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Research Article

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Maximum principle for higher order operators in general domains

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Abstract: We first prove De Giorgi type level estimates for functions in $W^{1,t}(\Omega)$, $\Omega \subset \mathbb{R}^N$, with $t > N \geq 2$. This augmented integrability enables us to establish a new Harnack type inequality for functions which do not necessarily belong to De Giorgi's classes as obtained in Di Benedetto–Trudinger [10] for functions in $W^{1,2}(\Omega)$. As a consequence, we prove the validity of the strong maximum principle for uniformly elliptic operators of any even order, in fairly general domains in dimension two and three, provided second order derivatives are taken into account.

Keywords: Harnack's inequality, Higher order PDEs, Polyharmonic operators, Positivity preserving property

MSC: 35J30, 35J48, 35B50

Dedicated to In loving memory of Louis Nirenberg

1 Introduction

One of the most powerful tools in the study of partial differential equations and nonlinear analysis is without any doubts the Maximum Principle (MP in the sequel). It turns out to be fundamental in obtaining existence, uniqueness and regularity results in the theory of linear elliptic equations, as well as to establish qualitative properties of solutions to nonlinear equations. We mainly refer to [22] for classical results and historical development, where suitable applications also to the parabolic and hyperbolic cases are discussed. Let us merely mention that the roots of MP date back two centuries in the work of Gauss on harmonic functions, up to the ultimate version of Hopf [16], and then further extended in the seminal work of Nirenberg [20], Alexandrov [2] and Serrin [24], within the foundations of modern theory of PDEs.

The underlying idea is simple: positivity of a suitable set of derivatives of a function induces positivity of the function itself. This is elementary true for real functions of one variable which vanish at the endpoints of an interval where $-u''(x) \geq 0$ and the validity can be extended to second order uniformly elliptic operators for which a prototype is the Laplace operator:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \subset \mathbb{R}^N, \quad N \geq 2 \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

for which we have

$$f \geq 0 \Rightarrow u \geq 0 \text{ in } \Omega\,.$$

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Surprisingly, this is no longer true when considering higher order elliptic operators such as the biharmonic operator Δ^2:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

Indeed, in this case in general one has

$$f \geq 0 \not\Rightarrow u \geq 0 \text{ in } \Omega\,.$$

This is a well known fact as long as the domain Ω is not a ball, for which the positive Green function was computed by Boggio [6] and which keeps on being positive for slight deformations of the ball [25]. As deeply investigated in [11] and references therein, the lack of the positivity preserving property is due to the appearance of sets carrying small Hausdorff measure (see [15]) where $u < 0$ and apparently without robust physical motivations. Recently the loss of the MP has been established in [1] also in the case of higher order fractional Laplacians. This paper is a step forward a better understanding of this phenomena and at the same time gives some general principle in order to recover the validity of the MP in the higher order setting.

Let us briefly recall some physical interpretation of (1.1)–(1.2). Indeed, (1.1) is modeling, among many other things, a membrane whose profile is u which deflects under the charge load f and clamped along the boundary $\partial\Omega$. This is the case in which tension forces prevale on bending forces which can be neglected because of the "thin" membrane. However, the model does not suite the case of a "thick" plate in which bending forces have to be taken into account. Here higher order derivatives come into play which yield (1.2). As one expects for (1.1), and there this is true by the MP, upwards pushing of a plate, clamped along the boundary, should yield upwards bending: this is false for (1.2) in contrast to some heuristic evidences in applications (see e.g. [17] and references therein).

Our point of view here, roughly speaking, is that approaching the boundary, where the bending energy carries some minor effect because of the clamping condition, tensional forces can not be neglected for which the contribute of lower order derivatives may restore the validity of the MP. As a reference example, consider the following simple model:

$$\begin{cases} \Delta^2 u - \gamma\Delta u = f, & \text{in } \Omega \subset \mathbb{R}^N,\ \gamma \geq 0 \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{1.3}$$

Clearly for $\gamma = 0$ one has (1.2) whence formally as $\gamma \to \infty$, in a sense one may expect that (1.3) inherits some properties of (1.1).

As we are going to see, this is the case and for the more accurate model (1.3) surprisingly the MP holds true, for fairly general domains, provided $\gamma \geq \gamma_0 > 0$, which is essentially given in terms of Sobolev and Poincaré best constants. Let us state our main result in the case of (1.3) though it extends to cover the general case of uniformly elliptic operators of any even order, see Corollary 5.1.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be an open connected and bounded set, with sufficiently smooth boundary and which satisfies the interior sphere condition. Let $u \in H_0^2(\Omega)$ be a weak solution to (1.3), where $f \in L^2(\Omega)$, $f \geq 0$ in Ω and $|\{x : f(x) > 0\}| > 0$. Then, there exists $\gamma_0 > 0$ (which depends on the diameter of Ω, Sobolev and Poincaré best constants but does not depend on f), such that for $\gamma > \gamma_0$ one has $u > 0$ in Ω.*

As a consequence of Theorem 1.1, the operator $\Delta^2 - \gamma\Delta$, which in addition to (1.2) contains the contribute of lower order derivatives, turns out to be a more natural extension of (1.1) to the higher order setting.

Overview. In Section 2 we prove some preliminary estimates which will be the key ingredient to prove in Section 3 a new Harnack type inequality. Indeed, in the higher order case, it is well known how truncation methods fail [11]. Our approach here is to demand some extra integrability on the function entering the Harnack inequality in place of being solution to a PDE, which usually yields Caccioppoli's inequality and the solution belongs to the corresponding De Giorgi class. In [10] the authors prove a Harnack type inequality just for functions with membership in some De Giorgi classes. Here we drop this assumption though we assume more regularity in terms of integrability which however enables us to prove De Giorgi type pointwise

level estimates. In Section 5 we apply the results obtained to prove the strong maximum principle for polyharmonic operators of any order, which contain lower order derivatives, in sufficiently smooth bounded domains which enjoy the interior sphere condition. This is done by a limiting procedure starting from compactly supported functions and then extending the results and estimates to the solutions of higher order PDEs subject to Dirichlet boundary conditions. Those boundary conditions are in a sense the natural ones as the higher order operator in this case does not decouple into powers of a second order operator. In one hand the result we obtain is a first step towards the investigation of qualitative properties of higher order nonlinear PDEs, such as uniqueness, optimal regularity, symmetries and concentration phenomena [5, 7, 13, 18, 19, 21, 23, 26]. On the other hand, we are confident the tools introduced here may reveal useful also in different higher order contexts, such as parabolic problems, in the study of the sign of solutions to quasilinear equations and in the higher order fractional Laplacian setting [4, 8, 14].
This research started in 2010 when Theorem 1.1 was settled by the first named author in the form of conjecture in a conference in Pisa. New advances towards the results in this paper have been made in 2014 during the first visit of Louis Nirenberg in Varese, then in New York 2015, Pisa 2016 and Varese again in 2017 (his last trip), occasions in which Louis has further stimulated this research during long discussions of which we keep nostalgic memories. Goodbye Louis!

