consider first the following constrained minimization:

$$S^V_{2^*_s-\epsilon} := \inf_{u\in H^s_V(\mathbb{R}^N)} \{\|u\|^2_{s,V} : \|u\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} = 1\}. \tag{2.1}$$

In the special case $V(x) = 1$, minimizers for $S_{2^*_s-\epsilon}$ always exist and do not change sign, see e.g. [18, 23], Moreover, they are radially symmetric, see [24].

Theorem 4 *Assume* (V_1) *holds. Then,* $S^V_{2^*_s-\epsilon}$ *is achieved at some* $w_\epsilon \in H^s_V(\mathbb{R}^N)$.

Proof We assume that $V(x) \not\equiv V_\infty$, otherwise the result is obvious. Let $\{w_n\}$ be a minimizing sequence for $S^V_{2^*_s-\epsilon}$. Since $|w_n| \in H^s_V(\mathbb{R}^N)$ and $[|w_n|]_s \le [w_n]_s$, we may assume that w_n is nonnegative. Clearly, $\{w_n\}$ is bounded in $H^s_V(\mathbb{R}^N)$ and $\|w_n\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} = 1$. Therefore, up to subsequences if necessary, there exists $w \in H^s_V(\mathbb{R}^N)$ such that $w_n \rightharpoonup w$ in $H^s_V(\mathbb{R}^N)$ as $n \to +\infty$. Let $\ell = \|w\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}$, then $0 \le \ell \le 1$. We next claim that actually $\ell = 1$.

Indeed, let

$$S^\infty_{2^*_s-\epsilon} := \inf_{u\in H^s_{V_\infty}(\mathbb{R}^N)} \{\|u\|^2_{s,V_\infty} : \|u\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} = 1\}.$$

Then minimizer u for $S^\infty_{2^*_s-\epsilon}$ exists and does not change sign, see e.g. [18, 23]. Without loss of generality, we assume u is positive. Using this u as a test function we can show that if $V(x)$ is not identically equal to V_∞, then $S^V_{2^*_s-\epsilon} < S^\infty_{2^*_s-\epsilon}$.

Set $v_n = w_n - w$, then $v_n \rightharpoonup 0$ in $H^s_V(\mathbb{R}^N)$ and $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ as $n \to +\infty$ and by Bresiz-Lieb lemma, we have

$$\begin{aligned}\lim_{n\to+\infty} \|v_n\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} &= \lim_{n\to+\infty} \|w_n\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} - \|w\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} \\ &= 1 - \|w\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} \\ &= 1 - \ell.\end{aligned} \tag{2.2}$$

On the one hand we have

$$\begin{aligned}\lim_{n\to+\infty}\|w_n\|_{s,V}^2 &= \lim_{n\to+\infty}\|v_n\|_{s,V}^2+\|w\|_{s,V}^2\\ &\quad+2\lim_{n\to+\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(v_n(x)-v_n(y))(w(x)-w(y))}{|x-y|^{N+2s}}dxdy\\ &\quad+\lim_{n\to+\infty}\int_{\mathbb{R}^N}V(x)v_nwdx\\ &=\lim_{n\to+\infty}\|v_n\|_{s,V}^2+\|w\|_{s,V}^2.\end{aligned}\tag{2.3}$$

On the other hand, by (V_1) and $v_n\to0$ in $L^2_{loc}(\mathbb{R}^N)$ as $n\to+\infty$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}[V(x)-V_\infty]w_n^2dx=0.$$

Thus, we have

$$\lim_{n\to+\infty}\|v_n\|_{s,V(x)}^2=\lim_{n\to+\infty}\|v_n\|_{s,V_\infty}^2.\tag{2.4}$$

For $0\le\ell\le1$, by the definitions of $S^V_{2_s^*-\epsilon}$ and $S^\infty_{2_s^*-\epsilon}$, we have

$$\|w\|_{s,V(x)}^2\ge\ell^{\frac{2}{2_s^*-\epsilon}}S^V_{2_s^*-\epsilon}\tag{2.5}$$

and by (2.2), we get

$$\lim_{n\to+\infty}\|v_n\|_{s,V_\infty}^2\ge\lim_{n\to+\infty}\|v_n\|^2_{2_s^*-\epsilon}S^\infty_{2_s^*-\epsilon}=(1-\ell)^{\frac{2}{2_s^*-\epsilon}}S^\infty_{2_s^*-\epsilon}.\tag{2.6}$$

Therefore, by (2.3), (2.4), (2.5) and (2.6), we have

$$S^V_{2_s^*-\epsilon}\ge\ell^{\frac{2}{2_s^*-\epsilon}}S^V_{2_s^*-\epsilon}+(1-\ell)^{\frac{2}{2_s^*-\epsilon}}S^\infty_{2_s^*-\epsilon}\tag{2.7}$$

which gives

$$1-\ell^{\frac{2}{2_s^*-\epsilon}}\ge(1-\ell)^{\frac{2}{2_s^*-\epsilon}}.\tag{2.8}$$

Thus, from (2.8), we deduce that $\ell=0$ or $\ell=1$. If $\ell=0$, then from (2.7), we get $S^V_{2_s^*-\epsilon}\ge S^\infty_{2_s^*-\epsilon}$, which is a contradiction. Thus, $\ell=1$, that is, $\|w\|_{2_s^*-\epsilon}=1$ and thus w is a minimizer of $S^V_{2_s^*-\epsilon}$. □

Remark 2 Notice that in the proof of Theorem 6, condition $S^V_{2_s^*-\epsilon}<S^\infty_{2_s^*-\epsilon}$ plays an important role. This is guaranteed by condition (V_1) with $V(x)\not\equiv V_\infty$.

By the Lagrange multiplier rule, there exists some $\lambda_\epsilon>0$ such that w_ϵ is a solution of the following equation

$$(-\Delta)^su+V(x)u=\lambda_\epsilon u^{2_s^*-1-\epsilon}\ \text{ in }\ \mathbb{R}^N.\tag{2.9}$$

By the maximum principle $w_\epsilon>0$. In fact, $w_\epsilon\ge0$, and if there exists some x_0 such that $w_\epsilon(x_0)=0$, then

$$0\le(-\Delta)^sw_\epsilon(x_0)+V(x)w_\epsilon(x_0)=c_{N,s}\mathrm{PV}\int_{\mathbb{R}^n}\frac{-u(y)}{|x_0-y|^{N+2s}}dy<0,\tag{2.10}$$

thus a contradiction.

Remark 3 If $V(x)$ is radial, by means of symmetric rearrangement techniques, we may assume that w_n is radially symmetric (cf. [32]). Thus, the minimizer w is radial.

Next we proof a Pohozaev type identity for the nonlocal equation

$$(-\Delta)^s u = f(x,u) \ \text{ in } \ \mathbb{R}^N. \tag{2.11}$$

The argument is similar to [4, 33, 34], where the Pohozaev identity for autonomous nonlocal equations was established, hence we just stress the differences.

Theorem 5 (Pohozaev identity) *Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a positive solution to (2.11) and $F(x,t) \in L^1(\mathbb{R}^N)$, where $F(x,t) = \int_0^t f(x,s)ds$. Then we have*

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} f(x,u)u dx = \int_{\mathbb{R}^N} [NF(x,u) + (x\cdot\nabla_x F(x,u))]\, dx. \tag{2.12}$$

Proof Let u be a bounded weak nontrivial solution. Suppose that w is the harmonic extension of u, see e.g. [9]. Then, w satisfies

$$\begin{cases} -\text{div}(y^{1-2s}\nabla w) = 0 & \text{in} \quad \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^s} = f((\cdot,0), w(\cdot,0)) & \text{in } \ \mathbb{R}^N \times \{y=0\}. \end{cases} \tag{2.13}$$

For $r > 0$, let

$$B_r := \{(x,y) \in \mathbb{R}^{N+1} : |(x,y)| < r\}$$

and

$$B_r^+ = B_r \cap \mathbb{R}^{N+1}_+, \ \ Q_r = B_r^+ \cup (B_r \cap (\mathbb{R}^N \times \{0\})).$$

Let $\phi \in C_0^\infty(\mathbb{R}^{N+1})$ with $0 \le \phi \le 1$, $\phi = 1$ in B_1 and $\phi = 0$ in B_2^c, $|\nabla\phi| \le 2$. For $R > 0$, let

$$\varphi_R(x,y) = \varphi\left(\frac{(x,y)}{R}\right),$$

where $\varphi := \phi|_{\mathbb{R}^{N+1}_+}$.

Then, multiplying (2.13) by $((x,y)\cdot\nabla w)\varphi_R$ and integrating in $\mathbb{R}^{N+1}_+$, we have,

$$\int_{Q_{2r}} \text{div}(y^{1-2s}\nabla w)[((x,y)\cdot\nabla w)\varphi_R]dxdy = 0. \tag{2.14}$$

From (2.14), by integrating by parts, we get

$$
\begin{aligned}
&\int_{Q_{2r}} y^{1-2s}\nabla w\nabla[((x,y)\cdot\nabla w)\varphi_R]dxdy\\
&=\int_{\partial Q_{2r}} y^{1-2s}(\nabla w\cdot\mathbf{n})[((x,y)\cdot\nabla w)\varphi_R]dS\\
&=-\lim_{y\to 0^+}\int_{B_{2R}\cap(\mathbb{R}^N\times\{y\})} y^{1-2s}\frac{\partial w}{\partial y}[((x,y)\cdot\nabla w)\varphi_R]dx\\
&=k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}(x\cdot\nabla_x w)\varphi_R\frac{\partial w}{\partial\nu^s}dx\\
&=k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}(x\cdot\nabla_x w)\varphi_R f(x,w)dx\\
&=k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}(x\cdot\nabla F(x,u))\varphi_R dx-k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}(x\cdot\nabla_x F(x,u))\varphi_R dx\\
&=-Nk_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}F(x,u)\varphi_R dx-k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}F(x,u)(x\cdot\nabla_x\varphi_R)dx\\
&\quad -k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}(x\cdot\nabla_x F(x,u))\varphi_R dx.
\end{aligned}
\tag{2.15}
$$

For the second integral in the last equality in (2.15), we have

$$
\begin{aligned}
\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})}F(x,u)(x\cdot\nabla_x\varphi_R)dx &\le C\int_{(B_{2R}\setminus B_R)\cap(\mathbb{R}^N\times\{0\})}F(x,u)\frac{|x|}{R}dx\\
&\le C\int_{(B_{2R}\setminus B_R)\cap(\mathbb{R}^N\times\{0\})}F(x,u)dx\to 0,\ \text{as } R\to+\infty
\end{aligned}
\tag{2.16}
$$

since $F(x,t)\in L^1(\mathbb{R}^N)$. As a consequence, from (2.15) we have

$$
\begin{aligned}
&\lim_{R\to+\infty}\int_{Q_{2r}} y^{1-2s}\nabla w\nabla[((x,y)\cdot\nabla w)\varphi_R]dxdy\\
&=-k_s^{-1}\int_{\mathbb{R}^N}[NF(x,u)+(x\cdot\nabla_x F(x,u))]\,dx.
\end{aligned}
\tag{2.17}
$$

On the other hand, similar to the proof of Theorem A.1 in [4], we have

$$
\lim_{R\to+\infty}\int_{Q_{2r}} y^{1-2s}\nabla w\nabla[((x,y)\cdot\nabla w)\varphi_R]dxdy=\frac{2s-N}{2}\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla w|^2dxdy.
\tag{2.18}
$$

Thanks to (2.17) and (2.18), we have

$$
\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla w|^2dxdy=k_s^{-1}\int_{\mathbb{R}^N}[NF(x,u)+(x\cdot\nabla_x F(x,u))]\,dx.
\tag{2.19}
$$

Multiply (3.26) by $w\varphi_R$ and integrate by parts to get

$$\begin{aligned}\int_{Q_{2r}} y^{1-2s}\nabla w\nabla(w\varphi_R)dxdy &= \int_{\partial Q_{2r}} y^{1-2s}(\nabla w\cdot \mathbf{n})w\varphi_R dS\\ &= k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})} \frac{\partial w}{\partial \nu^s} w\varphi_R dx\\ &= k_s^{-1}\int_{B_{2R}\cap(\mathbb{R}^N\times\{0\})} f(x,u)u\varphi_R dx\end{aligned}$$

Proceed now as above to get

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 dxdy = k_s^{-1}\int_{\mathbb{R}^N} f(x,u)udx. \tag{2.20}$$

Combining (2.18) and (2.20), we deduce that

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} f(x,u)udx = \int_{\mathbb{R}^N}\left[NF(x,u)+(x\cdot\nabla_x F(x,u))\right]dx.$$

□

Finally, we prove a crucial local estimate. This type of estimate has been studied in Proposition 3.1 and Proposition 2.4 in [38, 39]. Their methods relies on a localization method introduced by Caffarelli and Silvestre in [9, 36] and the standard Moser iteration. However, these estimates contain the extension local domain Q_R, which has no clear interpretation in terms of the original problem in $\mathbb{R}^N$ that is our context. We now give another version of this estimate based on a more direct test function method and Moser's iteration.

Theorem 6 *Assume $a(x)\in L^t_{loc}(\mathbb{R}^N)$ for some $t>\frac{N}{2s}$ and that $u\geq 0$ satisfies*

$$(-\Delta)^s u \leq a(x)u,\quad x\in\mathbb{R}^N\,.$$

Then

$$\max_{B_r} u(x)\leq C\left(\int_{B_R}|u|^{2_s^*}dx\right)^{\frac{1}{2_s^*}},\quad 0<r<R, \tag{2.21}$$

where the constant $C>0$ depends only on N, s, R, t and $\|a(x)\|_{L^t_{loc}(\mathbb{R}^N)}$.

Proof For $\beta>1$ and $T>0$, define the function

$$\varphi(t)=\begin{cases} 0 & \text{if } \quad t\leq 0,\\ t^\beta & \text{if } \ 0<t\leq T,\\ \beta T^{\beta-1}(t-T)+T^\beta & \text{if } \quad t\geq T.\end{cases} \tag{2.22}$$

Notice that $\varphi(t)$ is a convex and differentiable function and thus

$$(-\Delta)^s\varphi(u)\leq\varphi'(u)(-\Delta)^s u. \tag{2.23}$$

Let $\eta(x)=\eta(|x|)$ be a smooth cut-off function satisfying $\eta(x)=1$ in B_r, $0\leq\eta(x)\leq 1$, $\eta(x)=0$ in B_R^c and $|\eta'|\leq\frac{C}{R-r}$ for some constant $C>0$, where $0<r<R$ has to be determined. For simplicity, in the following, we denote by $\varphi:=\varphi(u(x))$ and $\varphi':=\varphi_u(x)$.

