

Duality between k-essence and Rastall gravity

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Received: 30 January 2017 / Accepted: 7 June 2017 / Published online: 17 June 2017

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Abstract The k-essence theory with a power-law function of $(\partial\phi)^2$ and Rastall’s non-conservative theory of gravity with a scalar field are shown to have the same solutions for the metric under the assumption that both the metric and the scalar fields depend on a single coordinate. This equivalence (called k–R duality) holds for static configurations with various symmetries (spherical, plane, cylindrical, etc.) and all homogeneous cosmologies. In the presence of matter, Rastall’s theory requires additional assumptions on how the stress-energy tensor non-conservation is distributed between different contributions. Two versions of such non-conservation are considered in the case of isotropic spatially flat cosmological models with a perfect fluid: one (R1) in which there is no coupling between the scalar field and the fluid, and another (R2) in which the fluid separately obeys the usual conservation law. In version R1 it is shown that k–R duality holds not only for the cosmological models themselves but also for their adiabatic perturbations. In version R2, among other results, a particular model is singled out that reproduces the same cosmological expansion history as the standard Λ CDM model but predicts different behaviors of small fluctuations in the k-essence and Rastall frameworks.

1 Introduction

The century-old general relativity (GR) theory still successfully passes all local experimental tests. However, there are many reasons to consider this theory not as an ultimate theory of gravity but only as a reasonable approximation, working well in a large but finite range of length and energy scales. Among such reasons are the old problem of unifying gravity with other physical interactions and the difficulties in attempts to quantize GR. Other reasons for dealing with mod-

ifications of GR are the well-known problems experienced by the theory itself: its prediction of space-time singularities in the most physically relevant solutions, actually showing situations where the theory does not work any more, and its inability to explain the main observable features of the Universe without introducing so far invisible forms of matter, dark matter (DM) and dark energy (DE), which add up to as much as 95% of the energy content of the Universe.

The existing modifications and extensions of classical GR can be divided into two large classes. The first one changes the geometric content of the theory and includes, in particular, $f(R)$ theories, multidimensional theories and non-Riemannian geometries. The second class introduces new fundamental, non-geometric fields and includes, in particular, scalar-tensor theories, Horndeski theory [1] and vector-tensor theories. Of much interest are the cases where it is possible to establish connections between different representatives of the same class or even different classes of theories (possibly the best-known example of such a connection is the equivalence of $f(R)$ theories with a certain subclass of scalar-tensor theories; see, e.g., [2,3]). In the present paper we discuss such an equivalence between large families of models of k-essence theories and Rastall’s non-conservative theories with a scalar field as a source of gravity.

The k-essence theories, introducing a non-standard form of the kinetic term of a scalar field [4,5], evidently belong to the class of theories with non-standard fundamental fields coupled to gravity. They proved to be a way of obtaining both early inflation and the modern accelerated expansion of the Universe [5–7] driven by a scalar kinetic term instead of a potential. Notably, a kind of k-essence structure also appears in string theories, for example, in the Dirac–Born–Infeld action, where the kinetic term of the scalar field has a structure similar to that of the Maxwell-like term in Born–Infeld electrodynamics [8].

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Rastall’s theory [9] is one more generalization of GR, which relaxes the conservation laws expressed by the zero divergence of the stress-energy tensor (SET) T_{μ}^{ν} of matter. In this theory, the quantity $\nabla_{\nu} T_{\mu}^{\nu}$ is linked to the gradient of the Ricci scalar, and in this way Rastall’s theory may be viewed as a phenomenological implementation of some quantum effects in a curved background. Rastall’s theory leads to results of interest in cosmology, e.g., the evolution of small DM fluctuations is the same as in the Λ CDM model, but DE is able to cluster. This might potentially provide an evolution of DM inhomogeneities in the non-linear regime different from the standard CDM model [10]. The whole success of the Λ CDM model is reproduced at the background and linear perturbation levels, but new effects are expected in the non-linear regime, where the Λ CDM model faces some difficulties [11, 12]. It has also been shown [13] that Rastall’s theory with the canonical SET of a scalar field, in the context of cosmological perturbations, is only consistent if matter is present. An interesting observation in this analysis is that scalar field coupling with gravity leads to equations very similar to those in some classes of Galileon theories.

A consideration of static, spherically symmetric solutions in k-essence theories with a power-law kinetic function [14] and similar solutions in Rastall’s theory in the presence of a free or self-interacting scalar field [15] has shown that some exact solutions of these two theories describe quite the same geometries, although the properties of the scalar fields are different. Also, a no-go theorem concerning the possible emergence of Killing horizons, proved in the k-essence framework [14], has its counterpart in the Rastall-scalar field system [15]. These similarities indicate a deeper relationship between the two theories, to be analyzed in this paper. We will show that in the absence of other matter than the (possibly self-interacting) scalar fields, the two theories lead to completely coinciding geometries (we will call this *k–R duality*) under the assumptions that the relevant quantities depend on a single spatial or temporal coordinate and that the k-essence theory is specified by a power-law function; then there emerges a simple relation between the numerical parameters of the theories. If there are other forms of matter, the situation is more involved and depends on how the non-conservation of the SET is distributed between different matter contributions in Rastall’s theory. We will discuss two variants of such non-conservation in the case of isotropic spatially flat cosmological models and show that k–R duality generically takes place.