Notation. In the sequel we will use the following basic definitions:
- $B(x_0, r)$ denotes the ball in $\mathbb{R}^N$ of center at x_0 and radius r;
- ω_N is the volume of the unit ball in $\mathbb{R}^N$;
- d_Ω denotes the diameter of the bounded set Ω in $\mathbb{R}^N$;
- $|\cdot|$ applied to sets denotes the Lebesgue measure in $\mathbb{R}^N$ otherwise it is the Euclidean norm in $\mathbb{R}^N$ with scalar product $(\cdot\,,\,\cdot)$;
- $A^+(x_0, k, r) := \{x : x \in B(x_0, r),\ u(x) > k\}$;
- $(u-k)^+ := \max\{u-k,\ 0\}$;
- $\{f > 0\}$ denotes the set $\{x \in \Omega : f(x) > 0\}$;
- Ω satisfies the *interior sphere condition* if for all $x \in \partial\Omega$ there exists $y \in \Omega$ and $r_0 > 0$ such that $B(y, r_0) \subset \Omega$ and $x \in \partial B(y, r_0)$;
- c and C denote positive constants which may change from line to line and which do not depend on the other quantities involved unless explicitly emphasized;
- $W^{m,p}(\Omega)$ is the standard Sobolev space endowed with the norm $\|\cdot\|^p_{m,p} = \sum_{0\le|\alpha|\le m} \|D^\alpha u\|^p_p$;
- $W^{m,p}_0(\Omega)$ is the completion of smooth compactly supported functions with respect to the norm $\|\cdot\|_{m,p}$;
- the critical Sobolev exponent $p^\star := \frac{Np}{N-mp}$, $1 < p < N/m$.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be an open bounded set with sufficiently smooth boundary. The following holds true

Lemma 2.1. *Let $u \in W^{1,t}(\Omega)$, $t > N$ and $1 < s < N$. Then there exists $c(s,t) > 0$ such that for all $k \in \mathbb{R}$, $x_0 \in \Omega$ and $\rho \in (0, r)$ where $0 < r < dist(x_0, \partial\Omega)$ the following holds*

$$\int_{A^+(x_0,k,\rho)} (u-k)^{s^\star}\ dx \le \frac{c(s,t,N)}{(r-\rho)^{s^\star}}\, |A^+(x_0,k,r)|^{(1-\frac{s}{t})\frac{s^\star}{s}} \left[\ \int_{A^+(x_0,k,r)} (u-k)^t\ dx + r^t \int_{A^+(x_0,k,r)} |\nabla u|^t\ dx\right]^{\frac{s^\star}{t}}. \tag{2.1}$$

Proof. Consider a standard cut-off function $\Theta \in C^\infty_0(\mathbb{R}^N)$ given by

$$\Theta(x) = \begin{cases} 1, & x \in B(x_0,\rho) \\ \\ 0, & x \notin B(x_0, r), \end{cases} \tag{2.2}$$

such that $0 \le \Theta(x) \le 1$ and $|\nabla\Theta(x)| \le \dfrac{c}{r-\rho}$.
As $W^{1,t}(\Omega) \hookrightarrow W^{1,s}(\Omega)$, one has from Sobolev's emedding and Hölder's inequality

$$\begin{aligned}
\int_{A^+(x_0,k,\rho)} (u-k)^{s^*}\, dx &\le \int_{A^+(x_0,k,r)} |(u-k)\Theta(x)|^{s^*}\, dx \\
&\le c(s)\left[\int_{A^+(x_0,k,r)} |\nabla[(u-k)\,\Theta]|^s\, dx\right]^{\frac{s^*}{s}} \\
&\le c(s)\,|A^+(x_0,k,r)|^{[1-\frac{s}{t}]\frac{s^*}{s}}\left[\int_{A^+(x_0,k,r)} |\nabla[(u-k)\,\Theta]|^t\, dx\right]^{\frac{s^*}{t}} \\
&\le c(s,t)\,|A^+(x_0,k,r)|^{[1-\frac{s}{t}]\frac{s^*}{s}} \\
&\quad \cdot\left[\frac{c}{(r-\rho)^t}\int_{A^+(x_0,k,r)} (u-k)^t\, dx + \int_{A^+(x_0,k,r)} |\nabla u|^t\, dx\right]^{\frac{s^*}{t}}
\end{aligned}$$

□

Remark 2.1. *The condition $0 < r < dist(x_0, \partial\Omega)$, namely that x_0 lies in the interior of Ω, is crucial to extend to the whole $\mathbb{R}^N$ the function $(u-k)\Theta$. Therefore when x_0 approaches $\partial\Omega$, necessarily $r = r(x_0)$ tends to zero.*

Lemma 2.2. *Let $u \in W^{1,t}(\Omega)$, $t > N \ge 2$ and $1 < s < N$. Let $l, k \in \mathbb{R}$ such that $l > k$, $x_0 \in \Omega$ and $r < dist(x_0, \partial\Omega)$. Then for all $\rho \in (0, r)$ one has*

$$\begin{aligned}
\int_{A^+(x_0,l,\rho)} (u-l)^2\, dx &\le \frac{c(t)\ |A^+(x_0,k,r)|^{\beta}}{(r-\rho)^{\frac{2(p-1)}{p}}}\left[\int_{A^+(x_0,k,r)} (u-k)^2\, dx\right]^{\frac{1}{p}} \\
&\quad\cdot\left[\int_{A^+(x_0,k,r)} (u-k)^t\, dx + r^t\int_{A^+(x_0,k,r)} |\nabla u|^t\, dx\right]^{\frac{2(p-1)}{pt}},
\end{aligned} \tag{2.3}$$

where $\beta = 1 - \dfrac{2}{q} + \left(1 - \dfrac{s}{t}\right)\dfrac{s^*}{s}\,\dfrac{2p-q}{pq}$, $s = \dfrac{2qN(p-1)}{N(2p-q) + 2q(p-1)}$ *and* $2 < q < 2p$, $p > 1$.

Proof. Let $x_0 \in \Omega$, $r < dist(x_0, \partial\Omega)$ and for simplicity let us write $A^+(k,r)$ in place of $A^+(x_0,k,r)$.
For $l, k \in \mathbb{R}$ and $\rho \in (0,r)$, since $A^+(l,\rho) \subset A^+(k,\rho)$ we have

$$\int_{A^+(l,\rho)} (u-l)^2\, dx \le \int_{A^+(k,\rho)} (u-k)^2\, dx. \tag{2.4}$$

Let $q > 2$ for which one has

$$\int_{A^+(k,\rho)} (u-k)^2\, dx \le |A^+(k,\rho)|^{1-\frac{2}{q}}\left[\int_{A^+(k,\rho)} (u-k)^q\, dx\right]^{\frac{2}{q}}. \tag{2.5}$$

Let now $p > 1$ and $2 < q < 2p$ and estimate by Hölder's inequality

$$\int_{A^+(k,\rho)} (u-k)^q\, dx \le \left[\int_{A^+(k,\rho)} (u-k)^2\, dx\right]^{\frac{q}{2p}}\left[\int_{A^+(k,\rho)} (u-k)^{\frac{2q(p-1)}{(2p-q)}}\, dx\right]^{\frac{2p-q}{2p}}$$

$$\leq \frac{c(p,q,t)}{(r-\rho)^{\frac{q(p-1)}{p}}} |A^+(k,r)|^{(1-\frac{s}{t})\frac{q(p-1)}{sp}} \left[\int\limits_{A^+(k,\rho)} (u-k)^2 \ dx \right]^{\frac{q}{2p}}$$

$$\cdot \left[\int\limits_{A^+(k,r)} (u-k)^t \ dx + r^t \int\limits_{A^+(k,r)} |\nabla u|^t \ dx \right]^{\frac{q(p-1)}{tp}}, \quad (2.6)$$

where in the last inequality we have used Lemma 2.1 with $s^\star = \frac{2q(p-1)}{2p-q}$. Combine (2.5) and (2.6) to get

$$\int\limits_{A^+(k,\rho)} (u-k)^2 \ dx \ \leq \ \frac{c(p,q,t,N)}{(r-\rho)^{2\frac{p-1}{p}}} |A^+(k,r)|^{(1-\frac{2}{q})+(1-\frac{s}{t})\frac{2(p-1)}{sp}} \left[\int\limits_{A^+(k,\rho)} (u-k)^2 \ dx \right]^{\frac{1}{p}}$$

$$\cdot \left[\int\limits_{A^+(k,r)} (u-k)^t \ dx + r^t \int\limits_{A^+(k,r)} |\nabla u|^t \ dx \right]^{\frac{2(p-1)}{tp}}. \quad (2.7)$$

□

In what follows we will use the following result from [3] in order to prove a version of the well known Poincaré inequality.

