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Cooking pasta with Lie groups

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Abstract

We extend the (gauged) Skyrme model to the case in which the global isospin group (which usually is taken to be $SU(N)$) is a generic compact connected Lie group G. We analyze the corresponding field equations in (3+1) dimensions from a group theory point of view. Several solutions can be constructed analytically and are determined by the embeddings of three dimensional simple Lie groups into G, in a generic irreducible representation. These solutions represent the so-called nuclear pasta state configurations of nuclear matter at low energy. We employ the Dynkin explicit classification of all three dimensional Lie subgroups of exceptional Lie group to classify all such solutions in the case G is an exceptional simple Lie group, and give all ingredients to construct them explicitly. As an example, we construct the explicit solutions for $G = G_2$. We then extend our ansatz to include the minimal coupling of the Skyrme field to a $U(1)$ gauge field. We extend the definition of the topological charge to this case and then concentrate our attention to the electromagnetic case. After imposing a "free force condition" on the gauge field, the complete set of coupled field equations corresponding to the gauged Skyrme model minimally coupled to an Abelian gauge field is reduced to just one linear ODE keeping alive the topological charge. We discuss the cases in which such ODE belongs to the (Whittaker-)Hill and Mathieu types.

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1. Introduction

Nuclear pasta is a phase of matter that appears organized in some ordered structures when a large number of Baryons is confined in a finite volume [1], [2], [3], [4], [5], [6]. These configurations appear, for instance, in the crust of neutron stars. Such aggregations of Baryons may take the form of tubular structures, called Spaghetti states, or layers having a finite width, called Lasagna states, or even globular shape, the gnocchi. Until very recently, it was always tacitly assumed that nuclear pasta phase is the prototypical situation in which it is impossible to reach a good analytic grasp. This is related to the fact that such structures appear in the low energy limit of Quantum Chromodynamics (QCD) in which perturbation theory does not work and, at a first glance, the strong non-linear interactions prevent any attempt to find exact solutions. Now, the low energy limit of QCD is described by the Skyrme model [7] at the leading order in the 't Hooft expansion (see [8], [9], [10], [11], [12], [13], as well as [14], [15] and references therein). Unsurprisingly, the highly non-linear character of the Skyrme field equations discouraged any mathematical description of this kind of structures. Consequently, as the above references show, numerical methods (which, computationally, are quite demanding) are dominating in this regime. The situation is even worse when one wants to analyze the electromagnetic field generated in the nuclear pasta phase as, when the minimal coupling with the $U(1)$ gauge field is taken into account; even the available numerical methods are not effective.

On the other hand, one may ask: is the mathematical dream of an analytic description of nuclear pasta structure really out of reach? Analytical methods to infer the general dependence of the nuclear pasta phase on relevant physical parameters (such as the Baryon density) not only would greatly improve our understanding of the nuclear pasta phase itself, but they could also shed considerable light on the interactions of dense nuclear matter with the electromagnetic field.

From the mathematical viewpoint, the problem is very deep and yet simple to state: *can we find analytic solutions of the (gauged) Skyrme model able to describe typical configurations of the nuclear pasta phase?* Despite the fact that this model has been introduced in the early sixties, for several years only numerical solutions had been available (the only exceptions being [16], in which the authors constructed analytic solutions of the Skyrme field equations in a suitable fixed curved background). Nevertheless, the mathematical beauty of the Skyrme model attracted the attention of many leading mathematicians and physicists. In particular, in [17], [18], [19], [20] and [21], the authors were able to disclose the geometrical structures of configurations with two Skyrmions, to analyze the interaction energy of well separated solitons, to establish necessary conditions for the existence of Skyrmionic crystals and so on. All these remarkable results have been obtained without the availability of analytic solutions of the Skyrme field equations. These efforts (together with the comparison with Yang-Mills theory in which explicit solutions representing instantons and non-Abelian monopoles shed considerable light on the mathematical and physical properties of Yang-Mills theory itself) show very clearly the importance to search for new analytic tools to analyze the gauged Skyrme model in sectors with high Baryonic charge.

Quite recently, new methods have been introduced that allowed the construction of explicit analytical solutions of the Skyrme field equations. Such solutions are suitable to describe nuclear Lasagna and Spaghetti states, see [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], and [33]. Let us recall that the Skyrme model is a non-linear field theory for a scalar field U taking values in the $SU(N)$ Lie group, where N is the flavor number. This theory possesses a conserved topological charge (the third homotopy class) which physically is interpreted as the Baryonic charge of the configuration.

Most of the solutions found so far have been constructed by employing ad hoc ansätze adapted to the properties of the $SU(2)$ group, but soon it has been realized that particular group structures seem to be at the root of the solvability of the Skyrme field equations. For example, the exponentiation of certain linear functions taking value in the Lie algebra lead to Spaghetti-like configurations, while Euler parameterization of the field U, with suitable linear exponents, lead to Lasagna-like solutions. In all these cases, the solutions are also topologically non-trivial with arbitrary Baryonic charge. A proper mathematical understanding and generalization of the strategy devised in [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32] and [33], offers the unique opportunity to disclose the deep connections of the nuclear pasta phase with the theory of Lie groups; two topics which (until very recently) could have been considered extremely far from each other. *The present paper is devoted to this opportunity:* to provide nuclear pasta configurations of Lasagna and Spaghetti types with the mathematical basis of Lie group theory.

A first step in this direction was to link certain properties of the semi-simple Lie group to the possibility of getting explicit solutions of the Skyrme equations in the Lasagna configurations for the case of $SU(N)$ groups with arbitrary N [33]. More in general, using the methods developed in [26], [27], [28], [30], [31], [33], with the generalization of the Euler angles to $SU(N)$ of [34], [35], [36], it has been possible to construct non-embedded multi-Baryonic solutions of nuclear Spaghetti and nuclear Lasagna, at least for the case for the $SU(N)$ groups, see [37].

A fundamental ingredient in the theory of Lie groups with relevant applications in the Skyrme model is the concept of *non-embedded solutions* introduced in [11] and [12]. These are solutions of the $SU(N)$-Skyrme model which cannot be written as trivial embeddings of $SU(2)$ in $SU(N)$. However, the techniques used to get such results, for example in [33], where quite specific of the group $SU(N)$. In fact, as we will show in the present manuscript, there is a very interesting relation between Lie group theory and such families of solutions, which allows to generalize the above results in a much more general setting and to classify the solutions: this is exactly the main goal of the present paper.

1.1. Resume of the results

Firstly, we will prove that, having fixed a compact connected Lie group G with a given irreducible representation (irrep), the solutions are determined in general by deformations of embeddings of three dimensional Lie groups into G.

Secondly, we will prove that inequivalent families of solutions correspond to inequivalent embeddings (not related by conjugation in G). The problem of determine all possible three dimensional subgroups of a simple Lie group has been solved by E. B. Dynkin in [38]. In particular, in that paper, all possible three dimensional subalgebras of the exceptional Lie algebras are written down.

Thirdly, we will show that such classification also classifies the Spaghetti and Lasagna solutions determined via group theory methods. The difference between Spaghetti and Lasagna depends on the realization of the subgroup element of G: if it is generated by the exponentiation of a linear combination of the generators of a three-dimensional subalgebra of $\mathfrak{g} = \mathcal{L}ie(G)$, then we get Spaghetti-like solutions, while if the realization is through Euler parameterization we get Lasagna-like solutions. Then, we will compute explicitly relevant quantities such as the energy of these configurations.

Fourthly, we will extend this classification to the case of the gauged Skyrme model minimally coupled to Maxwell theory. In particular, we will extend the definition of topological (Baryonic) charge to this case. We will reduce the complete set of coupled field equations both in the gauged Lasagna case and in the gauged Spaghetti case to a single linear equation and we will analyze the integrable cases which correspond to Whittaker-Hill and Mathieu types linear differential equations.

1.2. Main tools employed in the analysis

In the present work, we will employ abstract techniques and general properties of semi-simple Lie groups in order to investigate their relation with solvability of the Skyrme equations. This allows to extend all the results found in [33] for the special unitary groups to an arbitrary semi-simple compact Lie group. Indeed, all results will be based on the properties of the roots and weights of the associated Lie algebras, while a generalized Euler parameterization of the Skyrme field U, taking values in G, will lead in general to Lasagna configurations. Similarly, the direct exponentiation of the algebra, as discussed above, will lead us to Spaghetti structures extending the results of [37]. In any case, we will compute the energy of such configurations and will show that they have always a non-trivial Baryon (topological) charge. Interestingly enough, a strategy for constructing non-trivial non-$SU(2)$ solutions in the sense of [11] and [12] will result to be strictly related to the classification of all three dimensional groups in any given simple Lie group, provided by Dynkin in his PhD thesis work, see [38]. As an application of our general analysis, we will show how to construct all non-trivial Lasagna and Spaghetti configurations in any exceptional Lie group, making very explicit the case of $G = G_2$.

The generalization of our ansätze which allows to include the minimal coupling of the model to a $U(1)$ electromagnetic field will be introduced as follows. As usual, the gauge field will work as a connection making all derivative covariant under the action of the $U(1)$ gauge field, while their dynamics is expressed by the usual Maxwell action (although our methods also work in the Yang-Mills case). The covariant derivatives break the topological nature of the original term expressing the Baryonic charge. Therefore, generalizing the result in [8], we will deform the Baryonic density expression in order to recover topological invariance.

The introduction of the electromagnetic field makes the field equations of the gauged Skyrme model minimally coupled to Maxwell theory extremely more complicated than in the Skyrme case. Nevertheless, quite surprisingly, the equations will be separable (in a suitable sense) and once again solvable, after imposing the *free force conditions* on the gauge field. This condition appears quite naturally in Plasma physics (see [39], [40], [41], [42], [43] and references therein). Quite interestingly, such condition implies that the gauge field disappears from the gauged Skyrme field equations (without being a trivial gauge field, of course) and therefore, in this way the gauged Skyrme field equations can be solved as in the ungauged case. It is a very non-trivial result that the remaining field equations (which correspond to the Maxwell equations with the source term arising from the gauged Skyrme model) reduce just to one linear equation for a suitable component of the gauge field in which the Skyrmion act as a source-like term. We will analyze the integrable cases in which this last remaining equation takes the form of a Hill equation for the case of Lasagna states, while a Schrödinger equation with a bi-periodic potential of finite type in the Spaghetti case.

Interestingly enough, for the Lasagna case another nice coincidence shows up here: the relevant solutions we need are exactly the periodic solutions whose existence has been investigated in [44], and which explicit form for the case of a Whittaker-Hill equation has been determined in [45].

It is a truly remarkable result that such a complicated phase such as the nuclear pasta phase of the low energy limit of QCD (even taking into account the minimal coupling with Maxwell theory) can be understood so cleanly in terms of the theory of Lie group.

1.3. Notations and conventions

Our conventions are as follows. The action of the Skyrme model in $(3+1)$ dimensions is

$$I = \int d^4x\sqrt{-g}\left[\frac{K}{4}\mathrm{Tr}\left(\mathcal{L}_\mu\mathcal{L}^\mu + \frac{\lambda}{8}G_{\mu\nu}G^{\mu\nu}\right)\right], \tag{1.1}$$
$$\mathcal{L}_\mu = U^{-1}\partial_\mu U, \qquad G_{\mu\nu} = [\mathcal{L}_\mu, \mathcal{L}_\nu], \qquad U(x) \in G,$$

where K and λ are positive coupling constants and g is the metric determinant. The Skyrme field U is a map

$$U : \mathbb{R}^{1,3} \longrightarrow G$$

where G is semi-simple compact Lie group, so that

$$\mathcal{L}_\mu = \sum_{i=1}^{dim(G)} \mathcal{L}_\mu^i T_i,$$

where $\{T_i\}$ is a basis for the Lie algebra $\mathfrak{g} = Lie(G)$.

The system is confined in a box of finite volume with a flat metric. For Lasagna states we will use a metric of the form

$$ds^2 = -dt^2 + L_r^2 dr^2 + L_\gamma^2 d\gamma^2 + L_\phi^2 d\phi^2, \tag{1.2}$$

where the adimensional spatial coordinates have the ranges

$$0 \le r \le 2\pi, \quad 0 \le \gamma \le 2\pi, \quad 0 \le \phi \le 2\pi, \tag{1.3}$$

so that the solitons are confined in a box of volume $V = (2\pi)^3 L_r L_\gamma L_\phi$.
For nuclear Spaghetti we will use the metric ansatz

$$ds^2 = -dt^2 + L_r^2 dr^2 + L_\theta^2 d\theta^2 + L_\phi^2 d\phi^2, \tag{1.4}$$

with adimensional coordinates ranging in

$$0 \le r \le 2\pi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi, \tag{1.5}$$

and a total volume $V = 4\pi^3 L_r L_\theta L_\phi$.

The energy-momentum tensor associated to the Skyrme field is given by

$$T_{\mu\nu} \;=\; -\frac{K}{2}\mathrm{Tr}\left(\mathcal{L}_\mu\mathcal{L}_\nu - \frac{1}{2}g_{\mu\nu}\mathcal{L}_\alpha\mathcal{L}^\alpha + \frac{\lambda}{4}(g^{\alpha\beta}G_{\mu\alpha}G_{\nu\beta} - \frac{1}{4}g_{\mu\nu}G_{\alpha\beta}G^{\alpha\beta})\right). \tag{1.6}$$

The topological charge is defined by (see Proposition 3)

$$B = \frac{1}{24\pi^2}\int_{\mathcal{V}} \mathrm{Tr}(\mathcal{L}\wedge\mathcal{L}\wedge\mathcal{L}), \tag{1.7}$$

where $\mathcal{V}$ is the spatial region spanned by the coordinates at any fixed time t, $\mathcal{L} = U^{-1}dU$ and Tr is the trace over the matrix indices.

2. Lasagna groups

In [46] it has been shown how Lasagna configurations can be determined as solutions of the Skyrme equations realized as Euler parameterizations of three dimensional cycles in $SU(N)$. Indeed, these cycles result to be suitable deformations of different non-trivial embeddings of $SU(2)$ into $SU(N)$. Here we want to prove that such construction can be easily extended to any simple Lie group (at least for the case of the undeformed embedding). Recall that in the case of $SU(N)$ the embedding was defined [33] by the generalized Euler map

$$U(t,r,\gamma,\phi)=e^{\left(\frac{t}{L_\phi}-\phi\right)\sigma\kappa}e^{h(r)}e^{m\gamma\kappa}, \tag{2.1}$$

where σ is a constant, m is an integer, κ a suitable matrix in $SU(N)$ and $h(r)$ results to be a linear function of r with values in the Cartan algebra H. Indeed, the main trick was to determine a suitable matrix κ able to make everything easily computable and to grant periodicity of $e^{m\gamma\kappa}$. The convenient strategy has been composed in two steps: first we have taken a basis of eigenmatrices of the simple roots, λ_j, $j=1,\dots,r$, where $r=N-1$ is the rank of the group, and defined the matrix

$$\kappa=\sum_{j=1}^{r}(c_j\lambda_j-c_j^*\lambda_j^\dagger), \tag{2.2}$$

where $\dagger$ means hermitian conjugate and c_j are complex constants. The second step consisted in determining the allowed values for the c_j. We want to do the same with a generic simple Lie group G replacing $SU(N)$. The first problem we ran into is the following. If $\lambda\in\mathfrak{g}_{\mathbb{C}}$ is an eigenmatrix of a root α of the Lie algebra $\mathfrak{g}$ of G (so it belongs to the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$), in general $\lambda^\dagger$ doesn't belong to $\mathfrak{g}_{\mathbb{C}}$ if $G\neq SU(N)$ for some N. So in general κ defined above is not a matrix of $\mathfrak{g}$.

In order to overcome this problem, we notice that a compact simple Lie group G always contain a *split* maximal subgroup [47], which is a maximal subgroup K with the property that $2\dim(K)+r=\dim(G)$ and that there exists a Cartan subalgebra H of $\mathfrak{g}$ all contained in $\mathfrak{p}$, the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$ (w.r.t. the Killing product):

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}. \tag{2.3}$$

Of course $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, while $\mathfrak{p}$ is not, since

$$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k},\qquad[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p},\qquad[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}, \tag{2.4}$$

which says that $\mathfrak{p}$ is a representation space for G and K is an isotropy group for $\mathfrak{p}$. One can easily show that a root matrix λ, associated to a root α, must have the form

$$\lambda=k+ip,\quad k\in\mathfrak{k},\ p\in\mathfrak{p},\ p\neq 0. \tag{2.5}$$

Then, $k-ip$ also is a root matrix, corresponding to the root $-\alpha$. We replace the hermitian conjugation with the $\sim$ conjugation defined by

$$\widetilde{k+ip}\equiv(k+ip)^\sim:=k-ip. \tag{2.6}$$

This way, if λ_j, $j=1,\dots,r$ are matrix roots corresponding to the simple roots of $\mathfrak{g}$, then

$$\kappa=\sum_{j=1}^{r}(c_j\lambda_j+c_j^*\tilde{\lambda}_j)\in\mathfrak{g}. \tag{2.7}$$

Notice that for $G = SU(N)$ we have $\tilde{\lambda} = -\lambda^\dagger$.
If we choose normalizations as in Appendix A, we can use the matrices J_k to decompose $h(z) = \sum_{j=1}^r y_j(z) J_j$. The properties of the roots can be inferred case by case from the lists in Appendix A. Exactly the same calculations as in [46] show that the field equations for the Skyrme field are equivalent to the system

$$h'' + \frac{\lambda m^2}{2L_\gamma^2} \sum_{j=1}^{N-1} \alpha_j(h'')|c_j|^2 J_j = 0, \tag{2.8}$$

$$\sum_{j<k} \Big(\alpha_j(h')^2 - \alpha_k(h')^2 - i\left(\alpha_j(h'') - \alpha_k(h'')\right)\Big) c_j c_k (\alpha_j|\alpha_k) \lambda_{\alpha_j+\alpha_k} = 0. \tag{2.9}$$

The first equation has solution $h'' = 0$, as a consequence of the strict positivity of the Cartan matrix for each simple group. The second system, using that the $\lambda_{\alpha_j+\alpha_k}$ are independent, reduces to the set of equations

$$\alpha_j(h')^2 - \alpha_k(h')^2 = 0, \qquad j < k, \quad s.t.\ (\alpha_j|\alpha_k) \neq 0. \tag{2.10}$$

Since $(\alpha_j|\alpha_k) \neq 0$ if and only if α_j and α_k are linked and since there are $r-1$ links in a connected Dynkin diagram, these are exactly $r-1$ equations. These are independent and assuming $a = \alpha_1(h) \neq 0$ have the general solution

$$\alpha_j(h') = \epsilon_j a, \quad j = 2, \dots, r, \tag{2.11}$$

where ϵ_j are signs. As in [46], we can solve it by writing

$$h' = a \sum_{j=1}^r 2 \frac{w_j}{(\alpha_j|\alpha_j)} J_j. \tag{2.12}$$

Applying α_k to both hands and defining $\epsilon_1 = 1$ we get

$$\epsilon_k = \sum_{j=1}^r 2 \frac{w_j}{(\alpha_j|\alpha_j)} \alpha_k(J_j) = \sum_{j=1}^r w_j C^G_{jk}, \tag{2.13}$$

where C^G is the Cartan matrix associated to G. The Cartan matrix is positive definite and is therefore always invertible, so that

$$w_k = \sum_{j=1}^r \epsilon_j (C^G)^{-1}_{j,k}. \tag{2.14}$$

Therefore, we have proven the following generalization of Proposition 2 in [46].

Proposition 1. *All local solutions of the Skyrme field equations of the form determined by the ansatz (2.1), (2.7), with metric*

$$ds^2 = -dt^2 + L_r^2 dr^2 + L_\gamma^2 d\gamma^2 + L_\phi^2 d\phi^2, \tag{2.15}$$

are given by

$$h(r) = a r v_\epsilon, \tag{2.16}$$

$$v_\epsilon = \sum_{j,k} (C^G)^{-1}_{j,k} \epsilon_j \frac{2}{\|\alpha_k\|^2} J_k, \tag{2.17}$$

where a is a real constant and ϵ_j are signs, with $\epsilon_1 = 1$.

Now we have to discuss which choices of the coefficients c_j are allowed. To this hand, we have that the solution must cover a topological cycle entirely. First, we notice that as a consequence of our normalizations, if we want to get it with r varying in $[0, 2\pi]$, we must take

$$a = \frac{1}{2}, \tag{2.18}$$

see [46], Proposition 3.
The second step is to grant periodicity of $e^{\gamma\kappa}$. This is the difficult part and determines the allowed values for the c_j. Notice that κ is diagonalizable (over $\mathbb{C}$). Indeed, since G is compact, in the adjoint representation κ results to be antihermitian and then diagonalizable with imaginary eigenvalues. It follows that it is diagonalizable in any representation with purely imaginary eigenvalues. If N is the dimension of the representation, then the eigenvalues $i\mu_1, \dots, i\mu_N$ must be in rational ratios, which means that for any $\mu_a \neq 0$ it must exist integers $n_a \neq 0$ such that

$$\mu_a n_b = \mu_b n_a, \tag{2.19}$$

or, equivalently, that it exists a non-vanishing real number μ and N integers $n_a \in \mathbb{Z}$, such that

$$\mu_a = \mu n_a. \tag{2.20}$$

This condition in general will depend on N, G and the constants c_j. In [46] this problem has been shown to have a set of solutions for the particular case of $G = SU(N)$ in the fundamental representation. Here we have to generalize that procedure without exploiting a very explicit realization. Indeed, we can prove that there are solutions with all c_j different from zero by following a strategy developed by Dynkin in [38], that we will recall in the next section. Let us choose f in the Cartan subalgebra, such that $\alpha_j(f) = b$, a positive constant independent on j, so that

$$[f, \lambda_j] = ib\lambda_j, \qquad [f, \tilde{\lambda}_j] = -ib\tilde{\lambda}_j. \tag{2.21}$$

We can easily determine it as follows. If $h_j = i[\lambda_j, \tilde{\lambda}_j]$, then set $f = \sum_{k=1}^r p_k h_k$. Thus, the above condition is equivalent to

$$b = \sum_{k=1}^r p_k \alpha_j(h_k) = \sum_{k=1}^r p_k (\alpha_j|\alpha_k) = \sum_{k=1}^r p_k \frac{\|\alpha_k\|^2}{2} C^G_{kj}. \tag{2.22}$$

from which we immediately get

$$p_j = b\frac{2}{\|\alpha_j\|^2} \sum_{k=1}^r (C^G)^{-1}_{kj}. \tag{2.23}$$

By the properties of the Cartan matrix it follows that p_j are all positive. Finally, we set

$$c_j = e^{i\psi_j}\sqrt{\frac{b}{2}p_j}. \tag{2.24}$$

Then, we have the following proposition:

Proposition 2. *If κ is constructed with the above choice of c_j, then $e^{\kappa z}$ is periodic with period $n\frac{2\pi}{b}$ where n may be 1 or 2 depending on the representation, $n = 1$ for the adjoint representation.*

Proof. We first show that periodicity is independent on the phases of c_j. If $e^{\kappa z}$ is periodic, then, for any fixed $g \in G$, $g e^{\kappa z} g^{-1}$ is also periodic with the same period. Since the simple roots α_j are linearly independent, for any fixed j we can find an element h_j of the Cartan algebra such that $\alpha_k(h_j) = \delta_{kj}$. Let us set $g = e^{\psi h_j}$. Then,

$$\begin{aligned} g\kappa g^{-1} &= \sum_{k=1}^{r} (c_k e^{\psi h_j} \lambda_k e^{-\psi h_j} + c_k^* e^{\psi h_j} \tilde{\lambda}_k e^{-\psi h_j}) = \sum_{k=1}^{r} (c_k e^{i\psi \alpha_k(h_j)} \lambda_k + c_k^* e^{-i\psi \alpha_k(h_j)} \tilde{\lambda}_k) \\ &= \sum_{k \neq j} (c_k \lambda_k + c_j^* \tilde{\lambda}_k + c_j e^{i\psi} \lambda_j + c_j^* e^{-i\psi} \tilde{\lambda}_j), \end{aligned} \tag{2.25}$$

which shows that $g\kappa g^{-1}$ differs from κ only by the phase of c_j. This proves our assert. So, it is sufficient to prove the proposition for $\psi_j = 0$. In this case, $T_3 := f$, $T_1 := \kappa$ and $T_2 = \frac{1}{b}[f, \kappa]$ form an A_1 subalgebra of $\mathfrak{g}$, and κ is conjugate to f in G. Indeed, $[T_i, T_j] = b\epsilon_{ijk} T_k$ from which

$$T_1 = e^{\frac{\pi}{2} T_2} T_3 e^{-\frac{\pi}{2} T_2}. \tag{2.26}$$

Therefore, as before, the periodicity of $e^{\kappa z}$ is equivalent to the periodicity of e^{fz}. But

$$e^{fz} h_j e^{-fz} = h_j, \tag{2.27}$$

and

$$e^{fz} \lambda_\alpha e^{-fz} = e^{i\alpha(f)z} \lambda_\alpha. \tag{2.28}$$

Now, any given root α is

$$\alpha = \sum_j n_j \alpha_j, \tag{2.29}$$

with the n_j all non-negative or all non-positive integers. Therefore,

$$e^{i\alpha(f)z} = e^{i\sum_j n_j bz}. \tag{2.30}$$

All these exponentials are therefore periodic, with the longest period determined by the simple roots, for which $e^{i\alpha(f)z} = e^{ibz}$, which has period $T = 2\pi/b$. But

$$e^{fT} \lambda_\alpha e^{-fT} = \lambda_\alpha \tag{2.31}$$

for any root α implies that $g = e^{fT}$ is in the center of the group. Since the center of a simple compact group is finite, this means that $g^n = I$ is the unit matrix for some integer n. Now, since κ is not in the Cartan subalgebra, it follows from [48], Section VII, Theorem 8.5 (see also [47]) that $n = 1$ or $n = 2$ depending on the specific representation. □

This shows that there exist always at least a set of solutions with all non- vanishing c_j, parametrized by a torus of phases. In [46] it has been shown that indeed, for the case of $SU(N)$ in the smallest irreducible representation, there is a further set of deformations that has been called a moduli space. This is a very difficult task to be investigated in general and we will not consider it here.