The paper is organized as follows. In the next section we discuss vacuum solutions of k-essence and Rastall theories. In Sect. 3, isotropic cosmological models are analyzed with a matter source in the form of a perfect fluid. Some considerations on the speed of sound are presented in Sect. 4, while Sect. 5 is devoted to some special values of the numerical parameters of both theories. Some concrete cosmological

configurations with dustlike matter are discussed in Sect. 6. Our conclusions are presented in Sect. 7.

2 Scalar-vacuum space-times

2.1 k-essence

The k-essence theories can be defined as general relativity with generalized forms of scalar fields minimally coupled to gravity. In the absence of matter nonminimally coupled to gravity, the most general Lagrangian is

$$\mathcal{L} = \sqrt{-g}[R + F(X, \phi) + L_m], \tag{1}$$

with

$$X = \eta\phi_{\mu}\phi^{\mu}, \tag{2}$$

where $\phi_{\mu} = \partial_{\mu}\phi$, $F(X, \phi)$ is an arbitrary function, and $\eta = \pm 1$ is used to make X positive since otherwise in the cases like general power-law dependence F will be ill defined for $X < 0$; L_m is the Lagrangian density of other kinds of matter having no direct coupling to the curvature or the ϕ field. We are using the system of units where $c = 8\pi G = 1$

Variation of the Lagrangian (1) with respect to the metric and the scalar field leads to the field equations

$$G_{\mu}^{\nu} \equiv R_{\mu}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R = -T_{\mu}^{\nu}[\phi] - T_{\mu}^{\nu}[m], \tag{3}$$

$$T_{\mu}^{\nu}[\phi] \equiv \eta F_X\phi_{\mu}\phi^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}F, \tag{4}$$

$$\eta\nabla_{\alpha}(F_X\phi^{\alpha}) - \frac{1}{2}F_{\phi} = 0, \tag{5}$$

where G_{μ}^{ν} is the Einstein tensor, $F_X = \partial F/\partial X$, $F_{\phi} = \partial F/\partial\phi$, and $T_{\mu}^{\nu}[m]$ is the SET of matter due to L_m .

Now, let us make the following assumptions:

- (i) The k-essence Lagrangian is

$$F(X, \phi) = F_0X^n - 2V(\phi), \tag{6}$$

where $n = \text{const} \neq 0$ and $V(\phi)$ is an arbitrary function (the potential).

- (ii) $\phi = \phi(u)$, where u is one of the coordinates, which may be temporal or spatial.
- (iii) The metric has the form

$$ds^2 = \eta e^{2\alpha(u)}du^2 + h_{ik}dx^i dx^k, \tag{7}$$

where i, k are the numbers of coordinates other than u , and the determinant of h_{ik} has the factorized structure

$$\det(h_{ik}) = e^{2\sigma(u)}h_1(x^i). \tag{8}$$

In this case, we have $X = e^{-2\alpha(u)}\phi_u^2$ (the index u means d/du), and the SET of the ϕ field has the following nonzero components:

$$T_u^u[\phi] = (n - \frac{1}{2})F_0X^n + V, \tag{9}$$

$$T_i^i[\phi] = -\frac{1}{2}F_0X^n + V \tag{10}$$

(there is no summing over an underlined index). The scalar field equation has the form

$$e^{-2n\alpha}\phi_u^{2n-2}[(2n - 1)\phi_{uu} + \sigma_u\phi_u - (2n - 1)\alpha_u\phi_u] = -\frac{1}{nF_0}\frac{dV}{d\phi}. \tag{11}$$

2.2 Rastall’s theory with a scalar field

Rastall’s theory of gravity is characterized by the following equations [9]:

$$R_\mu^\nu - \frac{\lambda}{2}\delta_\mu^\nu R = -T_\mu^\nu, \tag{12}$$

$$\nabla_\nu T_\mu^\nu = \frac{\lambda - 1}{2}\partial_\mu R, \tag{13}$$

where λ is a free parameter and T_μ^ν is the SET of matter. At $\lambda = 1$, GR is recovered.

These equations can be rewritten as

$$G_\mu^\nu = -\left\{T_\mu^\nu - \frac{b-1}{2}\delta_\mu^\nu T\right\} \equiv -\tilde{T}_\mu^\nu, \tag{14}$$

$$\nabla_\nu T_\mu^\nu = \frac{b-1}{2}\partial_\mu T, \quad b := \frac{3\lambda - 2}{2\lambda - 1}, \quad T = T_\alpha^\alpha. \tag{15}$$

In this parametrization, GR is recovered if $b = 1$.