Theorem 2.1 (Theorem A. 28, p. 184 in [3]). *Let $u \in W^{1,1}(B_r)$, such that $u \geq 0$ and $|\{x : u(x) = 0\}| \geq \frac{|B_r|}{2}$. Then*

$$\left(\int\limits_{B_r} u^{1^\star} \ dx \right)^{\frac{1}{1^\star}} \leq c \int\limits_{B_r} |\nabla u| \ dx, \quad (2.8)$$

where $c = c(N)$ depends only on the dimension N.

Lemma 2.3. *Let $u \in W^{1,p}(B_r)$ be such that $|\{x : u(x) = 0\}| \geq \frac{|B_r|}{2}$, with $p \geq \frac{N}{N-1}$ and $N \geq 2$. Then, the following holds*

$$\left(\int\limits_{B_r} |u|^p \ dx \right)^{\frac{1}{p}} \leq c\, \omega_N^{\frac{1}{N}}\, p\, \frac{N-1}{N}\, r \left(\int\limits_{B_r} |\nabla u|^p \ dx \right)^{\frac{1}{p}}, \quad (2.9)$$

where $c = c(N)$ is the constant in (2.8).

Proof. Apply Theorem 2.1 to the function $|u|^p$, taking $p \geq \frac{N}{N-1}$ and $N \geq 2$, to get

$$\int\limits_{B_r} |u|^p \ dx = \int\limits_{B_r} \left(|u|^{p\frac{N-1}{N}} \right)^{\frac{N}{N-1}} \ dx \leq c^{\frac{N}{N-1}} \left[\int\limits_{B_r} \left|\nabla \left(|u|^{p\frac{N-1}{N}} \right) \right| \ dx \right]^{\frac{N}{N-1}}$$

$$= c^{\frac{N}{N-1}} \left[\int\limits_{B_r} p \frac{N-1}{N} |u|^{p\frac{N-1}{N}-1} |\nabla u| \, dx \right]^{\frac{N}{N-1}}$$

$$= c^{\frac{N}{N-1}}\, p^{\frac{N}{N-1}} \left(\frac{N-1}{N} \right)^{\frac{N}{N-1}} \left[\int\limits_{B_r} |u|^{p\frac{N-1}{N}-1} |\nabla u| \, dx \right]^{\frac{N}{N-1}}$$

$$= c^{\frac{N}{N-1}} p^{\frac{N}{N-1}} \left(\frac{N-1}{N}\right)^{\frac{N}{N-1}} \left[\int_{B_r} |u|^{p(\frac{N-1}{N}-\frac{1}{p})} (|\nabla u|^p)^{\frac{1}{p}}\, 1^{\frac{1}{N}}\, dx\right]^{\frac{N}{N-1}}$$

$$\leq c^{\frac{N}{N-1}} p^{\frac{N}{N-1}} \left(\frac{N-1}{N}\right)^{\frac{N}{N-1}} \left[\int_{B_r} |u|^p\, dx\right]^{(\frac{N-1}{N}-\frac{1}{p})\frac{N}{N-1}} \cdot \left[\int_{B_r} |\nabla u|^p\, dx\right]^{\frac{1}{p}\frac{N}{N-1}} |B_r|^{\frac{1}{N-1}}.$$

Since

$$\left(\frac{N-1}{N} - \frac{1}{p}\right) \frac{N}{N-1} = 1 - \frac{1}{p}\frac{N}{N-1},$$

we have

$$\left[\int_{B_r} |u|^p\, dx\right]^{\frac{1}{p}\frac{N}{N-1}} \leq c^{\frac{N}{N-1}} p^{\frac{N}{N-1}} \left(\frac{N-1}{N}\right)^{\frac{N}{N-1}} \omega_N^{\frac{1}{N-1}} r^{\frac{N}{N-1}} \left[\int_{B_r} |\nabla u|^p\, dx\right]^{\frac{1}{p}\frac{N}{N-1}}.$$

□

3 A Harnack type inequality

Next we derive a De Giorgi type level estimate (see [3, 12]) for functions $u \in W^{1,t}$, $t > N \geq 2$ which will be the key ingredient in establishing a new Harnack type inequality. Let us emphasize that in De Giorgi's theorem [9], level estimates hold for $u \in W^{1,2}$ which is a solution to a uniformly elliptic second order equation with bounded and measurable coefficients. As a consequence, Caccioppoli's inequality holds and $u \in W^{1,2}$ belongs to the corresponding so-called De Giorgi class. Later, Di Benedetto and Trudinger relaxed the framework and in [10] they merely assume $u \in W^{1,2}$ belonging to some De Giorgi class. Here, we further improve the setting, without requiring any of those previous assumptions, though demanding for some augmented integrability which turns out to be necessary, as it is well known, functions in $W^{1,N}(\Omega)$, $\Omega \subset \mathbb{R}^N$, may not be bounded.

Theorem 3.1. *Let $u \in W^{1,t}(\Omega)$, $t > N \geq 2$, $\Omega \subset \mathbb{R}^N$ be open and bounded set with sufficiently smooth boundary $\partial\Omega$. For all $k \in \mathbb{R}$, $y \in \Omega$, $r > 0$ such that $r < dist(y, \partial\Omega)$, the following holds*

$$\sup_{B(y,\frac{r}{2})} u \leq k + d, \tag{3.1}$$

where

$$d = \frac{c}{r^{\frac{\xi(p-1)}{\eta p}}} \left(\int_{A^+(k,r)} |u(x) - k|^t\, dx + r^t \int_{A^+(k,r)} |\nabla u(x)|^t\, dx\right)^{\frac{\xi(p-1)}{tp\eta}} \left(\int_{A^+(k,r)} |u(x) - k|^2\, dx\right)^{\frac{\xi(\theta-1)}{2\eta}} |A^+(k,r)|^{\frac{\theta-1}{2}}.$$

Remark 3.1. *Here we write for simplicity $A^+(k,\rho)$ in place of $A^+(y,k,\rho)$ and $c = c(t,p,\xi,\eta,\theta)$ is a positive constant which depends on the parameters $p > 1$, $t > N$ obtained in Lemma 2.2, while $\xi > 0$, $\eta > 0$, $\theta > 1$ are defined by suitable equations stated in the proof.*