2.1. On the physical meaning of the time-dependence in the ansatz

It is worth to discuss the physical meaning of the time-dependent ansatz in Eq. (2.1) for the Lasagna-type configurations as well as the one in Eqs. (3.1), (3.2) and (3.3) for the spaghetti-type configurations. First of all, despite the time-dependence of the ansatz of the U field, the energy-momentum tensor is still stationary (so that it describes a static distribution of energy and momentum). This approach is inspired by the usual time-dependent ansatz that is used for Bosons stars [58,59] (and generalize it to arbitrary Lie group) in which the $U(1)$ charged scalar field depends on time in such a way to avoid the Derrick theorem (see [60]). Secondly, the peculiar time-dependence is chosen in order to simplify as much as possible the field equations without loosing the topological charge (as, until very recently, the Skyrme field equations have always been considered a very hard nut to crack from the analytic viewpoint). Thirdly (as it will be discussed in the next sections on the minimal coupling with Maxwell), the present ansatz (both for lasagna and spaghetti type configurations) produces $U(1)$ currents associated to the minimal coupling with Maxwell with a manifest superconducting current. Indeed (as it is clear from Eqs. (6.69) and (6.130)), the present $U(1)$ current always has the form

$$J_\mu = \Omega(\partial_\mu \Phi - 2A_\mu) , \tag{2.32}$$

where Ω depends on either the Lasagna or the spaghetti profiles (see Eqs. (6.69) and (6.130)) while Φ is a field which is defined modulo 2π. Consequently, the following observations are important.

1) The current does not vanish even when the electromagnetic potential vanishes ($A_\mu = 0$).

2) Such a "left over"

$$J^{(0)}_\mu = J_\mu\big|_{A_\mu=0} = \Omega\partial_\mu\Phi , \tag{2.33}$$

is maximal where Ω is maximal (and this corresponds to the local maxima of the energy density: see Eqs. (6.69) and (6.70).

3) *$J_{(0)\mu}$ cannot be turned off continuously*. One can try to eliminate $J_{(0)\mu}$ either deforming the profiles appearing in Ω integer multiples of π (but this is impossible as such a deformation would kill the topological charge as well) or deforming Φ to a constant (but also this deformation cannot be achieved for the same reason). Moreover, as it is the case in [57], Φ is only defined modulo 2π. Consequently, $J_{(0)\mu}$ defined in Eq. (2.33) is *a superconducting current supported by the present gauged configurations*.

These are the three of the main physical reasons to choose this peculiar time-dependent ansatz. On the other hand, it is worth to emphasize that the peculiar time-dependence we have chosen (for the reasons explained above) prevents one from using the usual techniques (see, for instance, [13]) to "quantize" the present topologically non-trivial solutions. In particular, the typical hypothesis of a static $SU(N)$-valued field U is violated in our case (since, as it has been already emphasize, the requirement to have a static $T_{\mu\nu}$ which describes a stationary distribution of energy and momentum does not imply that U itself is static). Therefore, to estimate the "classical isospin" of the present configurations we will proceed in a different manner in the next sections.

2.2. Energy and Baryon number

The energy of these solutions can be easily computed by means of Proposition 6 in Appendix B. We get

$$E = L_r L_\gamma L_\phi \|\underline{c}\|^2 \frac{K}{2}\pi^3 \Bigg[16\frac{\sigma^2}{L_\phi^2} + \frac{\|v_\epsilon\|^2}{\|\underline{c}\|^2 L_r^2} + \frac{\sigma^2\lambda}{L_\phi^2 L_r^2}$$
$$+4\frac{m^2}{L_\gamma^2}\Bigg(2 + \frac{\lambda}{8L_r^2} + \frac{\lambda\sigma^2}{L_\phi^2\|\underline{c}\|^2}\Bigg(\sum_{j=1}^{r} \|\alpha_j\|^2 |c_j|^4$$
$$+\sum_{j<k} |c_j|^2|c_k|^2(\alpha_j|\alpha_k)\Big(2\epsilon_j\epsilon_k + (\alpha_j|\alpha_k)(1-\epsilon_j\epsilon_k)\Big)\Bigg)\Bigg)\Bigg], \tag{2.34}$$

with

$$\|v_\epsilon\|^2 = -\mathrm{Tr} v_\epsilon^2, \tag{2.35}$$

$$\|\underline{c}\|^2 = \sum_{j=1}^{r} |c_j|^2, \tag{2.36}$$

and where σ depends on the representation and has to be chosen so that the solution correctly covers a cycle when $m = 1$ and ϕ varies from 0 to 2π. To specify it, let us investigate the Baryon number integral. To this hand, let us look better at Proposition 2. The fact that $n = 1$ or 2 obviously distinguishes the $SO(3)$-type solutions from the $SU(2)$-type ones (see [46]), since only in the first case the period remains invariant when passing to the adjoint representation. The right ranges are then understood by considering the correct Euler parameterizations for $SO(3)$ and for $SU(2)$. If we write it generically as

$$U(x, y, z) = e^{xT_3}e^{yT_1}e^{zT_3}, \tag{2.37}$$

one finds that, if T is the period of the exponential functions, in both cases z must vary in a period and y in a range of $T/4$. The difference is in x, which has to vary in a period for $SO(3)$ and half a period for $SU(2)$, for example, see Appendix C in [46]. If we set $x = \sigma\phi$, $y = r$ and $z = m\gamma$ and we want to normalize the ranges of the coordinates ϕ, r, γ, so that all vary in $[0, 2\pi]$, we see that we always have to require

$$b = n, \qquad \sigma = \frac{n}{2}, \tag{2.38}$$

where n is an integer. With these conventions we can state the following proposition.

Proposition 3. *The Baryonic topological charge is*

$$B = \frac{1}{24\pi^2}\int \varepsilon^{ijk}\mathrm{Tr}(\mathcal{L}_i\mathcal{L}_j\mathcal{L}_k)\sqrt{g}\, dr d\gamma d\phi = mn\|\underline{c}\|^2, \tag{2.39}$$

where $\mathcal{L}_i = U^{-1}\partial_i U$.

The proof is exactly the same as in Appendix F of [46], so we omit it.
The energy per Baryon $g = E/B$ is therefore

$$g = L_r L_\gamma L_\phi \frac{K}{2mn}\pi^3 \Bigg[16\frac{\sigma^2}{L_\phi^2} + \frac{\|v_\epsilon\|^2}{\|\underline{c}\|^2 L_r^2} + \frac{\sigma^2\lambda}{L_\phi^2 L_r^2}$$
$$+ 4\frac{m^2}{L_\gamma^2}\Bigg(2 + \frac{\lambda}{8L_r^2} + \frac{\lambda\sigma^2}{L_\phi^2\|\underline{c}\|^2}\Bigg(\sum_{j=1}^{r} \|\alpha_j\|^2 |c_j|^4$$

$$+\sum_{j<k}|c_j|^2|c_k|^2(\alpha_j|\alpha_k)\left(2\epsilon_j\epsilon_k+(\alpha_j|\alpha_k)(1-\epsilon_j\epsilon_k)\right)\Big)\Big)\Big]. \tag{2.40}$$

3. Spaghetti groups

Another kind of configurations is obtained by starting from a different ansatz, which leads to Spaghetti like solutions. Spaghetti can be parameterized by the following ansatz:

$$U(t,r,\theta,\phi)=\exp(\chi(r)\tau_1), \tag{3.1}$$

where $\tau_1=\vec{n}\cdot\vec{T}=n_1T_1+n_2T_2+n_3T_3$ is defined by

$$\vec{T}=(T_1,T_2,T_3), \qquad \vec{n}=(\sin\Theta\cos\Phi,\sin\Theta\sin\Phi,\cos\Theta) \tag{3.2}$$

$$\Theta=q\theta, \quad \Phi=p\left(\frac{t}{L_\phi}-\phi\right), \quad q=2v+1, \quad p,v\in\mathbb{N}, \quad p\neq 0\,. \tag{3.3}$$

In the ansatz, T_i are matrices of a given representation of the Lie algebra of G and are required to define a three dimensional subalgebra that we can choose to normalize so that

$$[T_j,T_k]=\varepsilon_{jkm}T_m, \tag{3.4}$$

and satisfy

$$\mathrm{Tr}(T_jT_k)=-2I_{G,\rho}\delta_{jk}, \tag{3.5}$$

where $I_{G,\rho}$ is the Dynkin index of $su(2)$ in G (see [38]), that is the coefficient relating the trace product in the representation ρ of $Lie(G)$ to the Killing product of $su(2)$. We also define

$$\tau_2=\partial_\Theta\tau_1, \tag{3.6}$$

$$\tau_3=\frac{1}{\sin\Theta}\partial_\Phi\tau_1. \tag{3.7}$$

Together with τ_1, they satisfy

$$[\tau_j,\tau_k]=\epsilon_{jkm}\tau_m. \tag{3.8}$$

With these rules, we get for $\mathcal{L}_\mu=U^{-1}\partial_\mu U$:

$$\mathcal{L}_r=\tau_1\ \chi'(r). \tag{3.9}$$

For the other terms, set $\alpha=\Theta,\Phi$ and using

$$U^{-1}\partial_\alpha U=\chi\int_0^1 e^{-\sigma\chi\tau_1}\partial_\alpha\tau_1 e^{\sigma\chi\tau_1}, \tag{3.10}$$

$$e^{-\sigma\chi\tau_1}\tau_2e^{\sigma\chi\tau_1}=\cos(\sigma\chi)\tau_2-\sin(\sigma\chi)\tau_3, \tag{3.11}$$

$$e^{-\sigma\chi\tau_1}\tau_3e^{\sigma\chi\tau_1}=\sin(\sigma\chi)\tau_2+\cos(\sigma\chi)\tau_3, \tag{3.12}$$

we get

$$\mathcal{L}_\Theta=\sin\chi\ \tau_2-(1-\cos\chi)\ \tau_3, \tag{3.13}$$

$$\mathcal{L}_\Phi=\sin\Theta(\sin\chi\ \tau_3+(1-\cos\chi)\ \tau_2), \tag{3.14}$$

and

$$\mathcal{L}_t = \frac{p}{L}\mathcal{L}_\Phi, \tag{3.15}$$

$$\mathcal{L}_\theta = q\mathcal{L}_\Theta, \tag{3.16}$$

$$\mathcal{L}_\phi = -p\mathcal{L}_\Phi. \tag{3.17}$$

This shows that the expression of the $\mathcal{L}_\mu$ is universal (depend only on the algebra of the τ_j), so the field equations are always the same for any choice of the group. These are

$$4\chi''(r)\left(\lambda q^2 \sin^2\left(\frac{\chi}{2}\right) + L_\theta^2\right) - q^2 \sin\chi\left(4L_r^2 - \lambda\chi'^2\right) = 0\,. \tag{3.18}$$

What is expected to change is just the topological charge and the energy. Given this universality property, we see immediately that, for any given group G, these kinds of solutions are classified by all possible ways of finding a three dimensional simple subalgebra of the lie algebra $\mathfrak{g}$. Luckily, we don't need to tackle such a program, since has already been solved by E. B. Dynkin in [38], chapter III. This work as follows.

First, it is convenient to complexify the algebra, recombine and normalize the generators f, e_+, e_- of the subgroup so that

$$[e_+, e_-] = -if, \qquad [f, e_\pm] = \pm 2ie_\pm. \tag{3.19}$$

Each complex three dimensional simple algebra is isomorphic to this. However, we must consider as equivalent only the ones which are isomorphic through an automorphism of the group. Let α_j, $j = 1, \ldots, r$ be simple roots defined from a cartan subalgebra containing f. Then, it results that $(\alpha_j|f)$ must be integer numbers that can assume only the values 0,1,2. The set of numbers $d_j = \alpha_j(f)$ are called the Dynkin characteristic of the subgroup. The main result of [38] is that the three dimensional simple subalgebras A are in one to one correspondence with the characteristics and one can indeed classify the characteristics. A subalgebra is said to be regular if its roots are indeed roots of $\mathfrak{g}$. The subalgebra A is said to be integral if the projection of the roots of $\mathfrak{g}$ along the direction of the roots of A are integer multiples of the simple root α_A of A. Since $\alpha_A(f) = 2$, we see that the dual of α_A in the Cartan subalgebra H is

$$h_{\alpha_A} = \frac{2}{(f|f)}f. \tag{3.20}$$

From this it follows immediately that A is integral if and only if all the numbers of the Dynkin characteristic $\chi_A = (d_1, \ldots, d_r)$ of A are even (so are 0 and 2). All inequivalent characteristics for the exceptional Lie groups are listed in [38]. Furthermore, given such a characteristic $\chi = (d_1, \ldots, d_r)$, there it is explained how to construct explicitly the associated subalgebra. First, if J_j is the dual of α_j in H, write

$$f = \sum_{j=1}^{r} p_j J_j, \tag{3.21}$$

and choose p_j so that $\alpha_j(f) = d_j$. This gives

$$d_j = \sum_{k=1}^{r} p_k(\alpha_k|\alpha_j) = \sum_{k=1}^{r} \frac{\|\alpha_k\|^2}{2} p_k C_{kj}^G. \tag{3.22}$$

From this we get

$$p_k = \sum_{j=1}^{r} d_j (C^G)^{-1}_{jk} \frac{2}{\|\alpha_k\|^2}. \tag{3.23}$$

As usual, C^G is the Cartan matrix. In general, the construction of the remaining generators is non-trivial. To do it, one has to consider the subset of the root system Σ defined by

$$\Sigma_{\chi_G} = \{\alpha \in \Sigma | \alpha(f) = 2\}. \tag{3.24}$$

Then, all roots are positive. If λ_α are the corresponding eigenmatrices (normalized so that $\mathrm{Tr}(\tilde{\lambda}_\alpha \lambda_\alpha) = -1$), one then has to look for real coefficients k_α such that, setting

$$e_+ = \sum_{\alpha \in \Sigma_{\chi_G}} k_\alpha \lambda_\alpha, \tag{3.25}$$

$$e_- = \sum_{\alpha \in \Sigma_{\chi_G}} k_\alpha \tilde{\lambda}_\alpha, \tag{3.26}$$

then $[e_+, e_-] = -if$. If χ_G is an admissible characteristic, then in general there are infinite solutions, but we know that are all equivalent so it is sufficient to choose one, all the other ones being related to it by conjugation with elements of the group. Notice that the resulting equations are in general

$$\sum_{\alpha \neq \beta \in \Sigma_{\chi_G}} k_\alpha k_\beta [\lambda_\alpha, \tilde{\lambda}_\beta] = 0, \tag{3.27}$$

$$\sum_{\beta \in \Sigma_{\chi_G}} k_\beta^2 n_{\beta,j} = p_j, \tag{3.28}$$

where we used that any positive root can be written as

$$\beta = \sum_{j=1}^{r} n_{\beta,j} \alpha_j, \tag{3.29}$$

with $n_{\beta,j}$ non-negative integers, and

$$[\tilde{\lambda}_\beta, \lambda_\beta] = i \sum_{j=1}^{r} n_{\beta,j} J_j. \tag{3.30}$$

In the particular case when $d_j = 2$ for all j, Σ_{χ_G} consists of all simple roots and the solution is easily obtained as

$$e_+ = \sum_{j=1}^{r} \sqrt{p_j} \lambda_j, \qquad e_- = \sum_{j=1}^{r} \sqrt{p_j} \tilde{\lambda}_j. \tag{3.31}$$

Finally, we can go back to our real case by taking

$$T_1 = \frac{1}{2}(e_+ + e_-), \quad T_2 = \frac{1}{2i}(e_+ - e_-), \quad T_3 = \frac{1}{2} f. \tag{3.32}$$

Notice that this is the same construction we used to get a periodic generator κ for the Lasagna configurations. This also shows that indeed we can construct a κ matrix for each three dimensional subalgebra.

3.1. Energy density and Baryon charge

Let us determine the energy density and the Baryon charge. The energy density is defined by the T_{tt} component of the energy-momentum tensor

$$T_{\mu\nu} = -\frac{K}{2}\mathrm{Tr}(T_i T_j)\left[\mathcal{L}^i_\mu \mathcal{L}^j_\nu - \frac{1}{2}g_{\mu\nu}\mathcal{L}^{\rho i}\mathcal{L}_\rho{}^j + \frac{\lambda}{4}\left(g^{\rho\sigma}G^i_{\mu\rho}G^j_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}G^{\rho\sigma i}G^j_{\rho\sigma}\right)\right]. \tag{3.33}$$

A direct computation gives

$$T_{tt} = 2I_{G,\rho}\frac{Kp}{4L^2_\phi L_r L_\theta}\left[\rho_0 + 2\sin^2(q\theta)\rho_1\right], \tag{3.34}$$

with

$$\rho_0 = \frac{L^2_\phi}{p}\left[4L^2_r q^2\sin^2\left(\frac{\chi}{2}\right) + \left(L^2_\theta + q^2\lambda\sin^2\left(\frac{\chi}{2}\right)\right)\chi'^2\right], \tag{3.35}$$

$$\rho_1 = p\sin^2\left(\frac{\chi}{2}\right)\left[4L^2_r\left(L^2_\theta + q^2\lambda\sin^2\left(\frac{\chi}{2}\right)\right) + L^2_\theta\lambda\chi'^2\right]. \tag{3.36}$$

$I_{G,\rho}$ is the Dynkin index and can be computed as follows. First, observe that a generic root has the form

$$\beta(f) = \sum_{j=1}^r n_{\beta,j}\alpha_j(f) = \sum_{j=1}^r n_{\beta,j}d_j. \tag{3.37}$$

By using (3.22), (3.28) and the definition of Σ_{χ_G}, we get

$$\begin{aligned}\mathrm{Tr}(ff) &= -\sum_{j=1}^r\sum_{k=1}^r p_j p_k(\alpha_k|\alpha_j) = -\sum_{j=1}^r\sum_{k=1}^r \frac{\|\alpha_k\|^2}{2}p_j p_k C^G_{kj} \\ &= -\sum_{j=1}^r p_j d_j = -\sum_{j=1}^r\sum_{\beta\in\Sigma_{\chi_G}} k^2_\beta n_{\beta,j}d_j = -\sum_{\beta\in\Sigma_{\chi_G}} k^2_\beta \beta(f) = -2\sum_{\beta\in\Sigma_{\chi_G}} k^2_\beta\end{aligned} \tag{3.38}$$

$$\mathrm{Tr}(e_+e_-) = -\sum_{\beta\in\Sigma_{\chi_G}} k^2_\beta \tag{3.39}$$

Therefore,

$$\mathrm{Tr}(T_iT_j) = -\frac{\delta_{ij}}{2}\sum_{\beta\in\Sigma_{\chi_G}} k^2_\beta, \tag{3.40}$$

and so

$$I_{G,\rho} = \sum_{\beta\in\Sigma_{\chi_G}}\frac{k^2_\beta}{4}. \tag{3.41}$$

The Baryon charge can be written as

$$B=\frac{1}{24\pi^2}\int\rho_B\sqrt{g}drd\theta d\phi, \tag{3.42}$$

in which ρ_B is the Baryonic density charge

$$\rho_B=\epsilon^{ijk}\mathrm{Tr}(\mathcal{L}_i\mathcal{L}_j\mathcal{L}_k). \tag{3.43}$$

Recalling the ranges (1.5) for the coordinates and that $q=2v+1$ and $\chi(0)=0$, we get

$$B=\frac{2p}{\pi}I_{G,\rho}\chi(2\pi). \tag{3.44}$$

The boundary conditions on $\chi(r)$ depend on the periodicity of τ_1, which corresponds to the periodicity of T_3 ($T_3=\tau_1(\Theta=\pi)$). We must have $\chi(2\pi)=n\pi T_{G,\rho}$, so that

$$B=2npI_{G,\rho}T_{G,\rho}, \tag{3.45}$$

where $T_{G,\rho}=1$ for representations with even dimension and $T_{G,\rho}=2$ for representations with odd dimension.

3.2. *On the "classical" isospin of these configurations*

We have shown in previous sections that the inclusion of a suitable time-dependence in the ansätze, both for lasagna and spaghetti phases (see Eqs. (2.1) and (3.1)), is one of the key ingredients that allows the field equations to be considerably reduced, leading to a single integrable ODE equation for the profiles. This time-dependence offers a nice short-cut to estimate the "classical Isospin" of the configurations analyzed in the present paper (a relevant question is whether or not the classical Isospin is large when the Baryonic charge is large). In particular, one may evaluate the "cost" of removing such time-dependence. Such a cost is related to the internal Isospin symmetry of the theory. This is like trying to estimate the angular momentum of a spinning top by evaluating the cost to make the spinning top to stop spinning. In the present case, the time-dependence of the configurations can be removed from the ansätze by introducing a Isospin chemical potential; then the isospin chemical potential needed to remove such time-dependence is a measure of the classical Isospin of the present configurations. We will see how this works for the simplest $SU(2)$ case, where the generators are $T_j=i\sigma_j$, being σ_j the Pauli matrices (general group G behave in a similar way).