Let us consider matter in the form of a minimally coupled scalar field ψ , so that

$$T_\mu^\nu[\psi] = \epsilon(\psi_\mu\psi^\nu - \frac{1}{2}\delta_\mu^\nu\psi_\alpha\psi^\alpha) + \delta_\mu^\nu W(\psi), \tag{16}$$

where $\epsilon = \pm 1$, indicating an ordinary (+1) or phantom (−1) nature of the ψ field, $\psi_\mu \equiv \partial_\mu\psi$, and $W(\psi)$ is a potential. The scalar field equation follows from (15) and has the form

$$\square\psi + (b - 1)\frac{\psi^\mu\psi^\nu\nabla_\mu\psi_\nu}{\psi^\alpha\psi_\alpha} = -\epsilon(3 - 2b)\frac{dW}{d\psi}. \tag{17}$$

Let us now, in full similarity with what was done for k-essence theory, assume that $\psi = \psi(u)$ and the metric has the form (7). Then the nonzero components of the modified scalar field SET in the right-hand side of Eq. (14) are

$$\tilde{T}_u^u[\psi] = \frac{1}{2}\epsilon b\eta e^{-2\alpha}\psi_u^2 + (3 - 2b)W(\psi), \tag{18}$$

$$\tilde{T}_i^i[\psi] = \frac{1}{2}\epsilon(b - 2)\eta e^{-2\alpha}\psi_u^2 + (3 - 2b)W(\psi), \tag{19}$$

while the scalar field equation (17) takes the form

$$e^{-2\alpha}[b\psi_{uu} + \psi_u(\sigma_u - b\alpha_u)] = -\epsilon\eta(3 - 2b)W_\psi, \tag{20}$$

where $W_\psi \equiv dW/d\psi$ and, as before, $\eta = \text{sign } g_{uu}$.

2.3 Comparison

We assume that in the k-essence system there is no other matter than the scalar field ϕ and in the Rastall system there is no other matter than the scalar ψ . Let us see under which conditions the right-hand sides of the Einstein equations (9) and (10) coincide with those of the effective Einstein equations of Rastall’s theory, (18) and (19). This will guarantee that the solutions for the metric are also the same.

To begin with, we identify the potentials:

$$V(\phi) = (3 - 2b)W(\psi). \tag{21}$$

Next, we equate the ratios of the kinetic parts of $T_\mu^\nu[\phi]$ and $\tilde{T}_\mu^\nu[\psi]$, to obtain

$$\frac{2n - 1}{-1} = \frac{b}{b - 2} \Rightarrow (2 - b)n = 1. \tag{22}$$

Then, equating the kinetic parts themselves, we find that

$$\psi_u^2 = \epsilon\eta n F_0\phi_u^{2n} e^{2(1-n)\alpha}. \tag{23}$$

Under the three conditions (21)–(23), the metric field equations of the two theories completely coincide, therefore their sets of solutions are also identical. Substituting (23) to (20), one can easily verify that under these conditions the scalar field equations (11) and (20) are also equivalent.

This general result covers many static symmetries (spherical, plane, cylindrical, etc.), homogeneous cosmologies (FRW, all Bianchi types, Kantowski–Sachs) and even inhomogeneous ones if their metrics are of the form (7), (8).

Here and in most of the paper we consider the generic values of the parameters n and b and exclude from consideration their special values that require a separate analysis, such as, for example, $b = 0$, $b = 3/2$ and $n = 1/2$. Some remarks on these special cases will be made in Sect. 5.

3 Cosmology with matter

When, besides the scalar field, matter is present, it is better, for evident technical reasons, to restrict ourselves from the beginning to a certain type of metrics. We will consider cosmological FLRW spatially flat metrics

$$ds^2 = dt^2 - a(t)^2[dx^2 + dy^2 + dz^2], \tag{24}$$

so that in (7) and (8) we have $\eta = 1$, $e^\alpha = 1$, and $e^\sigma = a(t)^3$. Matter will be taken in the form of a perfect fluid, so that

$$T_\mu^\nu[m] = \text{diag}(\rho, -p, -p, -p), \tag{25}$$

where ρ is the density and p is the pressure.

3.1 k-essence cosmology

In the FLRW metric (24) and with $\phi = \phi(t)$, the field equations (3)–(5) with matter (where we denote $\rho = \rho_k$, $p = p_k$) take the form

$$3H^2 = \frac{1}{2}(2n - 1)F_0\dot{\phi}^{2n} + V(\phi) + \rho_k, \tag{26}$$

$$2\dot{H} + 3H^2 = -\frac{1}{2}F_0\dot{\phi}^{2n} + V(\phi) - p_k, \tag{27}$$

$$\dot{\phi}^{2n-2}[(2n - 1)\ddot{\phi} + 3H\dot{\phi}] = -\frac{1}{nF_0}V_\phi, \tag{28}$$

where $H = \dot{a}/a$ is the Hubble parameter and $V_\phi \equiv dV/d\phi$. The SET of matter satisfies the conservation law $\nabla_\nu T_\mu^\nu[m] = 0$, whence