Proof of Theorem 3.1. Let us set:

$$I(l,\rho) \quad = \quad \int_{A^+(l,\rho)} |u - l|^2\, dx\,,$$

$$M(r,k,t,p) \;=\; c(t)\left(\int\limits_{A^+(k,r)} |u-k|^t\ dx + r^t \int\limits_{A^+(k,r)} |\nabla u|^t\ dx\right)^{\frac{2(p-1)}{pt}},$$

where $c(t)$ is the constant of (2.3). For all $l, k \in \mathbb{R}$, such that $l > k$ and for all $\rho \in (0, r)$, one has

$$|A^+(l,\rho)| \leq \frac{1}{(l-k)^2}\, I(k,\rho) \tag{3.2}$$

and clearly $|A^+(l,\rho)| \leq |A^+(k,\rho)|$, for $l > k$.
Set

$$\Phi(l,\rho) \;=\; I(l,\rho)^{\xi}\ |A^+(l,\rho)|^{\eta}, \tag{3.3}$$

then from (3.2) and (2.3) we have

$$\Phi(l,\rho) \;\leq\; \frac{1}{(r-\rho)^{2\xi\frac{p-1}{p}}\ (l-k)^{2\eta}}\, \Phi(k,r)^{\theta}\, M(r,k,t,p)^{\xi}, \tag{3.4}$$

where $\eta, \xi, \theta > 0$ satisfy the following algebraic equations

$$\begin{cases} \dfrac{\xi}{p} + \eta \;=\; \theta\,\xi \\[2ex] \beta\,\xi \;=\; \theta\,\eta \end{cases} \tag{3.5}$$

from which we have $\theta^2 - \theta/p - \beta = 0$ and we take $\theta = \theta_1$ given by

$$\theta_1 \;=\; \frac{1/p + \sqrt{1/p^2 + 4\beta}}{2}\,. \tag{3.6}$$

As one can easily check $\theta_1 > 1$, for all $2 < q < 2p$, $t > N$ and $1 < s < N$.
From (3.4) we are done provided we prove that for all $k \in \mathbb{R}$ and $r < dist(y, \partial\Omega)$ there exists $d > 0$ satisfying

$$\Phi\left(k+d, \frac{r}{2}\right) = 0,$$

which in turn by (3.3) yields

$$\left|A^+\left(k+d, \frac{r}{2}\right)\right| \;=\; 0\,.$$

Next we proceed by using the iterative scheme from the proof of De Giorgi's theorem. For $m \in \mathbb{N}$ set

$$r_m = \frac{r}{2} + \frac{r}{2^{m+1}}, \qquad k_m = k_0 + d - \frac{d}{2^m},$$

where the parameter $d > 0$ has to be chosen in the sequel and $k_0 = k$. The idea is to exploit the inequality (3.4) with $r = r_m$ and $\rho = r_{m+1}$ where the sequence $\{r_m\}_{m\in\mathbb{N}}$ is decreasing so that $B(r_{m+1}) \subset B(r_m)$. On the other hand $\{k_m\}_{m\in\mathbb{N}}$ is increasing, and we set in (3.4) $l = k_{m+1}$ and $k = k_m$. With this choice we obtain from (3.4) the following inequality

$$\Phi(k_{m+1}, r_{m+1}) \leq \frac{2^{2\frac{(p-1)}{p}(m+2)\xi + 2(m+1)\eta}}{r^{2\frac{(p-1)}{p}\xi}\ d^{2\eta}}\, \Phi(k_m, r_m)^{\theta}\, M(r_m, k_m, t, p)^{\xi}. \tag{3.7}$$

Now multiply (3.7) by $2^{\mu(m+1)}$, $\mu > 0$ and set

$$\Psi_m = 2^{\mu m}\Phi(k_m, r_m) \tag{3.8}$$

to obtain form (3.7)

$$\Psi_{m+1} \leq \left[\frac{2^{2\frac{(p-1)}{p}(m+2)\xi + 2(m+1)\eta}}{r^{2\frac{(p-1)}{p}\xi}\ d^{2\eta}}\ 2^{\mu[1+m(1-\theta)]}\right] \Psi_m^{\theta}\, M(r_m, k_m, t, p)^{\xi}. \tag{3.9}$$

Let us choose $\mu > 0$ to avoid the dependence on m in the first factor in the right hand side of (3.9), namely

$$\mu = \frac{2\frac{(p-1)}{p}\xi + 2\eta}{\theta - 1}, \tag{3.10}$$

and thus (3.9) becomes

$$\Psi_{m+1} \leq \frac{2^{4\frac{(p-1)}{p}\xi+2\eta+\mu}}{r^{2\frac{(p-1)}{p}\xi}\, d^{2\eta}}\, \Psi_m^\theta\, M(r_m, k_m, t, p)^\xi \leq \frac{2^{4\frac{(p-1)}{p}\xi+2\eta+\mu}}{r^{2\frac{(p-1)}{p}\xi}\, d^{2\eta}}\, \Psi_m^\theta\, M(r, k_0, t, p)^\xi .$$

Set

$$A = \frac{2^{4\frac{(p-1)}{p}\xi+2\eta+\mu}}{r^{2\frac{(p-1)}{p}\xi}}\, M(r, k_0, t, p)^\xi,$$

so that for all $m \in \mathbb{N}$ one has

$$\Psi_{m+1} \leq \frac{A}{d^{2\eta}}\, \Psi_m^\theta .$$

At this point we choose $d > 0$ such that

$$\frac{A}{d^{2\eta}}\, \Psi_0^{\theta-1} = 1, \tag{3.11}$$

and by induction on $m \in \mathbb{N}$ we have

$$\Psi_m \leq \Psi_0, \qquad \text{for all } m \in \mathbb{N} .$$

Finally by (3.8) we obtain

$$\Phi(k_m, r_m) \leq \frac{1}{2^{\mu m}}\, \Phi(k_0, r)$$

and the proof is complete by letting $m \to \infty$.

□

Remark 3.2. *It is important to note that in Lemma 2.2 as $t \to N$ one has $\beta \to (p-1)/p$ so that $\theta_1 \to 1$ in* (3.6). *As a consequence, from* (3.10) *one has $\mu \to \infty$ and this, as expected, prevents the result to hold.*

Next we prove the following Harnack type inequality

Theorem 3.2. *Let $u \in W^{1,t}(\Omega)$, $t > N \geq 2$, and $\Omega \subset \mathbb{R}^N$ be an open bounded set with sufficiently smooth boundary $\partial\Omega$. Let $B(x_0, r) \subset \Omega$, then there exists a constant $c > 0$, which depends only on N, such that*

$$\sup_{B(x_0,\frac{r}{2})} u \leq \inf_{B(x_0,r)} u + c\, r^{[(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)]} \left(\int_{B(x_0,r)} |\nabla u|^t\, dx \right)^{\frac{\xi}{\eta}\frac{p-1}{tp}} \left(\int_{B(x_0,r)} |\nabla u|^2\, dx \right)^{\frac{\xi(\theta-1)}{2\eta}} . \tag{3.12}$$

Proof. Set $M = \sup_{B(x_0,r)} u$, $m = \min_{B(x_0,r)} u$, and let

$$I_1 = \left\{ k : k \in (m, M) : \left|\{x : x \in B(x_0, r),\, u(x) > k\}\right| < \frac{|B(x_0, r)|}{2} \right\},$$

$$I_2 = \left\{ k : k \in (m, M) : \left|\{x : x \in B(x_0, r),\, u(x) \geq k\}\right| \geq \frac{|B(x_0, r)|}{2} \right\} .$$

If $I_1 \neq \emptyset$ then we prove for all $k \in I_1$ the following

$$\sup_{B(x_0,\frac{r}{2})} u \leq k + c\, r^{[(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)]} \left(\int_{B(x_0,r)} |\nabla u|^t\, dx \right)^{\frac{\xi}{\eta}\frac{p-1}{tp}} \left(\int_{B(x_0,r)} |\nabla u|^2\, dx \right)^{\frac{\xi(\theta-1)}{2\eta}} . \tag{3.13}$$

Indeed, by Theorem 3.1 we have for all $k \in I_1$

$$\sup_{B(x_0,\frac{r}{2})} u \le k + c\, r^{-\frac{\xi(p-1)}{\eta p}} |A^+(k,r)|^{\frac{\theta-1}{2}} \left(\int_{A^+(k,r)} |u-k|^t \, dx + r^t \int_{A^+(k,r)} |\nabla u|^t \, dx \right)^{\frac{\xi}{\eta}\frac{p-1}{tp}} \left(\int_{A^+(k,r)} |u-k|^2 \, dx \right)^{\frac{\xi(\theta-1)}{2\eta}} . \tag{3.14}$$