As it is well known, the effects of the Isospin chemical potential can be taken into account by using the following covariant derivative

$$\nabla_\mu\to D_\mu=\nabla_\mu+\bar{\mu}[T_3,\cdot]\delta_{\mu 0}\ . \tag{3.46}$$

Now, we will use exactly the same ansatz as before in the spaghetti $SU(2)$ case, *but this time without the time dependence*:

$$U=e^{\chi(x)\,(\vec{n}\cdot\vec{T})}\ ,$$
$$\vec{n}=(\sin\Theta\sin\Phi,\sin\Theta\cos\Phi,\cos\Theta)\ ,$$

where

$$\chi=\chi\,(r)\ ,\quad \Theta=q\theta\ ,\quad \Phi=p\phi\ ,$$
$$q=\frac{1}{2}(2v+1)\ ,\quad p,v\in\mathbb{N}\ ,\quad p\neq 0\ ,$$

together with the introduction of the Isospin chemical potential in Eq. (3.46) in the theory. One can check directly that the complete set of Skyrme equations can still be reduced to the same ODE for the profile $\chi(r)$ in the case of the spaghetti phase in Eq. (3.18) *only provided the Isospin chemical potential for the spaghetti phase is given by*

$$\bar{\mu}_{\mathrm{S}} = \frac{p}{L_\phi} \ . \tag{3.47}$$

In other word, the cost to eliminate the time-dependence is to introduce an Isospin chemical potential which is large when the Baryonic charge of the spaghetti is large. Something similar happens in the case of the lasagna phase. Let us consider the ansatz in terms of the Euler angles *but without the time-dependence* for the $SU(2)$ case:

$$U_L = e^{\Phi T_3} e^{H T_2} e^{\Theta T_3} \ ,$$

where

$$\Phi = p\phi \ , \quad H = h(r) \ , \quad \Theta = m\theta \ , \qquad p, m \in \mathbb{N} \ .$$

Let us introduce the Isospin chemical potential, demanding that the profile $h(r)$ should be the same as before. Then, as in the spaghetti case, the Skyrme field equations with chemical potential can still be satisfied by the very same profile $h(r)$ *provided we fix the Isospin chemical potential as*

$$\bar{\mu}_{\mathrm{L}} = \frac{pm}{(p^2 L_\phi^2 + m^2 L_\theta^2)^{\frac{1}{2}}} \ . \tag{3.48}$$

At this point it is important to remember that in the $SU(2)$ case the lasagna and spaghetti type solutions have the following values for the topological charges

$$B_{\mathrm{S}} = np \ , \qquad B_{\mathrm{L}} = mp \ ,$$

see [25] and [26] for more details. These arguments show that the “classical Isospin” of configurations with high Baryonic charge is large. Finally, it is important to point out that the large Isospin case corresponds to either neutron rich or proton rich matter and due to Coulomb effects (not taken into account in this model), the neutron rich solution is preferred. This fact is very convenient as far as the physics of neutron stars is concerned.

4. Examples: exceptional pasta

As an example we can consider the “basic exceptional Skyrmions”, that are solutions in lowest dimensional representation when G is one of the exceptional Lie groups. There are five cases that we now recall according to the dimension of the group. For each of them we know all inequivalent three dimensional subalgebras, each one determined by the Dynkin characteristic $\chi_I(d_1, \ldots, d_r)$, where I is the Dynkin index and d_j are the coefficients of the characteristic, ordered as the simple root listed in Appendix A.

The smallest exceptional group is G_2, a 14 dimensional group of rank 2 whose smallest irrep is 7 dimensional. There are four different three dimensional subalgebras. It contains four 3D subalgebras, having characteristics

$$\chi_1 = (0, 1), \quad \chi_3 = (1, 0), \quad \chi_4 = (0, 2), \quad \chi_{28} = (2, 2).$$

χ_1 and χ_2 are regular but not integral, while χ_4 and χ_{28} are not regular but are integral. In particular, the minimal regular subalgebra containing χ_4 is $\chi_1 \oplus \chi_3$, while χ_{28} is maximal so that the smallest regular subalgebra containing it is G_2 itself.

The next group is F_4, a 52 dimensional group of rank 4. Its smallest irrep is 26 dimensional. It contains 15 $su(2)$ type subalgebras, whose characteristics are

$$\begin{aligned}
&\chi_1 = (1,0,0,0); \quad \chi_2 = (0,0,0,1); \quad \chi_3 = (0,1,0,0); \quad \chi_4 = (2,0,0,0);\\
&\chi_6 = (0,0,1,0); \quad \chi_8 = (0,0,0,2); \quad \chi_9 = (0,1,0,1); \quad \chi_{10} = (2,0,0,1);\\
&\chi_{11} = (1,0,1,0); \quad \chi_{12} = (0,2,0,0); \quad \chi_{28} = (2,2,0,0); \quad \chi_{35} = (1,0,1,2);\\
&\chi_{36} = (0,2,0,2); \quad \chi_{60} = (2,2,0,2); \quad \chi_{156} = (2,2,2,2).
\end{aligned}$$

The regular subalgebras are χ_1 and χ_2, which are not integral. The integral subalgebras are χ_4, χ_8, χ_{12}, χ_{28}, χ_{36}, χ_{60} and χ_{156}. In particular, χ_{156} is maximal.

The third group is E_6, a 78 dimensional group of rank 6. Its smallest irrep is 27 dimensional. It contains 20 $su(2)$ type subalgebras, whose characteristics are

$$\begin{aligned}
&\chi_1 = (0,1,0,0,0,0); \quad \chi_2 = (1,0,0,0,0,1); \quad \chi_3 = (0,0,0,1,0,0);\\
&\chi_4 = (0,2,0,0,0,0); \quad \chi_5 = (1,1,0,0,0,1); \quad \chi_6 = (0,0,1,0,1,0);\\
&\chi_8 = (2,0,0,0,0,2); \quad \chi_9 = (1,0,0,1,0,1); \quad \chi_{10} = (1,2,0,0,0,1);\\
&\chi_{11} = (0,1,1,0,1,0); \quad \chi_{12} = (0,0,0,2,0,0); \quad \chi_{20} = (2,2,0,0,0,2);\\
&\chi_{21} = (1,1,1,0,1,1); \quad \chi_{28} = (0,2,0,2,0,0); \quad \chi_{30} = (1,2,1,0,1,1);\\
&\chi_{35} = (2,1,1,0,1,2); \quad \chi_{36} = (2,0,0,2,0,2); \quad \chi_{60} = (2,2,0,2,0,2);\\
&\chi_{84} = (2,2,2,0,2,2); \quad \chi_{156} = (2,2,2,2,2,2).
\end{aligned}$$

The only regular subalgebra is χ_1, which is not integral. The integral subalgebras are $\chi_4, \chi_8, \chi_{12}, \chi_{20}, \chi_{28}, \chi_{36}, \chi_{60}$ and χ_{156}. The last one is also maximal.

The third group is E_7, a 133 dimensional group of rank 7. Its smallest irrep is 58 dimensional. It contains 44 $su(2)$ type subalgebras, whose characteristics are

$$\begin{aligned}
&\chi_1 = (1,0,0,0,0,0,0); \quad \chi_2 = (0,0,0,0,0,1,0); \quad \chi_{3'} = (0,0,1,0,0,0,0);\\
&\chi_{3''} = (0,0,0,0,0,0,2); \quad \chi_{4'} = (2,0,0,0,0,0,0); \quad \chi_{4''} = (0,1,0,0,0,0,1);\\
&\chi_5 = (1,0,0,0,0,1,0); \quad \chi_6 = (0,0,0,1,0,0,0); \quad \chi_7 = (0,2,0,0,0,0,0);\\
&\chi_8 = (0,0,0,0,0,2,0); \quad \chi_9 = (0,0,1,0,0,1,0); \quad \chi_{10} = (2,0,0,0,0,1,0);\\
&\chi_{11'} = (1,0,0,1,0,0,0); \quad \chi_{11''} = (2,0,0,0,0,0,2); \quad \chi_{12'} = (0,0,2,0,0,0,0)\\
&\chi_{12''} = (1,0,0,0,1,0,1); \quad \chi_{13} = (0,1,1,0,0,0,1); \quad \chi_{14} = (0,0,0,1,0,1,0);\\
&\chi_{15} = (0,0,0,0,2,0,0) \quad \chi_{20} = (2,0,0,0,0,2,0); \quad \chi_{21} = (1,0,0,1,0,1,0);\\
&\chi_{24} = (0,0,0,2,0,0,0); \quad \chi_{28} = (2,0,2,0,0,0,0); \quad \chi_{29} = (2,1,1,0,0,0,1);\\
&\chi_{30} = (2,0,0,1,0,1,0); \quad \chi_{31} = (2,0,0,0,2,0,0); \quad \chi_{35'} = (1,0,0,1,0,2,0);\\
&\chi_{35''} = (2,0,0,0,0,2,2); \quad \chi_{36'} = (0,0,2,0,0,2,0); \quad \chi_{36''} = (1,0,0,1,0,1,2);\\
&\chi_{38} = (0,1,1,0,1,0,2); \quad \chi_{39} = (0,0,0,2,0,0,2); \quad \chi_{56} = (0,0,0,2,0,2,0);\\
&\chi_{60} = (0,0,2,0,0,2,0); \quad \chi_{61} = (2,1,1,0,1,1,0) \quad \chi_{62} = (2,1,1,0,1,0,2);\\
&\chi_{63} = (2,0,0,2,0,0,2); \quad \chi_{84} = (2,0,0,2,0,2,0); \quad \chi_{110} = (2,1,1,0,1,2,2);
\end{aligned}$$

$$\chi_{111} = (2, 0, 0, 2, 0, 2, 2); \quad \chi_{156} = (2, 0, 2, 2, 0, 2, 0); \quad \chi_{159} = (2, 2, 2, 0, 2, 0, 2);$$
$$\chi_{231} = (2, 2, 2, 0, 2, 2, 2) \quad \chi_{399} = (2, 2, 2, 2, 2, 2, 2).$$

The only regular subalgebra is χ_1, which is not integral. The integral subalgebras are $\chi_{3''}$, $\chi_{4'}$, χ_7, χ_8, $\chi_{11''}$, $\chi_{12'}$, χ_{15}, χ_{20}, χ_{24}, χ_{28}, χ_{31}, $\chi_{35''}$, $\chi_{36'}$, χ_{39}, χ_{56}, χ_{60}, χ_{63}, χ_{84}, χ_{111}, χ_{156}, χ_{159}, χ_{231} and χ_{399}. The last one is also maximal.

The third group is E_8, a 248 dimensional group of rank 8. Its smallest irrep is 248 dimensional. It contains 70 $su(2)$ type subalgebras, whose characteristics are

$$\begin{aligned}
&\chi_1 = (0, 0, 0, 0, 0, 0, 0, 1); && \chi_2 = (1, 0, 0, 0, 0, 0, 0, 0); && \chi_3 = (0, 0, 0, 0, 0, 0, 1, 0);\\
&\chi_{4'} = (0, 1, 0, 0, 0, 0, 0, 0); && \chi_{4''} = (0, 0, 0, 0, 0, 0, 0, 2); && \chi_5 = (1, 0, 0, 0, 0, 0, 0, 1);\\
&\chi_6 = (0, 0, 0, 0, 0, 1, 0, 0); && \chi_7 = (0, 0, 1, 0, 0, 0, 0, 0); && \chi_8 = (2, 0, 0, 0, 0, 0, 0, 0);\\
&\chi_9 = (1, 0, 0, 0, 0, 0, 1, 0); && \chi_{10'} = (2, 0, 0, 0, 0, 0, 0, 1); && \chi_{10''} = (0, 0, 0, 0, 1, 0, 0, 0);\\
&\chi_{11} = (0, 0, 0, 0, 0, 1, 0, 1); && \chi_{12'} = (0, 0, 0, 0, 0, 0, 2, 0); && \chi_{12''} = (0, 0, 1, 0, 0, 0, 0, 1);\\
&\chi_{13} = (0, 1, 0, 0, 0, 0, 1, 0); && \chi_{14} = (1, 0, 0, 0, 0, 1, 0, 0); && \chi_{15} = (0, 0, 0, 1, 0, 0, 0, 0);\\
&\chi_{16} = (0, 2, 0, 0, 0, 0, 0, 0); && \chi_{20'} = (1, 0, 0, 0, 1, 0, 0, 0); && \chi_{20''} = (2, 0, 0, 0, 0, 0, 0, 2);\\
&\chi_{21} = (1, 0, 0, 0, 0, 1, 0, 1); && \chi_{22} = (0, 1, 0, 0, 0, 0, 0, 1); && \chi_{24} = (0, 0, 0, 0, 0, 2, 0, 0);\\
&\chi_{25} = (0, 0, 1, 0, 0, 1, 0, 0); && \chi_{28} = (0, 0, 0, 0, 0, 0, 2, 2); && \chi_{29} = (0, 1, 0, 0, 0, 0, 1, 2);\\
&\chi_{30'} = (1, 0, 0, 0, 0, 1, 0, 2); && \chi_{30''} = (0, 0, 0, 1, 0, 0, 1, 0); && \chi_{31} = (0, 0, 0, 1, 0, 0, 0, 2);\\
&\chi_{32} = (0, 2, 0, 0, 0, 0, 0, 2); && \chi_{34} = (0, 0, 1, 0, 0, 1, 0, 1); && \chi_{35} = (2, 0, 0, 0, 0, 1, 0, 1);\\
&\chi_{36'} = (1, 0, 0, 1, 0, 0, 0, 1); && \chi_{36''} = (2, 0, 0, 0, 0, 0, 2, 0); && \chi_{37} = (1, 0, 0, 0, 1, 0, 1, 0);\\
&\chi_{38} = (0, 1, 1, 0, 0, 0, 1, 0); && \chi_{39} = (0, 0, 0, 1, 0, 1, 0, 0); && \chi_{40} = (0, 0, 0, 0, 2, 0, 0, 0);\\
&\chi_{56} = (2, 0, 0, 0, 0, 2, 0, 0); && \chi_{57} = (1, 0, 0, 1, 0, 1, 0, 0); && \chi_{60} = (2, 0, 0, 0, 0, 0, 2, 2);\\
&\chi_{61} = (1, 0, 0, 0, 1, 0, 1, 2); && \chi_{62} = (0, 1, 1, 0, 0, 0, 1, 2); && \chi_{63} = (0, 0, 0, 1, 0, 1, 0, 2);\\
&\chi_{64} = (0, 0, 0, 0, 2, 0, 0, 2); && \chi_{70} = (1, 0, 0, 1, 0, 1, 0, 1); && \chi_{84'} = (1, 0, 0, 1, 0, 1, 1, 0);\\
&\chi_{84''} = (2, 0, 0, 0, 0, 2, 0, 2); && \chi_{85} = (1, 0, 0, 1, 0, 1, 0, 2); && \chi_{88} = (0, 0, 0, 2, 0, 0, 0, 2);\\
&\chi_{110} = (2, 1, 1, 0, 0, 0, 1, 2); && \chi_{111} = (2, 0, 0, 1, 0, 1, 0, 2); && \chi_{112} = (2, 0, 0, 0, 2, 0, 0, 2);\\
&\chi_{120} = (0, 0, 0, 2, 0, 0, 2, 0); && \chi_{156} = (2, 0, 0, 0, 0, 2, 2, 2); && \chi_{157} = (1, 0, 0, 1, 0, 1, 2, 2);\\
&\chi_{159} = (0, 1, 1, 0, 1, 0, 2, 2); && \chi_{160} = (0, 0, 0, 2, 0, 0, 2, 2); && \chi_{166} = (1, 0, 1, 1, 0, 0, 2, 2);\\
&\chi_{182} = (2, 1, 1, 0, 1, 1, 0, 1); && \chi_{184} = (2, 0, 0, 2, 0, 0, 2, 0); && \chi_{231} = (2, 1, 1, 0, 1, 0, 2, 2);\\
&\chi_{232} = (2, 0, 0, 2, 0, 0, 2, 2); && \chi_{280} = (2, 0, 0, 2, 0, 2, 0, 2); && \chi_{399} = (2, 1, 1, 0, 1, 1, 2, 2);\\
&\chi_{400} = (2, 0, 0, 2, 0, 2, 2, 2); && \chi_{520} = (2, 2, 2, 0, 2, 0, 2, 2); && \chi_{760} = (2, 2, 2, 0, 2, 2, 2, 2);\\
&\chi_{1240} = (2, 2, 2, 2, 2, 2, 2, 2).
\end{aligned}$$

The only regular subalgebra is χ_1, which is not integral. The integral subalgebras are $\chi_{4''}$, χ_8, $\chi_{12'}$, χ_{16}, $\chi_{20''}$, χ_{24}, χ_{28}, χ_{32}, $\chi_{36''}$, χ_{40}, χ_{56}, χ_{60}, χ_{64}, $\chi_{84''}$, χ_{112}, χ_{120}, χ_{156}, χ_{160}, χ_{184}, χ_{232}, χ_{280}, χ_{400}, χ_{520}, χ_{760} and χ_{1240}. The last one is also maximal.

As an example, we will finally construct the explicit solutions for G_2, which we can call "G_2 exceptional pasta".

4.1. G_2 exceptional Spaghetti

Here we consider explicit solutions case by case. Our deduction will be quite general and independent on the specific realization in terms of matrices, but just on the chosen representation. Nevertheless, for sake of completeness, in Appendix D we will provide an explicit matrix realization of the subalgebras in the lowest fundamental representation.

4.1.1. χ_1-Spaghetti

Since $\chi_1 = (0, 1)$, we get $p \equiv (p_1, p_2) = (6, 4)$. The only root satisfying $\alpha(f) = 2$ is $\alpha_6 = 3\alpha_1 + 2\alpha_2$. Therefore, $e_+ = k\lambda_6$ and equation (3.27) is trivial, while (3.28) gives

$$k = \sqrt{2}. \tag{4.1}$$

Therefore, the Spaghetti solution is determined by the matrices

$$T_1^{(1)} = \frac{\sqrt{2}}{2}(\lambda_6 + \tilde{\lambda}_6), \tag{4.2}$$

$$T_2^{(1)} = \frac{\sqrt{2}}{2i}(\lambda_6 - \tilde{\lambda}_6), \tag{4.3}$$

$$T_3^{(1)} = 3J_1 + 2J_2. \tag{4.4}$$

Notice that up to now this is independent on the choice of the irrep. The choice of the representation allows to further specify the type of solution. The fundamental representations of G_2 are the **7**, with maximal weight α_4, whose seven weight are on the small hexagon given by $\pm\alpha_a$, $a = 1, 3, 4$, plus one vanishing weight, and the **14** which is the adjoint representation, with maximal weight α_6 and with all roots as weight. The action of $\pm\alpha_6$ on the small hexagon shows that if we choose to work with the **7**, then $\mathbb{R}^7$ decomposes as $\mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ under χ_1, so that is an $SU(2)$ type solution.
For **14**, we see that the action of $\pm\alpha_6$ decomposes $\mathbb{R}^{14}$ into $\mathbf{3} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$, which is again an $SU(2)$ type solution.

4.1.2. χ_3-Spaghetti

Since $\chi_3 = (1, 0)$, we get $p \equiv (p_1, p_2) = (12, 6)$. The only root satisfying $\alpha(f) = 2$ is $\alpha_4 = 2\alpha_1 + \alpha_2$. Therefore, $e_+ = k\lambda_4$ and equation (3.27) is trivial, while (3.28) gives

$$k = \sqrt{6}. \tag{4.5}$$

Therefore, the Spaghetti solution is determined by the matrices

$$T_1^{(3)} = \frac{\sqrt{6}}{2}(\lambda_4 + \tilde{\lambda}_4), \tag{4.6}$$

$$T_2^{(3)} = \frac{\sqrt{6}}{2i}(\lambda_4 - \tilde{\lambda}_4), \tag{4.7}$$

$$T_3^{(3)} = 6J_1 + 3J_2. \tag{4.8}$$

The action of $\pm\alpha_6$ on the small hexagon shows that if we choose to work with the **7**, then $\mathbb{R}^7$ decomposes as $\mathbf{3} \oplus \mathbf{2} \oplus \mathbf{2}$ under χ_3, so that is an $SU(2)$ type solution.
For **14**, we see that the action of $\pm\alpha_4$ decomposes $\mathbb{R}^{14}$ into $\mathbf{4} \oplus \mathbf{4} \oplus \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$, which is again an $SU(2)$ type solution, since it contains even dimensional subrepresentations.

4.1.3. χ_4-Spaghetti

Since $\chi_4 = (0, 2)$, we get $p \equiv (p_1, p_2) = (12, 8)$. This time there are four roots satisfying the condition $\alpha(f) = 2$, which are $\alpha_2, \alpha_3, \alpha_4$ and α_5. Thus, we can put $e_+ = \sum_{j=2}^{5} k_j \lambda_j$ and $e_- = \sum_{j=2}^{5} k_j \tilde{\lambda}_j$. Using the results of Appendix D, we see that equations (3.27) and (3.28) become

$$\frac{1}{\sqrt{2}} k_2 k_3 + \sqrt{\frac{2}{3}} k_3 k_4 + \frac{1}{\sqrt{2}} k_4 k_5 = 0, \tag{4.9}$$

$$k_3^2 + 2k_4^2 = 3k_5^2 = 12, \tag{4.10}$$

$$k_2^2 + k_3^2 + k_4^2 + k_5^2 = 8. \tag{4.11}$$

There are several solutions of this system, but we know that we just need to find one. A very simple choice is

$$k_3 = k_4 = 0, \qquad k_2 = k_5 = 2. \tag{4.12}$$

Therefore, the Spaghetti solution is determined by the matrices

$$T_1^{(4)} = \lambda_2 + \tilde{\lambda}_2 + \lambda_5 + \tilde{\lambda}_5, \tag{4.13}$$

$$T_2^{(4)} = -i(\lambda_2 - \tilde{\lambda}_2 + \lambda_5 - \tilde{\lambda}_5), \tag{4.14}$$

$$T_3^{(4)} = 6J_1 + 4J_2. \tag{4.15}$$

To understand the type of solution, we notice that the action of T_1 and T_2 leave invariant the subspaces $\langle \lambda_3, \tilde{\lambda}_1, \tilde{\lambda}_4 \rangle$ and $\langle \tilde{\lambda}_3, \lambda_1, \lambda_4 \rangle$, so that in the representation **7**, $\mathbb{R}^7$ decomposes as $\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1}$. We see that it is a $SO(3)$-type solution.
Starting from the adjoint, we see that the action $\tilde{\lambda}_2 + \tilde{\lambda}_5$ applied repeatedly to λ_6 generates a combination of λ_2 and λ_5, then an element of H, then a combination of $\tilde{\lambda}_2$ and $\tilde{\lambda}_5$, and finally $\tilde{\lambda}_6$. This shows that working with **14**, $\mathbb{R}^{14}$ decomposes as $\mathbf{5} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1}$. Again, it is an $SO(3)$ type solution.

4.1.4. χ_{28}-Spaghetti

This is the principal case, with $\chi_{28} = (2, 2)$. Therefore $p \equiv (p_1, p_2) = (36, 20)$. We already know the solution in this case. The Spaghetti solution is

$$T_1^{(28)} = 3(\lambda_1 + \tilde{\lambda}_1) + \sqrt{5}(\lambda_2 + \tilde{\lambda}_2), \tag{4.16}$$

$$T_2^{(28)} = -i3(\lambda_1 - \tilde{\lambda}_1) - i\sqrt{5}(\lambda_2 - \tilde{\lambda}_2), \tag{4.17}$$

$$T_3^{(28)} = 18J_1 + 10J_2. \tag{4.18}$$

Because of Proposition 2, we already know that working in the adjoint the solution is of $SO(3)$-type. In the representation **7**, it is sufficient to verify that for $T_- = 3\tilde{\lambda}_1 + \sqrt{5}\tilde{\lambda}_2$, and v the maximal vector of **7**, then the vectors $\rho_7^k(T_-)(v)$, $k = 0, \ldots, 6$ are all linearly independent. Here $\rho_7 : G_2 \to End(\mathbb{R}^7)$ is the representation map of the algebra. This is proved in Appendix D and proves that $\mathbb{R}^7$ is irreducible under χ_{28}. Since it is odd dimensional, it is of $SO(3)$-type.