$$\dot{\rho}_k + 3H(\rho_k + p_k) = 0. \tag{29}$$

3.2 Rastall cosmology with a scalar field and matter

The Rastall equations have the form (14) and (15), where now T_μ^ν is the total energy-momentum tensor,

$$T_\mu^\nu = T_\mu^\nu[\psi] + T_\mu^\nu[m], \tag{30}$$

with $T_\mu^\nu[\psi]$ given by (16) and $T_\mu^\nu[m]$ by (25). In Eqs. (14), the modified energy-momentum tensor \tilde{T}_μ^ν is then a sum of $\tilde{T}_\mu^\nu[\psi]$ given by (18) and (19) and $\tilde{T}_\mu^\nu[m]$ with the components

$$\begin{aligned} \tilde{T}_t^t[m] &= \frac{1}{2}[(3 - b)\rho + 3(b - 1)p] \equiv \tilde{\rho}, \\ \tilde{T}_i^i[m] &= \frac{1}{2}[(1 - b)\rho + (3b - 5)p] \equiv -\tilde{p} \end{aligned} \tag{31}$$

(we preserve the notation ρ and p without indices for matter in Rastall gravity). Hence the Rastall equations read

$$3H^2 = \frac{1}{2}\epsilon b\dot{\psi}^2 + (3 - 2b)W + \tilde{\rho}, \tag{32}$$

$$2\dot{H} + 3H^2 = \frac{1}{2}\epsilon(b - 2)\dot{\psi}^2 + (3 - 2b)W - \tilde{p}, \tag{33}$$

while the equation for ψ depends on further assumptions on how the non-conservation of the full SET according to (15) is distributed between ψ and matter. One notices that

$$\tilde{\rho} + \tilde{p} = \rho + p \tag{34}$$

and

$$\tilde{\rho} - \rho = p - \tilde{p} = \frac{1 - b}{2}(\rho - 3p). \tag{35}$$

Let us consider two (of an infinite number of) alternatives in incorporating matter to Rastall’s theory with a scalar field:

- R1: The SETs of ψ and matter obey (15) each separately, so there is no mixing between the two sources of gravity;
- R2: The SET of matter is conservative, so that

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{36}$$

3.3 Case R1: no mixing of scalar field and matter

In this case we have

$$\nabla_\nu T_\mu^\nu[\psi] = \frac{b - 1}{2}\partial_\mu T[\psi], \tag{37}$$

$$\nabla_\nu T_\mu^\nu[m] = \frac{b - 1}{2}\partial_\mu T[m]. \tag{38}$$

The first of these conditions leads to the scalar field equation (20), which in the present case reads

$$b\ddot{\psi} + 3H\dot{\psi} = -\epsilon(3 - 2b)W_\psi. \tag{39}$$

With (34), the condition (38) has the form

$$\dot{\tilde{\rho}} + 3H(\rho + p) = 0. \tag{40}$$

The full set of equations consists of (32), (33), (39), and (40), with the definitions (31).

From (34) it follows that if matter satisfies the null energy condition (NEC), then the same is true for the “effective” density and pressure ($\tilde{\rho}$ and \tilde{p}) in Rastall’s theory. However, from (31) it can be verified that the positivity of the energy density (or pressure) is not guaranteed in the k-essence case if it is imposed in the Rastall theory, and vice versa.

It is easy to see that the right-hand sides of Eqs. (32) and (33) coincide with those of (26) and (27) if, in addition to the relationships (21)–(23) for scalar variables, we identify

$$\rho_k = \tilde{\rho}, \quad p_k = \tilde{p}. \tag{41}$$

The correctness of this identification is confirmed by the identity of the conservation laws (29) and (40). Thus, as in the vacuum case, the parameters n and b are related by (22), that is, $n(2 - b) = 1$, and the scalar fields ϕ and ψ are related by Eq. (23) which now reads

$$\dot{\psi}^2 = \epsilon n F_0 \dot{\phi}^{2n}. \tag{42}$$

3.4 Case R2: conservative matter

We now have $\nabla_\nu T_\mu^\nu[m] = 0$. This condition is particularly important for the structure formation in the Universe for the case of a pressureless fluid since ordinary matter must agglomerate.

In this case, for the scalar field SET we have

$$\nabla_\nu T_\mu^\nu[\psi] = \frac{1}{2}(b-1)(\partial_\mu T[\psi] + \partial_\mu T[m]), \tag{43}$$

which leads to the scalar field equation

$$\begin{aligned} \dot{\psi}(b\ddot{\psi} + 3H\dot{\psi}) &= -\epsilon(3-2b)W_\psi\dot{\psi} \\ &+ \frac{1}{2}\epsilon(b-1)(\dot{\rho} - 3\dot{p}). \end{aligned} \tag{44}$$

The full set of equations consists of (32), (33), (44), and (29), with the definitions (31). Note that Eq. (44) mixes the scalar field and the matter fluid even though the fluid is conserved as in GR.

This conservation lets us identify the matter SET components in the k-essence and Rastall theories: $\rho = \rho_k, p = p_k$. As a result, identification of the other parts of the total SET is only partly the same as in the previous case.