Since $k \in I_1$ one has

$$|\{x : (u(x)-k)^+ = 0\}| \ge \frac{|B(x_0,r)|}{2},$$

and apply Lemma 2.3 to the function $(u(x)-k)^+$ to get

$$\left(\int_{A^+(k,r)} |u(x)-k|^2 \, dx \right)^{\frac{1}{2}} = \left(\int_{B(x_0,r)} |(u(x)-k)^+|^2 \, dx \right)^{\frac{1}{2}} \le c(N) r \left(\int_{B(x_0,r)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}},$$

$$\left(\int_{A^+(k,r)} |u(x)-k|^t \, dx \right)^{\frac{1}{t}} = \left(\int_{B(x_0,r)} |(u(x)-k)^+|^t \, dx \right)^{\frac{1}{t}} \le c(N) r \left(\int_{B(x_0,r)} |\nabla u|^t \, dx \right)^{\frac{1}{t}}.$$

In the case $I_2 \ne \emptyset$, for all $k \in I_2$ set $h = -k$ and $v(x) = -u(x)$. Thus $h \in (-M, -m)$ and the following holds

$$\begin{aligned} \left|\{x : x \in B(x_0,r) : u(x) \ge k\}\right| &= \left|\{x : x \in B(x_0,r) : -u(x) \le -k\}\right| \\ &= \left|\{x : x \in B(x_0,r) : v(x) \le h\}\right| \ge \frac{|B(x_0,r)|}{2}. \end{aligned}$$

Therefore, the function v enjoys (3.13), namely

$$\sup_{B(x_0,\frac{r}{2})} v \le h + c\, r^{[(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)]} \left(\int_{B(x_0,r)} |\nabla v|^t \, dx \right)^{\frac{\xi}{\eta}\frac{p-1}{tp}} \left(\int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{\xi(\theta-1)}{2\eta}} . \tag{3.15}$$

From

$$\sup_{B(x_0,\frac{r}{2})} v = -\inf_{B(x_0,\frac{r}{2})} u$$

and (3.15) we have

$$-\inf_{B(x_0,\frac{r}{2})} u \le -k + c\, r^{[(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)]} \left(\int_{B(x_0,r)} |\nabla u|^t \, dx \right)^{\frac{\xi}{\eta}\frac{p-1}{tp}} \left(\int_{B(x_0,r)} |\nabla u|^2 \, dx \right)^{\frac{\xi(\theta-1)}{2\eta}} .$$

As a consequence, for all $k \in I_2$ we get

$$k \le \inf_{B(x_0,\frac{r}{2})} u + c\, r^{[(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)]} \left(\int_{B(x_0,r)} |\nabla u|^t \, dx \right)^{\frac{\xi}{\eta}\frac{p-1}{tp}} \left(\int_{B(x_0,r)} |\nabla u|^2 \, dx \right)^{\frac{\xi(\theta-1)}{2\eta}} . \tag{3.16}$$

Next we distinguish three cases, precisely:

i) $I_1 \ne \emptyset$ and $I_2 = \emptyset$. In this case any $k \in (m, M)$ belongs to I_1, for which (3.13) which holds for all $k \in I_1$, it holds for $k = m$ as well;

ii) $I_1 = \emptyset$ and $I_2 \ne \emptyset$. In this case any $k \in (m, M)$ belongs to I_2, and thus (3.16) which holds for all $k \in I_2$, in particular holds for $k = M$;

iii) $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. In this case we consider $\inf I_1$ and $\sup I_2$ and it is standard to prove there exists a unique $k_0 = \inf I_1 = \sup I_2$ which enjoys both (3.13) and (3.16) and the theorem follows.

□

In order to state the next result let us introduce the following

Definition 3.1. *Let Ω be an open set in $\mathbb{R}^N$, $N \geq 2$ with non-empty and sufficiently smooth boundary and which enjoys the interior sphere condition. Let $x \in \partial\Omega$ and consider balls of radius r, $B_x(r) \subset \Omega$ which are tangent in the interior to $\partial\Omega$ at point x and let $\delta(x) = \sup r$. We define the narrowness index of Ω as follows:*

$$\delta = \inf_{x \in \partial\Omega} \delta(x) . \tag{3.17}$$

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be open, connected, with sufficiently smooth boundary and which enjoys the interior sphere condition. Let δ be the narrowness index of Ω as in Definition 3.17. Let x_{max} and x_{min} be respectively a local maximum and local minimum for $u \in W^{1,t}(\Omega)$, $t > N$.*
Then, there exists $h \in \mathbb{N}$ and $r \in (0, \delta)$, such that

$$u(x_{max}) \leq u(x_{min}) + c\ h\, r^{(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)} \left(\int_\Omega |\nabla u(x)|^t\ dx \right)^{\frac{\xi(p-1)}{tp\eta}} \left(\int_\Omega |\nabla u(x)|^2\ dx \right)^{\frac{\xi(\theta-1)}{2\eta}} , \tag{3.18}$$

with $c = c(N)$ provided by Thorem 3.2 and where in particular h depends only on $dist(x_{max}, \partial\Omega)$, $dist(x_{min}, \partial\Omega)$ and δ.

Proof. Let $r > 0$ be such that:

i) for all $x \in B(x_{min}, r) \subset \Omega$ one has $u(x) \geq u(x_{min})$;

ii) $\overline{B(x_{min}, r)} \subset \Omega$;

iii) $\overline{B(x_{max}, r)} \subset \Omega$.

Consider the arc $g : [0, 1] \to \Omega$ such that $g(0) = x_{min}$ and $g(1) = x_{max}$. Let $t_0 = 0 < \ldots t_h = 1$ be a partition of $[0, 1]$ such that setting $x_i = g(t_i)$ one has

$$B\left(x_i, \frac{r}{2}\right) \cap B\left(x_{i+1}, \frac{r}{2}\right) \neq \emptyset, \quad i = 0, \ldots, h-1 \tag{3.19}$$

and where r is such that $B(x_i, r) \subset \Omega$.
By Theorem 3.2 we have

$$\sup_{B(x_0, \frac{r}{2})} u \leq u(x_{min}) + c\, r^{(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)} \left(\int_{B(x_0,r)} |\nabla u(x)|^t\ dx \right)^{\frac{\xi(p-1)}{tp\eta}} \left(\int_{B(x_0,r)} |\nabla u(x)|^2\ dx \right)^{\frac{\xi(\theta-1)}{2\eta}} ,$$

which we rewrite in the following form

$$\forall x \in B\left(x_0, \frac{r}{2}\right), \quad u(x) \leq u(x_{min}) + N_0, \tag{3.20}$$

where we have set for $i = 0, \ldots, h$

$$N_i := c \left(\int_{B(x_i,r)} |\nabla u(x)|^t\ dx \right)^{\frac{\xi(p-1)}{tp\eta}} \left(\int_{B(x_i,r)} |\nabla u(x)|^2\ dx \right)^{\frac{\xi(\theta-1)}{2\eta}} .$$

Now inequality (3.20) in particular holds for

$$x \in B\left(x_1, \frac{r}{2}\right) \cap B\left(x_0, \frac{r}{2}\right)$$

and thus

$$\inf_{B(x_1, \frac{r}{2})} u \leq u(x) \leq u(x_{min}) + N_0 . \tag{3.21}$$

By applying iteratively Theorem 3.2 we end up with

$$\sup_{B(x_{h+1},\, \frac{r}{2})} u \le u(x_{min}) + N_h + \cdots + N_1 + N_0 \,,$$

which completes the proof.