4.2. G_2 exceptional Lasagna

For the exceptional Lasagna we can use Proposition 2. Since we already know that $n = b$ must be equal to 1, we get that $(p_1, p_2) = (18, 10)$, and, if we fix $\psi_j = 0$ for simplicity, then

$$\kappa = T_1^{(28)} = 3(\lambda_1 + \tilde{\lambda}_1) + \sqrt{5}(\lambda_2 + \tilde{\lambda}_2). \tag{4.19}$$

Moreover, from Proposition 1 we get

$$h(z) = \frac{z}{2} T_3^{(28)}. \tag{4.20}$$

This defines the simplest exceptional G_2 Lasagna.

5. Extended ansatz

In order to allow for further generalizations, it is convenient to employ the Euler parameterization in a more general ansatz, after fixing the matrices κ and f. Let us consider the Skyrmionic field[1]

$$U(t,r,\phi,\gamma) = e^{\Phi(t,r,\phi,\gamma)\kappa} e^{\chi(t,r,\phi,\gamma) f} e^{\Theta(t,r,\phi,\gamma)\kappa}, \tag{5.1}$$

where κ is specified in (2.2), and f has the same properties as in (2.21). One of the aims of this generalization is to provide a description of different *pasta states* without specifying them *a priori*. This could lead to a comprehensive description of Skyrmions in a finite volume and to an analytical definition of other possible states (such as *gnocchi states*) and the transitions between them. In this work, we did not analyze all these possibilities and all the limits of these models, but we outline the main properties which characterize them, namely the wave equations, the topological charge and the energy density. If we define

$$\alpha = \frac{1}{2}(\Theta - \Phi) \qquad \text{and} \qquad \xi = \frac{1}{2}(\Phi + \Theta), \tag{5.2}$$

then

$$U(t,r,\phi,\gamma) = e^{-\alpha(t,r,\phi,\gamma)\kappa} e^{\xi(t,r,\phi,\gamma)\kappa} e^{\chi(t,r,\phi,\gamma) f} e^{\xi(t,r,\phi,\gamma)\kappa} e^{\alpha(t,r,\phi,\gamma)\kappa}. \tag{5.3}$$

This gives

$$\mathcal{L}_\mu = e^{-\alpha\kappa} e^{-\xi\kappa} \left[\partial_\mu \alpha(\kappa - \hat{\kappa}) + \partial_\mu \xi(\kappa + \hat{\kappa}) + \partial_\mu \chi f\right] e^{\xi\kappa} e^{\alpha\kappa}, \tag{5.4}$$

where we introduced the matrix function

$$\hat{\kappa} = e^{-\chi f} \kappa e^{\chi f}. \tag{5.5}$$

Since

$$\mathrm{tr}(\lambda_j \lambda_k) = 0, \qquad \mathrm{tr}(\lambda_j \tilde{\lambda}_k) = -\delta_{jk}, \tag{5.6}$$

we have

$$\mathrm{tr}(\kappa^2) = \mathrm{tr}(\hat{\kappa}^2) = -2\|c\|^2, \tag{5.7}$$

and f can be normalized so that

$$\mathrm{tr}(f^2) = \mathrm{tr}(\kappa^2). \tag{5.8}$$

This leads to the condition (2.24) and, in particular, $|c_j|^2 = \frac{b}{2} p_j$. Using these conventions we can now write the Skyrme equation explicitly.

[1] In this section we will use the coordinates $\{t, r, \phi, \gamma\}$, however the results are applicable for both the lasagna and the spaghetti phases.

5.1. Non-linear wave equations

We call wave equations to the field equations for the functions α, ξ and χ. These result to be

$$
\begin{aligned}
&\partial_\mu\partial^\mu\chi\left\{1+b^2\lambda\left[\partial_\mu\alpha\partial^\mu\alpha\sin^2\left(\frac{b\chi}{2}\right)+\partial_\mu\xi\partial^\mu\xi\cos^2\left(\frac{b\chi}{2}\right)\right]\right\}\\
&-b\sin(b\chi)\left(1-\frac{b^2\lambda}{4}\partial_\mu\chi\partial^\mu\chi\right)\left(\partial_\nu\alpha\partial^\nu\alpha-\partial_\nu\xi\partial^\nu\xi\right)\\
&-b^3\lambda\sin(b\chi)\cos(b\chi)\left[\partial_\mu\alpha\partial^\mu\alpha\partial_\nu\xi\partial^\nu\xi-\left(\partial_\mu\alpha\partial^\mu\xi\right)^2\right]\\
&-b^2\lambda\left\{\sin^2\left(\frac{b\chi}{2}\right)\partial_\mu\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\chi+\cos^2\left(\frac{b\chi}{2}\right)\partial_\mu\partial^\mu\xi\partial_\nu\xi\partial^\nu\chi\right.\\
&+\sin^2\left(\frac{b\chi}{2}\right)\left[\partial_\mu\alpha\partial^\mu\left(\partial_\nu\alpha\partial^\nu\chi\right)-\partial_\mu\chi\partial^\mu\left(\partial_\nu\alpha\partial^\nu\alpha\right)\right]\\
&\left.+\cos^2\left(\frac{b\chi}{2}\right)\left[\partial_\mu\xi\partial^\mu\left(\partial_\nu\xi\partial^\nu\chi\right)-\partial_\mu\chi\partial^\mu\left(\partial_\nu\xi\partial^\nu\xi\right)\right]\right\}\\
&-\frac{b^3\lambda}{4}\sin(b\chi)\left[\left(\partial_\mu\alpha\partial^\mu\chi\right)^2-\left(\partial_\mu\xi\partial^\mu\chi\right)^2\right]=0,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
&4\cos\left(\frac{b\chi}{2}\right)\left\{\cos\left(\frac{b\chi}{2}\right)\left\{\partial_\mu\partial^\mu\alpha\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]\right.\right.\\
&\left.-\frac{b^2\lambda}{4}\left[\partial_\mu\partial^\mu\chi\partial_\nu\chi\partial^\nu\alpha+\partial_\mu\chi\partial^\mu\left(\partial_\nu\chi\partial^\nu\alpha\right)-\partial_\mu\alpha\partial^\mu\left(\partial_\nu\chi\partial^\nu\chi\right)\right]\right\}\\
&-\sin\left(\frac{b\chi}{2}\right)\left\{\frac{b^2\lambda}{2}\sin(b\chi)\partial_\mu\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\xi\right.\\
&-\frac{b^2\lambda}{2}\sin(b\chi)\left[\partial_\mu\partial^\mu\xi\partial_\nu\alpha\partial^\nu\alpha+\partial_\mu\xi\partial^\mu\left(\partial_\nu\alpha\partial^\nu\alpha\right)-\partial_\mu\alpha\partial^\mu\left(\partial_\nu\alpha\partial^\nu\xi\right)\right]\\
&\left.\left.+b^3\lambda\cos(b\chi)\left[\partial_\mu\chi\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\xi-\partial_\mu\chi\partial^\mu\xi\partial_\nu\alpha\partial^\nu\alpha\right]+b\partial_\mu\chi\partial^\mu\xi\right\}\right\}\\
&+4\sin\left(\frac{b\chi}{2}\right)\left\{\sin\left(\frac{b\chi}{2}\right)\left\{\partial_\mu\partial^\mu\xi\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]\right.\right.\\
&\left.-\frac{b^2\lambda}{4}\left[\partial_\mu\partial^\mu\chi\partial_\nu\chi\partial^\nu\xi+\partial_\mu\chi\partial^\mu\left(\partial_\nu\chi\partial^\nu\xi\right)-\partial_\mu\xi\partial^\mu\left(\partial_\nu\chi\partial^\nu\chi\right)\right]\right\}\\
&-\cos\left(\frac{b\chi}{2}\right)\left\{\frac{b^2\lambda}{2}\sin(b\chi)\partial_\mu\partial^\mu\xi\partial_\nu\alpha\partial^\nu\xi\right.\\
&-\frac{b^2\lambda}{2}\sin(b\chi)\left[\partial_\mu\partial^\mu\alpha\partial_\nu\xi\partial^\nu\xi+\partial_\mu\alpha\partial^\mu\left(\partial_\nu\xi\partial^\nu\xi\right)-\partial_\mu\xi\partial^\mu\left(\partial_\nu\alpha\partial^\nu\xi\right)\right]\\
&\left.\left.+b^3\lambda\cos(b\chi)\left[\partial_\mu\chi\partial^\mu\xi\partial_\nu\alpha\partial^\nu\xi-\partial_\mu\chi\partial^\mu\alpha\partial_\nu\xi\partial^\nu\xi\right]-b\partial_\mu\chi\partial^\mu\alpha\right\}\right\}=0,
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
&4\sin\left(\frac{b\chi}{2}\right)\Bigg\{\cos\left(\frac{b\chi}{2}\right)\Big\{\partial_\mu\partial^\mu\alpha\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]\\
&-\frac{b^2\lambda}{4}\left[\partial_\mu\partial^\mu\chi\partial_\nu\chi\partial^\nu\alpha+\partial_\mu\chi\partial^\mu\left(\partial_\nu\chi\partial^\nu\alpha\right)-\partial_\mu\alpha\partial^\mu\left(\partial_\nu\chi\partial^\nu\chi\right)\right]\Big\}\\
&+\sin\left(\frac{b\chi}{2}\right)\Big\{\frac{b^2\lambda}{2}\sin(b\chi)\partial_\mu\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\xi\\
&-\frac{b^2\lambda}{2}\sin(b\chi)\left[\partial_\mu\partial^\mu\xi\partial_\nu\alpha\partial^\nu\alpha+\partial_\mu\xi\partial^\mu\left(\partial_\nu\alpha\partial^\nu\alpha\right)-\partial_\mu\alpha\partial^\mu\left(\partial_\nu\alpha\partial^\nu\xi\right)\right]\\
&+b^3\lambda\cos(b\chi)\left[\partial_\mu\chi\partial^\mu\alpha\partial_\nu\alpha\partial^\nu\xi-\partial_\mu\chi\partial^\mu\xi\partial_\nu\alpha\partial^\nu\alpha\right]+b\partial_\mu\chi\partial^\mu\xi\Big\}\Bigg\}\\
&-4\cos\left(\frac{b\chi}{2}\right)\Bigg\{\sin\left(\frac{b\chi}{2}\right)\Big\{\partial_\mu\partial^\mu\xi\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]\\
&-\frac{b^2\lambda}{4}\left[\partial_\mu\partial^\mu\chi\partial_\nu\chi\partial^\nu\xi+\partial_\mu\chi\partial^\mu\left(\partial_\nu\chi\partial^\nu\xi\right)-\partial_\mu\xi\partial^\mu\left(\partial_\nu\chi\partial^\nu\chi\right)\right]\Big\}\\
&+\cos\left(\frac{b\chi}{2}\right)\Big\{\frac{b^2\lambda}{2}\sin(b\chi)\partial_\mu\partial^\mu\xi\partial_\nu\alpha\partial^\nu\xi\\
&-\frac{b^2\lambda}{2}\sin(b\chi)\left[\partial_\mu\partial^\mu\alpha\partial_\nu\xi\partial^\nu\xi+\partial_\mu\alpha\partial^\mu\left(\partial_\nu\xi\partial^\nu\xi\right)-\partial_\mu\xi\partial^\mu\left(\partial_\nu\alpha\partial^\nu\xi\right)\right]\\
&+b^3\lambda\cos(b\chi)\left[\partial_\mu\chi\partial^\mu\xi\partial_\nu\alpha\partial^\nu\xi-\partial_\mu\chi\partial^\mu\alpha\partial_\nu\xi\partial^\nu\xi\right]-b\partial_\mu\chi\partial^\mu\alpha\Big\}\Bigg\}=0. \qquad (5.11)
\end{aligned}
$$

5.2. Energy density

The energy-momentum tensor takes the form

$$
\begin{aligned}
T_{\mu\nu}=&\frac{K}{2}\|c\|^2\Bigg\{8\left[\partial_\mu\alpha\partial_\nu\alpha\sin^2\left(\frac{b\chi}{2}\right)+\partial_\mu\xi\partial_\nu\xi\cos^2\left(\frac{b\chi}{2}\right)\right]+2\partial_\mu\chi\partial_\nu\chi\\
&-g_{\mu\nu}4\left[\partial_\rho\alpha\partial^\rho\alpha\sin^2\left(\frac{b\chi}{2}\right)+\partial_\rho\xi\partial^\rho\xi\cos^2\left(\frac{b\chi}{2}\right)\right]-g_{\mu\nu}\partial_\rho\chi\partial^\rho\chi\Bigg\}\\
&+\frac{K}{2}\|c\|^2(2b^2\lambda)\Big\{\Big[\partial_\mu\xi\partial_\nu\xi\partial_\rho\alpha\partial^\rho\alpha+\partial_\mu\alpha\partial_\nu\alpha\partial_\rho\xi\partial^\rho\xi\\
&-\left(\partial_\mu\alpha\partial_\nu\xi+\partial_\mu\xi\partial_\nu\alpha\right)\partial_\rho\alpha\partial^\rho\xi\Big]\sin^2(b\chi)\\
&-\left[\partial_\mu\alpha\partial_\nu\alpha\sin^2\left(\frac{b\chi}{2}\right)+\partial_\mu\xi\partial_\nu\xi\cos^2\left(\frac{b\chi}{2}\right)\right]\partial_\rho\chi\partial^\rho\chi\\
&-\left[\partial_\rho\alpha\partial^\rho\alpha\sin^2\left(\frac{b\chi}{2}\right)+\partial_\rho\xi\partial^\rho\xi\cos^2\left(\frac{b\chi}{2}\right)\right]\partial_\mu\chi\partial_\nu\chi\\
&-\sin^2\left(\frac{b\chi}{2}\right)\left(\partial_\mu\alpha\partial_\nu\chi+\partial_\mu\chi\partial_\nu\alpha\right)\partial_\rho\alpha\partial^\rho\chi\\
&-\cos^2\left(\frac{b\chi}{2}\right)\left(\partial_\mu\xi\partial_\nu\chi+\partial_\mu\chi\partial_\nu\xi\right)\partial_\rho\xi\partial^\rho\chi\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{K}{2}\|c\|^2 b^2 \lambda g_{\mu\nu}\Bigg\{\left[\partial_\rho\xi\,\partial^\rho\xi\,\partial_\sigma\alpha\,\partial^\sigma\alpha-\left(\partial_\rho\alpha\,\partial^\rho\xi\right)^2\right]\sin^2(b\chi)\\
&+\left[\partial_\rho\alpha\,\partial^\rho\alpha\,\sin^2\left(\frac{b\chi}{2}\right)+\partial_\rho\xi\,\partial^\rho\xi\,\cos^2\left(\frac{b\chi}{2}\right)\right]\partial_\sigma\chi\,\partial^\sigma\chi\\
&-\left[\left(\partial_\rho\alpha\,\partial^\rho\chi\right)^2\sin^2\left(\frac{b\chi}{2}\right)+\left(\partial_\rho\xi\,\partial^\rho\chi\right)^2\cos^2\left(\frac{b\chi}{2}\right)\right]\partial_\sigma\chi\,\partial^\sigma\chi\Bigg\}.
\end{aligned}
\tag{5.12}
$$

From this, we can obtain the energy density as $\rho_E = T_{tt}$.

5.3. Baryon charge

The Baryon charge is

$$
B=\frac{1}{24\pi^2}\int \rho_B\, dr\, d\gamma\, d\phi, \tag{5.13}
$$

with

$$
\rho_B=-12\|c\|^2\varepsilon^{ijk}\partial_i\alpha\,\partial_j\xi\,\partial_k\cos(b\chi). \tag{5.14}
$$

Up to now we have just written local expressions, but in order to compute the Baryonic charge it is necessary to define the ranges of α, ξ and χ. Proposition 2 tells us that the period of $e^{g\kappa}$ is $T_\kappa=\eta n\frac{2\pi}{b}$, where $\eta=1,2$ depending on the representation, while $n\in\mathbb{Z}$. Following [46], the ranges must be

$$
0\le\alpha\le\eta\sigma n\frac{2\pi}{b},\qquad 0\le\chi\le\frac{\pi}{b}\qquad\text{and}\qquad 0\le\xi\le\eta m\frac{2\pi}{b}, \tag{5.15}
$$

where $\sigma=1$ for odd-dimensional representations and $\frac{1}{2}$ for even-dimensional representations and m, n are both integer. The integration of the density charge leads to

$$
B=2mn\sigma\eta^2\frac{\|c\|^2}{b^2}. \tag{5.16}
$$

We can compute the ratio $\frac{\|c\|^2}{b^2}$ in the following way. From (5.8), we get

$$
-2\|c\|^2=\mathrm{Tr}(f^2)=\sum_{j=1}^{r}p_j\,\mathrm{Tr}(h_j f), \tag{5.17}
$$

where the definition $f=\sum_{j=1}^{r}p_j h_j$ has been used. Now, we can replace the coefficients p_j with (2.23) and $\mathrm{Tr}(h_j f)=\alpha_j(f)=ib$ to get

$$
\|c\|^2=\sum_{j=1}^{r}p_j\,\mathrm{Tr}(h_j f)=\sum_{j=1}^{r}\frac{b^2}{\|\alpha_j\|^2}\sum_{k=1}^{r}(C^G)^{-1}_{jk}. \tag{5.18}
$$

The Baryon charge takes the form

$$
B=2mn\sigma\eta^2\sum_{j,k=1}^{r}\frac{1}{\|\alpha_j\|^2}(C^G)^{-1}_{jk}. \tag{5.19}
$$

5.4. Example: the Lasagna case

Let us now compare the results obtained in this section with the previous ones. Our quantities can be written in terms of the Lasagna ansatz as follows

$$\alpha = -\frac{\sigma t}{2L_\phi} + \sigma\frac{\phi}{2} + m\frac{\gamma}{2} \qquad \text{and} \qquad \xi = \frac{\sigma t}{2L_\phi} - \sigma\frac{\phi}{2} + m\frac{\gamma}{2}. \tag{5.20}$$

Moreover, the *profile function* depends only on the parameter r ($\chi = \chi(r)$). This leads to the following relations

$$\begin{aligned}
&\partial_\mu\chi\partial^\mu\alpha = \partial_\mu\chi\partial^\mu\xi = 0, \qquad \partial_\mu\partial^\mu\alpha = \partial_\mu\partial^\mu\xi = 0\\
&\partial_\mu\alpha\partial^\mu\alpha = \partial_\mu\xi\partial^\mu\xi = \partial_\mu\alpha\partial^\mu\xi = \frac{1}{4L_\gamma^2}.
\end{aligned}$$

With these choices the equations (5.10) and (5.11) are automatically satisfied. The equation (5.9) becomes

$$\chi''(r)\left(1 + \frac{b^2\lambda}{4L_\gamma^2}\right) = 0, \tag{5.21}$$

which leads to the solution

$$\chi(r) = \frac{r}{2b}, \tag{5.22}$$

where the boundary conditions $\chi(0) = 0$ and $\chi(2\pi) = \frac{\pi}{b}$ have been used. Now, it is easy to compute the energy density, which results

$$\rho_E = \frac{K}{2}\|c\|^2\left\{\frac{2}{L_\phi^2} + \frac{1}{L_\gamma^2} + \frac{1}{4b^2L_r^2}\left(1 + \frac{b^2\lambda}{4L_\gamma^2}\right) + \frac{b^2\lambda}{8L_\phi^2L_\gamma^2}\left[4\sin^2\left(\frac{r}{2}\right) - 1\right]\right\}. \tag{5.23}$$

The integration over the volume of the box gives the total energy of the Lasagna

$$E = 4L_\phi L_r L_\gamma \pi^3 K\frac{\|c\|^2}{b^2}\left\{\frac{2}{L_\phi^2} + \frac{1}{L_\gamma^2} + \frac{1}{4b^2L_r^2}\left(1 + \frac{b^2\lambda}{4L_\gamma^2}\right) + \frac{b^2\lambda}{8L_\phi^2L_\gamma^2}\right\}. \tag{5.24}$$

6. Coupling with $U(1)$ gauge field

By employing the generalization presented in the previous section, it is now easy to couple the Skyrmion field to an electromagnetic field A_μ. To this aim we introduce the action

$$\mathcal{A} = \int d^4x\sqrt{-g}\,\mathrm{Tr}\left[\frac{K}{2}\left(\hat{\mathcal{L}}_\mu\hat{\mathcal{L}}^\mu + \frac{\lambda}{8}\hat{G}_{\mu\nu}\hat{G}^{\mu\nu}\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{6.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.2}$$

and the *hat* stands for the replacement of the partial derivative with a covariant derivative

$$D_\mu = \partial_\mu - A_\mu[T, \cdot], \tag{6.3}$$

which means that

$$\hat{\mathcal{L}}_\mu = U^{-1} D_\mu U = U^{-1}\left(\partial_\mu U - A_\mu [T, U]\right) \qquad \text{and} \qquad \hat{G}_{\mu\nu} = \left[\hat{\mathcal{L}}_\mu, \hat{\mathcal{L}}_\nu\right]. \tag{6.4}$$

Here T is any element of the Lie algebra of the group G, representing the direction of the $U(1)$ gauge field. Later, we will identify T with the generator T_3. The action (6.1) is now invariant under gauge transformation

$$U \to e^{-\beta T} U e^{\beta T}, \qquad A_\mu \to A_\mu + \partial_\mu \beta. \tag{6.5}$$

The gauge invariance appears also in the fact that the theory depends on A_μ through the quantity $\partial_\mu \alpha - A_\mu$, which is invariant for gauge transformations.

6.1. Covariant Baryonic charge

As in [49], in order to determine a topological invariant, one is tempted to start directly generalizing (1.7) to the expression

$$\hat{B} = \frac{1}{24\pi^2} \int_{\mathcal{V}} \mathrm{Tr}[\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}],$$

which, however, is not a topological invariant if the field-strength F is non-vanishing. Nevertheless, a topological invariant can be constructed after a simple subtraction, even for a non-Abelian gauge field. Indeed, we have:

Proposition 4. *Let $\mathcal{S}$ be a three dimensional closed compact manifold,*

$$U : \mathcal{S} \longrightarrow G \tag{6.6}$$

a differentiable map from S to the Lie group G, $\hat{\mathcal{L}}_\mu = U^{-1} D_\mu U$, $\hat{\mathcal{R}}_\mu = D_\mu U U^{-1}$, with a non necessarily Abelian connection ω, and Ω the curvature of ω,

$$D_\mu U = \partial_\mu U + [\omega, U], \tag{6.7}$$

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{6.8}$$

Hence

$$\hat{B} = \frac{1}{24\pi^2} \int_{\mathcal{S}} \mathrm{Tr}\left[\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} - 3\hat{\mathcal{L}} \wedge \Omega - 3\hat{\mathcal{R}} \wedge \Omega\right], \tag{6.9}$$

is a topological invariant. Moreover, if $H_2(\mathcal{S}) = 0$ and A is Abelian, then $\hat{B} = B$.