Identifying, as before, the potentials according to (21) (that is, $V = (3-2b)W$) and comparing the Friedmann-like equations (32) and (33) with their k-essence counterparts (26) and (27), we obtain, as before,

$$\epsilon\dot{\psi}^2 = nF_0\dot{\phi}^{2n}. \tag{45}$$

The correctness of this identification is verified by substituting (45) into the scalar field equation (44): indeed, since we have now, due to (35),

$$\epsilon\dot{\psi}^2 \left[b - \frac{2n-1}{n} \right] = (b-1)(\rho - 3p), \tag{46}$$

this substitution leads precisely to the scalar field equation (28) of the k-essence theory.

It is important that in the case of conservative matter, a comparison between the two theories does not lead to a direct relationship like (22) between their numerical parameters n and b . Instead, we have the equality (46), from which (22) is restored only in the special case $\rho = 3p$ (zero trace of the matter SET, radiation).

3.5 Further consequences of matter conservation

Equation (46) creates a connection between the temporal behavior of ψ and the matter content. Indeed, inserting (46) to (44) with zero or constant potential, we find

$$\dot{F} + \frac{6n}{2n-1}HF = 0, \tag{47}$$

where $F = \dot{\psi}^2$. This leads to

$$\dot{\psi}^2 = \psi_0 a^{-6n/(2n-1)}, \tag{48}$$

where ψ_0 is an integration constant.

From Eq. (46) it is clear that the matter density and pressure must also evolve by a power law as functions of the scale factor. Hence, only an equation of state (EoS) of the type $p = w\rho$, with $w = \text{const}$, is possible. In this case,

$$\rho = \rho_0 a^{-3(1+w)}, \quad \rho_0 = \text{const}, \tag{49}$$

implying the relation between w and n

$$w = \frac{1}{2n-1}. \tag{50}$$

We see that a substitution of $p = w\rho$ into the scalar field equation relates the EoS factor w with the k-essence power n , while the Rastall constant b remains arbitrary. Moreover, Eqs. (26) and (27) show that the pure k-essence scalar field ϕ behaves as a perfect fluid with the same EoS factor (50) (see also [16]).

In other words, assuming a zero or constant potential $V = (3-2b)W$ and conservative matter in the Rastall framework, we find that *the k-R duality is only possible if matter is a perfect fluid with the linear EoS $p = w\rho$, coinciding with the effective EoS of the scalar field ϕ .*

With $p = w\rho$ Eq. (46) gives

$$\epsilon\dot{\psi}^2 = k\rho, \quad k = \frac{n(b-1)(1-3w)}{bn-2n+1}. \tag{51}$$

Inserting this to the Friedmann-like equation (32), we obtain in terms of n or w

$$\begin{aligned} 3H^2 &= V + \frac{\rho}{2n-1} \left\{ \frac{2nb(b-1)(n-2)}{bn-2n+1} \right. \\ &\quad \left. + 2(2b-3) + 2n(3-b) \right\} \\ &= V + \frac{\rho}{2} \left\{ \frac{b(b-1)(1+w)(1-3w)}{(b-2)(1+w) + 2w} \right. \\ &\quad \left. + 3 - b + 3w(b-1) \right\}. \end{aligned} \tag{52}$$

The right-hand side must be positive. Therefore, given n and V (or alternatively w and V), we obtain a restriction on b . For example, if $V = 0$ and $w = 0$ (dust, $n \rightarrow \infty$),¹ we have either $b < 3/2$ or $b > 2$ (provided $\rho > 0$). For $w = 1$ (stiff matter, $n = 1$), there is no restriction on b , and we obtain $H^2 = V = \text{const} > 0$, hence a de Sitter expansion, $a(t) \propto e^{Ht}$. In this case, stiff matter precisely cancels the contribution from the scalar field ψ or ϕ .

If we introduce a variable potential or a more complex equation of state, the situation becomes much more involved. It must be stressed that the EoS $p = w\rho$ with $w = \text{const}$ covers most of the interesting cases in cosmology. Moreover,

¹ This relation makes sense even if the Lagrangian formulation becomes ill defined; see Ref. [16].

we expect that the perturbative behavior may be very different in the two theories even in this case.

There emerge two more natural questions. First, we have found that in k-essence theory there are simultaneously a scalar field and a perfect fluid with the same EoS and hence the same time evolution of their densities and pressures. Can we unify them by, for example, redefining the scalar field? A probable answer is “no” because these two kinds of matter are expected to behave quite differently at the perturbative level.

Another question is: how is it possible to have a completely definite situation in k-essence theory but arbitrariness in the parameter b in the dual solution of Rastall’s theory? An answer is that this arbitrariness is compensated by the corresponding non-conservative behavior of the scalar field ψ .