Choose as test function $\phi(x)=\eta^2\varphi\varphi'$ to obtain

$$\int_{\mathbb{R}^N}\eta^2\varphi\varphi'(x)(-\Delta)^s udx\leq\int_{\mathbb{R}^N}\eta^2\varphi\varphi' audx. \tag{2.24}$$

However, by (2.23), we have

$$\int_{\mathbb{R}^N} \eta^2\varphi(-\Delta)^s\varphi dx \leq \int_{\mathbb{R}^N} \eta^2\varphi\varphi'(-\Delta)^s u dx. \tag{2.25}$$

Using (2.24) and (2.25), the fact $u\varphi' \leq \beta\varphi$, by Sobolev embedding theorem and Cauchy inequality, we get

$$\begin{aligned} S(n,s)\|\eta\varphi\|_{2_s^*}^2 &\leq \int_{\mathbb{R}^N} \eta\varphi(-\Delta)^s[\eta\varphi]dx \\ &= \int_{\mathbb{R}^N} \eta^2\varphi(-\Delta)^s\varphi dx + \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y|^2}{|x-y|^{N+2s}}\varphi(x)\varphi(y)dxdy \\ &\leq \beta\int_{\mathbb{R}^N} a\eta^2\varphi^2 dx + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y|^2}{|x-y|^{N+2s}}\varphi(x)^2 dxdy. \end{aligned} \tag{2.26}$$

In what follows, we assume $u \in L^{\frac{2\beta t}{t-1}}(\mathbb{R}^N)$ where β has to be chosen later on. From $a(x) \in L^t_{loc}(\mathbb{R}^N)$ and $\varphi(t) \leq t^\beta$, we have

$$\int_{\mathbb{R}^N} a\eta^2 u^{2\beta} dx \leq \left[\int_{\mathbb{R}^N} (\eta a)^t dx\right]^{\frac{1}{t}} \left[\int_{\mathbb{R}^N} (\eta u^{2\beta})^{\frac{t}{t-1}} dx\right]^{\frac{t-1}{t}}. \tag{2.27}$$

Set

$$\begin{aligned} &\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx \\ &= \int_{|x|\leq R}\left(\int_{\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx + \int_{|x|\geq R}\left(\int_{\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx \\ &:= I_1 + I_2. \end{aligned} \tag{2.28}$$

We obtain

$$\begin{aligned} I_1 &\leq \frac{C^t}{(R-r)^{2t}}\int_{|x|\leq R}\left(\int_{|y-x|\leq R} \frac{1}{|x-y|^{N+2s-2}}dy\right)^t dx \\ &\quad + C^t\int_{|x|\leq R}\left(\int_{|y-x|\geq R} \frac{1}{|x-y|^{N+2s}}dy\right)^t dx \\ &= \frac{(2-2s)^{-t}C^t}{(R-r)^{2t}}(N\omega_{N-1})^{t+1}R^{N+(2-2s)t} + (2s)^{-t}C^t(N\omega_{N-1})^{t+1}R^{N-2st} \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} I_2 &= \int_{|x|\geq R}\left(\int_{|y|\leq R} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx \\ &= \int_{R\leq|x|\leq 2R}\left(\int_{|y|\leq R} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx + \int_{|x|\geq 2R}\left(\int_{|y|\leq R} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx \\ &:= I_3 + I_4 \end{aligned} \tag{2.30}$$

The estimate of I_3 is similar to the one for I_1. Finally,

$$\begin{aligned} I_4 &\leq \int_{|x|\geq 2R}\left(\int_{|y|\leq R}\frac{1}{(|x|-R)^{N+2s}}dy\right)^t dx \\ &= (N\omega_{N-1})^t R^{(N-1)t}\int_{|x|\geq 2R}(|x|-R)^{-(N+2s)t}dx \\ &\leq CR^{N-(1+2s)t} \end{aligned} \tag{2.31}$$

By combining (2.25)-(2.31), we obtain

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx \leq \frac{C}{(R-r)^{2t}}.$$

Hence

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y|^2}{|x-y|^{N+2s}}\varphi(x)^2dxdy \leq \left[\left(\int_{\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx\right]^{\frac{1}{t}}\left(\int_{\mathbb{R}^N}\eta^2u^{2\beta\frac{t}{t-1}}dx\right)^{\frac{t-1}{t}}. \tag{2.32}$$

Set

$$C := \left(\int_{\mathbb{R}^N}(\eta a(x))^t dx\right)^{\frac{1}{t}} + \left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}dy\right)^t dx\right]^{\frac{1}{t}}.$$

Combining (2.26), (2.27), (2.30) and (2.32), we get

$$S(n,s)\|\eta\varphi\|^2_{2^*_s} \leq \frac{C}{(R-r)^2}\beta\left(\int_{\mathbb{R}^N}(\eta u^{2\beta})^{\frac{t}{t-1}}dx\right)^{\frac{t-1}{t}}. \tag{2.33}$$

Now let $T\to+\infty$, to obtain

$$\left(\int_{B_r}u^{\beta 2^*_s}dx\right)^{\frac{1}{\beta 2^*_s}} \leq \left[\frac{C\beta}{(R-r)^2}\right]^{\frac{1}{2\beta}}\left(\int_{B_R}u^{2\beta\frac{t}{t-1}}dx\right)^{\frac{t-1}{2t\beta}}. \tag{2.34}$$

Since $t > \frac{N}{2s}$, we set $\beta_i = (\frac{2^*_s(t-1)}{2t})^i$, $i = 1, 2, \cdots$, $r_i = r_0 + \frac{1}{2^i}$. By iterating, we get

$$\left(\int_{B_{r_m}}u^{\beta_m 2^*_s}dx\right)^{\frac{1}{\beta_m 2^*_s}} \leq \left[\frac{2^*_s(t-1)C}{2t}\right]^{\frac{1}{2}\sum_{i=1}^m i/\beta_i}\left(\int_{B_R}u^{2^*_s}dx\right)^{\frac{1}{2^*_s}} \tag{2.35}$$

Let $m\to\infty$ to have

$$\max_{B_{r_0}} u(x) \leq C\left(\int_{B_R}u^{2^*_s}dx\right)^{\frac{1}{2^*_s}}. \tag{2.36}$$

□

3 Asymptotic Behavior of Ground States

Let w_ϵ be a positive minimizer for $S^V_{2^*_s-\epsilon}$ obtained in Theorem 6. Then, by the Lagrange multiplier rule, there exists $\lambda_\epsilon > 0$ such that w_ϵ is a solution to the equation

$$(-\Delta)^s u + V(x)u = \lambda_\epsilon u^{2^*_s-1-\epsilon} \text{ in } \mathbb{R}^N. \tag{3.1}$$

By multiplying both sides of equation (3.1) by w_ϵ and then integrating, we get $\lambda_\epsilon = S^V_{2^*_s-\epsilon}$.

The energy associated with equation (3.1) is given by

$$J_\epsilon(u) = \frac{1}{2}\|u\|^2_{s,V} - \frac{1}{2^*_s-\epsilon} S^V_{2^*_s-\epsilon}\|u\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}. \tag{3.2}$$

Thus, on the one hand we have

$$J_\epsilon(w_\epsilon) = \frac{1}{2}\|w_\epsilon\|^2_{s,V} - \frac{1}{2^*_s-\epsilon} S^V_{2^*_s-\epsilon}\|w_\epsilon\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} = \frac{2^*_s-\epsilon-2}{2(2^*_s-\epsilon)} S^V_{2^*_s-\epsilon}. \tag{3.3}$$

On the other hand, if v is a nontrivial solution of (3.1), then it satisfies $\|v\|^2_{s,V} = S^V_{2^*_s-\epsilon}\|v\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}$ and thus

$$\begin{aligned} J_\epsilon(v) &= \frac{1}{2}\|v\|^2_{s,V} - \frac{1}{2^*_s-\epsilon} S^V_{2^*_s-\epsilon}\|v\|^{2^*_s-\epsilon}_{2^*_s-\epsilon} \\ &= \frac{2^*_s-\epsilon-2}{2(2^*_s-\epsilon)} S^V_{2^*_s-\epsilon}\|v\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}. \end{aligned} \tag{3.4}$$

Besides, we have

$$S^V_{2^*_s-\epsilon} \le I_\epsilon(v) = \frac{\|v\|^2_{s,V}}{\|v\|^2_{2^*_s-\epsilon}} = \frac{S^V_{2^*_s-\epsilon}\|v\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}}{\|v\|^2_{2^*_s-\epsilon}} = S^V_{2^*_s-\epsilon}\|v\|^{2^*_s-\epsilon-2}_{2^*_s-\epsilon}, \tag{3.5}$$

which yields that $\|v\|_{2^*_s-\epsilon} \ge 1$. Thus, we have $J_\epsilon(v) \ge \frac{2^*_s-\epsilon-2}{2(2^*_s-\epsilon)} S^V_{2^*_s-\epsilon}$ by (3.4). This fact together with (3.3) implies that w_ϵ is a ground state of equation (3.1). Furthermore, if we set

$$u_\epsilon = \left(S^V_{2^*_s-\epsilon}\right)^{-\frac{1}{1-2^*_s+\epsilon}} w_\epsilon$$

then, u_ϵ is a ground state of equation (1.1). Observe that $I_\epsilon(u_\epsilon) = S^V_{2^*_s-\epsilon}$.

For each fixed $\epsilon \in (0, 2^*_s-2)$, by means of the mountain-pass theorem, (1.1) admits a positive ground state (see e.g. Theorem 1.4 in [23]). However, we don't know whether the mountain-pass solution and the minimal solution u_ϵ obtained above do agree since uniqueness is not known. Anyway, in what follows, we will focus on the minimal solution u_ϵ. We remark that in the special case $V(x) = 1$, the ground state is unique and radially symmetric, see [24].

Lemma 1 *For any fixed $\epsilon \in (0, 2^*_s-2)$, any nontrivial u_ϵ of (1.1) satisfies*

$$\|u_\epsilon\|_\infty \ge V_0^{\frac{1}{2^*_s-2}}. \tag{3.6}$$

Proof Since u_ϵ enjoys (1.1), we have

$$\|u_\epsilon\|^2_{s,V_0} \le \|u_\epsilon\|^2_{s,V} = \|u_\epsilon\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}$$

which yields $V_0\|u_\epsilon\|^2_2 \le \|u_\epsilon\|^{2^*_s-\epsilon}_{2^*_s-\epsilon}$, that is,

$$\int_{\mathbb{R}^N} u_\epsilon^2(V_0 - u_\epsilon^{2^*_s-2-\epsilon})dx \le 0.$$

Thus, we get $\|u_\epsilon\|_\infty \ge V_0^{\frac{1}{2^*_s-2-\epsilon}} \ge V_0^{\frac{1}{2^*_s-2}}$, and the result follows. □

We next need the following result proved in [14].

Lemma 2 [14] *The infimum in* (1.5) *is attained, that is*

$$S = \frac{[\widetilde{u}]_s^2}{\|\widetilde{u}\|_{2_s^*}^2},$$

where

$$\widetilde{u}(x) := \kappa(\mu^2 + |x - x_0|^2)^{\frac{2s-N}{2}}, \quad x \in \mathbb{R}^N$$

with $\kappa \in \mathbb{R} \setminus \{0\}$, $\mu > 0$ *and* $x_0 \in \mathbb{R}^N$ *fixed constant. Equivalently, the function* $\overline{u}$ *defined by* $\overline{u}(x) := \frac{\widetilde{u}}{\|\widetilde{u}\|_{2_s^*}}$ *is such that*

$$S = \inf_{u \in D^s(\mathbb{R}^N)} \{[u]_s^2 : \|u\|_{2_s^*} = 1\}. \tag{3.7}$$

Furthermore, the function

$$u^*(x) := \overline{u}\left(S^{-\frac{1}{2s}} x\right), \quad x \in \mathbb{R}^N$$

is a solution of

$$(-\Delta u)^s u = |u|^{2_s^*-2} u, \quad in \ \mathbb{R}^N$$

satisfying the property

$$\|u\|_{2_s^*}^{2_s^*} = S^{\frac{N}{2s}}.$$

Proposition 1 $\lim_{\epsilon \to 0^+} S_{2_s^*-\epsilon}^V = S.$

Proof Choose $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$ such that $\inf_{0<\epsilon<2_s^*-2} \|\phi\|_{2_s^*-\epsilon} > 0$, to get

$$0 < S_{2_s^*-\epsilon}^V \leq \frac{\|\phi\|_{s,V}^2}{\inf_{0<\epsilon<2_s^*-2} \|\phi\|_{2_s^*-\epsilon}^2} < +\infty. \tag{3.8}$$

This means that $\{S_{2_s^*-\epsilon}^V\}$ is uniformly bounded with respect to ϵ. Next, we further prove that $\lim_{\epsilon \to 0^+} S_{2_s^*-\epsilon}^V = S$.

Let $w \in H_V^s(\mathbb{R}^n)$ be such that $\|w\|_{2_s^*-\epsilon} = 1$ and $\|w\|_{s,V}^2 = S_{2_s^*-\epsilon}^V$. Then,

$$\|w\|_2^2 \leq \|w\|_s^2 \leq \max\left\{1, \frac{1}{V_0}\right\} \|w\|_{s,V}^2 =: C S_{2_s^*-\epsilon}^V. \tag{3.9}$$

By Hölder's inequality we have

$$1 = \|w\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} \leq \|w\|_2^{\frac{2\epsilon}{2_s^*-2}} \|w\|_{2_s^*}^{\frac{2_s^*(2_s^*-2-\epsilon)}{2_s^*-2}} \leq (C S_{2_s^*-\epsilon}^V)^{\frac{\epsilon}{2_s^*-2}} \|w\|_{2_s^*}^{\frac{2_s^*(2_s^*-2-\epsilon)}{2_s^*-2}}. \tag{3.10}$$

Thanks to (3.8) and (3.10), we get

$$1 \leq \liminf_{\epsilon \to 0^+} \|w\|_{2_s^*}. \tag{3.11}$$

On the other hand, by (1.5), we have

$$S \leq \frac{[w]_s^2}{\|w\|_{2_s^*}^2} \leq \frac{\|w\|_{s,V}^2}{\|w\|_{2_s^*}^2} = \frac{S_{2_s^*-\epsilon}^V}{\|w\|_{2_s^*}^2}, \tag{3.12}$$

Thanks to (3.11) and (3.12), we have

$$S \le \liminf_{\epsilon\to 0^+} S^V_{2^*_s-\epsilon}\,. \tag{3.13}$$

Next we prove that

$$\limsup_{\epsilon\to 0^+} S^V_{2^*_s-\epsilon} \le S. \tag{3.14}$$

Once (3.14) is proved, the result follows from (3.13).