Proof. In order to prove the proposition, we have to prove that the first variation of $\hat{B}$ w.r.t. U and ω (independently) vanishes at any functional point, that is independently if U and ω are constrained by some equations of motion. Notice that in taking variations, $\delta\omega$ is a well defined 1-form on $\mathcal{S}$ despite ω could not be. To keep notation compact we will use bold round brackets to indicate a trace $\boldsymbol{(}M\boldsymbol{)} \equiv \mathrm{Tr(M)}$. Moreover, we first recall the following properties. If a_j, $j = 1, \dots, k$ are Lie algebra valued 1-forms then

$$\boldsymbol{(}a_1 \wedge \cdots \wedge a_{k-1} \wedge a_k\boldsymbol{)} = (-1)^{k-1} \boldsymbol{(}a_k \wedge a_1 \wedge \cdots \wedge a_{k-1}\boldsymbol{)}. \qquad \text{(cyclicity (c))} \tag{6.10}$$

If [,] indicates the Lie product (commutator) of matrix valued forms and a, b, c are three differential forms of degree k_a k_b and k_c respectively, then

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{k_a k_b} b \wedge [a, c]. \qquad \text{(graded algebraic derivative (gad))} \tag{6.11}$$

Since

$$([a, b]) = 0, \tag{6.12}$$

in particular, we have that, if a is a 1-form, then

$$([a, b] \wedge c) = (-1)^{k_b} (b \wedge [a, c]), \qquad \text{(algebraic integration by parts (aip))} \tag{6.13}$$

and

$$(Db) = (db). \qquad \text{(algebraic trivialization (at))}. \tag{6.14}$$

Using these properties, taking a variation δU of U we can write

$$\begin{aligned}\frac{1}{3}\delta_U(\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) =& (\delta_U \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) = (-U^{-1}\delta U \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} + U^{-1} D\delta U \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) \\ =& (-U^{-1}\delta U \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} + D(U^{-1}\delta U) \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} - D(U^{-1})\delta U \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) \\ =& (D(U^{-1}\delta U) \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}})\end{aligned} \tag{6.15}$$

where we used $D(U^{-1}) = -U^{-1}DUU^{-1}$ and cyclicity in the last term of the second line. Hence,

$$\begin{aligned}\frac{1}{3}\delta_U(\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) =& (D[U^{-1}\delta g \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}]) - (U^{-1}\delta U[U^{-1}D(DU) \wedge \hat{\mathcal{L}} - \hat{\mathcal{L}} \wedge U^{-1}D(DU)]) \\ =& d(U^{-1}\delta g \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) - (U^{-1}\delta U[U^{-1}[\Omega, U] \wedge \hat{\mathcal{L}} - \hat{\mathcal{L}} \wedge U^{-1}[\Omega, U]),\end{aligned} \tag{6.16}$$

where we used (at) in the first term and $D(DU) = [\Omega, U]$ in the other ones. Now, let us consider

$$\begin{aligned}\delta_U(\Omega \wedge \hat{\mathcal{L}}) =& -(\Omega \wedge U^{-1}\delta U \hat{\mathcal{L}}) + (\Omega \wedge U^{-1} D\delta U) \\ =& -(\delta U U^{-1}(D\Omega)U^{-1}) + (D[\Omega U^{-1}\delta U]) + (\Omega \wedge \hat{\mathcal{L}} U^{-1}\delta U) \\ =& -(\delta U \hat{\mathcal{L}} \wedge \Omega U^{-1}) + d(\Omega U^{-1}\delta U) + (\Omega \wedge \hat{\mathcal{L}} U^{-1}\delta U),\end{aligned} \tag{6.17}$$

where again we used (c) and (at). In the same way

$$\delta_U(\Omega \wedge \hat{\mathcal{R}}) = -(\delta U g^{-1}\Omega \wedge \hat{\mathcal{R}}) + d(\delta U U^{-1}\Omega) + (\delta U U^{-1}\hat{\mathcal{R}} \wedge \Omega). \tag{6.18}$$

Subtracting (6.17) and (6.18) to (6.16), and multiplying times 3, we get

$$\delta_U(\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} - 3\hat{\mathcal{L}} \wedge \Omega - 3\hat{\mathcal{R}} \wedge \Omega) = 3d(U^{-1}\delta U \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} - \Omega U^{-1}\delta_U - \Omega \delta U U^{-1}). \tag{6.19}$$

Since the r.h.s. is the differential of a globally well defined 2-form and $\mathcal{S}$ is a smooth closed compact manifold, it follows from Stokes theorem that the first variation of $\hat{B}$ under variation of U vanishes.

As a second step, let us consider a variation $\delta\omega$ of ω. The strategy is the same as above. For

$$\mu \equiv \frac{1}{3}(\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} - 3\hat{\mathcal{L}} \wedge \Omega - 3\hat{\mathcal{R}} \wedge \Omega) \tag{6.20}$$

we get

$$\begin{aligned}\delta_\omega\mu =&\big(U^{-1}[\delta\omega, U]\wedge\hat{\mathcal{L}}\wedge\hat{\mathcal{L}}\big) - \big(U^{-1}[\delta\omega, U]\wedge\Omega\big) - \big(\hat{\mathcal{L}}\wedge(d\delta\omega + [\omega,\delta\omega])\big)\\ &- \big([\delta\omega, U]U^{-1}\wedge\Omega\big) - \big(\hat{\mathcal{R}}\wedge(d\delta\omega + [\omega,\delta\omega])\big)\\ =&\big(U^{-1}[\delta\omega, U]\wedge\hat{\mathcal{L}}\wedge\hat{\mathcal{L}}\big) - \big(U^{-1}[\delta\omega, U]\wedge\Omega\big) + d\big(\hat{\mathcal{L}}\wedge\delta\omega\big) - \big(D\hat{\mathcal{L}}\wedge\delta\omega\big)\\ &- \big([\delta\omega, U]U^{-1}\wedge\Omega\big) + d\big(\hat{\mathcal{R}}\wedge\delta\omega\big) - \big(D\hat{\mathcal{R}}\wedge\delta\omega\big).\end{aligned} \tag{6.21}$$

Now,

$$\begin{aligned}D\hat{\mathcal{L}} &= d(U^{-1}DU) + [\omega, U^{-1}DU] = -\hat{\mathcal{L}}\wedge\hat{\mathcal{L}} + U^{-1}(d[\omega, U] + [\omega,[\omega, U]] - \omega\wedge dU)\\ &= -\hat{\mathcal{L}}\wedge\hat{\mathcal{L}} + U^{-1}[\Omega, U],\end{aligned} \tag{6.22}$$

and similarly

$$D\hat{\mathcal{R}} = \hat{\mathcal{R}}\wedge\hat{\mathcal{R}} + [\Omega, U]U^{-1}. \tag{6.23}$$

Finally, noticing that

$$\big(U^{-1}[\delta\omega, U]\wedge\hat{\mathcal{L}}\wedge\hat{\mathcal{L}}\big) = \big(\delta\omega\wedge\hat{\mathcal{R}}\wedge\hat{\mathcal{R}}\big) - \big(\delta\omega\wedge\hat{\mathcal{L}}\wedge\hat{\mathcal{L}}\big) \tag{6.24}$$

and putting all together, we get

$$\delta_\omega\mu = d\big((\hat{\mathcal{L}} + \hat{\mathcal{R}}\wedge\delta\omega)\big), \tag{6.25}$$

which, as above, it proves invariance also under variations of the connection. Thus, $\hat{B}$ is topological invariant.
Now, we have to prove the second part. To this end it is convenient to introduce some further notation. After fixing a basis $\{T_a\}_a$ of $Lie(G)$, with structure constants $f^{\,a}_{\,bc}$ defined by[2]

$$[T_b, T_c] = f^{\,a}_{\,bc}T_a, \tag{6.26}$$

it is convenient to define

$$\tilde{T}_a := U^{-1}T_a U, \tag{6.27}$$

$$\tau_a := T_a - \tilde{T}_a, \tag{6.28}$$

$$\check{\tau}_a := T_a + \tilde{T}_a, \tag{6.29}$$

so that, writing $\omega = \omega^a\tau_a$, we have

$$\hat{\mathcal{L}} = \mathcal{L} - \omega^a\tau_a \tag{6.30}$$

and also

$$\big(\hat{\mathcal{L}}\wedge\Omega\big) + \big(\hat{\mathcal{R}}\wedge\Omega\big) = \Omega^a\wedge\big(\hat{\mathcal{L}}\check{\tau}_a\big). \tag{6.31}$$

Therefore,

$$\begin{aligned}&\big(\hat{\mathcal{L}}\wedge\hat{\mathcal{L}}\wedge\hat{\mathcal{L}}\big) - 3\big((\hat{\mathcal{L}} + \hat{\mathcal{R}})\wedge\Omega\big)\\ &\quad = \big(\mathcal{L}\wedge\mathcal{L}\wedge\mathcal{L}\big) - 3\omega^a\wedge\big(\tau_a\mathcal{L}\wedge\mathcal{L}\big) + 3\omega^a\wedge\omega^b\wedge\big(\tau_a\tau_b\mathcal{L}\big)\\ &\qquad - \omega^a\wedge\omega^b\wedge\omega^c\big(\tau_a\tau_b\tau_c\big) - 3\Omega^a\wedge\big(\mathcal{L}\check{\tau}_a\big) + 3\Omega^a\wedge\omega^b\big(\tau_b\check{\tau}_a\big).\end{aligned} \tag{6.32}$$

By the Maurer-Cartan equation $d\mathcal{L} = -\frac{1}{2}[\mathcal{L},\mathcal{L}]$, and we can write,

[2] We use the Einstein's convention on sums.

$$\begin{aligned}-\omega^a\wedge(\tau_a\mathcal{L}\wedge\mathcal{L})-\Omega^a\wedge(\mathcal{L}\check{\tau}_a)&=\omega^a\wedge(\tau_a d\mathcal{L})-d\omega^a\wedge(\mathcal{L}\check{\tau}_a)-\frac{1}{2}f^a_{\ bc}\omega^b\wedge\omega^c\wedge(\mathcal{L}\check{\tau}_a)\\&=\omega^a\wedge(\tau_a d\mathcal{L})-d[\omega^a\wedge(\mathcal{L}\check{\tau}_a)]-\omega^a\wedge d(\mathcal{L}\check{\tau}_a)-\frac{1}{2}f^a_{\ bc}\omega^b\wedge\omega^c\wedge(\mathcal{L}\check{\tau}_a),\end{aligned}\tag{6.33}$$

which, using

$$\begin{aligned}\omega^a\wedge(\tau_a d\mathcal{L})-\omega^a\wedge d(\mathcal{L}\check{\tau}_a)&=-2\omega^a\wedge(\tilde{T}_a d\mathcal{L})-\omega^a\wedge(\mathcal{L}\wedge d\tilde{T}_a)\\&=-2\omega^a\wedge(\tilde{T}_a d\mathcal{L})-\omega^a\wedge(\mathcal{L}\wedge(-[\mathcal{L},\tilde{T}_a])=0,\end{aligned}\tag{6.34}$$

because of the Maurer-Cartan equations, becomes

$$-\omega^a\wedge(\tau_a\mathcal{L}\wedge\mathcal{L})-\Omega^a\wedge(\mathcal{L}\check{\tau}_a)=-d[\omega^a\wedge(\mathcal{L}\check{\tau}_a)]-\frac{1}{2}f^a_{\ bc}\omega^b\wedge\omega^c\wedge(\mathcal{L}\check{\tau}_a).\tag{6.35}$$

Next, we rewrite

$$\begin{aligned}\omega^a\wedge\omega^b\wedge(\tau_a\tau_b\mathcal{L})=&\omega^a\wedge\omega^b\wedge(T_aT_b\mathcal{L})+\omega^a\wedge\omega^b\wedge(\tilde{T}_a\tilde{T}_b\mathcal{L})-\omega^a\wedge\omega^b\wedge(T_a\tilde{T}_b\mathcal{L})\\&-\omega^a\wedge\omega^b\wedge(\tilde{T}_aT_b\mathcal{L})\\=&\frac{1}{2}\omega^a\wedge\omega^b\wedge([T_a,T_b]\mathcal{L})+\frac{1}{2}\omega^a\wedge\omega^b\wedge([\tilde{T}_a,\tilde{T}_b]\mathcal{L})\\&-\omega^a\wedge\omega^b\wedge([T_a,\tilde{T}_b]\mathcal{L})\\=&\frac{1}{2}\omega^a\wedge\omega^b\wedge(f^c_{\ ab}T_c\mathcal{L})+\frac{1}{2}\omega^a\wedge\omega^b\wedge(f^c_{\ ab}\tilde{T}_c\mathcal{L})\\&-\omega^a\wedge\omega^b\wedge(T_a[\tilde{T}_b,\mathcal{L}])\\=&\frac{1}{2}f^c_{\ ab}\omega^a\wedge\omega^b\wedge(\check{\tau}_c\mathcal{L})-\omega^a\wedge\omega^b\wedge(T_a d\tilde{T}_b)\\=&\frac{1}{2}f^c_{\ ab}\omega^a\wedge\omega^b\wedge(\check{\tau}_c\mathcal{L})-d[\omega^a\wedge\omega^b\wedge(T_a\tilde{T}_b)]\\&+d[\omega^a\wedge\omega^b]\wedge(T_a\tilde{T}_b)\\=&\frac{1}{2}f^c_{\ ab}\omega^a\wedge\omega^b\wedge(\check{\tau}_c\mathcal{L})-d[\omega^a\wedge\omega^b\wedge(T_a\tilde{T}_b)]\\&+d\omega^a\wedge\omega^b\wedge(T_a\tilde{T}_b-\tilde{T}_aT_b).\end{aligned}\tag{6.36}$$

Since $(\tilde{T}_a\tilde{T}_b)=(T_aT_b)$, we also have

$$\Omega^a\wedge\omega^b(\tau_b\check{\tau}_a)=-\Omega^a\wedge\omega^b(T_a\tilde{T}_b-\tilde{T}_aT_b).\tag{6.37}$$

Finally, using also $(\tilde{T}_a\tilde{T}_b\tilde{T}_c)=(T_aT_bT_c)$,

$$\begin{aligned}\omega^a\wedge\omega^b\wedge\omega^c(\tau_a\tau_b\tau_c)=&3\omega^a\wedge\omega^b\wedge\omega^c(\tilde{T}_a\tilde{T}_bT_c-T_aT_b\tilde{T}_c)\\=&\frac{3}{2}\omega^a\wedge\omega^b\wedge\omega^c([\tilde{T}_a,\tilde{T}_b]T_c-[T_a,T_b]\tilde{T}_c)\\=&\frac{3}{2}\omega^a\wedge\omega^b\wedge\omega^c f^d_{\ ab}(\tilde{T}_dT_c-T_d\tilde{T}_c).\end{aligned}\tag{6.38}$$

After replacing (6.35), (6.36), (6.37) and (6.38) in (6.32) we get

$$(\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) - 3((\hat{\mathcal{L}} + \hat{\mathcal{R}}) \wedge \Omega) = (\mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L}) - 3d[\omega^a \wedge (\mathcal{L}\check{\tau}_a) + \omega^a \wedge \omega^b \wedge (T_a \tilde{T}_b)]. \tag{6.39}$$

In particular, if the connection is Abelian,

$$(\hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \wedge \hat{\mathcal{L}}) - 3((\hat{\mathcal{L}} + \hat{\mathcal{R}}) \wedge \Omega) = (\mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L}) - 3d[\omega^a \wedge (\mathcal{L}\check{\tau}_a)]. \tag{6.40}$$

In this case $\Omega = d\omega$ and, if $H_2(\mathbb{S}) = 0$ so that $H^2(\mathbb{S}) = 0$, then Ω is exact and ω is well defined everywhere on $\mathcal{S}$. Therefore, $3d[\omega^a \wedge (\mathcal{L}\check{\tau}_a)]$ is an exact form and Stokes theorem ensures that under these hypotheses $\hat{B} = B$. □

Notice that with our conventions in (6.4), we have to make the identifications

$$\omega = -A, \qquad \Omega = -F, \qquad F = dA - \frac{1}{2}[A, A]\,. \tag{6.41}$$

From (6.39) we then see that

$$\hat{B} = \frac{1}{24\pi^2} \int_{\mathcal{S}} \hat{\rho}_B \, dr d\gamma d\phi, \tag{6.42}$$

with

$$\hat{\rho}_B = \rho_B + 3\varepsilon^{ijk} \partial_i [A^a_j \mathrm{Tr}(\mathcal{L}_k (T_a + U^{-1} T_a U)) - A^a_j A^b_k \mathrm{Tr}(T_a U^{-1} T_b U)], \tag{6.43}$$

where, according to the conventions in [46], the orientation of the coordinates is such that $\varepsilon^{r\gamma\phi} = 1$. In our case $\mathcal{S}$ is a closed three dimensional manifold in a semisimple compact Lie group G. Since in this case $H_2(G, \mathbb{Q}) = 0$, we get that the correction to the density does not contribute to the integral and we expect $\hat{B} = B$ always.

It is worth to mention that in the construction of the solutions of the Skyrme equations, however, $\mathcal{S}$ is replaced by $\mathcal{V}$ that is compact but it is not a closed smooth manifold but a hyperrectangle with boundary. Therefore, the above integral does not define a topological invariant unless we impose suitable boundary conditions. To understand which are the most suitable ones, let us first analyze the case $A = 0$. In this case the map U maps the hyperrectangle in a closed smooth submanifold of G, so $\hat{\mathcal{L}}$ is the pull-back of a 1-form well defined on a closed compact manifold (indeed, the left-invariant Maurer-Cartan form) and this is the reason we get a topological invariant. This suggests the boundary conditions we are looking for. They have to be imposed so that also A is the pull-back of a well defined 1-form over G (or the image of the hyperrectangle in G).

Under these conditions, the quantities A_μ are not independent, due to the fact that Φ, Θ and χ defines a map $M : \mathbb{R}^{3+1} \mapsto \mathbb{R}^3$. Locally, the embedding takes the form $A = A_\Phi d\Phi + A_\Theta d\Theta + A_\chi d\chi$, equivalent to

$$A_\mu = A_\Phi \partial_\mu \Phi + A_\Theta \partial_\mu \Theta + A_\chi \partial_\mu \chi\,. \tag{6.44}$$

6.2. Example: Lasagna states coupled to an electromagnetic field

To be explicit, we now work out the example of Lasagna states. For this case we choose $T = \kappa$. The covariant derivative determines the coupling of the gauge field to the Skyrmions, which appears in the definition of $\hat{\mathcal{L}}_\mu$

$$\hat{\mathcal{L}}_\mu = e^{-\alpha\kappa} e^{-\xi\kappa} \left[\left(\partial_\mu \alpha - A_\mu\right)(\kappa - \hat{\kappa}) + \partial_\mu \xi (\kappa + \hat{\kappa}) + \partial_\mu \chi f\right] e^{\xi\kappa} e^{\alpha\kappa}. \tag{6.45}$$

Notice that the introduction of the gauge field in the direction κ causes a shift in $\mathcal{L}_\mu$ given by $\partial_\mu\alpha \to \partial_\mu\alpha - A_\mu$. It results that all the quantities we computed in the previous section are shifted by this quantity when the Skyrmions are coupled to a Maxwell field and it is really easy to convert the uncoupled theory with the coupled one. The covariant Baryon density charge now becomes

$$\begin{aligned}\hat{\rho}_B &= \rho_B + 3\varepsilon^{ijk}\partial_i[A_j \mathrm{Tr}(\mathcal{L}_k(\kappa + U^{-1}\kappa U))] \\ &= \rho_B + 3\varepsilon^{ijk}\partial_i \mathrm{Tr}\left[\left(\partial_k\alpha(\kappa-\hat{\kappa}) + \partial_k\xi(\kappa+\hat{\kappa}) + \partial_k\chi f\right)(\kappa+\hat{\kappa})\right].\end{aligned} \tag{6.46}$$

Using that

$$\mathrm{Tr}[(\kappa-\hat{\kappa})(\kappa+\hat{\kappa})] = \mathrm{Tr}[f(\kappa+\hat{\kappa})] = 0, \tag{6.47}$$

and

$$\mathrm{Tr}[(\kappa-\hat{\kappa})(\kappa+\hat{\kappa})] = 2(1+\cos(b\chi))\mathrm{Tr}\kappa^2 = -4\|c\|^2(1+\cos(b\chi)), \tag{6.48}$$

$$\hat{\rho}_B = \rho_B - 12\|c\|^2\varepsilon^{ijk}\partial_i\left[A_j\partial_k\xi(1+\cos(b\chi))\right], \tag{6.49}$$

where ρ_B is the uncoupled density. The correction to ρ_B is a total derivative, so it depends only on the boundary conditions, as discussed above. Differently from [49], our system lives in a box, so, the electromagnetic field is not constrained to zero at the boundaries. Therefore, the Baryonic charge is not necessarily a topological invariant and not even expected to be an integer. As we said above, we can fix this problem by requiring for A to be the pull-back of a well defined potential over the homology cycle of G selected by the map U. This is easily accomplished by looking at the form of the ansatz for the Lasagna states. As t is irrelevant, we fix $t=0$ to simplify the expressions:

$$U(r,\gamma,\phi) = e^{-\phi\sigma\kappa}e^{\chi(r)f}e^{m\gamma\kappa} = e^{-\phi\sigma\kappa}e^{m\gamma\hat{\kappa}(r)}e^{\chi(r)f}. \tag{6.50}$$

Since $b\chi(2\pi)=\pi$, we see that $\hat{\kappa}(0) = -\hat{\kappa}(2\pi) = \kappa$, so that, if, for a generic fixed r, $U(r,\gamma,\phi)$ defines a two dimensional surface in G, for $r=0,2\pi$ it collapses down to one dimensional circles:

$$U(0,\gamma,\phi) = e^{(m\gamma-\sigma\phi)\kappa} = e^{\xi\kappa}, \tag{6.51}$$

$$U(2\pi,\gamma,\phi) = e^{-(m\gamma+\sigma\phi)\kappa}e^{\chi(2\pi)f} = e^{-\alpha\kappa}e^{\chi(2\pi)f}. \tag{6.52}$$

This degeneration means that well defined 1-forms on the whole manifold must have components only along the direction on the degeneration submanifolds, which in our case means

$$A_\alpha(r=0) = 0, \tag{6.53}$$

$$A_\xi(r=2\pi) = 0, \tag{6.54}$$

which in the original coordinates becomes

$$\frac{1}{2m}A_\gamma(r=0) + \frac{1}{2\sigma}A_\phi(r=0) = 0, \tag{6.55}$$

$$\frac{1}{2m}A_\gamma(r=2\pi) - \frac{1}{2\sigma}A_\phi(r=2\pi) = 0. \tag{6.56}$$

Also, one between ϕ and γ has to be identified periodically, while the other one is periodic or "antiperiodic"[3] according to the cases if the cycle is of $SO(3)$ or $SU(2)$, respectively. Therefore,

[3] The antiperiodicity is not exact and in general some matrix components are periodic and other are antiperiodic. However, what happens is that points are identified in the image, in such a way to respect orientation, so the corresponding differential forms are periodically identified.

in any case, the 1-forms in the image of the embedding have to be periodically identified so that the integrals at the "boundaries" $\phi = 0$ and $\phi = 2\pi$ cancel out and the same happens for the boundaries at $\gamma = 0$ and $\gamma = 2\pi$. So, the only boundaries that may contribute are the ones at $r = 0$ and $r = 2\pi$, which we collectively call $\partial^r B$. Therefore, the Baryonic charge results

$$\begin{aligned}\hat{B} =& B - \frac{\|c\|^2}{2\pi^2} \int_{\partial B} (1 + \cos(b\chi)) A \wedge d\xi \\ =& B + \frac{\|c\|^2}{2\pi^2} \int_{\partial^r B} (1 + \cos(b\chi))(\sigma A_\gamma + m A_\phi) d\gamma \wedge d\phi \\ =& B - \frac{\|c\|^2}{\pi^2} \int_{[0,2\pi]\times[0,2\pi]} (\sigma A_\gamma(0,\gamma,\phi) + m A_\phi(0,\gamma,\phi)) d\gamma d\phi = B \end{aligned} \tag{6.57}$$

because of the above boundary conditions, and we used that $A_\alpha = \sigma A_\gamma + m A_\phi$.