4 Perturbations and the speed of sound

A power-law k-essence model with $V = 0$ is equivalent in the cosmological framework (such that $(\partial_\mu \phi)^2 > 0$) to a perfect fluid with the equation of state $p = w\rho$, where the constant w is related to the power n :

$$w = \frac{1}{2n - 1}. \tag{54}$$

In a fluid, adiabatic perturbations propagate as a sound with the speed v_s such that

$$v_s^2 = dp/d\rho = w. \tag{55}$$

Scalar field perturbations for general Lagrangians of the form $F(X, \phi)$ have been treated in detail in Refs. [6, 17]. It has been shown there that a k-essence theory implies

$$v_s^2 = \frac{F_X}{F_X + 2XF_{XX}}, \tag{56}$$

and this expression is valid even if there is an arbitrary potential term $V(\phi)$. In particular, for the theory (6), where $F(X) = F_0X^n - 2V(\phi)$, we find again

$$v_s^2 = \frac{1}{2n - 1} \tag{57}$$

in full agreement with (55). Thus there is a complete equivalence between a perfect fluid and k-essence without a potential not only for a cosmological background but even on the perturbative level as far as adiabatic perturbations are concerned. In particular, if $w < 0$, corresponding to $n < 1/2$, the model is perturbatively unstable since it implies $v_s^2 < 0$.

This is true both for a perfect fluid and for k-essence. Moreover, although the presence of a potential changes the scalar field dynamics, the propagation speed of its perturbations, coinciding with the derived speed of sound [17], is still the same as with $V = 0$.

In Rastall’s theory things may be different. The speed of sound for a scalar field is given by [13, 18]

$$v_s^2 = \frac{2 - b}{b}. \tag{58}$$

In scalar vacuum and in the R1 case (matter obeys the non-conservation equation (37)), we have the relation (22), $(2 - b)n = 1$, which makes (57) and (58) identical. Furthermore, the fluids in the corresponding models obey different equations of state; see (41). However, in the Rastall model we can still characterize the fluid by the “effective” density and pressure, $\tilde{\rho}$ and \tilde{p} , the SET written in their terms is conservative, hence the squared speed of sound of the Rastall fluid is equal to $d\tilde{p}/d\tilde{\rho} = dp_k/d\rho_k$. Thus we can conclude, even without performing a complete perturbation analysis, that the models belonging to the two theories coincide not only at the background level but also at the level of adiabatic perturbations.

In the case R2 (conserved matter), we have another relation, Eq. (46), between the parameters b and n , without such a simple connection. As a consequence, in principle it is possible that an unstable model in k-essence theory may correspond to a stable model in Rastall’s theory, or vice versa, since, as shown above, Eq. (22) does not hold, b being now essentially independent of n up to some possible restrictions on their range. In fact, in this case, even non-adiabatic perturbations may appear, due to the coupling between the scalar field and matter.

5 Some special cases

5.1 $n = 1/2$

In this case, the k-essence scalar field equation takes the form

$$3H = -2F_0^{-1}V_\phi. \tag{59}$$

Thus if $V = \text{const}$, we have $H = 0$, hence $a = \text{const}$, and flat space-time is obtained. One can certainly obtain $H = 0$ in Rastall gravity under special assumptions, but the question of k-R duality looks meaningless in this trivial case.

If $V = V(\phi)$, the ϕ field has no dynamics of its own, but Eq. (59) expresses it in terms of H , and the Friedmann equations (26) and (27) are meaningful.

The Rastall counterpart in the scalar-vacuum and R1 cases is then obtained with $b = 0$, $V = 3W$ (according to (22) and (21)) and

$$2\epsilon\dot{\psi}^2 = F_0\dot{\phi}. \tag{60}$$

In the case R2 (conserved matter), b remains arbitrary, but k–R duality still holds. Indeed, if we substitute (60) into Eq. (44) and use the Friedmann-like equations (32), (33) to calculate $\dot{\rho} - 3\dot{p}$, we obtain Eq. (59).

The main feature in this case is that nontrivial solutions with $n = 1/2$ and k–R duality are only achieved in the presence of a variable potential.

5.2 $b = 3/2$

With this value of b , the potential disappears from Rastall’s gravity, hence the k–R duality implies $V = 0$, and we deal with zero potentials. In other respects, the situation is described as in the general case.

A feature of interest is that with $b = 3/2$ Eq. (31) leads to $\tilde{\rho} = 3\tilde{p}$. Therefore, in the scalar-vacuum and R1 cases, the dual k-essence counterpart of this Rastall model contains matter with $\rho_k = 3p_k$ (see (41)). Due to (22), in addition, $n = 2$, so that the ϕ field also behaves as radiation.

In the case R2 (conserved matter), Eqs. (41) and (22) are no more valid, and the general description is applicable.

5.3 $b = 2$

Equations (54) and (58) give zero values of pressure and the speed of sound of a scalar field in Rastall’s theory. The corresponding expression (57) in k-essence theory leads to $n \rightarrow \infty$ according to the general relation (22).

If there is conserved matter (case R2), then (unless this conserved matter is pure radiation, $\rho = 3p$) Eq. (22) is no more valid, so that the speeds of sound of scalar fields are different in the two theories. It means that k–R duality does not exist for perturbations even though it does exist for the isotropic background.

5.4 $b = 0$

In the scalar-vacuum and R1 cases we return to the above description for $n = 1/2$.