Let

$$U_\epsilon(x) = \epsilon^{\frac{2s-N}{2}} u^*(x/\epsilon),$$

where u^* is defined in Lemma 12. Furthermore, let $\eta(x) \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \eta(x) \le 1$ in $\mathbb{R}^N$, $\eta(x) \equiv 1$ in $B_{1/2}$ and $\eta(x) \equiv 0$ in B_1^c. Set $u_\epsilon(x) := \eta(x)U_\epsilon(x)$, $x \in \mathbb{R}^N$. Then, as $\epsilon \to 0^+$ we have

$$[u_\epsilon]_s^2 \le S^{\frac{N}{2s}} + O(\epsilon^{N-2s}), \tag{3.15}$$

$$\int_{\mathbb{R}^N} V(x)|u_\epsilon(x)|^2 dx = C_s\epsilon^{2s} + O(\epsilon^{N-2s}), \quad \text{if } N > 4s \tag{3.16}$$

and

$$\|u_\epsilon(x)\|_{2^*_s}^{2^*_s} = S^{\frac{N}{2s}} + O(\epsilon^N), \tag{3.17}$$

for some positive constant C_s depending only on s, see Propositions 21 and 22 in [35] or Lemma 2.4 in [20]. By Taylor's expansion we get

$$\|u_\epsilon(x)\|^2_{2^*_s-\epsilon} = \|u_\epsilon(x)\|^2_{2^*_s} + O(\epsilon). \tag{3.18}$$

Hence, we deduce from (3.15)–(3.18) that

$$\limsup_{\epsilon\to 0^+} S^V_{2^*_s-\epsilon} \le \limsup_{\epsilon\to 0^+} \frac{\|u_\epsilon\|^2_{s,V}}{\|u_\epsilon\|^2_{2^*_s-\epsilon}} \le S, \tag{3.19}$$

which implies (3.14). □

Recalling that u_ϵ is a solution to (1.1) and that u_ϵ attains $S^V_{2^*_s-\epsilon}$, we get

$$\|u_\epsilon\|^2_{s,V} = \|u_\epsilon\|^{2^*_s-\epsilon}_{2^*_s-\varepsilon} \quad\text{and}\quad \|u_\epsilon\|^2_{s,V} = S^V_{2^*_s-\epsilon}\|u_\epsilon\|^2_{2^*_s-\varepsilon}. \tag{3.20}$$

So, we have

$$\|u_\epsilon\|_{s,V} = \left(S^V_{2^*_s-\varepsilon}\right)^{\frac{2^*_s-\epsilon}{2(2^*_s-2-\epsilon)}} \quad\text{and}\quad \|u_\epsilon\|_{2^*_s-\varepsilon} = \left(S^V_{2^*_s-\varepsilon}\right)^{\frac{1}{2^*_s-2-\epsilon}}. \tag{3.21}$$

These facts together with Lemma 13 imply the following

Lemma 3

$$\lim_{\epsilon\to 0^+} \|u_\epsilon\|_{s,V} = S^{\frac{N}{4s}}, \qquad \lim_{\epsilon\to 0^+} \|u_\epsilon\|_{2^*_s-\epsilon} = S^{\frac{N-2s}{4s}}. \tag{3.22}$$

Now let us prove that $\|u_\epsilon\|_\infty$ blows up as $\epsilon \to 0^+$, namely

Lemma 4 $\lim_{\epsilon\to 0^+} \|u_\epsilon\|_\infty = +\infty.$

Proof Suppose by contradiction the claim does not hold true. Then, there exists a sequence $\epsilon_j \to 0^+$ such that $\|u_{\epsilon_j}\|_\infty$ stays bounded. Let x_{ϵ_j} be a maximum point of u_{ϵ_j}. Define $w_{\epsilon_j}(x) = u_{\epsilon_j}(x + x_{\epsilon_j})$, then $\|w_{\epsilon_j}\|_\infty$ is bounded as well and

$$(-\Delta)^s w_{\epsilon_j} = -V(x + x_{\epsilon_j})w_{\epsilon_j} + w_{\epsilon_j}^{2^*_s-1-\epsilon_j} \quad \text{in } \mathbb{R}^N.$$

Now, since $V \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have that $\|(-\Delta)^s w_{\epsilon_j}\|_\infty$ is uniformly bounded with respect to ϵ_j. As a consequence of this fact and of standard regularity results (see e.g. Lemma 4.4 in [8]), we deduce that $\|w_{\epsilon_j}\|_{C^{2,\alpha}}$ is uniformly bounded with respect to ϵ_j, for some $\alpha \in (0, 1)$.

By (3.22), $[w_{\epsilon_j}]_s = [u_{\epsilon_j}]_s$ and $\|w_{\epsilon_j}\|_2 = \|u_{\epsilon_j}\|_2$ are bounded. Thus, $\{w_{\epsilon_j}\}$ is bounded in $H^s_V(\mathbb{R}^N)$. Up to extracting a subsequence, which we still denote by $\{w_{\epsilon_j}\}$, one has $w_{\epsilon_j} \rightharpoonup w_0$ in $H^s_V(\mathbb{R}^N)$, $w_{\epsilon_j} \to w_0$ a.e. in $\mathbb{R}^N$ and $w_{\epsilon_j} \to w_0$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$. Moreover, by (3.6) one has $w_0(0) \geq V_0^{\frac{1}{2^*_s-2}} > 0$.

Let us now distinguish two cases:

Case 1. $\{x_{\epsilon_j}\}_j$ is bounded. Up to a subsequence, we may assume that $x_{\epsilon_j} \to x_0$. Then, w_0 is a nonnegative classical solution of

$$(-\Delta)^s w_0 = -V(x + x_0)w_0 + w_0^{2^*_s-1} \quad \text{in} \quad \mathbb{R}^N. \tag{3.23}$$

It follows from the maximum principle that $w_0 > 0$. Thus, by Lemma 14 we have

$$S \leq \frac{[w_0]_s^2}{\|w_0\|_{2^*_s}^2} < \|w_0\|_{2^*_s}^{2^*_s-2} \leq \liminf_{j\to\infty} \|w_{\epsilon_j}\|_{2^*_s}^{2^*_s-2} = S, \tag{3.24}$$

which is a contradiction.

Case 2. $\{x_{\epsilon_j}\}_j$ is unbounded. Up to a subsequence, we may assume that $x_{\epsilon_j} \to \infty$. Then, by (3.20) and the dominated convergence theorem, we have

$$\begin{aligned} [w_0]_s^2 &\leq -V_0\|w_0\|_2^2 + \|w_0\|_{2^*_s}^{2^*_s} + \lim_{j\to\infty} \int_{\mathbb{R}^N} (V(x) - V(x + x_{\epsilon_j}))w_{\epsilon_j} w_0 dx \\ &\leq -V_0\|w_0\|_2^2 + \|w_0\|_{2^*_s}^{2^*_s}, \end{aligned} \tag{3.25}$$

which yields $[w_0]_s^2 < \|w_0\|_{2^*_s}^{2^*_s}$ and similarly to the proof of (3.24), we get a contradiction. □

As $\epsilon \to \epsilon_0 \in (0, \frac{2^*_s-2}{2})$, we have that $\{u_\epsilon\}$ is uniformly bounded with respect to ϵ, as established in the following

Lemma 5 *There exists $K > 0$, which does not depend on ϵ, such that any solutions u_ϵ of* (1.1) *satisfies $\|u_\epsilon\|_\infty \leq K$ as $\epsilon \to \epsilon_0$.*

Proof The claim can be achieved via Moser's iteration. Indeed, let w_ϵ be the harmonic extension of u_ϵ, see e.g. [9]. Then, w_ϵ satisfies

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_\epsilon) = 0 & \text{in} \quad \mathbb{R}^{N+1}_+, \\ \frac{\partial w_\epsilon}{\partial \nu^s} = -V(\cdot)w_\epsilon(\cdot, 0) + w_\epsilon^{2^*_s-1-\epsilon}(\cdot, 0) & \text{in} \ \ \mathbb{R}^N \times \{y = 0\}, \end{cases} \tag{3.26}$$

where

$$\frac{\partial w_\epsilon}{\partial \nu^s} := -\frac{1}{k_s} \lim_{y\to 0^+} y^{1-2s} \frac{\partial w_\epsilon}{\partial y}(x, y).$$

and $k_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

Following Corollary 2.1 in [1], for each $L > 0$, we set

$$w_{\epsilon,L}(x, y) = \begin{cases} w_\epsilon(x, y) & \text{if} \ \ w_\epsilon(x, y) \leq L, \\ L & \text{if} \ \ w_\epsilon(x, y) \geq L, \end{cases} \quad u_{\epsilon,L}(x) = w_{\epsilon,L}(x, 0), \tag{3.27}$$

and $\psi_{\epsilon,L}=w_{\epsilon,L}^{2(\beta-1)}w_\epsilon$, where $\beta>1$ to be determined later on. By testing with $\psi_{\epsilon,L}$, we get

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s}\nabla w_\epsilon \nabla \psi_{\epsilon,L}dxdy = k_s^{-1}\int_{\mathbb{R}^N}[-V(x)u_\epsilon(x)+u_\epsilon^{2_s^*-1-\epsilon}(x)]\psi_{\epsilon,L}(x,0)dx. \tag{3.28}$$

Thus,

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s}\nabla w_\epsilon \nabla (w_{\epsilon,L}^{2(\beta-1)}w_\epsilon)dxdy \le k_s^{-1}\int_{\mathbb{R}^N} u_\epsilon^{2_s^*-\epsilon}u_{\epsilon,L}^{2(\beta-1)}dx. \tag{3.29}$$

Note that

$$\nabla w_\epsilon \nabla (w_{\epsilon,L}^{2(\beta-1)}w_\epsilon) = \begin{cases} (2\beta-1)w_{\epsilon,L}^{2(\beta-1)}(x,y)|\nabla w_\epsilon|^2 & \text{if } \ w_\epsilon(x,y)\le L, \\ L^{2\beta-1}|\nabla w_\epsilon|^2 & \text{if } \ w_\epsilon(x,y)\ge L. \end{cases} \tag{3.30}$$

Thus, from (3.29), Sobolev imbedding (see e.g. (2.9) in [13]) and Hölder's inequality,

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|u_{\epsilon,L}^{\beta-1}u_\epsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}} &\le C(N,s)\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla(w_{\epsilon,L}^{\beta-1}w_\epsilon)|^2dxdy \\ &\le \beta C(N,s)\int_{\mathbb{R}^N} u_\epsilon^{2_s^*-2-\epsilon}u_\epsilon^2 u_{\epsilon,L}^{2(\beta-1)}dx \\ &\le \beta C(N,s)\left(\int_{\mathbb{R}^N}u_\epsilon^{2_s^*}dx\right)^{\frac{2_s^*-2-\epsilon}{2_s^*}}\left(\int_{\mathbb{R}^N}(u_{\epsilon,L}^{\beta-1}u_\epsilon)^{\frac{22_s^*}{2+\epsilon}}dx\right)^{\frac{2+\epsilon}{2_s^*}}. \end{aligned} \tag{3.31}$$

Since $\|u_\epsilon\|_{2_s^*}$ is bounded, from (3.31) we get

$$\|u_{\epsilon,L}^{\beta-1}u_\epsilon\|_{2_s^*}^2 \le \beta C(N,s)\left(\int_{\mathbb{R}^N}(u_{\epsilon,L}^{\beta-1}u_\epsilon)^{\frac{22_s^*}{2+\epsilon}}dx\right)^{\frac{2+\epsilon}{2_s^*}}. \tag{3.32}$$

As $u_\epsilon\in L^{\frac{22_s^*\beta}{\epsilon}}(\mathbb{R}^N)$, by using the fact that $w_{\epsilon,L}\le w_\epsilon$, we get

$$\|u_{\epsilon,L}^{\beta-1}u_\epsilon\|_{2_s^*}^2 \le \beta C(N,s)\left(\int_{\mathbb{R}^N}u_\epsilon^{\frac{22_s^*\beta}{2+\epsilon}}dx\right)^{\frac{2+\epsilon}{2_s^*}}. \tag{3.33}$$

Let $L\to+\infty$ and apply Fatou's lemma to get

$$\|u_\epsilon\|_{2_s^*\beta}^2 \le \beta^{\frac{1}{\beta}}C^{\frac{1}{\beta}}(N,s)\|u_\epsilon\|_{\frac{22_s^*}{2+\epsilon}\beta}^2. \tag{3.34}$$

The claim now follows by iteration: let $\beta_i=(\frac{2+\epsilon}{2})^i$, $i=1,2,\cdots$, then

$$\|u_\epsilon\|_{2_s^*\beta_{m+1}} \le \left(\frac{2+\epsilon}{2}\right)^{\frac{1}{2}\sum_{i=1}^m i(\frac{2+\epsilon}{2})^{-i}} C^{\frac{1}{2}\sum_{i=1}^m(\frac{2+\epsilon}{2})^{-i}}(N,s)\|u_\epsilon\|_{2_s^*}. \tag{3.35}$$

Passing to the limit as $m\to+\infty$ in (3.35), we have

$$\|u_\epsilon\|_\infty \le C\|u_\epsilon\|_{2_s^*}.$$

which concludes the proof. □

Lemma 6 *Let* $\epsilon_0>0$, *then* $\limsup_{\epsilon\to\epsilon_0} S_{2_s^*-\epsilon}^V \le S_{2_s^*-\epsilon_0}^V$.

Proof Let $\phi > 0$ be such that $S^V_{2^*_s-\epsilon_0} = \frac{\|\phi\|^2_{s,V}}{\|\phi\|^2_{2^*_s-\epsilon_0}}$. Then,

$$\int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon} dx = \int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon_0} dx + (\epsilon-\epsilon_0) \int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon_0+t(\epsilon_0-\epsilon)} \ln\phi\, dx, \tag{3.36}$$

where $t \in (0,1)$. Since $|\phi^\alpha \ln\phi| \le C$ for any $\alpha > 0$ as $\phi \to 0^+$ and $\ln\phi \le 1+\phi$ as $\phi \ge 1$, recalling that $\|\phi\|_\infty$ is bounded by Lemma 16, we get

$$\int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon} dx = \int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon_0} dx + O(\epsilon-\epsilon_0). \tag{3.37}$$

Now, by (3.37) and the definition of $S^V_{2^*_s-\epsilon}$, we have

$$\begin{aligned} \limsup_{\epsilon\to\epsilon_0} \left(S^V_{2^*_s-\epsilon}\right)^{\frac{2^*_s-\epsilon_0}{2}} &\le \limsup_{\epsilon\to\epsilon_0} \frac{\|\phi\|^{2^*_s-\epsilon_0}_{s,V}}{\|\phi\|^{2^*_s-\epsilon_0}_{2^*_s-\epsilon}} \\ &= \limsup_{\epsilon\to\epsilon_0} \frac{\|\phi\|^{2^*_s-\epsilon_0}_{s,V}}{[\|\phi\|_{2^*_s-\epsilon_0} + O(\epsilon-\epsilon_0)]^{2^*_s-\epsilon_0}} = \left(S^V_{2^*_s-\epsilon_0}\right)^{\frac{2^*_s-\epsilon_0}{2}}. \end{aligned} \tag{3.38}$$

The proof of Lemma 17 is complete. □

Let x_ϵ be the global maximum point of u_ϵ and let $\mu_\epsilon > 0$ be such that

$$u_\epsilon(x_\epsilon) = \|u_\epsilon\|_\infty = \mu_\epsilon^{-\frac{2s}{2^*_s-2-\epsilon}}.$$

Clearly, from Lemma 15 $\mu_\epsilon \to 0$ as $\epsilon \to 0^+$. Set

$$v_\epsilon(x) = \mu_\epsilon^{\frac{2s}{2^*_s-2-\epsilon}} u_\epsilon(x_\epsilon + \mu_\epsilon x).$$

Then $0 < v_\epsilon(x) \le 1$, $v_\epsilon(0) = 1$ and v_ϵ satisfies the following

$$(-\Delta)^s v_\epsilon + \mu_\epsilon^{2s} V(x_\epsilon + \mu_\epsilon x) v_\epsilon = v_\epsilon^{2^*_s-1-\epsilon} \quad \text{in } \mathbb{R}^N. \tag{3.39}$$

We have that $\|(-\Delta)^s v_\epsilon\|_\infty$ is uniformly bounded with respect to ϵ. As a consequence of this fact and regularity results, we deduce that also $\|v_\epsilon\|_{C^{2,\alpha}}$ is uniformly bounded with respect to ϵ, for some $\alpha \in (0,1)$. Similarly to the proof of Lemma 15, there exists a sequence ϵ, still denoted by v_ϵ, such that $v_\epsilon \to U$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, where U is the positive solution of equation

$$(-\Delta)^s u = u^{2^*_s-1} \quad \text{in } \mathbb{R}^N \tag{3.40}$$

and $U(0) = \|U\|_\infty = 1$. From Theorem 1.2 in [11],

$$U(x) = \left(1 + \frac{|x|^2}{\lambda^2}\right)^{\frac{2s-N}{2}}, \quad \text{where} \quad \lambda = 2\left(\frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)}\right)^{\frac{1}{2}}. \tag{3.41}$$

Since

$$S = \frac{[U]_s^2}{\|U\|^2_{2^*_s}} = \|U\|^{2^*_s-2}_{2^*_s} = [U]_s^{2-\frac{4}{2^*_s}},$$

we conclude that

$$\|U\|^{2^*_s}_{2^*_s} = [U]_s^2 = S^{\frac{N}{2s}}.$$

By Lemma 14, we have

$$
\begin{aligned}
S^{\frac{N}{2s}} &= [U]_s^2 \le \liminf_{\epsilon\to 0^+}[v_\epsilon]_s^2 \\
&\le \limsup_{\epsilon\to 0^+}[v_\epsilon]_s^2 \\
&\le \limsup_{\epsilon\to 0^+}\left[\int_{\mathbb{R}^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^2}{|x-y|^{N+2s}}dxdy+\int_{\mathbb{R}^N}V(x_\epsilon+\mu_\epsilon x)v_\epsilon^2dx\right] \\
&= \limsup_{\epsilon\to 0^+}\mu_\epsilon^{\frac{(N-2s)\epsilon}{2_s^*-2-\epsilon}}\left[\int_{\mathbb{R}^N}\frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{N+2s}}dxdy+\int_{\mathbb{R}^N}V(x)u_\epsilon^2dx\right] \\
&\le \limsup_{\epsilon\to 0^+}\left[\int_{\mathbb{R}^N}\frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{N+2s}}dxdy+\int_{\mathbb{R}^N}V(x)u_\epsilon^2dx\right] \\
&= S^{\frac{N}{2s}}.
\end{aligned}
\tag{3.42}
$$

Finally, form Lemma 14 and (3.42), we obtain the following convergences

Lemma 7 $[v_\epsilon - U]_s \to 0$, $\|v_\epsilon - U\|_{2_s^*} \to 0$, $[v_\epsilon]_s^2 \to S^{\frac{N}{2s}}$, *and* $\mu_\epsilon^\epsilon \to 1$ *as* $\epsilon \to 0^+$.