□

Remark 3.3. *One may wonder what happens if in the construction of Theorem 3.3 we consider a sequence of balls with increasing radius and center converging to a point on the boundary of Ω. For this purpose consider $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ converging to a point $x_\infty \in \partial\Omega$. Consider balls of center x_n and radius r_n such that:*
(i) $B(x_n, r_n) \subset \Omega$;
(ii) $r_n < dist(x_n, \partial\Omega) = dist(x_n, x_\infty)$;
(iii) $B(x_n, \frac{r_n}{2}) \cap B(x_{n+1}, \frac{r_{n+1}}{2}) \neq \emptyset$.

Applying to this sequence the reasoning carried out in the proof of Theorem 3.3 where $x_0 = x_{max}$, we get

$$u(x_{max}) \le u(x_\infty) + c \sum_{n=0}^{\infty} r_n^{(\frac{\xi}{\eta}+\frac{N}{2})(\theta-1)} \left(\frac{1}{\gamma} \int_{\Omega_1} f(x)\, dx \right)^{\frac{\xi}{2\eta}(\theta-1)} u(x_{max}).$$

We would get a contradiction if the above series converge. Actually as we are going to see this is not the case. Consider $B(x_n, \frac{r_n}{2})$ and $B(x_{n+1}, \frac{r_{n+1}}{2})$ and let $C \in \partial B(x_n, \frac{r_n}{2}) \cap \partial B(x_{n+1}, \frac{r_{n+1}}{2})$ and D its projection on the segment with endpoints $A = x_n$ and $B = x_{n+1}$. Set $AD = \rho_n$, $DB = \rho_{n+1}$, so that considering the triangles ADC and CDB one has $\frac{r_n^2}{4} - \rho_n^2 = \frac{r_{n+1}^2}{4} - \rho_{n+1}^2$, and then

$$\frac{\frac{r_n}{4} + \rho_n}{\frac{r_{n+1}}{4} + \rho_{n+1}} = \frac{\frac{r_{n+1}}{4} - \rho_{n+1}}{\frac{r_n}{4} - \rho_n}.$$

We can apply Kummer's test to the series with general terms $a_n = \left(\frac{r_n}{4} + \rho_n\right)^a$ and $b_n = \left(\frac{r_n}{4} - \rho_n\right)^a$, $a > 0$, from which since $\frac{a_n}{a_{n+1}} = \frac{b_{n+1}}{b_n}$, for all $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} \frac{1}{b_n} = +\infty$ we obtain $\sum_{n=0}^{\infty} a_n = +\infty$. From $a_n < r_n^a$ we have $\sum_{n=0}^{\infty} r_n^a = +\infty$.

4 Towards the Positivity Preserving Property

Next we apply the results so far obtained to prove the strong maximum principle for the biharmonic operator perturbed by the Laplacian for compactly supported data. As we are going to see, here it comes for the first time the restriction on the Euclidean dimension $N < 4$ and the fact that we deal with the solution to a PDE. Precisely, this section is devoted to proving the following

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ be an open and bounded set, with sufficiently smooth boundary and which enjoys the interior sphere condition. Let $u \in W^{4,2}(\Omega) \cap H_0^2(\Omega)$ be a solution to*

$$\Delta^2 u(x) - \gamma \Delta u(x) = f(x), \quad x \in \Omega, \tag{4.1}$$

where $\gamma > 0$, $f \in L^2(\Omega)$, $f \ge 0$ in Ω and $|\{x : f(x) > 0\}| > 0$. Moreover, $f(x) = 0$ on $\Omega \setminus \Omega_1$, with Ω_1 a bounded subset of Ω such that $dist(\partial\Omega_1, \partial\Omega) > 0$. Then, there exists $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ the solution to (4.1) satisfies $u(x) > 0$, for all $x \in \Omega$.

Assuming the hypotheses of Theorem 4.1 we have the following preliminary lemmas:

Lemma 4.1. *The following holds true*

$$\sup_{\Omega_1} u > 0 . \tag{4.2}$$

Proof. By multiplying (4.1) by u and integrating by parts

$$\int_\Omega |\Delta u(x)|^2 \, dx + \gamma \int_\Omega |\nabla u(x)|^2 \, dx = \int_\Omega f(x)\, u(x) \, dx \le \sup_{\Omega_1} u \int_{\Omega_1} f(x) \, dx \tag{4.3}$$

□

In order to apply the Harnack inequality established in Section 3 we next estimate first order derivatives of the solution to (4.1). Though from one side elliptic regularity yields enough summability, on the other side we need estimates which are uniform with respect to the parameter γ, and for this reason we restrict ourself to dimensions $N < 4$.

Lemma 4.2. *There exists a constant $c = c(N) > 0$ which does not depend on γ in (29) such that*

$$\|\nabla u\|_{L^t(\Omega)} \le c d_\Omega^{\frac{2}{t}(3-N)} \left(\int_\Omega f(x)\, u(x) \, dx \right)^{\frac{1}{2}} ,$$

for any $t > 2$ when $N = 2$ and for $t = 6$ when $N = 3$.

Proof. Since $u = \nabla u = 0$ on $\partial\Omega$, one has

$$\int_\Omega |\Delta u(x)|^2 \, dx = \sum_{i,j=1}^{n} \int_\Omega |D_{ij} u(x)|^2 \, dx = \int_\Omega \|D^2 u(x)\|^2 \, dx .$$

By Sobolev's embedding, Poincaré inequality and from (4.3), when $N = 3$ and $t = 6$ we have,

$$\|\nabla u\|_{L^t(\Omega)} \le \frac{c_S}{d_\Omega} \|\nabla u\|_{L^2(\Omega)} + c_S \|D^2 u\|_{L^2(\Omega)} \le c\|D^2 u\|_{L^2(\Omega)} = c\|\Delta u\|_{L^2(\Omega)} \le c \left(\int_\Omega f(x)\, u(x) \, dx \right)^{\frac{1}{2}} .$$

Similarly when $N = 2$ and $t \ge 1$ we obtain

$$\|\nabla u\|_{L^t(\Omega)} \le \frac{c_S}{d_\Omega^{1-\frac{2}{t}}} \|\nabla u\|_{L^2(\Omega)} + c_S d_\Omega^{\frac{2}{t}} \|D^2 u\|_{L^2(\Omega)} \le c d_\Omega^{\frac{2}{t}} \|D^2 u\|_{L^2(\Omega)} = c d_\Omega^{\frac{2}{t}} \|\Delta u\|_{L^2(\Omega)} \le c d_\Omega^{\frac{2}{t}} \left(\int_\Omega f(x) u(x) \, dx \right)^{\frac{1}{2}} .$$

□

Proof of Theorem 4.1. Let x_{max} be an absolute maximum point for u in $\overline{\Omega_1}$ and x_{min} a local minimum for u in Ω. Set

$$\mathbf{a} = \frac{\xi(\theta-1)}{2\eta} + \frac{\xi(p-1)}{2\eta p}, \ \mathbf{b} = \left(\frac{\xi}{\eta} + \frac{N}{2} \right)(\theta - 1), \ \mathbf{c} = \frac{\xi}{\eta} \frac{p-1}{p} .$$