With conserved matter (case R2), the Rastall-scalar field takes the form

$$3H\epsilon\dot{\psi}^2 = -3\dot{W} - \frac{1}{2}(\dot{\rho} - 3\dot{p}), \tag{61}$$

looking like a constraint equation since it contains only the first-order derivative. However, k–R duality still works, as before: thus, a substitution of (42) (what is important, with arbitrary $n \neq 0$) and $\rho - 3p$ from Eqs. (32), (33) into (61) leads to (28), which is a second-order equation unless $n = 1/2$.

6 Examples

Let us now consider some specific examples of the equivalence discussed above, assuming a zero or constant potential and dust as a possible matter contribution.

6.1 Scalar vacuum

Consider scalar vacuum with zero potential. The k-essence equations give

$$\dot{\phi} = \phi_0 a^{-3/(2n-1)}, \quad \phi_0 = \text{const}, \tag{62}$$

$$H^2 = \frac{2n-1}{6} F_0 \phi_0^{2n} a^{-6n/(2n-1)}. \tag{63}$$

In terms of cosmic time we obtain

$$a = a_0 t^{2/[3(1+w)]}, \quad a_0 = \text{const}, \tag{64}$$

$$\dot{\phi} = \phi_1 t^{-2w/(1+w)}, \tag{65}$$

where ϕ_1 is a combination of the previous constants, and we have written $w = 1/(2n - 1)$, thus identifying the k-essence with a perfect fluid with the EoS $p = w\rho$.

In Rastall’s theory, the dual solution contains the same $a(t)$, while the scalar field is given by

$$\dot{\psi} \propto a^{-3(1+w)/2} = a^{-3/b} \propto t^{-1}, \tag{66}$$

where now we should put $w = (2 - b)/b$. We notice that while the k-essence scalar field behavior depends on the EoS parameter w , the Rastall scalar is simply $\psi = \log t + \text{const}$.

Addition of a constant potential, equivalent to a cosmological constant, does not change the scalar field evolution laws (62) and (66) in terms of a but makes the time dependences more complex, not to be considered here.

In the presence of matter, as we saw above, the form of k–R duality depends on how matter couples to the scalar field.

6.2 Dust and Rastall-R1 models

Suppose that in k-essence theory, in addition to the scalar field ϕ , there is pressureless fluid (dust), so that

$$p_k = 0, \quad \rho_k = \rho_1 a^{-3}, \quad \rho_1 = \text{const}. \tag{67}$$

Then in the R1 version of Rastall’s theory, according to (41), we have the conditions

$$\begin{aligned} \frac{1}{2} \left[(3 - b)\rho + 3(b - 1)p \right] &= \tilde{\rho} = \rho_k, \\ \frac{1}{2} \left[(b - 1)\rho + (5 - 3b)p \right] &= \tilde{p} = 0, \end{aligned} \tag{68}$$

leading to the following relations for the density and pressure:

$$\begin{aligned} \rho &= \frac{5 - 3b}{2(3 - 2b)} \rho_k, \\ p &= \frac{1 - b}{5 - 3b} \rho = \frac{1 - b}{2(3 - 2b)} \rho_k, \end{aligned} \tag{69}$$

both evolving as $\rho \propto p \propto a^{-3}$. Thus in Rastall cosmology the fluid acquires pressure (except for the GR value $b = 1$). The scalar fields in both models satisfy the same relations as in the vacuum case, valid for any values of n and b such that $n(b - 2) = 1$.

Adding a constant potential $V = (3 - 2b)W$ does not change the relations (68) and (69) and introduces an effective cosmological constant. We then obtain a three-component model with matter, a cosmological constant and a scalar field whose behavior is determined by n or, equivalently, by $w = 1/(2n - 1)$. In the dual Rastall model, we have an effective pressure even though in the k-essence model $p_k = 0$.

If, on the contrary, we introduce matter with $p = 0$ in Rastall's (R1) theory, then in the dual k-essence model we have

$$\rho_k = \tilde{\rho} = \frac{1}{2}(3 - b)\rho, \quad p_k = \tilde{p} = \frac{1}{2}(b - 1)\rho, \tag{70}$$

and their evolution law agreeing with Eq. (40) reads

$$\rho_k \propto \rho \propto a^{-6/(3-b)} = a^{-3(1+w_k)}, \tag{71}$$

where the EoS parameter w_k of the fluid in the k-essence model is

$$w_k = \frac{b - 1}{3 - b} = \frac{n - 1}{n + 1} \tag{72}$$

(not to be confused with $w = 1/(2n - 1)$ characterizing the ϕ field behavior); as before, the relation $n(2 - b) = 1$ holds. The model thus obtained is quite different from the one with dust introduced in k-essence theory.