6.3. Decoupling of Skyrme equations and free-force conditions

To the Skyrmion equation coupled to a Maxwell field, obtained by shifting $\partial_\mu \alpha \to \partial_\mu \alpha - A_\mu$ in (5.9), (5.10) and (5.11), we have to add the Maxwell equations, which are given by

$$\nabla_\nu F^{\nu\mu} - \mathrm{Tr}\left\{ \frac{K}{2} D \left(\hat{\mathcal{R}}^\mu + \frac{\lambda}{4} \left[\hat{\mathcal{R}}_\nu, \hat{G}^{\mu\nu} \right] \right) \right\} = 0. \tag{6.58}$$

In the generic Euler parameterization, they become

$$\begin{aligned}&\partial_\nu \partial^\nu A_\mu - \partial_\mu \left(\partial_\nu \partial^\nu \alpha \right) - \frac{K}{2} \|c\|^2 \Big\{ \left(\partial_\mu \alpha - A_\mu \right) \Big[8 \sin^2 \left(\frac{a\chi}{2} \right) \left(1 + \frac{a^2 \lambda}{4} \partial_\nu \chi \partial^\nu \chi \right) \\ &\quad + 2a^2 \lambda \sin^2(a\chi) \partial_\nu \xi \partial^\nu \xi \Big] - 2a^2 \lambda \left[\partial_\mu \xi (\partial_\nu \alpha - A_\nu) \partial^\nu \xi + \partial_\mu \chi (\partial_\nu \alpha - A_\nu) \partial^\nu \chi \right] \Big\} = 0.\end{aligned} \tag{6.59}$$

To look for explicit solutions, we aim to decouple the Skyrme equations from the Maxwell field. Since A_μ appears in the products $\left(\partial_\mu \alpha - A_\mu\right)(\partial^\mu \alpha - A^\mu)$, $\left(\partial_\mu \alpha - A_\mu\right) \partial^\mu \xi$ and $\left(\partial_\mu \alpha - A_\mu\right) \partial^\mu \chi$ and in the derivative $\partial_\mu (\partial^\mu \alpha - A^\mu)$, we can separate the Skyrme equations from the rest by looking for solutions where these terms are a priori fixed functions

$$\begin{aligned}&\left(\partial_\mu \alpha - A_\mu\right)\left(\partial^\mu \alpha - A^\mu\right) = f(t,r,\theta,\phi), \qquad \partial_\mu \left(\partial^\mu \alpha - A^\mu\right) = g(t,r,\theta,\phi), \\ &\left(\partial_\mu \alpha - A_\mu\right) \partial^\mu \xi = p(t,r,\theta,\phi), \qquad \left(\partial_\mu \alpha - A_\mu\right) \partial^\mu \chi = q(t,r,\theta,\phi).\end{aligned} \tag{6.60}$$

Recall that, the quantity $\partial_\mu \alpha - A_\mu$ is gauge invariant.

6.3.1. Free-force conditions

Due to gauge invariance, we can introduce the new gauge field $\tilde{A}_\mu = A_\mu - \partial_\mu \alpha$. Imposing the conditions

$$f(t,r,\theta,\phi) = g(t,r,\theta,\phi) = p(t,r,\theta,\phi) = q(t,r,\theta,\phi) = 0 \quad \text{and} \quad \tilde{A}^\nu \partial_\nu \tilde{A}_\mu = 0, \tag{6.61}$$

the so called *free-force conditions* are satisfied [27], namely

$$\tilde{F}_{\mu\nu} J^\nu = 0, \qquad \tilde{J}^\nu = \nabla_\rho \tilde{F}^{\rho\nu}, \tag{6.62}$$

where $\tilde{F}_{\mu\nu}$ is the field-strength of $\tilde{A}_\mu$. The wave equations become

$$\begin{aligned}
&\partial_\mu\partial^\mu\chi\left[1+b^2\lambda\partial_\nu\xi\partial^\nu\xi\cos^2\left(\frac{b\chi}{2}\right)\right]+b\sin(b\chi)\left(1-\frac{b^2\lambda}{4}\partial_\mu\chi\partial^\mu\chi\right)\partial_\nu\xi\partial^\nu\xi\\
&\quad-b^2\lambda\Big\{\cos^2\left(\frac{b\chi}{2}\right)\partial_\mu\partial^\mu\xi\partial_\nu\xi\partial^\nu\chi\\
&\quad+\cos^2\left(\frac{b\chi}{2}\right)\left[\partial_\mu\xi\partial^\mu\left(\partial_\nu\xi\partial^\nu\chi\right)-\partial_\mu\chi\partial^\mu\left(\partial_\nu\xi\partial^\nu\xi\right)\right]\Big\}\\
&\qquad+\frac{b^3\lambda}{4}\sin(b\chi)\left(\partial_\mu\xi\partial^\mu\chi\right)^2=0, \qquad (6.63)\\
&4b\cos\left(\frac{b\chi}{2}\right)\partial_\mu\chi\partial^\mu\xi-4\sin\left(\frac{b\chi}{2}\right)\Big\{\partial_\mu\partial^\mu\xi\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]\\
&\qquad-\frac{b^2\lambda}{4}\left[\partial_\mu\partial^\mu\chi\partial_\nu\chi\partial^\nu\xi+\partial_\mu\chi\partial^\mu\left(\partial_\nu\chi\partial^\nu\xi\right)-\partial_\mu\xi\partial^\mu\left(\partial_\nu\chi\partial^\nu\chi\right)\right]\Big\}=0, \qquad (6.64)\\
&4b\sin\left(\frac{b\chi}{2}\right)\partial_\mu\chi\partial^\mu\xi-4\cos\left(\frac{b\chi}{2}\right)\Big\{\partial_\mu\partial^\mu\xi\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]\\
&\qquad-\frac{b^2\lambda}{4}\left[\partial_\mu\partial^\mu\chi\partial_\nu\chi\partial^\nu\xi+\partial_\mu\chi\partial^\mu\left(\partial_\nu\chi\partial^\nu\xi\right)-\partial_\mu\xi\partial^\mu\left(\partial_\nu\chi\partial^\nu\chi\right)\right]\Big\}=0. \qquad (6.65)
\end{aligned}$$

The last two equations imply that

$$\partial_\mu\chi\partial^\mu\xi=0, \qquad (6.66)$$

so

$$\partial_\mu\partial^\mu\xi\left[1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right]=0 \qquad (6.67)$$

gives $\partial_\mu\partial^\mu\xi=0$. With these solutions, the Eq. (6.63) takes the simpler form

$$\partial_\mu\partial^\mu\chi\left[1+b^2\lambda\partial_\nu\xi\partial^\nu\xi\cos^2\left(\frac{b\chi}{2}\right)\right]+b\sin(b\chi)\left(1-\frac{b^2\lambda}{4}\partial_\mu\chi\partial^\mu\chi\right)\partial_\nu\xi\partial^\nu\xi=0. \qquad (6.68)$$

We can also apply them to the Maxwell equations

$$\begin{aligned}
&\partial_\nu\partial^\nu A_\mu-\partial_\mu\left(\partial_\nu\partial^\nu\alpha\right)+\frac{K}{2}\|c\|^2\left(\partial_\mu\alpha-A_\mu\right)\Big[8\sin^2\left(\frac{b\chi}{2}\right)\left(1+\frac{b^2\lambda}{4}\partial_\nu\chi\partial^\nu\chi\right)\\
&\qquad+2b^2\lambda\sin^2(b\chi)\partial_\nu\xi\partial^\nu\xi\Big]=0.
\end{aligned} \qquad (6.69)$$

In particular, the energy density is

$$\begin{aligned}
\rho_E=4K\|c\|^2\Big\{&\left[\tilde{A}_t^2\sin^2\left(\frac{b\chi}{2}\right)+(\partial_t\xi)^2\cos^2\left(\frac{b\chi}{2}\right)\right]\left(1-\frac{b^2\lambda}{4}\partial_\rho\chi\partial^\rho\chi\right)\\
&+\frac{b^2\lambda}{4}\tilde{A}_t^2\partial_\rho\xi\partial^\rho\xi\sin^2(b\chi)\\
&+\left[1-b^2\lambda\partial_\rho\xi\partial^\rho\xi\cos^2\left(\frac{b\chi}{2}\right)\right]\frac{(\partial_t\chi)^2}{4}
\end{aligned} \qquad (6.70)$$

$$-\frac{g_{tt}}{2}\partial_\rho\xi\,\partial^\rho\xi\,\cos^2\left(\frac{b\chi}{2}\right)\left(1+\frac{b^2\lambda}{4}\partial_\sigma\chi\,\partial^\sigma\chi\right)-\frac{g_{tt}}{8}\partial_\rho\chi\,\partial^\rho\chi\Big\}$$

6.4. Example: Lasagna, again

We can use the results of this section in order to study the behavior of Lasagna when coupled to the $U(1)$ gauge field. To simplify the results, we use the free-force conditions

$$\begin{aligned}&\left(\partial_\mu\alpha-A_\mu\right)\left(\partial^\mu\alpha-A^\mu\right)=0, \qquad \partial_\mu\left(\partial^\mu\alpha-A^\mu\right)=0,\\ &\left(\partial_\mu\alpha-A_\mu\right)\partial^\mu\xi=0, \qquad \left(\partial_\mu\alpha-A_\mu\right)\partial^\mu\chi=0,\end{aligned} \tag{6.71}$$

and for the gauge field we make the ansatz

$$A_\mu=\left(A_t(r),0,A_\gamma(r),A_\phi(r)\right). \tag{6.72}$$

If we simply shift the gauge field, we can write

$$\tilde{A}_\mu\tilde{A}^\mu=0, \qquad \partial_\mu\tilde{A}^\mu=0, \qquad \tilde{A}_\mu\partial^\mu\xi=0, \qquad \tilde{A}_\mu\partial^\mu\chi=0. \tag{6.73}$$

These conditions are easily solved by using (5.20) (which also apply to $\tilde{A}_\mu$) together with (6.72). This leads to the solution $\tilde{A}_t=-\frac{\tilde{A}_\phi}{L_\phi}$ and $\tilde{A}_\gamma=0$ ($A_\gamma=\frac{m}{2}$); so, only one gauge field results to be independent, for instance we can take $\tilde{A}_\phi$. Thus, the wave equations and the Maxwell equation become

$$\frac{\chi''}{L_r^2}\left[1+\frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)\right]+\frac{b}{4L_\gamma^2}\sin(b\chi)\left(1-\frac{b^2\lambda}{4L_r^2}\chi'^2\right)=0, \tag{6.74}$$

$$\frac{\tilde{A}''_\phi}{L_r^2}+\frac{K}{2}\|c\|^2\tilde{A}_\phi\Big[8\sin^2\left(\frac{b\chi}{2}\right)\left(1+\frac{b^2\lambda}{4L_r^2}\chi'^2\right)+\frac{b^2\lambda}{2L_\gamma^2}\sin^2(b\chi)\Big]=0. \tag{6.75}$$

The first equation can be rewritten as

$$\frac{d}{dr}\left\{\frac{\chi'^2}{2L_r^2}\left[1+\frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)\right]-\frac{1}{2L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)\right\}=0, \tag{6.76}$$

so that

$$\chi'^2(r)=L_r^2\frac{M+\frac{1}{2L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)}{1+\frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)}, \tag{6.77}$$

where M is an integration constant. This determines the boundary values of χ' from the ones of χ

$$\chi'^2(0)=L_r^2\frac{M+\frac{1}{2L_\gamma^2}}{1+\frac{b^2\lambda}{4L_\gamma^2}}, \tag{6.78}$$

$$\chi'^2(2\pi)=L_r^2M. \tag{6.79}$$

Vice versa, we can write M in terms of $\chi'(0)$

$$M = \frac{\chi'^2(0)}{L_r^2}\left(1+\frac{b^2\lambda}{4L_\gamma^2}\right) - \frac{1}{2L_\gamma^2}, \tag{6.80}$$

so that

$$\chi'^2(2\pi) = \chi'^2(0)\left(1+\frac{b^2\lambda}{4L_\gamma^2}\right) - \frac{L_r^2}{2L_\gamma^2}. \tag{6.81}$$

We can put this result into the Maxwell equations, getting

$$\frac{\tilde{A}_\phi''}{L_r^2} + \frac{K}{2}\|c\|^2\tilde{A}_\phi \sin^2\left(\frac{b\chi}{2}\right) V_M(\chi) = 0, \tag{6.82}$$

where

$$V_M(\chi) = 8\left[1+\frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)\right] + 2b^2\lambda\frac{M+\frac{1}{2L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)}{1+\frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)}. \tag{6.83}$$

From (6.78) we can locally write r in terms of χ as

$$r(\chi) = \pm\frac{1}{L_r}\int_0^\chi \sqrt{\frac{1+\frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\hat{\chi}}{2}\right)}{M+\frac{1}{2L_\gamma^2}\cos^2\left(\frac{b\hat{\chi}}{2}\right)}}\,d\hat{\chi}, \tag{6.84}$$

which leads to a definition of $\tilde{A}_\phi$ in terms of χ, let us call $B(\chi) = \tilde{A}_\phi(r(\chi))$. This way, equation (6.82) can be entirely written in terms of χ

$$B''(\chi) - \frac{b^2\lambda}{16L_\gamma^2}B'(\chi)\sin(b\chi)\frac{1-\hat{M}+D_1(\chi)}{D_1(\chi)D_{\hat{M}}(\chi)} + K\|c\|^2 b^2\lambda B(\chi)\sin^2\left(\frac{b\chi}{2}\right)\frac{2D_1^2(\chi)+D_{\hat{M}}(\chi)}{D_{\hat{M}}(\chi)} = 0, \tag{6.85}$$

where a prime indicates derivative with respect to χ, and the following quantities have been introduced

$$\hat{M} = M\frac{b^2\lambda}{2}, \qquad D_a = a + \frac{b^2\lambda}{4L_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right). \tag{6.86}$$

Replacing

$$B(\chi) = C(\chi)\exp\int_0^\chi \frac{b^2\lambda}{32L_\gamma^2}\sin(b\hat{\chi})\frac{1-\hat{M}+D_1(\hat{\chi})}{D_1(\hat{\chi})D_{\hat{M}}(\hat{\chi})}d\hat{\chi}, \tag{6.87}$$

in (6.85), we get

$$C''(\chi) + W_{\hat{M}}(\chi)C(\chi) = 0, \tag{6.88}$$

with

$$
\begin{aligned}
W_{\hat{M}}(\chi) =& \frac{b^2\lambda}{32L_\gamma^2}\Bigg[\cos(b\chi)\frac{1-\hat{M}+D_1(\hat{\chi})}{D_1(\hat{\chi})D_{\hat{M}}(\hat{\chi})} \\
&\qquad + \frac{b\lambda}{8L_\gamma^2}\sin^2(b\chi)\frac{1}{D_{\hat{M}}^2}\left(1+\frac{1-\hat{M}}{D_1}\right)\left(b+1+\frac{1-\hat{M}}{D_1}\right)\Bigg] \\
&+ K\|c\|^2 b^2\lambda \sin^2\left(\frac{b\chi}{2}\right)\frac{2D_1^2(\chi)+D_{\hat{M}}(\chi)}{D_{\hat{M}}(\chi)}.
\end{aligned}
\tag{6.89}
$$

We can use

$$
\frac{1}{D_a} = \frac{1}{a}\sum_{n=0}^{\infty}\left[-\frac{b^2\lambda}{4aL_\gamma^2}\cos^2\left(\frac{b\chi}{2}\right)\right]^n, \tag{6.90}
$$

and

$$
\cos^{2n}(x) = 2^{-2n}\left\{\frac{(2n)!}{(n!)^2}+2\sum_{s=0}^{n-1}\binom{2n}{s}\cos\left[2(n-s)x\right]\right\}. \tag{6.91}
$$

to rewrite the potential (6.89) in series of $\cos(kb\chi)$, where k runs on all integers, and recognize (6.88) as a Hill equation. Notice that putting

$$
x = b\chi/2, \tag{6.92}
$$

we have that x must vary in the interval $[0, \pi/2]$. Also, we have seen that for the Lasagna states such interval must be one quarter of the period over the cycle, which means that the solution we are looking for must be periodic with period 2π as a function of x. Therefore, it is worth mentioning the following result [44] (see also [50]):

Proposition 5. *Let $y_1(x)$ and $y_2(x)$ be the solutions of the Hill equation*

$$
y'' + f(x)y = 0, \tag{6.93}
$$

where $f(x)$ is an even function of the form

$$
f(x) = 2\sum_{n=1}^{\infty} f_n\cos(2nx), \qquad \sum_{n=1}^{\infty}|f_n| < \infty, \tag{6.94}
$$

with Cauchy conditions

$$
y_1(0) = 1, \qquad y_1'(0) = 0, \tag{6.95}
$$

$$
y_2(0) = 0, \qquad y_2'(0) = 0. \tag{6.96}
$$

Then, equation (6.93) has:

1. *an even solution of period π if and only if $y_1'(\pi/2) = 0$;*
2. *an odd solution of period π if and only if $y_2(\pi/2) = 0$;*
3. *an even solution of period 2π if and only if $y_1(\pi/2) = 0$;*
4. *an odd solution of period 2π if and only if $y_2'(\pi/2) = 0$.*

Now, the boundary conditions for our Hill equation (6.82) are

$$\tilde{A}_\phi(0) = -m, \tag{6.97}$$
$$\tilde{A}_\phi(2\pi) = 0. \tag{6.98}$$

Therefore, if we call $y_{(j)}$, $j = 1, \dots, 4$ the solutions corresponding to the four points of the above proposition, if they exist, we get that the solutions of interest for us have the general form

$$\tilde{A}_\phi(r) = B(\chi) = -my_{(3)}(x) + \kappa y_{(2)}(x), \tag{6.99}$$

for κ an arbitrary constant. The question on the existence of such solutions is investigated in [44]. For example, the existence of solution $y_{(2)}$ is granted if and only if ω takes values for which the determinant of the infinite dimensional matrix

$$M_{ab} = \delta_{ab} + \frac{f_{a-b} - f_{a+b}}{\omega^2 - a^2}, \qquad a, b = 1, 2, 3, \dots, \tag{6.100}$$

vanishes.[4] However, for any practical purposes, such a way is impracticable for looking for explicit solutions in this very general case. Therefore, in place of pursuing this very general analysis, we move now to a particular but more tractable case.

6.4.1. Linear solution of the Skyrme equations

In the particular case when $M = \frac{2}{b^2\lambda}$, we can find a very simple solution:

$$\chi'^2(r) = \frac{2L_r^2}{b^2\lambda} \qquad \Rightarrow \qquad \chi(r) = \pm\sqrt{\frac{2L_r^2}{b^2\lambda}}r, \tag{6.101}$$

paying the price of fixing $\frac{L_r^2}{\lambda}$ (which corresponds to $\hat{M} = 1$). Indeed, the boundary conditions on χ give $\chi^2(2\pi) = \frac{2L_r^2}{b^2\lambda}4\pi^2 = \frac{\pi^2}{b^2}$. The Maxwell equation takes the form

$$\frac{\tilde{A}''_\phi}{L_r^2} + K\|c\|^2\tilde{A}_\phi \left[\left(3 + \frac{b^2\lambda}{8L_\gamma^2}\right) - 3\cos\left(\frac{r}{2}\right) - \frac{b^2\lambda}{8L_\gamma^2}\cos r \right] = 0, \tag{6.102}$$

which is a Whittaker-Hill equation [51,45], see also [52]. It is convenient to introduce the variable change $\hat{r} = \frac{r}{4}$, so $\hat{r}$ has range $0 \le \hat{r} \le \frac{\pi}{2}$. This way, equation (6.102) takes the canonical form

$$\tilde{B}''_\phi + 16KL_r^2\|c\|^2 \left[\left(3 + \frac{b^2\lambda}{8L_\gamma^2}\right) - 3\cos(2\hat{r}) - \frac{b^2\lambda}{8L_\gamma^2}\cos(4\hat{r}) \right] \tilde{B}_\phi = 0. \tag{6.103}$$

We can therefore determine the solutions $y_{(2)}$ and $y_{(3)}$, following [45]. Using the same notations of that paper, we can identify

$$\omega = 4\sqrt{K\lambda}L_r\|c\|\frac{b}{L_\gamma}, \tag{6.104}$$
$$\eta = 48KL_r^2\|c\|^2, \tag{6.105}$$
$$\rho = -12\sqrt{\frac{K}{\lambda}}L_rL_\gamma\frac{\|c\|}{b}. \tag{6.106}$$

[4] We use the notation $f_{a-b} = 0$ if $a - b \le 0$.

In particular, $\eta = -\omega\rho$. For the function $\phi_{(3)}$ we have to take (see [45])

$$\phi_{(3)}(x) = Re\left[e^{-\frac{i}{2}\omega\cos(2x)}\psi_{(3)}(x)\right], \tag{6.107}$$

where

$$\psi_{(3)}(x) = \sum_{n=0}^{\infty} A_n B_n \cos((2n+1)x), \tag{6.108}$$

with

$$A_0 = 1, \qquad A_n = \prod_{j=1}^{n}(\rho + 2ji), \quad j > 0, \tag{6.109}$$

and B_n solves the recursion relations

$$(2-\eta)B_0 - \frac{1}{\omega}(\eta^2 + 4\omega^2)B_1 = 0, \tag{6.110}$$

$$-\omega B_n + 2[(2n+1)^2 - \eta]B_{n+1} - \frac{1}{\omega}[\eta^2 + 4(n+1)^2]B_{n+2} = 0, \quad n \geq 1. \tag{6.111}$$

To find the periodic solution of period 2π, ω and η must be constrained by the following trascendental equation, expressed in terms of a continued fraction (see [45], formula (5.1))

$$1 - \frac{1}{2}\eta = \frac{\frac{1}{4}(\eta^2 + 4\omega^2)}{9-\eta-}\,\frac{\frac{1}{4}(\eta^2 + 16\omega^2)}{25-\eta-}\,\frac{\frac{1}{4}(\eta^2 + 36\omega^2)}{49-\eta-}\cdots, \tag{6.112}$$

which then gives the solution

$$\frac{B_n}{B_{n-1}} = \frac{\frac{1}{2}\omega}{(2n+1)^2 - \eta}\,\frac{\frac{1}{4}[\eta^2 + 4(n+1)^2\omega^2]}{(2n+3)^2 - \eta-}\cdots\frac{\frac{1}{4}[\eta^2 + 4(n+s-1)^2\omega^2]}{(2n+2s-1)^2 - \eta-}\cdots, \quad n \geq 1. \tag{6.113}$$

Finally, B_0 is fixed by the condition $\phi_{(3)}(0) = 1$.
As what concerns the solution $\kappa\phi_{(2)}(x)$, we have to consider

$$\kappa\phi_{(2)}(x) = Re\left[e^{-\frac{i}{2}\omega\cos(2x)}\psi_{(2)}(x)\right], \tag{6.114}$$

where

$$\psi_{(2)}(x) = \sum_{n=1}^{\infty} C_n D_n \sin(2nx), \tag{6.115}$$

with

$$C_1 = 1, \qquad C_n = \prod_{j=1}^{n-1}(\rho + (2j+1)i), \quad j > 1, \tag{6.116}$$

and D_n solves the recursion relations

$$(4-\eta)D_1 - \frac{1}{2\omega}(\eta^2 + 9\omega^2)D_2 = 0, \tag{6.117}$$

$$-\omega D_{n-1} + 2[4n^2 - \eta]D_n - \frac{1}{\omega}[\eta^2 + (2n+1)^2]D_{n+1} = 0, \quad n > 2. \tag{6.118}$$

To find the periodic solution of period π, ω and η must be constrained by the following trascendental equation, expressed in terms of a continued fraction (see [45], formula (5.3))

$$4-\eta=\frac{\frac{1}{4}(\eta^2+9\omega^2)}{16-\eta-}\,\frac{\frac{1}{4}(\eta^2+25\omega^2)}{36-\eta-}\,\frac{\frac{1}{4}(\eta^2+49\omega^2)}{64-\eta-}\cdots, \tag{6.119}$$

which then gives the solution

$$\frac{D_n}{D_{n-1}}=\frac{\frac{1}{2}\omega}{4n^2-\eta}\,\frac{\frac{1}{4}[\eta^2+(2n+1)^2\omega^2]}{4(n+1)^2-\eta-}\cdots\frac{\frac{1}{4}[\eta^2+(2n+2s-1)^2\omega^2]}{4(n+s-1)^2-\eta-}\cdots, \qquad n\geq 2. \tag{6.120}$$

Since κ is arbitrary, no normalization is required for D_1. However, notice that $\kappa\phi_{(2)}(x)$ can be considered only if equation (6.119) has common solutions with (6.112).