Proposition 2 *If* $x_\epsilon \to x_0$, *as* $\epsilon \to 0^+$. *Then*

$$|u_\epsilon|^{2_s^*}(x) \to S^{\frac{N}{2s}}\delta(x-x_0), \quad \epsilon \to 0$$

in the sense of distributions.

Proof For any $\phi \in C_0^\infty(\mathbb{R}^N)$, we get

$$
\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}|u_\epsilon|^{2_s^*}\phi dx &= \lim_{\epsilon\to 0^+}\left[\mu_\epsilon^{-\frac{\epsilon N(N-2s)}{4s-\epsilon(N-2s)}}\int_{\mathbb{R}^N}|v_\epsilon|^{2_s^*}\phi(x_\epsilon+\mu_\epsilon x)dx\right] \\
&= \phi(x_0)\int_{\mathbb{R}^N}|U|^{2_s^*}dx \\
&= \phi(x_0)S^{\frac{N}{2s}}.
\end{aligned}
\tag{3.43}
$$

□

Notice that up to now we do not know wether the global maximum point x_ϵ turns out to be bounded or unbounded.

Lemma 8 *Suppose that* $\{x_\epsilon\}$ *is bounded. Then,*

$$\sup_{\epsilon\in\left(0,\frac{2_s^*-2}{2}\right)}\int_{|x|\ge R}u_\epsilon^{2_s^*}dx \to 0 \text{ as } R \to +\infty. \tag{3.44}$$

Proof Assume by contradiction that Eq. 3.44 does not hold. Then, there exist two sequences $\epsilon_j \to \epsilon_0$ and $R_j \to +\infty$ such that

$$\int_{|x|\ge R_j}u_{\epsilon_j}^{2_s^*}dx \ge \delta, \tag{3.45}$$

for some $\delta > 0$ and $j = 1, 2, \cdots$. We distinguish two cases:

Case 1. $\varepsilon_0 > 0$. By (3.22), $\{u_\epsilon\}$ is bounded in $H_V^s(\mathbb{R}^N)$, passing to a subsequence $\{u_{\epsilon_j}\}$ if necessary, we may assume $u_{\epsilon_j} \rightharpoonup u_{\epsilon_0}$ in $H_V^s(\mathbb{R}^N)$. On the other hand, from Lemma 16, we know that $\|u_{\varepsilon_j}\|_\infty$ is bounded and by regularity we deduce that $\|u_{\epsilon_j}\|_{C^{2,\alpha}}$ is uniformly bounded with respect to ϵ_j, for some $\alpha \in (0, 1)$. Up to extracting again a subsequence, still denoted by $\{u_{\epsilon_j}\}$, we have $u_{\epsilon_j} \to u_{\epsilon_0}$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$. Thus, u_{ϵ_0} is a classical nonnegative solution of the equation

$$(-\Delta)^s u + V(x)u = u^{2_s^*-1-\epsilon_0} \ \text{ in } \ \mathbb{R}^N. \tag{3.46}$$

By (3.6), we get $u_{\epsilon_0}(x) \not\equiv 0$. Moreover, if there is $x_0 \in \mathbb{R}^N$ such that $u_{\epsilon_0}(x_0) = 0$, then from (3.46), $(-\Delta)^s u(x_0) = 0$. However, by the very definition

$$(-\Delta)^s u(x_0) = c_{N,s}\text{PV} \int_{\mathbb{R}^N} \frac{-u(y)}{|x_0 - y|^{N+2s}} dy < 0,$$

which is a contradiction. Thus, $u_{\epsilon_0}(x) > 0$ for all $x \in \mathbb{R}^N$.

Now, by (3.21) and Lemma 17, observe that

$$\begin{aligned} S^V_{2_s^*-\epsilon_0} &\le \frac{\|u_{\epsilon_0}\|^2_{s,V}}{\|u_{\epsilon_0}\|^2_{2_s^*-\epsilon_0}} = \|u_{\epsilon_0}\|_{s,V}^{2-\frac{4}{2_s^*-\epsilon_0}} \\ &\le \liminf_{j\to\infty} \|u_{\epsilon_j}\|_{s,V}^{2-\frac{4}{2_s^*-\epsilon_0}} \\ &= \liminf_{j\to\infty} S^V_{2_s^*-\epsilon_j} \\ &\le \limsup_{\epsilon\to\epsilon_0} S^V_{2_s^*-\epsilon} \\ &\le S^V_{2_s^*-\epsilon_0}. \end{aligned} \tag{3.47}$$

Therefore, we get

$$\lim_{j\to\infty} S^V_{2_s^*-\epsilon_j} = S^V_{2_s^*-\epsilon_0}. \tag{3.48}$$

Similarly to the proof of Lemma 17, from (3.48) we get $\|u_{\epsilon_j}\|_{s,V} \to \|u_{\epsilon_0}\|_{s,V}$ as $j \to +\infty$ and hence $u_{\epsilon_j} \to u_{\epsilon_0}$ in $L^{2_s^*}(\mathbb{R}^N)$ as $j \to +\infty$. This contradicts (3.45).

Case 2. $\varepsilon_0 = 0$. Thanks to Lemma 18, we obtain a contradiction from (3.45). Indeed, we have

$$\begin{aligned} \delta &\le \int_{|x|\ge R_j} u_{\epsilon_j}^{2_s^*} dx = \mu_{\epsilon_j}^{N-\frac{2s2_s^*}{2_s^*-2-\epsilon_j}} \int_{|x_{\epsilon_j}+\mu_{\epsilon_j}x|\ge R_j} v_{\epsilon_j}^{2_s^*}(x)dx \\ &\le (\mu_{\epsilon_j}^{\epsilon_j})^{\frac{N(N-2)}{4s-N\epsilon_j+2s\epsilon_j}} \int_{|x|\ge \frac{R_j-|x_{\epsilon_j}|}{\mu_{\epsilon_j}}} v_{\epsilon_j}^{2_s^*}(x)dx \\ &\to 0 \text{ as } j \to \infty, \end{aligned} \tag{3.49}$$

since $\mu_{\epsilon_j}^{\epsilon_j} \to 1$, $\frac{R_j-|x_{\epsilon_j}|}{\mu_{\epsilon_j}} \to +\infty$ and $v_{\epsilon_j} \to U$ in $L^{2_s^*}(\mathbb{R}^N)$ as $j \to \infty$.

The proof is now complete. □

Lemma 9 *Suppose that $\{x_\epsilon\}$ is unbounded. Then, for any fixed $R > 0$,*

$$\limsup_{\epsilon\to 0^+} \int_{|x-x_\epsilon|\ge R} u_\epsilon^{2_s^*} dx = 0. \tag{3.50}$$

Proof The proof is similar to Lemma 20. Suppose that the claim is not true. Then there exist a sequence $\epsilon_j \to \epsilon_0$ such that

$$\int_{|x-x_{\epsilon_j}|\geq R} u_{\epsilon_j}^{2_s^*} dx \geq \delta, \tag{3.51}$$

for some $\delta > 0$ and $j = 1, 2, \cdots$.

The proof of the case $\epsilon_0 > 0$ is similar to Lemma 20. For $\epsilon_0 = 0$, we have

$$\begin{aligned} \delta &\leq \int_{|x-x_{\epsilon_j}|\geq R} u_{\epsilon_j}^{2_s^*} dx = \mu_{\epsilon_j}^{N-\frac{2s2_s^*}{2_s^*-2-\epsilon_j}} \int_{|x|\geq \frac{R}{\mu_{\epsilon_j}}} v_{\epsilon_j}^{2_s^*}(x)dx \\ &= (\mu_{\epsilon_j}^{\epsilon_j})^{\frac{N(N-2)}{4s-N\epsilon_j+2s\epsilon_j}} \int_{|x|\geq \frac{R}{\mu_{\epsilon_j}}} v_{\epsilon_j}^{2_s^*}(x)dx \\ &\to 0, \text{ as } j \to \infty, \end{aligned} \tag{3.52}$$

since $\mu_{\epsilon_j}^{\epsilon_j} \to 1$, $\frac{R}{\mu_{\epsilon_j}} \to +\infty$ and $v_{\epsilon_j} \to U$ in $L^{2_s^*}(\mathbb{R}^N)$ as $j \to \infty$. □

The following lemmas will play an important role in our analysis.

Lemma 10 *Assume that* $\{x_\epsilon\}$ *stays bounded. Then, there exist constants* $C, R > 0$ *independent of* ϵ*, such that*

$$|u_\epsilon(x)| \leq \frac{C}{|x|^{N+2s}}, \quad \textit{for } |x| \geq R. \tag{3.53}$$

Proof We observe that

$$(-\Delta)^s u_\epsilon \leq u_\epsilon^{2_s^*-2-\epsilon} u_\epsilon .$$

Since $u_\epsilon^{2_s^*-2-\epsilon} \in L^t_{loc}(\mathbb{R}^N)$ for some $t > \frac{N}{2s}$, from Theorem 10 we have

$$\max_{B_r(y)} u_\epsilon(x) \leq C\left(\int_{B_R(y)} |u_\epsilon|^{2_s^*} dx\right)^{\frac{1}{2_s^*}}, \quad \forall y \in \mathbb{R}^N, 0 < r < R, \tag{3.54}$$

where C is independent of ϵ. Thus, we conclude from (3.44) and (3.54) that

$$\sup_{\epsilon\in\left(0,\frac{2_s^*-2}{2}\right)} u_\epsilon(y) \to 0, \quad \text{as } |y| \to +\infty. \tag{3.55}$$

This fact together with Lemma C.2 in [24] imply

$$|u_\epsilon(x)| \leq \frac{C}{|x|^{N+2s}}. \tag{3.56}$$

Actually, we first fix $\epsilon > 0$ to applying Lemma C.2 in [24], and then we take the supremum with respect to ϵ. Finally, from Lemmas 17 and 14, we get (3.56). See also [22]. □

Lemma 11 *Suppose that* $\{x_\epsilon\}$ *is unbounded. Then there exists a constant* $C > 0$ *independent of* ϵ *such that for small* $\epsilon > 0$,

$$|u_\epsilon(x)| \leq \frac{C}{|x - x_\epsilon|^{N+2s}}, \quad \textit{for } |x - x_\epsilon| \geq R, \tag{3.57}$$

for any $R > 0$.

Proof From (3.50), for any $\delta, R > 0$, there exists a small $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and $|y - x_\epsilon| \geq R$, then

$$u_\epsilon(y) \leq \delta. \tag{3.58}$$

Let $w_\epsilon(x) = u_\epsilon(y)$, $y = V_0^{-\frac{1}{2s}} x$. Then $w_\epsilon(x)$ enjoys the following

$$(-\Delta)^s w_\epsilon + V(y)V_0^{-1} w_\epsilon = V_0^{-1} w_\epsilon^{2_s^*-1-\epsilon}. \tag{3.59}$$

Furthermore, by condition (V_1), if we choose $\delta > 0$ sufficiently small and $R_1 > 0$ large enough, we have

$$(-\Delta)^s w_\epsilon + w_\epsilon = f_\epsilon(x) := \left[1 - V(y)V_0^{-1}\right] w_\epsilon(x) + V_0^{-1} w_\epsilon^{2_s^*-1-\epsilon}(x) \leq 0 \tag{3.60}$$

for small $\epsilon > 0$, $|x| \geq R_1$ and $|V_0^{-\frac{1}{2s}} x - x_\epsilon| \geq R$.

Borrowing some results from [23], we also have

$$w_\epsilon(x) = \mathcal{K} * f_\epsilon(x) = \int_{\mathbb{R}^N} \mathcal{K}(x-y) f_\epsilon(y) dy, \tag{3.61}$$

where $\mathcal{K}$ is the Bessel kernel and which enjoys the following properties:

(K_1) $\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$;

(K_2) There is $C_1, C_2 > 0$ such that

$$\mathcal{K}(x) \leq \frac{C_1}{|x|^{N+2s}}, \quad \text{if } |x| \geq 1 \tag{3.62}$$

and

$$\mathcal{K}(x) \leq \frac{C_2}{|x|^{N-2s}}, \quad \text{if } |x| \leq 1. \tag{3.63}$$

From (3.60) and (3.61) we have

$$w_\epsilon(x) \leq \int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x-y) f_\epsilon(y) dy + \int_{\{|y| \leq R_1\}} \mathcal{K}(x-y) f_\epsilon(y) dy \tag{3.64}$$

Note that $|V_0^{-\frac{1}{2s}} y - x_\epsilon| > R \Leftrightarrow |y - V_0^{\frac{1}{2s}} x_\epsilon| > V_0^{\frac{1}{2s}} R$. Since $\{x_\epsilon\}$ is unbounded, then there exists $0 < \epsilon_1 < \epsilon_0$ such that $|x_\epsilon| \geq R + R_1$ for $0 < \epsilon < \epsilon_1$. Thus, for $|y| \leq R_1$, we get $|y - x_\epsilon| \geq |x_\epsilon| - |y| \geq R$. So, from (3.59) and (3.60), we obtain

$$\int_{\{|y| \leq R_1\}} \mathcal{K}(x-y) f_\epsilon(y) dy \leq C \int_{\{|y| \leq R_1\}} \mathcal{K}(x-y) dy. \tag{3.65}$$

By (3.62) and (3.63), we have

$$\begin{aligned}\int_{\{|y| \leq R_1\}} \mathcal{K}(x-y) dy &= \int_{\{|y| \leq R_1, |x-y| < 1\}} \mathcal{K}(x-y) dy + \int_{\{|y| \leq R_1, |x-y| \geq 1\}} \mathcal{K}(x-y) dy \\ &\leq \frac{N\omega_{N-1} C_1}{2s} + C_2 N \omega_{N-1} R_1^N.\end{aligned} \tag{3.66}$$

Besides, for $R_2 > R$, $|x - V_0^{\frac{1}{2s}} x_\epsilon| > V_0^{\frac{1}{2s}} R_2$ and $|y - V_0^{\frac{1}{2s}} x_\epsilon| \leq V_0^{\frac{1}{2s}} R$, we get $|x-y| \geq \frac{R_2 - R}{R}|x - V_0^{\frac{1}{2s}} x_\epsilon|$ and thus

$$\begin{aligned}
&\int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x-y) f_\epsilon(y) dy \\
&\leq C \int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x-y) u_\epsilon^{2_s^* - 1 - \epsilon}(y) dy \\
&\leq C \|u_\epsilon\|_{2_s^* - \epsilon}^{2_s^* - 1 - \epsilon} \left(\int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x-y)^{\frac{2N - (N-2s)\epsilon}{N-2s}} dy \right)^{\frac{N-2s}{2N-(N-2s)\epsilon}} \\
&\leq \frac{C}{|V_0^{-\frac{1}{2s}} x - x_\epsilon|^{N+2s}}.
\end{aligned} \tag{3.67}$$

Combining (3.64)– (3.67), for $|V_0^{-\frac{1}{2s}} x - x_\epsilon| > R_2$ and small $\epsilon > 0$, we have

$$|w_\epsilon(x)| \leq \frac{C}{|V_0^{-\frac{1}{2s}} x - x_\epsilon|^{N+2s}}.$$

That is,

$$|u_\epsilon(x)| \leq \frac{C}{|x - x_\epsilon|^{N+2s}}, \quad |x - x_\epsilon| > R_2.$$

Since R is arbitrary, as well as R_2 is arbitrary, the proof is complete. □

Remark 4 By using standard comparison arguments as in [23], we can prove results similar to (3.44) and (3.50). However, the constant C obtained there may depend on ϵ.