From (30) of Theorem 3.3, (36) and Lemma 4.2 we have

$$u(x_{max}) \le u(x_{min}) + ch\, r^{\mathbf{b}} \, d_\Omega^{[\frac{2}{t}(3-N)]\mathbf{c}} \, \frac{\left(\int_\Omega f(x) u(x) \, dx\right)^{\mathbf{a}}}{\gamma^{\frac{\xi}{2\eta}(\theta-1)}} \tag{4.4}$$

where $\mathbf{a} = \dfrac{\xi(\theta-1)}{2\eta} + \dfrac{\xi(p-1)}{2\eta p} < 1$. If $\sup_{\Omega_1} u \ge 1$ then we have

$$u(x_{max}) \le u(x_{min}) + ch\, r^{\mathbf{b}} \, d_\Omega^{[\frac{2}{t}(3-N)]\mathbf{c}} \, \frac{\left(\int_\Omega f(x) \, dx\right)^{\mathbf{a}} u(x_{max})}{\gamma^{\frac{\xi}{2\eta}(\theta-1)}} .$$

The thesis follows as γ is large enough. If $\sup_{\Omega_1} u < 1$, let $k > 0$ be such that $k \sup_{\Omega_1} u \geq 1$. Set $w_k(x) := k\, u(x)$, which satisfies

$$\begin{cases} w_k \in W^{4,2}(\Omega) \cap H_0^2(\Omega) \\ \\ \Delta^2 w_k(x) - \gamma \Delta w_k(x) = k f(x), \quad x \in \Omega, \end{cases} \tag{4.5}$$

Peforming the change of variable $x = sy$, with $s > 0$, $v_k(y) = w_k(sy)$, $g(y) = f(sy)$, $y_{min} = \frac{x_{min}}{s}$, $y_{max} = \frac{x_{max}}{s}$, we obtain

$$\begin{cases} v_k \in W^{4,2}(\Omega_s) \cap H_0^2(\Omega_s) \\ \\ \Delta^2 v_k(y) - \gamma\, s^{-2}\, \Delta v_k(y) = s^{-4}\, k\, g(y), \quad y \in \Omega_s, \end{cases} \tag{4.6}$$

where $\Omega_s = \{y : \ y = x/s, \ \ x \in \Omega\}$. Next apply (4.4) to the solution of (4.6) to get

$$v_k(y_{max}) \leq v_k(y_{min}) + ch \left(\frac{r}{s}\right)^{\mathbf{b}} \left(\frac{d_\Omega}{s}\right)^{[\frac{2}{t}(3-N)]\mathbf{c}} \frac{\left(\int_{\Omega_s} g(x)\, dx\right)^{\mathbf{a}} v_k(y_{max})}{\gamma^{\frac{\xi}{2\eta}(\theta-1)}} \frac{k^{\mathbf{a}}}{s^{3\mathbf{a}}}\,. \tag{4.7}$$

With respect to the original variables it reads as follows

$$u(x_{max}) \leq u(x_{min}) + ch \left(\frac{r}{s}\right)^{\mathbf{b}} \left(\frac{d_\Omega}{s}\right)^{[\frac{2}{t}(3-N)]\mathbf{c}} \frac{\left(\int_{\Omega} f(x)\, dx\right)^{\mathbf{a}} u(x_{max})}{\gamma^{\frac{\xi}{2\eta}(\theta-1)}} \frac{k^{\mathbf{a}}}{s^{(N+3)\mathbf{a}}}\,. \tag{4.8}$$

Let us now observe that thanks to the interior sphere condition, the number h of balls covering the path from y_{max} to y_{min} does not depend on the parameter s. The same happens for the parameter k. Thus we choose the parameter s such that

$$\frac{h\, k^{\mathbf{a}}}{s^{\mathbf{b}+[\frac{2}{t}(3-N)]\mathbf{c}+(N+2)\mathbf{a}}} = 1, \tag{4.9}$$

namely the thesis of the theorem follows for all

$$\gamma > c^{\frac{2\eta}{\xi(\theta-1)}} d_\Omega^{\frac{2\eta}{\xi(\theta-1)}\{[\frac{2}{t}(3-N)]\mathbf{c}+\mathbf{b}\}} \left(\int_\Omega f(x)\ dx\right)^{\mathbf{a}\frac{2\eta}{\xi(\theta-1)}}, \tag{4.10}$$

and thus γ_0 is the right hand side of (4.10) with optimal constant c. When $\gamma = \gamma_0$ we just get the weak inequality $u \geq 0$. □

5 The validity of the strong maximum principle for higher order elliptic operators

In this section we first prove Theorem 1.1 for which we have to remove the restriction to compactly supported data of Theorem 4.1. Then, we will extend the result obtained to polyharmonic operators and to more general uniformly elliptic operators of any even order with constant coefficients.

Proof of Theorem 1.1. Consider the following family of sets $\{\Omega_m\}_{m\in\mathbb{N}}$ such that for all $m \in \mathbb{N}$ satisfy:

i) $\overline{\Omega}_m \subset \Omega_{m+1} \subset \overline{\Omega}_{m+1} \subset \Omega$;
ii) $\cup_{m=1}^{\infty} \Omega_m = \Omega$;
iii) $\{x \in \Omega \ : \ |\{f > 0\}| > 0\} \cap \Omega_1 \neq \Omega_1$;
iv) $dist(\partial\Omega_m, \partial\Omega) \to 0$ as $m \to \infty$.

Let χ_m be the characteristic function of Ω_m

$$\chi_m(x) = \begin{cases} 1, & x \in \Omega_m, \\ 0, & x \notin \Omega_m. \end{cases}$$

and set

$$g_m(x) := \frac{1}{S(x)} \frac{\chi_m(x)}{m^2} f(x), \quad x \in \Omega, \tag{5.1}$$

where

$$S(x) = \sum_{m=1}^{+\infty} \frac{\chi_m(x)}{m^2},$$

converges pointwise on Ω. Moreover, notice that $g_m \in L^2(\Omega)$.
Next consider the following problems

$$\begin{cases} u_m \in W^{4,2} \cap H_0^2(\Omega) \\ \Delta^2 u_m(x) - \gamma \Delta u_m(x) = g_m(x), \quad x \in \Omega, \end{cases} \tag{5.2}$$

where by construction $g_m(x) = 0$ for $x \in \Omega \setminus \Omega_m$ and thus by Theorem 4.1 there exists $\gamma_m > 0$ such that for all $\gamma > \gamma_m$, one has $u_m(x) > 0$, for all $x \in \Omega$, $m \in \mathbb{N}$.
It is crucial here that by (4.9) and (4.10) the parameter γ_m does not depend on h, namely does not depend on the distance of the maximum point of u_m from the boundary (recall the proof of Theorem 3.3). Indeed, this prevents γ_m to blow up and actually remain bounded since from (4.10)

$$\gamma_m = c^{\frac{2\eta}{\xi(\theta-1)}} d_\Omega^{\frac{2\eta}{\xi(\theta-1)}\{[\frac{2}{t}(3-N)]\mathbf{c}+\mathbf{b}\}} \left(\int_{\Omega_m} g_m(x) \ dx \right)^{\mathbf{a}\frac{2\eta}{\xi(\theta-1)}} \le c^{\frac{2\eta}{\xi(\theta-1)}} d_\Omega^{\frac{2\eta}{\xi(\theta-1)}\{[\frac{2}{t}(3-N)]\mathbf{c}+\mathbf{b}\}} \left(\int_{\Omega} f(x) \ dx \right)^{\mathbf{a}\frac{2\eta}{\xi(\theta-1)}} = \gamma_\infty \, .$$