6.5. Example: Spaghetti states coupled to an electromagnetic field

In the case of Spaghetti, we do not use a parameterization of Euler type but the exponential parameterization of Section 3. Still, the analysis can be easily extended to this case. Following [53], the gauge field is described by

$$A^a_\mu T_a = A_\mu T_3. \tag{6.121}$$

Also in this case, the free-force conditions decouples the Skyrme equations from the Maxwell field. In particular, we take

$$A_\mu A^\mu=0, \qquad \partial_\mu A^\mu=0 \qquad A_\mu\partial^\mu\Phi=0, \qquad A_\mu\partial^\mu\Theta=0. \tag{6.122}$$

A reasonable explicit form of a gauge field with these properties is given by

$$A_\mu(t,r,\theta,\phi)=(A_t(r,\theta),0,0,-L_\phi A_t(r,\theta)). \tag{6.123}$$

From (6.121) we see that the equations of motion for the Skyrme field are

$$D^\mu\left(\hat{\mathcal{L}}_\mu+\frac{\lambda}{4}\hat{\mathcal{G}}_\mu\right)=0, \tag{6.124}$$

where $\hat{\mathcal{L}}_\mu$ is given by (6.4) with $T=T_3$, and

$$\hat{\mathcal{G}}_\mu=\left[\hat{\mathcal{L}}^\nu,\left[\hat{\mathcal{L}}_\mu,\hat{\mathcal{L}}_\nu\right]\right]. \tag{6.125}$$

Conditions (6.122) lead to

$$\partial^\mu\left(\mathcal{L}_\mu+\frac{\lambda}{4}\mathcal{G}_\mu\right)=0, \tag{6.126}$$

which are the uncoupled Skyrme equations. Notice that from (3.18) we get

$$\frac{d}{dr}\left[2\chi'^2\left(\lambda q^2\sin^2\left(\frac{\chi}{2}\right)+L_\theta^2\right)+4q^2L_r^2\cos\chi\right]=0, \tag{6.127}$$

which means that

$$2\chi'^2\left(\lambda q^2\sin^2\left(\frac{\chi}{2}\right)+L_\theta^2\right)+4q^2L_r^2\cos\chi=2Z, \tag{6.128}$$

where Z is a constant, which is equivalent to

$$\chi'^2(r) = \frac{Z - 2q^2 L_r^2 + 4q^2 L_r^2 \sin^2\left(\frac{\chi}{2}\right)}{L_\theta^2 + \lambda q^2 \sin^2\left(\frac{\chi}{2}\right)}. \tag{6.129}$$

The Maxwell equations for $A^\phi = -L_\phi A_t$ are

$$\frac{1}{L_r^2}\partial_r^2 A^\phi + \frac{1}{L_\theta^2}\partial_\theta^2 A^\phi + 2K I_{G,\rho} \sin^2(q\theta)\sin^2\left(\frac{\chi}{2}\right)\left(A^\phi - \frac{p}{L_\phi^2}\right)\left[1 + \frac{\chi'^2}{L_r^2} + \frac{4q^2}{L_\theta^2}\sin^2\left(\frac{\chi}{2}\right)\right] = 0. \tag{6.130}$$

This is a stationary Schrödinger equation with a double periodic potential of finite type. In particular, here one is interested in the zero eigenvalue case. Both the direct and inverse problem for this kind of equation is well studied and much more involuted than the one dimensional case (already highly non-trivial). Here we simply defer the reader to the literature (see [54] and reference therein), and limit ourselves to discuss the boundary conditions.

As we discussed in the previous sections, the boundary conditions on A_μ are outlined by the behavior of the Skyrme field in the edges of the box, namely (once again, we fix $t = 0$)

$$U(0,\theta,\phi) = 1, \tag{6.131}$$

$$U(2\pi,\theta,\phi) = e^{2\pi\tau_1}, \tag{6.132}$$

$$U(r,0,\phi) = e^{\chi T_3}, \tag{6.133}$$

$$U(r,\pi,\phi) = e^{\chi T_3}. \tag{6.134}$$

This requires the following constraints

$$A_\phi(0,\theta) = A_\phi(r,0) = A_\phi(r,\pi) = 0. \tag{6.135}$$

The contribution of the gauge field to the Baryonic density can be always computed from (6.43). This gives

$$\hat{\rho}_B = \rho_B + 3\partial_\theta\left[A_\phi \mathrm{Tr}\left(\mathcal{L}_r\left(T_3 + U^{-1}T_3U\right)\right)\right] - 3\partial_r\left[A_\phi \mathrm{Tr}\left(\mathcal{L}_\theta\left(T_3 + U^{-1}T_3U\right)\right)\right], \tag{6.136}$$

where $\mathcal{L}_r$ and $\mathcal{L}_\theta$ are specified in (3.9) and (3.16). We easily find

$$U^{-1}T_3U = T_3 + \sin(q\theta)\tau_2 - \sin(q\theta)\cos\chi\,\tau_2 + \sin(q\theta)\sin\chi\,\tau_3. \tag{6.137}$$

This leads to

$$\mathrm{Tr}(\mathcal{L}_\theta T_3) = \mathrm{Tr}(\mathcal{L}_\theta U^{-1}T_3U) = I_{G,\rho}\, q \sin\chi\,\sin(q\theta), \tag{6.138}$$

$$\mathrm{Tr}(\mathcal{L}_r T_3) = \mathrm{Tr}(\mathcal{L}_r U^{-1}T_3U) = -I_{G,\rho}\chi'\cos(q\theta). \tag{6.139}$$

Thus, we can check that the contribution to the Baryon charge becomes

$$\hat{B} = B + \frac{I_{G,\rho}}{4\pi}\int_0^{2\pi}\left(A_\phi(r,\pi) + A_\phi(r,0)\right)\chi' dr = B, \tag{6.140}$$

according to the boundary conditions specified above, and the general results following Proposition 4.

CRediT authorship contribution statement

All authors equally contributed to all phases in the preparation of the manuscript, checking all calculations and final reediting of the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Roots of simple algebras

Here we list the roots of all simple algebras.

A.1. $\boldsymbol{A_N}$

The corresponding complex algebra is $\mathfrak{sl}(N+1)$, while the compact form is $\mathfrak{su}(N+1)$. If e_a, $a = 1, \dots, N+1$, is the canonical basis[5] of $\mathbb{R}^{N+1}$, then the real linear space generated by the roots is isomorphic to a hyperplane in $\mathbb{R}^{N+1}$ in which all non-vanishing roots are represented by the vectors $\alpha_{j,k} = e_j - e_k$, $j \neq k$. The simple roots are $\alpha_j = e_j - e_{j+1}$, $j = 1, \dots, N$. If λ_j are the root matrices corresponding to the simple roots, then

$$[\lambda_j, \lambda_k] = 0 \quad \text{if} \quad j - k \neq \pm 1. \tag{A.1}$$

The split subalgebra is $\mathfrak{so}(N+1)$.

A.2. $\boldsymbol{B_N}$

The corresponding compact form is $\mathfrak{so}(2N+1)$. The real linear space generated by the roots is isomorphic to $\mathbb{R}^N$. If e_a, $a = 1, \dots, N$, is the canonical basis of $\mathbb{R}^N$, then all non-vanishing roots are represented by the vectors $e_j - e_k$, $j \neq k$, $\pm(e_j + e_k)$, $j < k$, $\pm e_j$. The simple roots are $\alpha_j = e_j - e_{j+1}$, $j = 1, \dots, N-1$ and $\alpha_N = e_N$. If λ_j are the root matrices corresponding to the simple roots, then

$$[\lambda_j, \lambda_k] = 0 \quad \text{if} \quad j - k \neq \pm 1. \tag{A.2}$$

The split subalgebra is $\mathfrak{so}(N) \oplus \mathfrak{so}(N+1)$.

[5] In the literature, the canonical basis e_a is also commonly denoted as L_a.

A.3. $\boldsymbol{C_N}$

The corresponding compact form is $\mathfrak{us}_\mathfrak{p}(2N)$, the compact symplectic algebra. The real linear space generated by the roots is isomorphic to $\mathbb{R}^N$. If e_a, $a = 1, \ldots, N$, is the canonical basis of $\mathbb{R}^N$, then all non-vanishing roots are represented by the vectors $e_j - e_k$, $j \neq k$, $\pm(e_j + e_k)$, $j < k$, $\pm 2e_j$. The simple roots are $\alpha_j = e_j - e_{j+1}$, $j = 1, \ldots, N-1$ and $\alpha_N = 2e_N$. If λ_j are the root matrices corresponding to the simple roots, then

$$[\lambda_j, \lambda_k] = 0 \quad \text{if} \quad j - k \neq \pm 1. \tag{A.3}$$

The split subalgebra is $\mathfrak{u}(N)$.

A.4. $\boldsymbol{D_N}$

The corresponding compact form is $\mathfrak{so}(2N)$. The real linear space generated by the roots is isomorphic to $\mathbb{R}^N$. If e_a, $a = 1, \ldots, N$, is the canonical basis of $\mathbb{R}^N$, then all non-vanishing roots are represented by the vectors $e_j - e_k$, $j \neq k$, $\pm(e_j + e_k)$, $j < k$. The simple roots are $\alpha_j = e_j - e_{j+1}$, $j = 1, \ldots, N-1$ and $\alpha_N = e_{N-1} + e_N$. If λ_j are the root matrices corresponding to the simple roots, the relevant non-vanishing commutators are

$$[\lambda_j, \lambda_{j+1}] \quad j = 1, \ldots, N-2, \qquad [\lambda_{N-2}, \lambda_N]. \tag{A.4}$$

The split subalgebra is $\mathfrak{so}(N) \oplus \mathfrak{so}(N)$.

A.5. $\boldsymbol{G_2}$

The corresponding compact form is $\mathfrak{g}_2$. The real linear space generated by the roots is isomorphic to a hyperplane in $\mathbb{R}^3$. If $e_a, a = 1, \ldots, 3$, is the canonical basis of $\mathbb{R}^3$, then all non-vanishing roots are represented by the vectors $e_j - e_k$, $j \neq k$, and $\pm(e_1 + e_2 + e_3 - 3e_s)$, s=1,2,3. The simple roots are $\alpha_1 = e_2 - e_3$ and $\alpha_2 = e_1 - 2e_2 + e_3$. If λ_j are the root matrices corresponding to the simple roots, the relevant non-vanishing commutator is $[\lambda_1, \lambda_2]$. The split subalgebra is $\mathfrak{so}(4)$.

A.6. $\boldsymbol{F_4}$

The corresponding compact form is $\mathfrak{f}_4$. The real linear space generated by the roots is isomorphic to $\mathbb{R}^4$. If $e_a, a = 1, \ldots, 4$, is the canonical basis of $\mathbb{R}^4$, then all non-vanishing roots are represented by the vectors $e_j - e_k$, $j \neq k$, $\pm(e_j + e_k)$, $j < k$, $\pm e_j$, $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. The simple roots are $\alpha_1 = e_2 - e_3$, $\alpha_1 = e_3 - e_4$, $\alpha_3 = e_4$, and $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$. If λ_j are the root matrices corresponding to the simple roots, the relevant non-vanishing commutators are

$$[\lambda_j, \lambda_{j+1}]. \tag{A.5}$$

The split subalgebra is $\mathfrak{us}_\mathfrak{p}(6) \oplus \mathfrak{us}_\mathfrak{p}(2)$.

A.7. $\boldsymbol{E_6}$

The corresponding compact form is $\mathfrak{e}_6$. The real linear space generated by the roots is isomorphic to $\mathbb{R}^6$. If $e_a, a = 1, \ldots, 6$, is the canonical basis of $\mathbb{R}^6$, then all non-vanishing roots are represented by the vectors $\pm(e_j - e_k)$, $j < k < 6$, $\pm(e_j + e_k)$, $j < k < 6$,

$$\pm\frac{1}{2}\{\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + \sqrt{6}\, e_6\},$$

where in the parenthesis only an even number of minus signs can appear. The simple roots are

$$\alpha_1 = \frac{1}{2}\{e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{6}\, e_6\}, \qquad \alpha_2 = e_1 + e_2, \qquad \alpha_k = e_{k-1} - e_{k-2}, \quad k = 3, \dots, 6.$$

If λ_j are the root matrices corresponding to the simple roots, the relevant non-vanishing commutators are

$$[\lambda_j, \lambda_{j+1}], \; j \neq 1, 2, \qquad [\lambda_1, \lambda_3], \quad [\lambda_2, \lambda_4]. \tag{A.6}$$

The split subalgebra is $\mathfrak{usp}(8)$.

A.8. $\boldsymbol{E_7}$

The corresponding compact form is $\mathfrak{e}_7$. The real linear space generated by the roots is isomorphic to $\mathbb{R}^7$. If e_a, $a = 1, \dots, 7$, is the canonical basis of $\mathbb{R}^7$, then all non-vanishing roots are represented by the vectors $\pm(e_j - e_k)$, $j < k < 7$, $\pm(e_j + e_k)$, $j < k < 7$, $\pm\sqrt{2}\, e_7$,

$$\pm\frac{1}{2}\{\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 + \sqrt{2}\, e_7\},$$

where in the parenthesis only an odd number of minus signs can appear. The simple roots are

$$\alpha_1 = \frac{1}{2}\{e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2}\, e_7\}, \qquad \alpha_2 = e_1 + e_2, \qquad \alpha_k = e_{k-1} - e_{k-2}, \quad k = 3, \dots, 7.$$

If λ_j are the root matrices corresponding to the simple roots, the relevant non-vanishing commutators are

$$[\lambda_j, \lambda_{j+1}], \; j \neq 1, 2, \qquad [\lambda_1, \lambda_3], \quad [\lambda_2, \lambda_4]. \tag{A.7}$$

The split subalgebra is $\mathfrak{su}(8)$.

A.9. $\boldsymbol{E_8}$

The corresponding compact form is $\mathfrak{e}_8$. The real linear space generated by the roots is isomorphic to $\mathbb{R}^8$. If e_a, $a = 1, \dots, 8$, is the canonical basis of $\mathbb{R}^8$, then all non-vanishing roots are represented by the vectors $\pm(e_j - e_k)$, $j < k$, $\pm(e_j + e_k)$, $j < k$,

$$\frac{1}{2}\{\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm\; e_8\},$$

where in the parenthesis all signs can appear. The simple roots are

$$\alpha_1 = \frac{1}{2}\{e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8\}, \qquad \alpha_2 = e_1 + e_2, \qquad \alpha_k = e_{k-1} - e_{k-2}, \quad k = 3, \dots, 8.$$

If λ_j are the root matrices corresponding to the simple roots, the relevant non-vanishing commutators are

$$[\lambda_j, \lambda_{j+1}], \ j \neq 1, 2, \qquad [\lambda_1, \lambda_3], \quad [\lambda_2, \lambda_4]. \tag{A.8}$$

The split subalgebra is $\mathfrak{so}(16)$.

A.10. Resuming

In conclusion, we see that the commutators we need are strictly related to the Dynkin diagram of the algebra: a commutator between eigenmatrices of two simple roots is non zero only if the roots are linked, that is if the scalar product is not zero. This is simply related to the fact that, with obvious notation, the commutators $[\lambda_{\alpha_i}, \lambda_{\alpha_j}]$ or is an eigenmatrix for $\alpha_i + \alpha_j$, or it vanishes. We also recall here some very well known facts. The fact that Dynkin diagrams have no loops allows to choose the normalization of the matrices λ_j so that

$$[\lambda_{\alpha_j}, \lambda_{\alpha_k}] = \mathrm{sign}(k-j)(\delta_{jk}(\alpha_j|\alpha_j) - (\alpha_j|\alpha_k))\lambda_{\alpha_j+\alpha_k} \tag{A.9}$$

t.i. $[\lambda_{\alpha_j}, \lambda_{\alpha_k}] = -(\alpha_j|\alpha_k)\lambda_{\alpha_j+\alpha_k}$ if $j < k$ and with the opposite sign if we change j and k. Here $(|)$ is the scalar product in the space of roots. Notice that also

$$[\tilde{\lambda}_{\alpha_j}, \tilde{\lambda}_{\alpha_k}] = \mathrm{sign}(k-j)(\delta_{jk}(\alpha_j|\alpha_j) - (\alpha_j|\alpha_k))\tilde{\lambda}_{\alpha_j+\alpha_k}. \tag{A.10}$$

Remember that the trace product is proportional to the Killing product and that the only non-Killing orthogonal root spaces are the ones corresponding to opposite roots. This allows to fix a global normalization so that

$$\mathrm{Tr}(\tilde{\lambda}_j \lambda_k) = -\delta_{jk}. \tag{A.11}$$

We also have, for the simple roots α_j,

$$[\tilde{\lambda}_j, \lambda_k] = -i\delta_{jk} J_j, \tag{A.12}$$

where J_j are in a Cartan algebra. From the fact that the simple roots are linearly independent, it easily follows that the J_j, $j = 1, \ldots, r$, are a basis for the Cartan subalgebra. This is also sufficient to fix the scalar product in the space of roots so that

$$(\alpha_j|\alpha_k) = \alpha_j(J_k), \tag{A.13}$$

if the roots are defined as

$$[h, \lambda_j] = i\alpha_j(h)\lambda_j, \tag{A.14}$$

for any $h \in H$. In particular, using ad invariance of the trace product we get

$$\mathrm{Tr}(J_j J_k) = i\mathrm{Tr}([\tilde{\lambda}_j, \lambda_j]J_k) = i\mathrm{Tr}(\tilde{\lambda}_j[\lambda_j, J_k]) = \alpha_j(J_k)\mathrm{Tr}(\tilde{\lambda}_j\lambda_j),$$

so that

$$\mathrm{Tr}(J_j J_k) = -(\alpha_j|\alpha_k). \tag{A.15}$$

Finally, recall that any given simple Lie algebra is characterized by the $r \times r$ Cartan matrix

$$C^G_{jk} = 2\frac{(\alpha_j|\alpha_k)}{(\alpha_j|\alpha_j)}. \tag{A.16}$$

With this we can rewrite the normalization conditions as

$$\mathrm{Tr}(\tilde{\lambda}_j \lambda_k) = -\delta_{jk}, \tag{A.17}$$

$$\mathrm{Tr}(J_j J_k) = -C^G_{jk} \frac{(\alpha_j|\alpha_j)}{2}. \tag{A.18}$$

The Cartan matrices of simple Lie groups can be found, for example, in [55], Table 6.

Appendix B. A proposition

Proposition 6. *Let $\kappa = \sum_{j=1}^r (c_j \lambda_j + c_j^* \tilde{\lambda}_j)$, and $h \in H$ a matrix such that $\alpha_j(h) = \varepsilon_j a$, where ε_j is a sign, $j = 1, \dots, r$, and set $x := e^{-hz} \kappa e^{hz}$, $z \in \mathbb{R}$. Then*

$$\mathrm{Tr}\kappa^2 = -2\|\underline{c}\|^2, \tag{B.1}$$

$$\mathrm{Tr}([h,\kappa][h,\kappa]) = \mathrm{Tr}([h,x][h,x]) = -2a^2\|\underline{c}\|^2, \tag{B.2}$$

and

$$\begin{aligned} &\mathrm{Tr}([x,\kappa][x,\kappa]) \\ &\quad = -4\sin^2(az)\left(\sum_{j=1}^r \|\alpha_j\|^2 |c_j|^4 + \sum_{j<k} |c_j|^2|c_k|^2(\alpha_j|\alpha_k)[2\varepsilon_j\varepsilon_k + (\alpha_j|\alpha_k)(1-\varepsilon_j\varepsilon_k)]\right). \end{aligned} \tag{B.3}$$

Proof. Let us start with

$$\begin{aligned} \mathrm{Tr}\kappa^2 &= \sum_{j=1}^r \sum_{k=1}^r \{c_j c_k \mathrm{Tr}(\lambda_j\lambda_k) + c_j^* c_k^* \mathrm{Tr}(\tilde{\lambda}_j\tilde{\lambda}_k) + c_j^* c_k \mathrm{Tr}(\tilde{\lambda}_j \lambda_k) + c_j c_k^* \mathrm{Tr}(\lambda_j \tilde{\lambda}_k)\} \\ &= \sum_{j=1}^r \sum_{k=1}^r \{c_j^* c_k(-\delta_{jk}) + c_j c_k^*(-\delta_{jk})\} = -2\sum_{j=1}^r |c_j|^2, \end{aligned} \tag{B.4}$$

where we used the normalization in the previous section. This proves (B.1).
For (B.2), we first note that

$$[h,x] = [h, e^{-hz}\kappa e^{hz}] = e^{-hz}[h,\kappa]e^{hz}. \tag{B.5}$$

Thus,

$$\mathrm{Tr}([h,x][h,x]) = \mathrm{Tr}(e^{-hz}[h,\kappa][h,\kappa]e^{hz}) = \mathrm{Tr}([h,\kappa][h,\kappa]). \tag{B.6}$$

Now, we can use

$$[h,\kappa] = \sum_{j=1}^r (c_j[h,\lambda_j] + c_j^*[h,\tilde{\lambda}_j]) = i\sum_{j=1}^r \alpha_j(h)(c_j\lambda_j - c_j^*\tilde{\lambda}_j). \tag{B.7}$$

to get

$$\begin{aligned} \mathrm{Tr}([h,\kappa][h,\kappa]) &= -\mathrm{Tr}\left(\sum_{j=1}^r (i\alpha_j(h)\lambda_j c_j - i\alpha_j(h)c_j^*\tilde{\lambda}_j) \sum_{k=1}^r (i\alpha_k(h)\lambda_k c_k - i\alpha_k(h) c_k^* \tilde{\lambda}_k)\right) \\ &= -2\sum_{j=1}^r \alpha_j(h)^2 c_j c_j^* = -2a^2\|\underline{c}\|^2, \end{aligned} \tag{B.8}$$

where we used that $\alpha_j(h')^2 = (\varepsilon_j a)^2 = a^2$, and that the only non-vanishing traces are $\mathrm{Tr}(\tilde{\lambda}_m \lambda_n) = -\delta_{mn}$. This proves (B.2).
For (B.3), we use that

$$x = \sum_{j=1}^{r} (c_j e^{-hz} \lambda_j e^{hz} + c_j e^{-hz} \tilde{\lambda}_j e^{hz}) = \sum_{j=1}^{r} (c_j e^{-\alpha_j(h)z} \lambda_j + c_j^* e^{\alpha_j(h)z} \tilde{\lambda}_j)$$
$$= \sum_{j=1}^{r} (c_j e^{-i\varepsilon_j az} \lambda_j + c_j^* e^{i\varepsilon_j az} \tilde{\lambda}_j). \tag{B.9}$$

Therefore,

$$[x,\kappa] = \sum_{j,k} \Big(c_j c_k e^{-i\varepsilon_j az} [\lambda_j, \lambda_k] + c_j^* c_k^* e^{i\varepsilon_j az} [\tilde{\lambda}_j, \tilde{\lambda}_k] + c_j c_k^* e^{-i\varepsilon_j az} [\lambda_j, \tilde{\lambda}_k]$$
$$+ c_j^* c_k e^{i\varepsilon_j az} [\tilde{\lambda}_j, \lambda_k] \Big). \tag{B.10}$$