Lemma 12 *There exists a positive constant C independent of ϵ, such that*

$$v_\epsilon(x) \leq C U(x), \quad x \in \mathbb{R}^N. \tag{3.68}$$

Proof Note that we do not assume that $\{x_\epsilon\}$ is bounded or unbounded. From the definition of v_ϵ and U, $v_\epsilon(0) = U(0) = 1$, and since $v_\epsilon(x) \in C^{2,\alpha}$, by choosing some large C, (3.68) holds in a neighborhood of zero. Therefore, it is enough to establish (3.68) ifor $|x|$ bounded away from zero. For this purpose, let $\Phi_\epsilon(x)$ be the Kelvin transform of v_ϵ, namely

$$\Phi_\epsilon(x) = |x|^{2s-N} v_\epsilon\left(\frac{x}{|x|^2}\right). \tag{3.69}$$

Then, Φ_ϵ satisfies

$$(-\Delta)^s \Phi_\epsilon + \mu_\epsilon^{2s} |x|^{-4s} V\left(x_\epsilon + \mu_\epsilon \frac{x}{|x|^2}\right) \Phi_\epsilon = |x|^{(2s-N)\epsilon} \Phi_\epsilon^{2_s^* - 1 - \epsilon} \text{ in } \mathbb{R}^N. \tag{3.70}$$

Now, we aim at proving that $\{\Phi_\epsilon\}$ is uniformly bounded with respect to ϵ in a neighborhood of zero, and this will imply (3.68) by (3.69).

From (3.70), we obtain

$$(-\Delta)^s \Phi_\epsilon \leq |x|^{(2s-N)\epsilon} \Phi_\epsilon^{2_s^* - 1 - \epsilon} := a(x) \Phi_\epsilon, \quad a(x) = |x|^{(2s-N)\epsilon} \Phi_\epsilon^{2_s^* - 2 - \epsilon}. \tag{3.71}$$

Claim: $a(x) \in L_{loc}^t(\mathbb{R}^N)$ with some $t > \frac{N}{2s}$.

Assume for the moment the claim holds true and let us complete the proof. By Theorem 10, for any compact set K, we have

$$\begin{aligned}\max_K \Phi_\epsilon(x) &\le C\left(\int_K |\Phi_\epsilon|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \le C\left(\int_{\mathbb{R}^N} |v_\epsilon|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \\ &\le C(\mu_\epsilon^\epsilon)^{\frac{N}{2_s^*(2_s^*-2-\epsilon)}}\left(\int_{\mathbb{R}^N} |u_\epsilon|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \le C.\end{aligned} \tag{3.72}$$

The last inequality follows from the facts $\mu_\epsilon^\epsilon \to 1$ and $\|u_\epsilon\|_{2_s^*} \le C\|u_\epsilon\|_{s,V(x)} \to CS^{\frac{N}{4s}}$ as $\epsilon \to 0^+$.

Thus, it remains to prove the claim. On the one hand, for $r > 0$ we get

$$\begin{aligned}\int_{\mu_\epsilon^2 \le |x| \le r} a(x)^t dx &\le (\mu_\epsilon^\epsilon)^{2(2s-N)t}\int_{\mu_\epsilon^2 \le |x| \le r} \Phi_\epsilon^{(2_s^*-2-\epsilon)t}dx \\ &\le (\mu_\epsilon^\epsilon)^{2(2s-N)t}|B_r|^{1-\frac{(2_s^*-2-\epsilon)t}{2_s^*}}\left(\int_{B_r} \Phi_\epsilon^{2_s^*}dx\right)^{\frac{(2_s^*-2-\epsilon)t}{2_s^*}} \le C,\end{aligned} \tag{3.73}$$

since $\mu_\epsilon^\epsilon \to 1$ as $\epsilon \to 0^+$ and $\Phi_\epsilon \to \overline{U}$ in $L^{2_s^*}(\mathbb{R}^N)$, where $\overline{U}(x) = |x|^{2s-N}U\left(\frac{x}{|x|^2}\right)$.

On the other hand, if $\{x_\epsilon\}$ is bounded and $|x| \le \frac{\mu_\epsilon}{R-|x_\epsilon|}$, or if $\{x_\epsilon\}$ is unbounded and $|x| \le \frac{\mu_\epsilon}{R}$, by Lemmas 22 and 23, we have

$$\begin{aligned}\Phi_\epsilon(x) &= |x|^{2s-N}v_\epsilon\left(\frac{x}{|x|^2}\right) \\ &= \mu_\epsilon^{\frac{2s}{2_s^*-2-\epsilon}}|x|^{2s-N}u_\epsilon\left(x_\epsilon + \mu_\epsilon\frac{x}{|x|^2}\right) \\ &\le C\mu_\epsilon^{\left[\frac{2s}{2_s^*-2-\epsilon}-(N+2s)\right]}|x|^{4s}.\end{aligned} \tag{3.74}$$

Thus, we have

$$\int_{|x|\le\mu_\epsilon^2} a(x)^t dx \le C\mu_\epsilon^{\left[\frac{2s}{2_s^*-2-\epsilon}-(N+2s)\right](2_s^*-2-\epsilon)t}\int_{|x|\le\mu_\epsilon^2} |x|^{[4s(2_s^*-2-\epsilon)+(2s-N)\epsilon]t}dx \le C \tag{3.75}$$

and the proof is complete. □

Proposition 3 *Assume $N > 4s$ and suppose that $x_\epsilon \to x_0$ as $\epsilon \to 0^+$. Then,*

$$\lim_{\epsilon\to 0^+} \epsilon\|u_\epsilon\|_\infty^{\frac{4s}{N-2}} = A_{N,s}\left[V(x_0) + \frac{1}{2s}x_0 \cdot \nabla V(x_0)\right],$$

where

$$A_{N,s} = \frac{2^{2(N+1)}N^2\pi^{\frac{N}{2}}\Gamma\left(\frac{N-4s}{2}\right)}{(N-2s)^2\Gamma(N-2s)}S^{-\frac{N}{2s}}.$$

Proof By Pohozaev identity (2.12), we have

$$\begin{aligned}
&\left(\frac{1}{2_s^*-\epsilon}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} u_\epsilon^{2_s^*-\epsilon}dx\\
&=\int_{\mathbb{R}^N}\left[V(x)+\frac{1}{2s}x\cdot\nabla V(x)\right]u_\epsilon^2dx\\
&=\mu_\epsilon^N\int_{\mathbb{R}^N}\left[V(x_\epsilon+\mu_\epsilon x)+\frac{1}{2s}(x_\epsilon+\mu_\epsilon x)\cdot\nabla V(x_\epsilon+\mu_\epsilon x)\right]u_\epsilon^2(x_\epsilon+\mu_\epsilon x)dx\\
&=\mu_\epsilon^{N-\frac{4s}{2_s^*-2-\epsilon}}\int_{\mathbb{R}^N}\left[[V(x_\epsilon+\mu_\epsilon x)+\frac{1}{2s}(x_\epsilon+\mu_\epsilon x)\cdot\nabla V(x_\epsilon+\mu_\epsilon x)\right]v_\epsilon^2dx.
\end{aligned} \tag{3.76}$$

Since $N>4s$, by Lemma 25 and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{\epsilon\to0^+}\int_{\mathbb{R}^N}\left[V(x_\epsilon+\mu_\epsilon x)+\frac{1}{2s}(x_\epsilon+\mu_\epsilon x)\cdot\nabla V(x_\epsilon+\mu_\epsilon x)\right]v_\epsilon^2dx\\
&=\left[V(x_0)+\frac{1}{2s}x_0\cdot\nabla V(x_0)\right]\int_{\mathbb{R}^N}U^2dx.
\end{aligned} \tag{3.77}$$

By direct calculations, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^N}U^2dx&=\lambda^{2N}\int_{\mathbb{R}^N}(1+|x|^2)^{2s-N}dx\\
&=\lambda^{2N}\omega_N\int_0^\infty(1+r^2)^{2s-N}r^{N-1}dr\\
&=\frac{1}{2}\lambda^{2N}\omega_N\int_0^\infty(1+s)^{2s-N}s^{-1+\frac{N}{2}}ds\\
&=\lambda^{2N}\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}B\left(\frac{N}{2},\frac{N}{2}-2s\right)\\
&=2^{2N}\pi^{\frac{N}{2}}\left[\frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)}\right]^N\frac{\Gamma\left(\frac{N-4s}{2}\right)}{\Gamma(N-2s)}.
\end{aligned} \tag{3.78}$$

Finally, combine (3.76)–(3.78) to have as $\epsilon\to0$

$$\begin{aligned}
\epsilon\mu_\epsilon^{2s}&=\left(\frac{2N}{N-2s}\right)^2\left[V(x_0)+\frac{1}{2s}x_0\cdot\nabla V(x_0)\right]S^{-\frac{N}{2s}}\int_{\mathbb{R}^N}U^2dx+o(1)\\
&=\frac{2^{2(N+1)}N^2\pi^{\frac{N}{2}}\Gamma\left(\frac{N-4s}{2}\right)}{(N-2s)^2\Gamma(N-2s)}\left[\frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)}\right]^N\left[V(x_0)+\frac{1}{2s}x_0\cdot\nabla V(x_0)\right]S^{-\frac{N}{2s}}\\
&\quad+o(1).
\end{aligned} \tag{3.79}$$

This concludes the proof of Lemma 26. □

Remark 5 From the proof of Proposition 26, assuming $N>4s$, no matter x_ϵ stays bounded or not, we still have

$$\epsilon=O(\mu_\epsilon^{2s}).$$

Proof of Theorem 1. The conclusions (1) and (2) in Theorem 1 follow from Propositions 13 and 26. Clearly, Corollary 2 is a particular case of Theorem 1.

4 Localizing Blow up Points

We next recall for convenience of the reader a few basic facts on fractional Sobolev spaces. Let $\beta > 0$ and $p \in [1, \infty)$,

$$\mathcal{W}^{\beta,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : \mathscr{F}^{-1}[(1+|\xi|^\beta)\hat{u}] \in L^p(\mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_{\mathcal{W}^{\beta,p}(\mathbb{R}^N)} = \|\mathscr{F}^{-1}[(1+|\xi|^\beta)\hat{u}]\|_p .$$

We refer to [23] for the following results.

Proposition 4 *The following properties hold true:*

(1) *If* $0 < \beta < 1$, $1 < p \le q \le \frac{Np}{N-\beta p} < \infty$ *or* $p = 1$ *and* $1 \le q < \frac{N}{N-\beta}$, *then* $\mathcal{W}^{\beta,p}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$.

(2) *Assume that* $0 \le \beta \le 2$ *and* $\beta > \frac{N}{p}$. *If* $\beta - \frac{N}{p} > 1$ *and* $0 < \mu \le \beta - 1 - \frac{N}{p}$, *then* $\mathcal{W}^{\beta,p}(\mathbb{R}^N)$ *is continuously embedded in* $C^{1,\mu}(\mathbb{R}^N)$. *If* $\beta - \frac{N}{p} < 1$ *and* $0 < \mu \le \beta - \frac{N}{p}$, *then* $\mathcal{W}^{\beta,p}(\mathbb{R}^N)$ *is continuously embedded in* $C^{0,\mu}(\mathbb{R}^N)$.

For $p \in [1, +\infty)$ and $\beta > 0$, consider the Bessel potential space

$$\mathcal{L}^{\beta,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : \mathscr{F}^{-1}[(1+|\xi|^2)^{\frac{\beta}{2}}\hat{u}] \in L^p(\mathbb{R}^N)\}.$$

Then, $\mathcal{L}^{\beta,p}(\mathbb{R}^N) = \mathcal{W}^{\beta,p}(\mathbb{R}^N)$, see Theorem 3.1 in [23]. On the other hand, from Theorem 5 in Chapter V of [37], for $p \in [2, \infty)$ and $0 < \beta < 1$, one has $\mathcal{W}^{\beta,p}(\mathbb{R}^N) \subset W^{\beta,p}(\mathbb{R}^N)$, where $W^{\beta,p}(\mathbb{R}^N)$ is the usual fractional Sobolev space defined by

$$W^{\beta,p}(\mathbb{R}^N) = \left\{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\beta p}}dxdy\right\}.$$

Our next target is to identify the location of the blow up points. For this purpose we adapt the method developed in [28], where the basic idea is to get an asymptotic expansion of the ground state energy and then to compare it with an upper bound of $S^V_{2^*_s-\epsilon}$. This method has been used to deal with the localization of blow-up points of ground states to semilinear problems in [40].

Let us begin with establishing an upper bound for $S^V_{2^*_s-\epsilon}$.