Therefore, for all $\gamma > \gamma_\infty$ and for all $m \in \mathbb{N}$ one has

$$u_m(x) > 0, \quad x \in \Omega \tag{5.3}$$

Finally, we prove that the function

$$v(x) = \sum_{m=1}^{\infty} u_m(x) \tag{5.4}$$

solves the following

$$\begin{cases} v \in W^{4,2} \cap H_0^2(\Omega) \\ \Delta^2 v(x) - \gamma \Delta v(x) = f(x), \quad x \in \Omega \end{cases} \tag{5.5}$$

and thus by (5.3) we conclude that for all $\gamma > \gamma_\infty$ and for all $x \in \Omega$ one has

$$v(x) > 0 \, .$$

By uniqueness of the solution to the Dirichlet problem (4.1) the Theorem follows. Hence, it remains to show that $v_m \to v \in W^{4,2} \cap H_0^2(\Omega)$ which is a solution to (5.5).
Set

$$f_m = \sum_{i=1}^{m} g_i, \qquad v_m = \sum_{i=1}^{m} u_i \, .$$

By Lebesgue's dominated convergence $f_m \to f$ in $L^2(\Omega)$ and notice that v_m solves the following

$$\begin{cases} v_m \in W^{4,2} \cap H_0^2(\Omega) \\ \\ \Delta^2 v_m(x) - \gamma \Delta v_m(x) = f_m(x), \quad x \in \Omega\,. \end{cases} \tag{5.6}$$

Thus for all $m, l \in \mathbb{N}$ we have

$$\begin{cases} v_m - v_l \in W^{4,2} \cap H_0^2(\Omega) \\ \\ \Delta^2[v_m(x) - v_l(x)] - \gamma\Delta[v_m(x) - v_l(x)] = f_m(x) - f_l(x) \end{cases} \tag{5.7}$$

and multiplying by $v_m - v_l$ and integrating by parts we get

$$\int_\Omega |\Delta[v_m(x) - v_l(x)]|^2 \, dx \le \int_\Omega |f_m(x) - f_l(x)|^2 \, dx$$

which together with the equation (5.6) yields

$$\int_\Omega |\Delta^2[v_m(x) - v_l(x)]|^2 \, dx \le c\,(2\gamma^2 + 2) \int_\Omega |f_m(x) - f_l(x)|^2 \, dx\,.$$

Thus $\{v_m\}$ is a Cauchy sequence in $W^{4,2}(\Omega)$ which converges to $v \in W^{4,2}(\Omega)$, the solution to (5.5). □

Remark 5.1. *Observe that in* (5.5) *the solution can be normalized dividing the equation by* $\int_\Omega f\,dx > 0$, *so that the parameter* γ_0 *identified in* (4.10) *does not depend effectively on* f.

What we have seen so far naturally extends to polyharmonic operators of any order and more in general to uniformly elliptic operators of any even order as established in the following

Corollary 5.1. *Let* $u \in W^{2m,2} \cap W_0^{m,2}(\Omega)$, $m \ge 2$ *be the solution to the following equation*

$$(-1)^m \mathcal{A}_{2m}(D)u(x) - \gamma \mathcal{A}_2(x, D)u(x) = f(x), \quad x \in \Omega, \tag{5.8}$$

where $f \in L^2(\Omega)$,

$$\mathcal{A}_{2m}(D) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta}$$

and

$$\mathcal{A}_2(x, D) = \sum_{i,j=1}^n D_i[a_{ij}(x) D_j],$$

are uniformly elliptic operators on Ω, *namely there exist* $\nu_m > 0$ *and* $\nu_1 > 0$ *such that for all* $\xi \in \mathbb{R}^N$ *and* $x \in \Omega$

$$\nu_m \|\xi\|^{2m} \le \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \xi^{\alpha+\beta}, \quad \nu_1 \|\xi\|^2 \le \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j,$$

with $a_{\alpha\beta} \in \mathbb{R}$ *and* $a_{ij}(x) \in L^\infty(\Omega)$. *Then, there exists* $\gamma_0 > 0$ *such that for all* $\gamma > \gamma_0$ *one has* $u(x) > 0$ *for all* $x \in \Omega$.

Proof. We have to estimate intermediate derivatives of suitable order avoiding the dependance on γ. Multiplying the equation (5.8) by u and integrating by parts we get

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u(x)\, D^\beta u(x) \, dx + \gamma \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_j u(x) D_i u(x) \, dx \le \int_\Omega f(x)\, u(x) \, dx \le \|f\|_{L^2(\Omega)} \|u\|_{W_0^m(\Omega)}\,.$$

By the ellipticity condition and Gårding's inequality one has

$$\nu_m \|u\|^2_{W_0^m(\Omega)} + \gamma\nu_1 \|u\|^2_{W_0^{1,2}(\Omega)} \leq \int_\Omega |f(x)|\,|u(x)| \ dx$$

together with Poincaré's inequality

$$\nu_m \|u\|^2_{W_0^m(\Omega)} \leq c(N,\Omega) \int_\Omega |f(x)|^2 \ dx\,.$$

We conclude by the Sobolev embedding theorem as follows:

– If $N \leq 2(m-1)$ one has $\nabla u \in L^t(\Omega)$, for all $t \geq 1$ and in particular for $t > N$ and

$$\|\nabla u\|_{L^t(\Omega)} \leq c\|u\|_{W^{m,2}(\Omega)} \leq c\|f\|_{L^2(\Omega)}\,;$$

– If $N = 2m-1$ one has $\nabla u \in L^t(\Omega)$ with $t = 4m-2$ and

$$\|\nabla u\|_{L^t(\Omega)} \leq c\|u\|_{W^{m,2}(\Omega)} \leq c\|f\|_{L^2(\Omega)}\,,$$

where the constant c does not depend on γ.

□

It is well known from [11] that the positivity preserving property of the ball for polyharmonic operators carries over to small deformations of the ball. Actually on those domains what we have proved yields the positivity preserving property of the γ-perturbed polyharmonic operator for all $\gamma \geq 0$. For simplicity let us state the result in the case of the biharmonic operator:

Corollary 5.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain such that for all $f \in L^2(\Omega)$ with $f \geq 0$ and $|\{x \in \Omega : f(x) = 0\}| = 0$, the solution $u \in W^{4,2} \cap W_0^{2,2}(\Omega)$ to*

$$\Delta^2 u = f \tag{5.9}$$

enjoys $u(x) > 0$, a.e. in Ω. If there exists $\gamma_0 > 0$ such that the solution $v \in W^{4,2} \cap W_0^{2,2}(\Omega)$ to

$$\Delta^2 v - \gamma_0 \Delta v = f \tag{5.10}$$

enjoys $v(x) > 0$, a.e. in Ω, then for all $\gamma \in [0,\gamma_0]$ the solution $w \in W^{4,2} \cap W_0^{2,2}(\Omega)$ to $\Delta^2 w_\gamma - \gamma\Delta w_\gamma = f$ enjoys $w_\gamma(x) > 0$, a.e. in Ω.

Proof. Set $w_\tau = \tau v + (1-\tau)u$, $\gamma_\tau = \tau\gamma_0$, hence $\Delta^2 w_\tau - \gamma_\tau \Delta w_\tau = f$. For $\tau = 0$, w_τ is a solution to (5.9) whence for $\tau = 1$, w_τ enjoys (5.10) and then $w_\tau > 0$ a.e. in Ω for all $\tau \in [0,1]$. By uniqueness of the Dirichlet problem $w_\tau = w_\gamma$ and the claim follows.

□

From Corollary 5.2 we also have

Corollary 5.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain such that for all $f \in L^2(\Omega)$ with $f \geq 0$ and $|\{x \in \Omega : f(x) = 0\}| = 0$, the solution $u \in W^{4,2} \cap W_0^{2,2}(\Omega)$ to $\Delta^2 u = f$ enjoys $u(x) > 0$, a.e. in Ω. Then, if $N = 2, 3$, we have for all $\gamma \in [0, +\infty)$ that $w_\gamma > 0$.*

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