Using (A.9), (A.10) and (A.12), it can be rewritten as

$$[x,\kappa] = -\sum_{j<k} \Big(c_j c_k (\alpha_j|\alpha_k)(e^{-i\varepsilon_j az} - e^{-i\varepsilon_k az}) \lambda_{\alpha_j+\alpha_k}$$
$$+ c_j^* c_k^* (\alpha_j|\alpha_k)(e^{i\varepsilon_j az} - e^{i\varepsilon_k az}) \tilde{\lambda}_{\alpha_j+\alpha_k} \Big)$$
$$- i \sum_{j=1}^{r} |c_j|^2 (e^{-i\varepsilon_j az} - e^{i\varepsilon_j az}) J_j$$
$$= -\sum_{j<k} \Big(c_j c_k (\alpha_j|\alpha_k)(e^{-i\varepsilon_j az} - e^{-i\varepsilon_k az}) \lambda_{\alpha_j+\alpha_k}$$
$$+ c_j^* c_k^* (\alpha_j|\alpha_k)(e^{i\varepsilon_j az} - e^{i\varepsilon_k az}) \tilde{\lambda}_{\alpha_j+\alpha_k} \Big)$$
$$- 2 \sum_{j=1}^{r} |c_j|^2 \sin(\varepsilon_j az) J_j. \tag{B.11}$$

With our normalization for the scalar products we get

$$\mathrm{Tr}([x,\kappa][x,\kappa]) = -2 \sum_{j<k} |c_j|^2 |c_k|^2 |(\alpha_j|\alpha_k)|^2 \left| e^{-i\varepsilon_j az} - e^{-i\varepsilon_k az} \right|^2$$
$$- 4 \sum_{j=1}^{r} |c_j|^4 \|\alpha_j\|^2 \sin^2(\varepsilon_j az)$$
$$- 8 \sum_{j<k} |c_j|^2 |c_k|^2 (\alpha_j|\alpha_k) \sin(\varepsilon_j ar) \sin(\varepsilon_k az). \tag{B.12}$$

Next, we use

$$\left| e^{-i\varepsilon_j az} - e^{-i\varepsilon_k az} \right|^2 = 4 \sin^2 \left(az \frac{\varepsilon_j - \varepsilon_k}{2} \right) = 2 \sin^2(az)(1 - \varepsilon_j \varepsilon_k), \tag{B.13}$$

where we used that $(\varepsilon_j - \varepsilon_k)/2 = 0, \pm 1$. Finally, using

$$\sin(\varepsilon_j az)\sin(\varepsilon_k az) = \sin^2(az)\varepsilon_j\varepsilon_k, \tag{B.14}$$

we get (B.3). $\square$

Appendix C. Connection between Lasagna and Spaghetti

It is interesting to compare the results obtained from Lasagna and Spaghetti parameterization. We can do this by determining the relation between the two parameterizations via de identification

$$U \equiv e^{\chi(\sin\Theta\cos\Phi T_1+\sin\Theta\sin\Phi T_2+\cos\Theta T_3)} = e^{\Psi(\chi,\Theta,\Phi)T_1}e^{H(\chi,\Theta,\Phi)T_3}e^{\Gamma(\chi,\Theta,\Phi)T_1}. \tag{C.1}$$

A priori one could expect this parameterization to be dependent on the representation the T_j are belonging to, since of course the exponentials do. Nevertheless, in both cases the left invariant current $\mathcal{L}_\mu = U^{-1}\partial_\mu U$ for fixed coordinates is independent on the representation but depends only on the normalizations. If we normalize the matrices as in the previous sections, after writing $U^{-1}dU|_{Exp} = U^{-1}dU|_{Eul}$ we get the differential equation relating the exponential coordinates to the Euler ones. These are easily obtained, but writing them is not helpful since it would be quite difficult to solve them by brute force. Instead, we can find a solution without knowing them. The shown independence on the representation suggests to write down (C.1) in the lowest representation. This is achieved by choosing

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{C.2}$$

With this choice (C.1) becomes

$$\begin{aligned}
&\cos\frac{\chi}{2}\,\mathbb{I}_2 + 2\sin\frac{\chi}{2}\sin\Theta\cos\Phi\;T_1 + 2\sin\frac{\chi}{2}\sin\Theta\sin\Phi\;T_2 + 2\sin\frac{\chi}{2}\cos\Theta\;T_3 \\
&\quad = \cos\frac{H}{2}\cos\frac{\Psi+\Gamma}{2}\mathbb{I}_2 + 2\cos\frac{H}{2}\sin\frac{\Psi+\Gamma}{2}T_1 + 2\sin\frac{H}{2}\sin\frac{\Psi-\Gamma}{2}T_2 \\
&\qquad + 2\sin\frac{H}{2}\cos\frac{\Psi-\Gamma}{2}T_3.
\end{aligned} \tag{C.3}$$

This gives

$$H = 2\arcsin(\sin\frac{\chi}{2}\sqrt{1+\sin^2\Phi}), \tag{C.4}$$

$$\Psi+\Gamma = 2\arctan(\tan\frac{\chi}{2}\sin\Theta\cos\Phi), \tag{C.5}$$

$$\Psi-\Gamma = 2\arctan(\tan\Theta\sin\Phi), \tag{C.6}$$

and the inverse

$$\chi = 2\arccos(\cos\frac{H}{2}\,\cos\frac{\Psi+\Gamma}{2}), \tag{C.7}$$

$$\Theta = \arctan\left(\tan\frac{\Psi-\Gamma}{2}\sqrt{1+\frac{1}{\tan^2\frac{H}{2}}\frac{\sin^2\frac{\Psi+\Gamma}{2}}{\sin^2\frac{\Psi-\Gamma}{2}}}\right), \tag{C.8}$$

$$\Phi = \arctan\left(\tan\frac{H}{2}\,\frac{\sin\frac{\Psi-\Gamma}{2}}{\sin\frac{\Psi+\Gamma}{2}}\right). \tag{C.9}$$

For example, the Lasagna solutions have the form

$$\Psi = \frac{t}{L_\phi} - \phi, \tag{C.10}$$

$$H = ar, \tag{C.11}$$

$$\Gamma = m\theta, \tag{C.12}$$

so that in the exponential form they take the very complicated form

$$\chi(t, r, \theta, \phi) = 2\arccos\left(\cos\frac{ar}{2}\ \cos\frac{t - L_\phi(\phi - \theta)}{2L_\phi}\right), \tag{C.13}$$

$$\Theta(t, r, \theta, \phi) = \arctan\left(\tan\frac{t - L_\phi(\phi + \theta)}{2L_\phi}\ \sqrt{1 + \frac{1}{\tan^2\frac{ar}{2}}\ \frac{\sin^2\frac{t-L_\phi(\phi-\theta)}{2L_\phi}}{\sin^2\frac{t-L_\phi(\phi+\theta)}{2L_\phi}}}\right), \tag{C.14}$$

$$\Phi(t, r, \theta, \phi) = \arctan\left(\tan\frac{ar}{2}\ \frac{\sin\frac{t-L_\phi(\phi+\theta)}{2L_\phi}}{\sin\frac{t-L_\phi(\phi-\theta)}{2L_\phi}}\right). \tag{C.15}$$

Appendix D. Some technical details about G_2

There are different ways of constructing a convenient basis for the Lie algebra of G_2. We will refer to [56]. In that notation a basis is C_J, $J = 1, \ldots, 14$. The only maximal regular subgroup is $SO(4)$ generated by C_L, $L = 1, 2, 3, 8, 9, 10$. The remaining matrices generate $\mathfrak{p}$. A convenient Cartan subspace is thus

$$H = \langle C_5, C_{11}\rangle_{\mathbb{R}}. \tag{D.1}$$

As a basis, we take $h_1 = C_{11}$ and $h_2 = C_5$. One can easily diagonalize the action of $ad(H)$. If we set

$$\lambda_1 \equiv k_1 + ip_1 = \frac{1}{4\sqrt{2}}(\sqrt{3}C_3 - C_8) + i\frac{1}{2\sqrt{2}}C_{12}, \tag{D.2}$$

$$\lambda_2 \equiv k_2 + ip_2 = \frac{1}{8}(C_1 + C_2 - \sqrt{3}C_9 - \sqrt{3}C_{10}) + i\frac{1}{8}(C_6 + C_7 + \sqrt{3}C_{13} - \sqrt{3}C_{14}), \tag{D.3}$$

then, they satisfy $\mathrm{Tr}(\lambda_i\tilde{\lambda}_j) = -\delta_{ij}$ and

$$[h_1, \lambda_1] = i\frac{2}{\sqrt{3}}\lambda_1, \quad [h_2, \lambda_1] = 0, \tag{D.4}$$

$$[h_1, \lambda_2] = -i\sqrt{3}\lambda_2, \quad [h_2, \lambda_2] = i\lambda_2. \tag{D.5}$$

To keep contact with our conventions we have to redefine the basis for H as J_1 and J_2, defined by (notice that $\tilde{\lambda}_j$ is simply the complex conjugate of λ_j)

$$J_1 = -i[\tilde{\lambda}_1, \lambda_1] = -\frac{1}{2\sqrt{3}}C_{11}, \tag{D.6}$$

$$J_2 = -i[\tilde{\lambda}_2, \lambda_2] = \frac{\sqrt{3}}{4}C_{11} - \frac{1}{4}C_5. \tag{D.7}$$

This gives us the geometry of roots, so that

$$(\alpha_1|\alpha_1) = -\mathrm{Tr}(J_1 J_1) = \frac{1}{3}, \tag{D.8}$$

$$(\alpha_2|\alpha_2) = -\mathrm{Tr}(J_2 J_2) = 1, \tag{D.9}$$

$$(\alpha_1|\alpha_2) = -\mathrm{Tr}(J_1 J_2) = -\frac{1}{2}. \tag{D.10}$$

We can represent this vectors in the canonical euclidean $\mathbb{R}^2$ as the vectors

$$\alpha_1 \equiv \left(\frac{1}{\sqrt{3}}, 0\right), \qquad \alpha_2 \equiv \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right). \tag{D.11}$$

The corresponding Cartan matrix is

$$C^G = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \tag{D.12}$$

with inverse

$$(C^G)^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}. \tag{D.13}$$

It is also useful to determine the basis for all eigenspaces, in a convenient way, normalized so that $\mathrm{Tr}(\tilde{\lambda}_\alpha \lambda_\alpha) = -1$ for any root. After setting

$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_4 = 2\alpha_1 + \alpha_2, \quad \alpha_5 = 3\alpha_1 + \alpha_2, \quad \alpha_6 = 3\alpha_1 + 2\alpha_2, \tag{D.14}$$

we can state the following proposition.

Proposition 7. *A suitable choice of the eigenmatrices associated to the roots α_j, $j = 3, 4, 5, 6$, is given by*

$$\lambda_3 = \sqrt{2}[\lambda_1, \lambda_2], \quad \lambda_4 = \sqrt{\frac{3}{2}}[\lambda_1, \lambda_3], \quad \lambda_5 = \sqrt{2}[\lambda_1, \lambda_4], \quad \lambda_6 = \sqrt{2}[\lambda_3, \lambda_4]. \tag{D.15}$$

Moreover, $\tilde{\lambda}_j$ is the complex conjugate of λ_j.

Proof. We know from the general theory that if λ_a and λ_b are eigenmatrices of the roots α_a and α_b respectively, and if $\alpha_a + \alpha_b$ is also root, than the eigenmatrices of $\alpha_a + \alpha_b$ have the form $\mu[\lambda_a, \lambda_b]$ for any given constant μ. Since λ_1 and λ_2 are eigenmatrices for the fundamental roots α_1 and α_2, we have that the matrices λ_j specified above are surely eigenmatrices for the corresponding roots α_j, $j = 3, 4, 5, 6$. We have only to explain the choices of the constant factors. These are chosen to be real and such that $\mathrm{Tr}(\lambda_j \tilde{\lambda}_j) = -1$. To prove it, first notice that necessarily $[\tilde{\lambda}_i, \tilde{\lambda}_j]$ are eigenmatrices for $-\alpha_i - \alpha_j$, so we can identify $\tilde{\lambda}_3 = \sqrt{2}[\tilde{\lambda}_1, \tilde{\lambda}_2]$ and so on. The last part of the proposition then follows from the fact that it is true for λ_1 and λ_2 and that all the coefficient we chosen for defining the remaining λ_j is real. Then, using ad-invariance of the trace product, that is $\mathrm{Tr}([A, B]C) = \mathrm{Tr}(A, [B, C])$, and the Jacobi identity $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$, we have

$$\begin{aligned} \mathrm{Tr}(\lambda_3 \tilde{\lambda}_3) =& \mathrm{Tr}(2[\lambda_1, \lambda_2][\tilde{\lambda}_1, \tilde{\lambda}_2]) = 2\mathrm{Tr}(\lambda_1[\lambda_2, [\tilde{\lambda}_1, \tilde{\lambda}_2]]) \\ =& 2\mathrm{Tr}(\lambda_1([[\lambda_2, \tilde{\lambda}_1], \tilde{\lambda}_2] + [\tilde{\lambda}_1, [\lambda_2, \tilde{\lambda}_2]])). \end{aligned} \tag{D.16}$$

Since α_1 and α_2 are simple roots, we have $[\lambda_2, \tilde{\lambda}_1] = 0$. From (D.6) we see that $[\lambda_2, \tilde{\lambda}_2] = iJ_2$ and since $[J_2, \tilde{\lambda}_1] = -i\alpha_1(J_2)\tilde{\lambda}_1 = -i(\alpha_1|\alpha_2)\tilde{\lambda}_1 = \frac{i}{2}\tilde{\lambda}_1$, we get

$$\mathrm{Tr}(\lambda_3\tilde{\lambda}_3) = \mathrm{Tr}(\lambda_1\tilde{\lambda}_1) = -1. \tag{D.17}$$

Next, consider

$$\begin{aligned}
\mathrm{Tr}(\lambda_4\tilde{\lambda}_4) =& \mathrm{Tr}(\frac{3}{2}[\lambda_1, \lambda_3][\tilde{\lambda}_1, \tilde{\lambda}_3]) = -\frac{3}{2}\mathrm{Tr}(\lambda_3[\lambda_1, [\tilde{\lambda}_1, \tilde{\lambda}_3]]) \\
=& -\frac{3}{2}\mathrm{Tr}(\lambda_3[[\lambda_1, \tilde{\lambda}_1], \tilde{\lambda}_3]) - \frac{3}{2}\mathrm{Tr}(\lambda_3[\tilde{\lambda}_1, [\lambda_1, \tilde{\lambda}_3]]) \\
=& -\frac{3}{2}\mathrm{Tr}(\lambda_3[iJ_1, \tilde{\lambda}_3]) - \frac{3}{2}\mathrm{Tr}([\tilde{\lambda}_1, \lambda_3][\lambda_1, \tilde{\lambda}_3]) \\
=& -\frac{3}{2}i(-i\alpha_3(J_1))\mathrm{Tr}(\lambda_3\tilde{\lambda}_3) - \frac{3}{2}\mathrm{Tr}([\tilde{\lambda}_1, \lambda_3][\lambda_1, \tilde{\lambda}_3]) \\
=& \frac{3}{2}(\alpha_3|\alpha_1) - \frac{3}{2}\mathrm{Tr}([\tilde{\lambda}_1, \lambda_3][\lambda_1, \tilde{\lambda}_3]).
\end{aligned} \tag{D.18}$$

Since $\alpha_3 = \alpha_1 + \alpha_2$, we have

$$(\alpha_3|\alpha_1) = (\alpha_1|\alpha_1) + (\alpha_2|\alpha_1) = \frac{1}{3} - \frac{1}{2}. \tag{D.19}$$

On the other hand

$$[\lambda_1, \tilde{\lambda}_3] = \sqrt{2}[\lambda_1, [\tilde{\lambda}_1, \tilde{\lambda}_2]] = \sqrt{2}[[\lambda_1, \tilde{\lambda}_1], \tilde{\lambda}_2] = i\sqrt{2}[J_1, \tilde{\lambda}_1] = i\sqrt{2}(-i\alpha_2(J_1))\tilde{\lambda}_2, \tag{D.20}$$

where we used again that $[\lambda_1, \tilde{\lambda}_2] = 0$, and, therefore,

$$\mathrm{Tr}([\tilde{\lambda}_1, \lambda_3][\lambda_1, \tilde{\lambda}_3]) = 2(\alpha_2(J_1))^2\mathrm{Tr}(\tilde{\lambda}_2\lambda_2) = -2(-1/2)^2, \tag{D.21}$$

and putting all together we get $\mathrm{Tr}(\lambda_4\tilde{\lambda}_4) = -1$.
The remaining two cases are proved exactly in the same way. □

D.1. The fundamental irreps of G_2

G_2 has 12 non null roots forming two concentric hexagons in H^*, plus two vanishing roots, like in Fig. 1.

$\alpha_1, \dots, \alpha_6$ are the positive roots. To each of them, α_j, it corresponds an eigenmatrix λ_j and to each negative root $-\alpha_j$ it corresponds $\tilde{\lambda}_j$. To the vanishing roots one associates the matrices $h_a = i[\lambda_a, \tilde{\lambda}_a]$, $a = 1, 2$. The 14 matrices h_a, $\lambda_j + \tilde{\lambda}_j$, $i(\lambda_j - \tilde{\lambda}_j)$, $a = 1, 2$, $j = 1, \dots, 6$ form a basis for the adjoint representation **14**, with maximal weight $\mu_1 = \alpha_6$.
The second fundamental irreducible representation has maximal weight $\mu_2 = \alpha_4$. The weights of such representation are $0, \pm\alpha_b$, $b = 1, 3, 4$, each one with multiplicity 1, so that it is a seven dimensional representation, **7**. It is depicted in Fig. 2.

The matrices in this representation are $\rho_7(h_a)$, $\rho_7(\lambda_j) + \rho_7(\tilde{\lambda}_j)$, $i(\rho_7(\lambda_j) - \rho_7(\tilde{\lambda}_j))$, $a = 1, 2$, $j = 1, \dots, 6$, and can be understood by noticing that $\rho_7(\tilde{\lambda}_j)$ shifts the weights by α_j, giving zero if and only if the result is not a weight, and similar for $\tilde{\lambda}_j$.

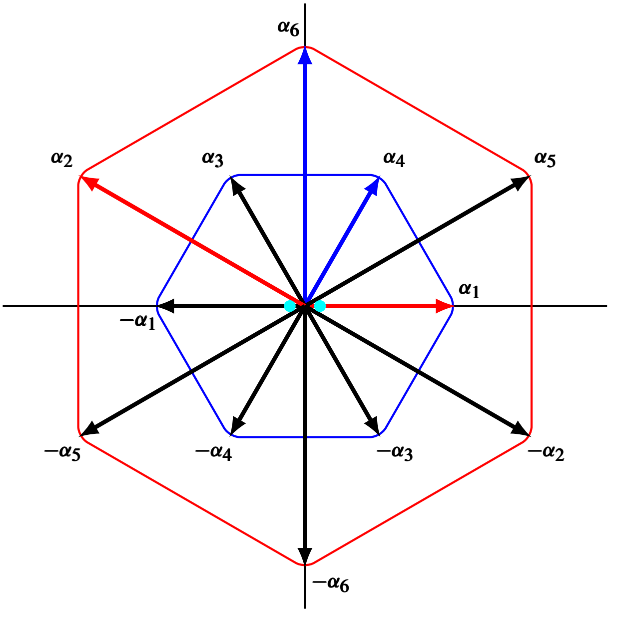

Fig. 1. The twelve roots of G_2, two of which are zero. α_1 and α_2 are the simple roots. α_6 and α_4 are the fundamental weights.

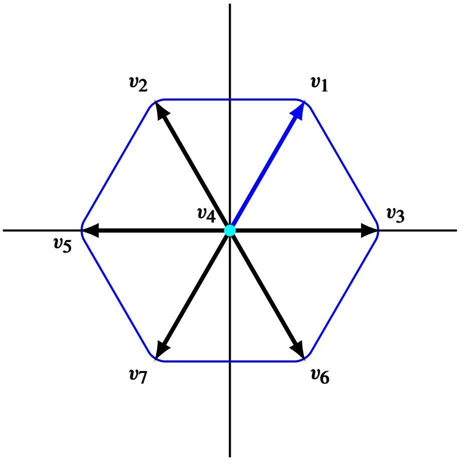

Fig. 2. The irrep **7** of G_2. To each weight it corresponds a vector of the basis defining a 7 dimensional vector space.

D.2. Irreducibility of χ_{28} in representation **7**

Let us consider $\rho_7(T_-) = 3\rho_7(\tilde{\lambda}_1) + \sqrt{5}\rho(\tilde{\lambda}_2)$ acting on the maximal vector v_1 of **7**. Since $\alpha_4 - \alpha_2$ is not a weight of **7**, while $\alpha_4 - \alpha_1 = \alpha_3$ is, we have that $\rho_7(T_-)v_1 = 3\rho(\tilde{\lambda}_1)v_1 = k_2 v_2$, with[6] $k_2 \neq 0$. In the same way we have the chain of relations:

[6] We could compute it explicitly, but it is not necessary for our purposes.

$$\rho_{\mathbf{7}}(T_-)v_2 = \sqrt{5}\rho_{\mathbf{7}}(\tilde{\lambda}_2)v_2 = k_3v_3,$$
$$\rho_{\mathbf{7}}(T_-)v_3 = 3\rho_{\mathbf{7}}(\tilde{\lambda}_1)v_3 = k_4v_4,$$
$$\rho_{\mathbf{7}}(T_-)v_4 = 3\rho_{\mathbf{7}}(\tilde{\lambda}_1)v_4 = k_5v_5,$$
$$\rho_{\mathbf{7}}(T_-)v_5 = \sqrt{5}\rho_{\mathbf{7}}(\tilde{\lambda}_2)v_5 = k_6v_6,$$
$$\rho_{\mathbf{7}}(T_-)v_6 = 3\rho_{\mathbf{7}}(\tilde{\lambda}_1)v_6 = k_7v_7,$$
$$\rho_{\mathbf{7}}(T_-)v_7 = 0,$$

with all k_j different from zero. Therefore, χ_{28} is a representation of spin 3 and **7** is irreducible under χ_{28}.

D.3. Explicit matrix realizations

Here we present the explicit matrix representation of the three dimensional subalgebras in the irrep **7** of G_2. These are

$$T_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \tag{D.22}$$

$$T_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.23}$$

$$T_3^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.24}$$

$$T_1^{(3)} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 & 0 \end{pmatrix}, \tag{D.25}$$

$$T_2^{(3)} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.26}$$

$$T_3^{(3)} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -3 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.27}$$

$$T_1^{(4)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{D.28}$$

$$T_2^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.29}$$

$$T_3^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.30}$$

$$T_1^{(28)} = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{5} & \sqrt{5} & 0 & 0 & 0 & 0 \\ -\sqrt{5} & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \\ -\sqrt{5} & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\sqrt{6} & \sqrt{5} & -\sqrt{5} \\ 0 & 0 & 0 & 2\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{5} & 0 & 0 & -\sqrt{6} \\ 0 & 0 & 0 & \sqrt{5} & 0 & \sqrt{6} & 0 \end{pmatrix}, \tag{D.31}$$

$$T_2^{(28)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & -2\sqrt{6} & -\sqrt{5} & \sqrt{5} \\ 0 & 0 & 0 & -\sqrt{5} & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & -\sqrt{5} & 0 & 0 & \sqrt{6} \\ 0 & \sqrt{5} & \sqrt{5} & 0 & 0 & 0 & 0 \\ 2\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{5} & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{5} & 0 & -\sqrt{6} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.32}$$

$$T_3^{(28)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -5 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{D.33}$$

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