Theorem 7 *Assume* $N > 4s$ *and that* u_{ϵ_j} *is a ground state of* (1.1) *satisfying* (1.6) *which has a maximum point* x_{ϵ_j} *which enjoys* $x_{\epsilon_j} \to x_0$ *as* $j \to \infty$. *Then*

$$\begin{aligned} S^V_{2^*_s-\epsilon_j} &\le S \\ &+\mu_j^{2s}\left\{S^{\frac{2s-N}{2s}}V(\hat{x}_0)\int_{\mathbb{R}^N}U^2dy - \widetilde{C}_{N,s}S\left[\frac{2}{(2^*_s)^2}\ln S^{\frac{N}{2s}} - \frac{2}{2^*_s}S^{-\frac{N}{2s}}\int_{\mathbb{R}^N}U^{2^*_s}\ln U dx\right]\right\} \\ &+o(\mu_j^{2s}), \end{aligned} \tag{4.1}$$

where $\hat{x}_0$ *is a global minimum point of* $V(x)$ *and*

$$\widetilde{C}_{N,s} = \Psi_{N,s}\left[V(x_0) + \frac{1}{2s}x_0\cdot\nabla V(x_0)\right].$$

Proof Let

$$\phi_j(x) = U\left(\frac{x - \hat{x}_0}{\mu_j}\right).$$

Then by inspection

$$[\phi_j]_s^2 = \mu_j^{N-2s}[U]_s^2 = \mu_j^{N-2s} S^{\frac{N}{2s}}, \tag{4.2}$$

and by dominated convergence, we have

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)\phi^2 dx &= \mu_j^N \int_{\mathbb{R}^N} V(\hat{x}_0 + \mu_j y)U^2 dy \\ &= V(\hat{x}_0)\mu_j^N \int_{\mathbb{R}^N} U^2 dy + o(\mu_j^N).\end{aligned} \tag{4.3}$$

By (3.79), we also get

$$\epsilon_j = \widetilde{C}_{N,s}\mu_j^{2s} + o(\mu_j^{2s}). \tag{4.4}$$

Thus, by using Taylor's formula, we get

$$\begin{aligned}&\mu_j^{\frac{-2N}{2_s^*-\epsilon_j}} \left(\int_{\mathbb{R}^N} |\phi|^{2_s^*-\epsilon_j} dx\right)^{\frac{2}{2_s^*-\epsilon_j}} \\ &= \left(\int_{\mathbb{R}^N} U^{2_s^*-\epsilon_j} dx\right)^{\frac{2}{2_s^*-\epsilon_j}} \\ &= \left[\int_{\mathbb{R}^N} (U^{2_s^*} - \epsilon_j U^{2_s^*} \ln U) dx + o(\epsilon_j)\right]^{\frac{2}{2_s^*-\epsilon_j}} \\ &= \left[\left(S^{\frac{N}{2s}} - \epsilon_j \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx\right)^{\frac{2}{2_s^*-\epsilon_j}} + o(\epsilon_j)\right] \\ &= \left[S^{\frac{N-2s}{2s}} + \epsilon_j S^{\frac{N-2s}{2s}} \left(\frac{2}{(2_s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2_s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx\right) + o(\epsilon_j)\right] \\ &= S^{\frac{N-2s}{2s}} \left[1 + \epsilon_j \left(\frac{2}{(2_s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2_s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx\right) + o(\epsilon_j)\right].\end{aligned} \tag{4.5}$$

So, by (4.4), $\mu_j^{\frac{2N}{2_s^*-\epsilon_j}} = \mu_j^{N-2s} + o(\epsilon_j)$, we have

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |\phi|^{2_s^*-\epsilon_j} dx\right)^{\frac{2}{2_s^*-\epsilon_j}} \\ &= \mu_j^{N-2s} S^{\frac{N-2s}{2s}} \left[1 + \widetilde{C}_{N,s}\mu_j^{2s} \left(\frac{2}{(2_s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2_s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx\right) + o(\mu_j^{2s})\right].\end{aligned} \tag{4.6}$$

By the very definition of $S^V_{2_s^*-\epsilon_j}$, we have

$$
\begin{aligned}
S^V_{2_s^*-\epsilon_j} &\le \frac{\|u_j\|^2_{s,V}}{\|u_j\|^2_{2_s^*-\epsilon_j}} \\
&\le \frac{\mu_j^{N-2s} S^{\frac{N}{2s}} + V(\hat{x}_0)\mu_j^N \int_{\mathbb{R}^N} U^2 dy + o(\mu_j^N)}{\mu_j^{N-2s} S^{\frac{N-2s}{2s}} \left[1 + \widetilde{C}_{N,s}\mu_j^{2s}\left(\frac{2}{(2_s^*)^2}\ln S^{\frac{N}{2s}} - \frac{2}{2_s^*}S^{-\frac{N}{2s}}\int_{\mathbb{R}^N} U^{2_s^*}\ln U dx\right) + o(\mu_j^{2s})\right]} \\
&= \left[S + \mu_j^{2s} S^{\frac{2s-N}{2s}} V(\hat{x}_0)\int_{\mathbb{R}^N} U^2 dy + o(\mu_j^{2s})\right] \\
&\quad \cdot \left[1 - \widetilde{C}_{N,s}\mu_j^{2s}\left(\frac{2}{(2_s^*)^2}\ln S^{\frac{N}{2s}} - \frac{2}{2_s^*}S^{-\frac{N}{2s}}\int_{\mathbb{R}^N} U^{2_s^*}\ln U dx\right) + o(\mu_j^{2s})\right]. \qquad (4.7)
\end{aligned}
$$

This concludes the proof. □

For simplicity, set $\mu_j := \mu_{\epsilon_j}$, $x_j := x_{\epsilon_j}$. For $v_j(x) = \mu_j^{\frac{2s}{2_s^*-2-\epsilon}} u_j(x_j + \mu_j x)$, let $v_j = U + \mu_j^{2s} w_j$, then by (3.39) we have

$$
(-\Delta)^s w_j - (2_s^* - 1)U^{2_s^*-2} w_j + V(x_j + \mu_j x)v_j = F(w_j) \ \text{ in } \mathbb{R}^N, \qquad (4.8)
$$

where

$$
F(w_j) = \mu_j^{-2s}\left[(U + \mu_j^{2s} w_j)^{2_s^*-1-\epsilon} - (2_s^* - 1)\mu_j^{2s} U^{2_s^*-2} w_j - U^{2_s^*-1}\right].
$$

Define the operator L as follows:

$$
L := (-\Delta)^s - (2_s^* - 1)U^{2_s^*-2}.
$$

Then (4.8) can be rewritten as

$$
Lw_j + V(x_j + \mu_j x)v_j = F(w_j)\,.
$$

Proposition 5 *Assume $N > 6s$. Then $w_j \to w$ in $L^\infty(\mathbb{R}^N)$ as $j \to \infty$, where w is a bounded solution of*

$$
(-\Delta)^s w - (2_s^* - 1)U^{2_s^*-2} w + V(x_0)U + \widetilde{C}(N, s)U^{2_s^*-1}\ln U = 0 \ \text{ in } \mathbb{R}^N. \qquad (4.9)
$$

In order to prove Proposition 30, we need the following result from [17]

Lemma 13 (Nondegeneracy) *The solution U is nondegenerate in the sense that all bounded solutions of equation*

$$
(-\Delta)^s \phi = (2_s^* - 1)U^{2_s^*-2}\phi \quad \text{in } \mathbb{R}^N
$$

are linear combinations of the functions

$$
\frac{N-2s}{2}U + x \cdot \nabla U, \quad \frac{\partial U}{\partial x_i}, \ i = 1, 2, \cdots, N.
$$

Let

$$
X = \operatorname{span}\left\{\frac{\partial U}{\partial x_1}, \cdots, \frac{\partial U}{\partial x_N}, \frac{N-2s}{2}U + x \cdot \nabla U\right\}
$$

Clearly, $X \subset L^p(\mathbb{R}^N)$ with $\frac{N}{N-2s} < p < +\infty$. For $1 < r < \frac{N}{2s}$, define

$$
Y_r := \left\{u \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N} uv dx = 0 \ \text{ for all } \ v \in X\right\}.
$$

Then $L^r(\mathbb{R}^N) = X \oplus Y_q$, where $\frac{N}{N-2s} < r < \frac{N}{2s}$.

Lemma 14 *Suppose $N > 4s$. Then for any $1 < q < \frac{N}{4s}$, there exists a constant $C > 0$ such that*

$$\|u\|_{\mathcal{W}^{2s,r}} \le C(\|Lu\|_r + \|Lu\|_q), \tag{4.10}$$

for all $u \in Y_r \cap \mathcal{W}^{2s,r}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ with $Lu \in L^q(\mathbb{R}^N)$, $\frac{1}{q} - \frac{2s}{N} = \frac{1}{r}$.

Proof It is enough to prove

$$\|u\|_r \le C(\|Lu\|_r + \|Lu\|_q).$$

In fact, by

$$(-\Delta)^s u + u = Lu + [1 - (2_s^* - 1)U^{2_s^*-2}]u$$

we get

$$\|u\|_{\mathcal{W}^{2s,r}} \le \|Lu\|_r + C\|u\|_r \le C(\|Lu\|_r + \|Lu\|_q).$$

Assume that $u \neq 0$. Otherwise, we are done. By homogeneity, we can replace u by $\frac{u}{\max\{\|u\|_{C^2}, \|u\|_r\}}$ in (4.10). Thus, assume that there exists a sequence $\{u_n\} \subset Y_r \cap \mathcal{W}^{2s,r}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ such that

$$\text{either } \|u_n\|_{C^2} = 1, \ \ \|u_n\|_r < 1, \ \text{ or } \|u_n\|_{C^2} < 1, \ \ \|u_n\|_r = 1, \tag{4.11}$$

and

$$\|Lu_n\|_q + \|Lu_n\|_r \to 0. \tag{4.12}$$

Then, there exists $u_\infty \in C^2(\mathbb{R}^N)$ such that after passing to a subsequence if necessary, $u_n \to u_\infty$ in $C^2_{loc}(\mathbb{R}^N)$ and in particular, $u_n \to u_\infty$ in $L^t_{loc}(\mathbb{R}^N)$, $r \le t < 2_s^*$. Let $I = (-\Delta)^{-s}$ the Riesz potentials defined by

$$(I * f)(x) = \frac{1}{\gamma(s)} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy$$

with

$$\gamma(s) = \pi^{\frac{N}{2}} 2^{2s} \Gamma(s)/\Gamma(N/2 - s).$$

See Chapter V in [37]. Then, we have

$$u_n - I * [(2_s^* - 1)U^{2_s^*-2} u_n] = I * Lu_n.$$

By Hardy-Littlewood-Sobolev inequality [26, 37], we have

$$\|I * Lu_n\|_r \le C\|Lu_n\|_q \to 0$$

and Hölder's inequality yields

$$\begin{aligned}
&\|I * [(2_s^* - 1)U^{2_s^*-2}(u_n - u_m)]\|_r \\
&= (2_s^* - 1)\left[\int_{B_R(0)} |I * U^{2_s^*-2}(u_n - u_m)|^r dx + \int_{B_R^c(0)} |I * U^{2_s^*-2}(u_n - u_m)|^r dx\right]^{\frac{1}{r}} \\
&\le C\|U^{2_s^*-2}(u_n - u_m)\|_{L^q(B_R(0))} + C\|u_n - u_m\|_r \left(\int_{|x|\ge R} U^{\frac{2N}{N-2s}} dx\right)^{\frac{2s}{N}} \\
&\le C\|u_n - u_m\|_{L^r(B_R(0))} + C\|u_n - u_m\|_r \left(\int_{|x|\ge R} U^{\frac{2N}{N-2s}} dx\right)^{\frac{2s}{N}},
\end{aligned} \tag{4.13}$$

where $\frac{1}{q} - \frac{2s}{N} = \frac{1}{r}$.

Thus, $\{I * [(2_s^* - 1)U^{2_s^*-2}u_n\}$ is a Cauchy sequence in $L^r(\mathbb{R}^N)$ and then $\{u_n\}$ is a Cauchy sequence in $L^r(\mathbb{R}^N)$. So, $u_\infty \in L^r(\mathbb{R}^N)$, $u_\infty \in Y_r$ and

$$(-\Delta)^s u_\infty - (2_s^* - 1)U^{2_s^*-2}u_\infty = 0 \quad \text{in } \mathbb{R}^N. \tag{4.14}$$

By (4.11), $u_\infty \in L^\infty(\mathbb{R}^N)$ and $u_\infty \in X$ by (4.14). But since $u_\infty \in Y_r$, we get $u_\infty \equiv 0$, which is a contradiction from (4.11). □

For fixed $j = 1, 2, \cdots$, we have

$$(-\Delta)^s v_j + v_j = (1 + \mu_j^{2s} V(x_j + \mu_j x))v_j + v_j^{2_s^*-1-\epsilon_j} \quad \text{in } \mathbb{R}^N. \tag{4.15}$$

Note that $0 \le v_j \le 1$. Thus, $v_j \in \mathcal{W}^{2s,p}(\mathbb{R}^N) \cap C^{2,\beta}(\mathbb{R}^N)$ for $p \in [2, +\infty)$ and $w_j \in \mathcal{W}^{2s,p}(\mathbb{R}^N) \cap C^{2,\beta}(\mathbb{R}^N)$ for $2 \le p < +\infty$.

Let

$$w_j = \sum_{i=1}^{N+1} a_{ij} e_i + z_j, \quad j = 1, 2, \cdots,$$

where $e_1 = \frac{\partial U}{\partial x_i}$, $i = 1, 2, \cdots, N$, $e_{N+1} = \frac{N-2s}{2}U + x \cdot \nabla U$, $z_j \in Y_q \cap C^{2,\beta}(\mathbb{R}^N)$, $\frac{1}{q} + \frac{1}{p} = 1$.

Lemma 15 *Assume $N > 6s$ and let $M_j = \max\{|a_{1j}|, |a_{2j}|, \cdots, |a_{(N+1)j}|\}$. Then M_j and $\|z_j\|_{\mathcal{W}^{2s,r}}$ are bounded as $j \to \infty$.*

Proof We may assume without loss of generality, $M_j \to +\infty$ as $j \to \infty$ and

$$\frac{1}{M_j}(a_{1j}, \cdots, a_{(N+1)j}) \to (b_1, \cdots, b_{N+1}) \ne 0, \quad \text{as } j \to \infty.$$

Then

$$(-\Delta)^s \frac{z_j}{M_j} = (2_s^* - 1)U^{2_s^*-2}\frac{z_j}{M_j} + \frac{1}{M_j}\left[F(w_j) - V(x_j + \mu_j x)v_j\right]. \tag{4.16}$$

Let us now now estimate the three terms in the right hand side of equation (4.16). We have

$$\begin{aligned} |\mu_j^{2s} F(w_j)| &= |(U + \mu_j^{2s} w_j)^{2_s^*-1-\epsilon} - (2_s^* - 1)\mu_j^{2s} U^{2_s^*-2} w_j - U^{2_s^*-1}| \\ &\le |U^{2_s^*-1} - U^{2_s^*-1-\epsilon_j}| + |(2_s^* - 1)\mu_j^{2s} U^{2_s^*-2} w_j - (2_s^* - 1 - \epsilon_j)\mu_j^{2s} U^{2_s^*-2-\epsilon_j} w_j| \\ &\quad + |(U + \mu_j^{2s} w_j)^{2_s^*-1-\epsilon_j} - U^{2_s^*-1-\epsilon_j} - (2_s^* - 1 - \epsilon_j)\mu_j^{2s} U^{2_s^*-2-\epsilon_j} w_j| \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.17}$$

Hence

$$I_1 = U^{2_s^*-1-\epsilon_j}|U^{\epsilon_j} - 1| = U^{2_s^*-1-\epsilon_j}|\epsilon_j \ln U + o(\epsilon_j)| \le \epsilon_j U^{2_s^*-1-\epsilon_j}(|\ln U| + 1) \tag{4.18}$$

and since $v_j \le CU$ and $\mu_j^{2s} w_j = v_j - U$, we get

$$\begin{aligned} I_2 &= \mu_j^{2s} U^{2_s^*-2-\epsilon_j} |w_j| |(2_s^*-1)U^{\epsilon_j} - (2_s^* - 1 - \epsilon_j)| \\ &\le C\epsilon_j \mu_j^{2s} |w_j| U^{2_s^*-2-\epsilon_j} (|\ln U| + 1) \\ &\le C\epsilon_j U^{2_s^*-1-\epsilon_j} (|\ln U| + 1). \end{aligned} \tag{4.19}$$