6.3 Dust and Rastall-R2 models

Let us again assume Eq. (67) but now consider version R2 of Rastall's theory, so that now $\rho \propto a^{-3}$ and

$$\epsilon \dot{\psi}^2 \left[b - \frac{2n - 1}{n} \right] = (b - 1)\rho \propto a^{-3}. \tag{73}$$

Then for $b \neq 1$ we find according to (45)

$$\begin{aligned} \dot{\psi} &\propto a^{-3/2}, \\ \dot{\phi}^n &\propto a^{-3/2} \Rightarrow \dot{\phi} \propto a^{-3/(2n)}. \end{aligned} \tag{74}$$

Combining this with the relation $\epsilon \dot{\psi}^2 = nF_0\dot{\phi}^{2n}$ and the field equation (28) with $V_\phi = 0$, we find that this situation

corresponds to the limit $n \rightarrow \infty$, which is, however, well defined. In this way we obtain $a \propto t^{2/3}$ as in the pure dust model of GR. For the scalar fields it follows in this limit that

$$\begin{aligned} \dot{\phi} &= \text{const} \Rightarrow \phi \propto t, \\ \dot{\psi} &\propto a^{-3/2} \propto 1/t. \end{aligned} \tag{75}$$

The condition for b is obtained from (73): writing $\dot{\psi} = \psi_0/t$ and $\rho = \rho_0/t^2$, we arrive at

$$\epsilon \psi_0^2 (b - 2) = (b - 1)\rho_0. \tag{76}$$

Thus the value of b is determined by the relative contributions of matter and the scalar field. Moreover, the speed of sound of the scalar field now does not follow the adiabatic relation verified in the R1 case.

A cosmological constant can be easily introduced in the form of $V = (3 - 2b)W = \text{const}$. The scalar field again follows the law (73), and the whole configuration reduces to the Λ CDM model where Λ is given by the constant potential and the matter component consists of the scalar field and ordinary matter. All background relations of the Λ CDM model are preserved in this case, but the degeneracy between the scalar field and usual matter is broken at the perturbative level. Due to the fact that the Λ CDM model is subject to problem at the perturbative level in the non-linear regime (see, e.g., [11, 12]), such a more complex configuration in k-essence and Rastall models may lead to interesting results, to be studied in the future.

7 Conclusion

We have studied the conditions of equivalence between the k-essence and Rastall theories of gravity in the presence a scalar field (k-R duality). These two theories have actually emerged in very different contexts, the k-essence theory being based on a generalization of the kinetic term of a scalar field, suggested by some fundamental theories, while Rastall's theory is a non-conservative theory of gravity which can be seen as a possible phenomenological implementation of quantum effects in gravitational theories. Such equivalence has been revealed in the case of static spherically symmetric models [14, 15], and it has been more explicitly stated here for all cases where the metric and scalar fields essentially depend on a single coordinate, and the k-essence theory is specified by a power-law function of the usual kinetic term, to which a potential term can be added. This generalization covers diverse static and cosmological models, including all homogeneous cosmologies.

We have discussed cosmological configurations with scalar fields and matter in the form of a perfect fluid whose evolution in Rastall's theory can follow one of two possible laws: one (R1) assumes no mixing between matter and

the scalar field, each of them separately obeying the non-conservation law (15), and the other (R2) ascribes the whole non-conservation to the scalar field while matter is conservative ($\nabla_\nu T_\mu^\nu[m] = 0$). Let us summarize the main results obtained in this context:

1. k–R duality has been established for version R1 of Rastall’s theory with an arbitrary EoS of matter. It has been found that the EoS of matter is different in the mutually dual k-essence and Rastall models; however, it is argued that the respective speeds of sound are the same. Since the speeds of sound characterizing the scalar fields (ϕ in k-essence theory and ψ in Rastall’s) also coincide, we conclude that k–R duality is maintained not only for the cosmological backgrounds but also for adiabatic perturbations.
2. For version R2 of Rastall’s theory, it has been found that k–R duality exists only with fluids having the EoS $p = w\rho$, $w = \text{const}$, which is the same for k-essence and Rastall models. Moreover, in the k-essence model the scalar field obeys the same effective EoS. However, on the perturbative level the mutually dual models behave, in general, differently.
3. Some special cases have been discussed, showing how there emerge some restrictions on the free parameters of each theory.
4. An example has been considered in which a cosmological model completely equivalent to the Λ CDM model of GR is obtained at the background level, but different features must appear at the perturbative level.

The equivalence between the two theories discussed here is somewhat surprising because of their basically different origin. A curious aspect is that the k-essence theory has a Lagrangian formulation unlike Rastall’s theory. It is possible that the equivalence studied here may lead to a restricted Lagrangian formulation of Rastall’s theory in the minisuperspace in terms of metric functions depending on a single variable. If this is true, it might suggest how to recover a complete Lagrangian formulation for Rastall’s theory in a more general framework.

Acknowledgements We thank CNPq (Brazil) and FAPES (Brazil) for partial financial support. KB thanks his colleagues from UFES for kind

hospitality and collaboration. The work of KB was performed within the framework of the Center FRPP supported by MEFH Academic Excellence Project (Contract No. 02.a03. 21.0005, 27.08.2013), within the RUDN University Program 5-100, and RFBR Project No. 16-02-00602.

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