Set $g(t) = (U + t\mu_j^{2s} w_j)^{2_s^*-1-\epsilon_j}$. Then, we obtain

$$\begin{aligned} I_3 &= |g(1) - g(0) - g'(0)| \\ &\le \int_0^1 t|g''(1-t)|dt \\ &\le C\int_0^1 t[U + (1-t)\mu_j^{2s} w_j]^{2_s^*-3-\epsilon_j} \mu_j^{4s} w_j^2 dt \\ &\le C\mu_j^{2s} |w_j||v_j - U| \int_0^1 t U^{2_s^*-3-\epsilon_j} dt \\ &\le C\mu_j^{2s} |w_j||v_j - U| U^{2_s^*-3-\epsilon_j}. \end{aligned} \tag{4.20}$$

Thus, by (4.17)–(4.20) and Remark 27, we get

$$|F(w_j)| \le C\left[U^{2_s^*-1-\epsilon_j}(|\ln U| + 1) + |w_j||v_j - U|U^{2_s^*-3-\epsilon_j}\right]. \tag{4.21}$$

So, by dominated convergence we obtain

$$\begin{aligned} \|F(w_j)\|_q &\le C\left[\|U^{2_s^*-1-\epsilon_j}(|\ln U| + 1)\|_q + \|w_j |v_j - U| U^{2_s^*-3-\epsilon_j}\|_q\right] \\ &\le C\left[\|U^{2_s^*-1-\epsilon_j}(|\ln U| + 1)\|_q + \|w_j\|_r \| |v_j - U| U^{2_s^*-3-\epsilon_j}\|_{\frac{N}{2s}}\right] \\ &\le C\left[1 + o(1)\|w_j\|_r\right]. \end{aligned} \tag{4.22}$$

Thus, we get

$$\frac{1}{M_j}\|F(w_j)\|_q \le C\left[o(1) + o(1)\left\|\frac{z_j}{M_j}\right\|_r\right]. \tag{4.23}$$

Again by dominated convergence we get

$$\begin{aligned} \|F(w_j)\|_r &\le C\left[\|U^{2_s^*-1-\epsilon_j}(|\ln U| + 1)\|_r + \|w_j |v_j - U| U^{2_s^*-3-\epsilon_j}\|_r\right] \\ &\le C\left[1 + o(1)\|w_j\|_r\right], \end{aligned} \tag{4.24}$$

which yields

$$\frac{1}{M_j}\|F(w_j)\|_r \le C\left[o(1) + o(1)\left\|\frac{z_j}{M_j}\right\|_r\right]. \tag{4.25}$$

By Lemma 32, for $\frac{N}{N-2s} < q < \frac{N}{4s}$ with $\frac{1}{q} - \frac{2s}{N} = \frac{1}{r}$ (Note that $N > 6s$ is needed), we get

$$\begin{aligned} \left\|\frac{z_j}{M_j}\right\|_{\mathcal{W}^{2s,r}} &\le \frac{C}{M_j}\left[\|V_j v_j\|_q + \|V_j v_j\|_r + \|F(w_j)\|_q + \|F(w_j)\|_r\right] \\ &\le \frac{C}{M_j}\left[1 + \|F(w_j)\|_q + \|F(w_j)\|_r\right] \\ &\le C\left[o(1) + o(1)\left\|\frac{z_j}{M_j}\right\|_r\right]. \end{aligned} \tag{4.26}$$

Thus, we have

$$\left\| \frac{z_j}{M_j} \right\|_{\mathcal{W}^{2s,r}} = o(1). \tag{4.27}$$

By Proposition 28, we get

$$\left\| \frac{z_j}{M_j} \right\|_{t} = o(1), \quad r \le t \le \frac{Nr}{N-2sr}. \tag{4.28}$$

By choosing r close to $\frac{N}{2s}$, t can be arbitrarily large. Besides, from (4.24), we have

$$\begin{aligned} \left| L\left(\frac{z_j}{M_j}\right)\right| &\le C\frac{1}{M_j}\left[U + U^{2_s^*-1-\epsilon_j}(|\ln U|+1) + |w_j||v_j-U|U^{2_s^*-2-\epsilon_j}\right] \\ &\le o(1)\left[U + U^{2_s^*-1-\epsilon_j}(|\ln U|+1) + U^{2_s^*-2-\epsilon_j}\sum_{i=1}^{N+1}|e_i|\right] + o(1)\left|\frac{z_j}{M_j}\right|, \end{aligned} \tag{4.29}$$

which yields that $L\left(\frac{z_j}{M_j}\right) \in L^t(\mathbb{R}^N)$. Thus, from (4.28), we get

$$\left\| \frac{z_j}{M_j} \right\|_{\mathcal{W}^{2s,t}} = o(1). \tag{4.30}$$

By Lemma 28, we have

$$\left\| \frac{z_j}{M_j} \right\|_{C^{0,\mu}} = o(1), \tag{4.31}$$

for some $0 < \mu < 1$. In particular we have $\left\|\frac{z_j}{M_j}\right\|_\infty \le C$ and from (4.29), $\left\|(-\Delta)^s\frac{z_j}{M_j}\right\|_\infty \le C$. From Lemma 4.4 in [8], $\left\|\frac{z_j}{M_j}\right\|_{C^{2,\beta}} \le C$. So, $\frac{z_j}{M_j} \to 0$ in $L^\infty(\mathbb{R}^N)$ and $C^1_{loc}(\mathbb{R}^N)$ as $j \to \infty$. Since $v_j(0) = U(0) = 1$ and both they achieve their maximum at 0, we get

$$\begin{aligned} 0 = w_j(0) &= M_j\left(\sum_{i=1}^{N+1} b_i e_i(0) + o(1)\right), \\ 0 = \nabla w_j(0) &= M_j\left(\sum_{i=1}^{N+1} b_i \nabla e_i(0) + o(1)\right). \end{aligned} \tag{4.32}$$

By direct calculations, it follows $(b_1, b_2, \cdots, b_{N+1}) = 0$, which is a contradiction.

Similarly, we can prove the remaining part of the Lemma. □

Lemma 16 *Assume $N > 6s$. Then $z_j \to z$ in $C^1_{loc}(\mathbb{R}^N)$, where z is radial and satisfies*

$$(-\Delta)^s z - (2_s^*-1)U^{2_s^*-2}z + V(x_0)U - \widetilde{C}(N,s)U^{2_s^*-1}\ln U = 0 \ \ in \ \ \mathbb{R}^N. \tag{4.33}$$

Proof By Lemma 33, there exists a subsequence $\{z_{jk}\}$ such that $z_{jk} \rightharpoonup z$ in $\mathcal{W}^{2s,r}$ and $z_{jk} \to z$ in $C^1_{loc}(\mathbb{R}^N)$, see also [8]. Since $\|z_j\|_\infty$ is bounded, from (4.19) and (4.20), we get

$$\frac{I_2 + I_3}{\mu_j^{2s}} = o(1) \tag{4.34}$$

and

$$\begin{aligned} \frac{1}{\mu_j^{2s}}(U^{2_s^*-1}-U^{2_s^*-1-\epsilon_j}) &= \frac{\epsilon_j \ln U + o(\epsilon_j)}{\mu_j^{2s}} U^{2_s^*-1} \\ &= \frac{\mu_j^{2s}\widetilde{C}(N,s)\ln U + o(\mu_j^{2s})}{\mu_j^{2s}} U^{2_s^*-1} \\ &= \widetilde{C}(N,s) U^{2_s^*-1}\ln U + o(1). \end{aligned} \tag{4.35}$$

Thus, z satisfies (4.33).

Since $z_{jk} \in Y_r$, we get $z \in Y_r$. Thus, (4.33) has at most one such solution, and $z_j \rightharpoonup z$ in $\mathcal{W}^{2s,r}$. Moreover, since $(-\Delta)^s$ is invariant with respect to the action of the orthogonal group $O(n)$ on $\mathbb{R}^N$ (see [15]), if T denotes a rotation in $\mathbb{R}^N$, since (4.33) is invariant under rotation, then $z(Tx) - z(x) \in X$. Consequently, $z(Tx) = z(x)$. This proves that z is radial. □

Lemma 17 *Assume $N > 6s$. Then $|a_{ij}| \to 0$, $i = 1, 2, \cdots, N$ and $a_{(N+1)j} \to -\frac{2}{N-2s} z(0)$ as $j \to \infty$.*

Proof Note that

$$\begin{aligned} 0 &= \sum_{i=1}^{N+1} a_{ij} e_i(0) + z_j(0), \\ 0 &= \sum_{i=1}^{N+1} a_{ij} \nabla e_i(0) + \nabla z_j(0), \end{aligned} \tag{4.36}$$

which gives

$$\begin{aligned} 0 &= \frac{N-2s}{2} a_{(N+1)j} + z_j(0), \\ 0 &= \sum_{i=1}^{N} b_i \nabla e_i(0) + \nabla z_j(0). \end{aligned} \tag{4.37}$$

Since $\nabla z(0) = 0$, we get the result. □

Lemma 18 *Assume $N > 6s$. Then $w_j \to w$ in $L^\infty(\mathbb{R}^N)$ as $j \to \infty$, where*

$$w = z - \frac{2}{N-2s} z(0) \left(\frac{N-2s}{2} U + x \cdot \nabla U \right).$$

Proof It sufficient to prove $z_j \to z$ in $L^\infty(\mathbb{R}^N)$ as $j \to \infty$. In fact, by consider $L(z_j - z)$, the proof is analogous to the proof of Lemma 33. □

Theorem 8 *Assume $N > 6s$, u_{ϵ_j} is a ground state of (1.1) satisfying (1.6) which has a maximum point x_{ϵ_j} satisfying $x_{\epsilon_j} \to x_0$ as $j \to \infty$. Then*

$$\begin{aligned} S^V_{2_s^*-\epsilon_j} &= S + S^{-\frac{N-2s}{2s}} \mu_j^{2s} \int_{\mathbb{R}^n} \left[\frac{2}{2_s^*} \widetilde{C}_{N,s} U^{2_s^*} \ln U + V(x_0) U^2 \right] dx \\ &\quad - \mu_j^{2s} \widetilde{C}_{N,s} \frac{2}{(2_s^*)^2} S \ln S^{\frac{N}{2s}} + o(\mu_j^{2s}). \end{aligned} \tag{4.38}$$

Proof By the very definition of $S^V_{2_s^*-\epsilon_j}$, we have

$$\begin{aligned}\left(S^V_{2_s^*-\epsilon_j}\right)^{\frac{2_s^*-\epsilon_j}{2_s^*-2-\epsilon_j}} &= \int_{\mathbb{R}^n} v_j^{2_s^*-\epsilon_j}dx \\ &= \int_{\mathbb{R}^n}\left(U+\mu_j^{2s}w_j\right)^{2_s^*-\epsilon_j}dx \\ &= \int_{\mathbb{R}^n}\Big(U^{2_s^*-\epsilon_j}+(2_s^*-\epsilon_j)U^{2_s^*-1-\epsilon_j}\mu_j^{2s}w_j \\ &\quad +\frac{1}{2}(2_s^*-\epsilon_j)(2_s^*-1-\epsilon_j)(U+t\mu_j^{2s}w_j)^{2_s^*-2-\epsilon_j}\mu_j^{4s}w_j^2\Big)dx, \quad t\in(0,1).\end{aligned} \tag{4.39}$$

Since $v_j \le CU$ and $v_j = U+\mu_j^{2s}w_j$, then by Lemma 18,

$$\begin{aligned}&\int_{\mathbb{R}^n}(U+t\mu_j^{2s}w_j)^{2_s^*-2-\epsilon_j}\mu_j^{4s}w_j^2dx \\ &\le C\|w_j\|_\infty\mu_j^{2s}\int_{\mathbb{R}^n}U^{2_s^*-2-\epsilon_j}|v_j-U|dx \\ &\le C\|w_j\|_\infty\mu_j^{2s}\left(\int_{\mathbb{R}^n}U^{\frac{2_s^*(2_s^*-2-\epsilon_j)}{2_s^*-1}}dx\right)^{\frac{2_s^*-1}{2_s^*}}\left(\int_{\mathbb{R}^n}|v_j-U|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \\ &= o(\mu_j^{2s}).\end{aligned} \tag{4.40}$$

By (4.39), we get

$$\begin{aligned}\left(S^V_{2_s^*-\epsilon_j}\right)^{\frac{2_s^*-\epsilon_j}{2_s^*-2-\epsilon_j}} &= \int_{\mathbb{R}^n}\left[U^{2_s^*-\epsilon_j}+(2_s^*-\epsilon)U^{2_s^*-1-\epsilon_j}\mu_j^{2s}w_j\right]dx+o(\mu_j^{2s}) \\ &= \int_{\mathbb{R}^n}\left[U^{2_s^*}-\epsilon_jU^{2_s^*}\ln U+(2_s^*-\epsilon_j)U^{2_s^*-1-\epsilon_j}\mu_j^{2s}w_j\right]dx+o(\epsilon_j)+o(\mu_j^{2s}) \\ &= \int_{\mathbb{R}^n}\left[U^{2_s^*}-\epsilon_jU^{2_s^*}\ln U+2_s^*U^{2_s^*-1}\mu_j^{2s}w_j\right]dx+o(\epsilon_j)+o(\mu_j^{2s}) \\ &= S^{\frac{N}{2s}}+\mu_j^{2s}\int_{\mathbb{R}^n}\left[-\widetilde{C}_{N,s}U^{2_s^*}\ln U+2_s^*U^{2_s^*-1}w\right]dx+o(\mu_j^{2s})\end{aligned} \tag{4.41}$$

By (4.15), we get

$$\begin{aligned}\int_{\mathbb{R}^n}U^{2_s^*-1}wdx &= \int_{\mathbb{R}^n}(-\Delta)^sUwdx \\ &= \int_{\mathbb{R}^n}(-\Delta)^swUdx \\ &= \int_{\mathbb{R}^n}[(2_s^*-1)U^{2_s^*-2}w-V(x_0)U-\widetilde{C}_{N,s}U^{2_s^*-1}\ln U]Udx.\end{aligned} \tag{4.42}$$

Thus,

$$(2_s^*-2)\int_{\mathbb{R}^n}U^{2_s^*-1}wdx = \int_{\mathbb{R}^n}[V(x_0)U+\widetilde{C}_{N,s}U^{2_s^*-1}\ln U]Udx. \tag{4.43}$$

So,

$$\begin{aligned}\left(S^V_{2_s^*-\epsilon_j}\right)^{\frac{2_s^*-\epsilon_j}{2_s^*-2-\epsilon_j}} &= S^{\frac{N}{2s}} + \mu_j^{2s}\int_{\mathbb{R}^n}\left[-\widetilde{C}_{N,s}U^{2_s^*}\ln U + \frac{2_s^*}{2_s^*-2}\left(V(x_0)U^2+\widetilde{C}_{N,s}U\ln U\right)\right]dx + o(\mu_j^{2s})\\ &= S^{\frac{N}{2s}} + \mu_j^{2s}\int_{\mathbb{R}^n}\left[\frac{2}{2_s^*-2}\widetilde{C}_{N,s}U^{2_s^*}\ln U + \frac{2_s^*}{2_s^*-2}V(x_0)U^2\right]dx + o(\mu_j^{2s})\,. \end{aligned} \tag{4.44}$$

Thus, we have

$$\begin{aligned} S^V_{2_s^*-\epsilon_j} &= S + \tfrac{2s}{N}S^{-\frac{N-2s}{2s}}\mu_j^{2s}\int_{\mathbb{R}^n}\left[\tfrac{2}{2_s^*-2}\widetilde{C}_{N,s}U^{2_s^*}\ln U + \tfrac{2_s^*}{2_s^*-2}V(x_0)U^2\right]dx\\ &\quad -\epsilon_j\frac{2}{(2_s^*)^2}S\ln S^{\frac{N}{2s}} + o(\mu_j^{2s}), \end{aligned} \tag{4.45}$$

which yields

$$\begin{aligned} S^V_{2_s^*-\epsilon_j} &= S + S^{-\frac{N-2s}{2s}}\mu_j^{2s}\int_{\mathbb{R}^n}\left[\tfrac{2}{2_s^*}\widetilde{C}_{N,s}U^{2_s^*}\ln U + V(x_0)U^2\right]dx\\ &\quad -\mu_j^{2s}\widetilde{C}_{N,s}\frac{2}{(2_s^*)^2}S\ln S^{\frac{N}{2s}} + o(\mu_j^{2s}). \end{aligned} \tag{4.46}$$

The proof is complete. □

Theorem 9 *Assume* (V_1), (V_2) *with* $\inf_{x\in\mathbb{R}^N} V(x) < \sup_{x\in\mathbb{R}^N} V(x)$, $N > 6s$ *and let* u_ϵ *be the ground state of* (1.1) *which has a maximum point at* x_ϵ. *Then, up to a subsequence,* $V(x_{\epsilon_j}) \to \min_{x\in\mathbb{N}} V(x)$ *as* $\epsilon_j \to 0^+$.

Proof By Theorems 29 and 37, it is sufficient to prove that x_ϵ remains bounded. We argue by contradiction. Assume that there exists a sequence $x_j \to \infty$ such that

$$\lim_{j\to\infty} V(x_j) = V_\infty > \inf_{x\in\mathbb{R}^N} V(x). \tag{4.47}$$

By Remark 27,

$$\epsilon_j = A\mu_j^{2s} + o(\mu_j^{2s})$$

for some $A > 0$. Analogous to the proof of Theorems 29 and 37 with $\widetilde{C}_{N,s}$ and $V(x_0)$ replaced by A and V_∞, we get $V_\infty \le \inf_{x\in\mathbb{R}^N} V(x)$, which contradicts (4.47). □

Proof of Theorem 3. It follows from Theorems 1, 29, 37 and 38.

5 Local Uniqueness: Proof of Theorem 4

Let us argue by contradiction. Suppose that there exists a sequence $\epsilon_j \to 0$ and two ground states far apart, namely $u_j^1 := u_{\epsilon_j}^1$ and $u_j^2 := u_{\epsilon_j}^2$. Set

$$v_j^i(x) := (\mu_j^i)^{\frac{2s}{2_s^*-2-\epsilon_j}} u_j^i(\mu_j^i x), \quad i = 1, 2.$$

Then $v_j^i \to U$ in $C^{2,\beta}_{loc}(\mathbb{R}^N)$ for $i = 1, 2$ as $j \to \infty$.

Assume further that $v_j^1 \neq v_j^2$. Set

$$\theta_j := v_j^1 - v_j^2, \quad \psi_j := \frac{v_j^1 - v_j^2}{\|v_j^1 - v_j^2\|_\infty}.$$

Then

$$(-\Delta)^s \psi_j + (\mu_j^1)^{2s} V(\mu_j^1 x) v_j^1 - (\mu_j^2)^{2s} V(\mu_j^2 x) v_j^2 = \Phi_n \psi_n \ \text{ in } \ \mathbb{R}^N, \tag{5.1}$$

where

$$\Phi_n = (2_s^* - 1 - \epsilon_j) \int_0^1 \Big[t v_j^1(x) + (1-t) v_j^2(x) \Big]^{2_s^* - 2 - \epsilon_j} dt. \tag{5.2}$$

Since $\|v_j^i\|_\infty = 1$, $i = 1, 2$, by standard regularity we have $\psi_j \to \psi$ in $C_{loc}^{2,\beta}(\mathbb{R}^N)$. By Lemma 18, we have that $\{\psi_j\}$ is uniformly bounded in $H_V^s(\mathbb{R}^N)$. Without loss of generality, we may assume that $\psi_n \rightharpoonup \psi$ in $H_V^s(\mathbb{R}^N)$.

From (5.1), we have

$$\begin{aligned} \textstyle\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \psi_j (-\Delta)^{\frac{s}{2}} \varphi dx \;=\; & -(\mu_j^1)^{2s} \textstyle\int_{\mathbb{R}^N} V(\mu_j^1 x) v_j^1 \varphi dx + (\mu_j^2)^{2s} \textstyle\int_{\mathbb{R}^N} V(\mu_j^2 x) v_j^2 \varphi dx \\ & + \textstyle\int_{\mathbb{R}^N} \Phi_n \psi_n \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{5.3}$$

Taking $j \to \infty$ in (5.3), we get

$$(-\Delta)^s \psi = (2_s^* - 1) U^{2_s^* - 2} \psi \ \text{ in } \ \mathbb{R}^N. \tag{5.4}$$

Note that $\|\psi_j\|_\infty = 1$ implies $\|\psi\|_\infty = 1$. By Lemma 31,

$$\psi \in X = \text{span} \left\{ \frac{\partial U}{\partial x_1}, \cdots, \frac{\partial U}{\partial x_N}, \frac{N-2s}{2} U + x \cdot \nabla U \right\}.$$

On the other hand, since v_j^i is radially symmetric, ψ is a radial function as well. Thus,

$$\psi(x) = c \frac{\lambda^2 - |x|^2}{(\lambda^2 + |x|^2)^{\frac{N-2s+2}{2}}}$$

for some constant $c \in \mathbb{R}$.

We next actually prove that $c = 0$. Indeed, otherwise assume for simplicity $c = 1$. By Pohozaev's identity, we have

$$\frac{1}{N} (\mu_j^i)^{2s} \int_{\mathbb{R}^N} V(\mu_j^i x) |v_j^i|^2 dx + \frac{1}{2N} (\mu_j^i)^{2s+1} \int_{\mathbb{R}^N} x \cdot \nabla V(\mu_j^i x) |v_j^i|^2 dx = \frac{\epsilon_j}{2_s^*(2_s^* - \epsilon_j)} \int_{\mathbb{R}^N} |v_j^i|^{2_s^* - \epsilon_j} dx, \ \ i = 1, 2. \tag{5.5}$$

By Remark 27, we get $(\mu_j^i)^{2s} \sim \epsilon_j$. Thus, from (5.5), we have

$$\frac{V_\infty}{N} \int_{\mathbb{R}^N} \psi_j (v_j^1 + v_j^2) dx + o(1) \geq \frac{2_s^* - \epsilon_j}{2_s^*(2_s^* - \epsilon_j)} \int_{\mathbb{R}^N} \psi_j \int_0^1 \Big[t v_j^1 + (1-t) v_j^2 \Big]^{2_s^* - 1 - \epsilon_j} dt dx. \tag{5.6}$$

Notice that

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} \psi_j (v_j^1 + v_j^2) dx = 2 \int_{\mathbb{R}^N} \psi U dx = 2 \lambda^{N-2s} \int_{\mathbb{R}^N} \frac{\lambda^2 - |x|^2}{(\lambda^2 + |x|^2)^{N-2s+1}} dx. \tag{5.7}$$

Direct calculations show that

$$\int_{\mathbb{R}^N} \frac{\lambda^2 - |x|^2}{(\lambda^2 + |x|^2)^{N-2s+1}} dx < 0. \tag{5.8}$$

Thus, from (5.7), we obtain

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\psi_j(v_j^1+v_j^2)dx<0. \tag{5.9}$$

On the other hand, we have

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\psi_j\int_0^1\left[tv_j^1+(1-t)v_j^2\right]^{2_s^*-1-\epsilon_j}dtdx=\int_{\mathbb{R}^N}\psi U^{2_s^*-1}. \tag{5.10}$$

By a suitable scaling we end up with

$$\int_{\mathbb{R}^N}\psi U^{2_s^*-1}\sim\int_0^{+\infty}\frac{(1-r^2)r^{N-1}}{(1+r^2)^{1+N}}dr=0. \tag{5.11}$$

By combining (5.6)–(5.11), we get a contradiction.

Thus, $c=0$ and $\psi_j\to 0$ in $\Omega\subset\subset\mathbb{R}^N$. If we let $y_j\in\mathbb{R}^N$ such that $\psi_j(y_j)=\|\psi_j\|_\infty=1$, then $y_j\to+\infty$ as $j\to\infty$. However, by Lemma 25, we get $v_j^i(x)\le C\frac{1}{|x|^{N-2s}}$, $i=1,2$ and thus $|\psi_j(x)|\le C\frac{1}{|x|^{N-2s}}$, which implies $\psi_j(x)\to 0$ as $|x|\to\infty$. This is a contradiction since $\psi_j(y_j)=1$.

The proof of Theorem 4 is now complete.

Funding Open access funding provided by Università degli Studi dell'Insubria within the CRUI-CARE Agreement.

References

1. Alves, C.O., Miyagaki, O.H.: Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method, Calc. Var. Partial Differential Equations **55**, 19 (2016)
2. Ambrosio, V.: Ground states for superlinear fractional Schrödinger equations in $\mathbb{R}^N$. Ann. Acad. Sci. Fen. Math. **41**, 745–756 (2016)
3. Atkinson, F., Peletier, L.: Elliptic equations with nearly critical growth. J. Differential Equations **70**, 349–365 (1987)
4. Bhakta, M., Mukherjee, D.: Semilinear nonlocal elliptic equations with critical and supercritical exponents. Commun. Pure Appl. Anal. **16**, 1741–1766 (2017)
5. Bhakta, M., Mukherjee, D.: Nonlocal scalar field equations: Qualitative properties, asymptotic profiles and local uniqueness of solutions. J. Differential Equations **266**, 6985–7037 (2019)
6. Brezis, H., Peletier, L.: Asymptotics for elliptic equations involving critical growth, Progr. Nonlinear Differential Equations Appl. **1**, 149–192 (1989)
7. Bucur, C., Medina, M.: A fractional elliptic problem in $\mathbb{R}^N$ with critical growth and convex nonlinearities. Manuscripta Math. **158**, 371–400 (2019)
8. Cabré, X., Sire, Y.: Nonlinear equations for fractional Laplacians, i: regularity, maximum principles, and Hamiltonian estimates. Ann. Inst. Henri Poincaré Anal. Nonliné,aire **31**, 23–53 (2014)
9. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations **32**, 1245–1260 (2007)

10. Chang, X.J., Wang, Z.Q.: Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. Nonlinearity **26**, 479–494 (2013)
11. Chen, W., Li, C., Ou, B.: Classification of solutions for an integral equation. Comm. Pure. Appl. Math. **59**, 330–343 (2006)
12. Chen, W., Li, C., Li, Y.: A direct method of moving planes for the fractional Laplacian. Adv. Math. **308**, 404–437 (2017)
13. Choi, W., Kim, S., Lee, K.: Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian. J. Funct. Anal. **266**, 6531–6598 (2014)
14. Cotsiolis, A., Tavoularis, N.: Best constants for Sobolev inequalities for higher order fractional derivatives. J. Math. Anal. Appl. **295**, 225–236 (2004)
15. Danielli, D., Pop, C.A.: New developments in the analysis of nonlocal operators. Contemp. Math. **723**, 191–214 (2019)
16. Dávila, J., Del Pino, M., Wei, J.: Concentrating standing waves for the fractional nonlinear schrödinger equations. J. Differential Equations **256**, 858–892 (2014)
17. Dávila, J., Del Pino, M., Sire, Y.: Nondegeneracy of the bubble in the critical case for nonlocal equations. Proc. Amer. Math. Soc. **141**, 3865–3870 (2013)
18. Dipierro, S., Palatucci, G., Valdinoci, E.: Existence and symmetry results for a schrödinger type problem involving the fractional Laplacian. Matematiche **68**, 201–216 (2013)
19. Dipierro, S., Medina, M., Peral, I., Valdinoci, E.: Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^N$. Manuscripta Math. **153**, 183–230 (2017)
20. do Ó, J.M., Miyagaki, O.H., Squassina, M.: Critical and subcritical fractional problems with vanishing potentials. Commun. Contemp. Math. **18**, 20 (2016)
21. Fall, M.M., Valdinoci, E.: Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}^N$ when s is close to 1. Comm. Math. Phy. **329**, 383–404 (2014)
22. Fall, M.M., Mahmoudi, F., Valdinoci, E.: Ground states and concentration phenomena for the fractional schrödinger equation. Nonlinearity **28**, 19–37 (2015)
23. Felmer, P., Quaas, A., Tan, J.: Positive solutions of Nonlinear schrödinger equation with the fractional Laplacian. Proc. Roy. Soc. Edinburgh Sect. A **142**, 1237–1262 (2012)
24. Frank, R.L., Lenzmann, E., Silvestre, L.: Uniqueness of radial solutions to fractional laplacian. Commun. Pure Appl. Math. **69**, 1671–1726 (2016)
25. Han, Z.C.: Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent. Ann. Inst. Henri Poincaré Anal. Nonlinéaire **8**, 159–174 (1991)
26. Lieb, E.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. Math. **118**, 349–374 (1983)
27. Mosconi, S., Squassina, M.: Nonlocal problems at nearly critical growth. Nonlinear Anal. **136**, 84–101 (2016)
28. Ni, W.M., Takagi, I.: Locating the peaks of least energy solutions to a semilinear Neumann problem. Duke Math. J. **70**, 247–281 (1993)
29. Palatucci, G., Pisante, A.: Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces. Calc. Var. Partial Differential Equations **50**, 799–829 (2014)
30. Palatucci, G., Pisante, A., Sire, Y.: Subcritical approximation of a Yamabe type non local equation: a Γ-convergence approach. Ann. Sc. Norm. Super Pisa Cl. Sci. **14**, 819–840 (2015)
31. Pan, X.B., Wang, X.: Blow-up behavior of ground states of semilinear elliptic equations in $\mathbb{R}^N$ involving critical Sobolev exponents. J. Differential Equations **99**, 78–107 (1992)
32. Park, Y.J.: Fractional Polya-Szego inequality. J. Chungcheong Math. Soc. **24**, 267–271 (2011)
33. Ros-Oton, X., Serra, J.: The Pohozaev identity for the fractional Laplacian. Arch. Ration. Mech. Anal. **213**, 587–628 (2014)
34. Ros-Oton, X., Serra, J.: Nonexistence results for nonlocal equations with critical and supercritical nonlinearities. Comm. Partial Differential Equations **40**, 115–133 (2015)
35. Servadei, R., Valdinoci, E.: The Brezis-Nirenberg result for the fractional Laplacian. Trans. Am. Math. Soc. **367**, 67–102 (2015)
36. Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. Comm. Pure. Appl. Math. **60**, 67–112 (2007)
37. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
38. Tan, J., Xiong, J., Harnack, A.: Inequality for fractional Laplace equations with lower order terms. Discrete Contin. Dyn. Syst. **31**, 975–983 (2011)

39. Tianling, J., Li, Y.Y., Xiong, J.: On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. J. Eur. Math. Soc. **16**, 1111–1171 (2014)
40. Wang, X.: On Location of blow-up of ground states of semilinear elliptic equations in $\mathbb{R}^N$ involving critical Sobolev exponents. J. Differential Equations **127**, 148–173 (1996)