# CAPACITIES AND CHOQUET AVERAGES OF ULTRAFILTERS

Abstract. We show that a normalized capacity $\nu : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ is invariant with respect to an ideal $\mathcal{I}$ on $\mathbf{N}$ if and only if it can be represented as a Choquet average of $\{0,1\}$-valued finitely additive probability measures corresponding to the ultrafilters containing the dual filter of $\mathcal{I}$. This is obtained as a consequence of an abstract analogue in the context of Archimedean Riesz spaces.

## 1. Introduction

In this paper, we study normalized capacities which could be defined on the quotient $\mathcal{P}(\mathbf{N})/\mathcal{I}$, where $\mathcal{I}$ is a given ideal on the positive integers $\mathbf{N}$. Our goal is to represent them as suitable averages of $\{0,1\}$-valued finitely additive probability measures corresponding to the ultrafilters containing the dual filter of $\mathcal{I}$. In what follows, we define the notions of normalized capacity, ideal, Choquet integral, and we introduce the topological requirements needed for our results. This will allow us to discuss them formally and prove them.

**Normalized capacities and ideals.** Given a measurable space $(S,\Sigma)$, where $\Sigma$ is a $\sigma$-algebra of subsets of $S$, a *normalized capacity* $\nu : \Sigma \to \mathbf{R}$ is a monotone set function (i.e., $\nu(A) \le \nu(B)$ for all $A, B \in \Sigma$ with $A \subseteq B$) such that $\nu(\emptyset) = 0$ and $\nu(S) = 1$. A family $\mathcal{I} \subseteq \mathcal{P}(\mathbf{N})$ is an *ideal* if it is closed under subsets, finite unions, and $\mathbf{N} \notin \mathcal{I}$. Unless otherwise stated, we assume that $\mathcal{I}$ contains the family Fin of finite sets. Let $\mathcal{I}^\star := \{A \subseteq \mathbf{N} : A^c \in \mathcal{I}\}$ be its dual filter. Note that $\mathcal{I}$ is maximal with respect to inclusion if and only if $\mathcal{I}^\star$ is a free ultrafilter on $\mathbf{N}$. Two important examples of ideals are: (i) the family of asymptotic density zero sets

$$\mathcal{Z} := \{A \subseteq \mathbf{N} : \mathsf{d}^\star(A) = 0\},$$

where $\mathsf{d}^\star$ is the upper asymptotic density defined by $\mathsf{d}^\star(A) = \limsup_n |A \cap [1,n]|/n$ for all $A \subseteq \mathbf{N}$; (ii) the summable ideal

$$\mathcal{I}_{1/n} := \left\{A \subseteq \mathbf{N} : \sum\nolimits_{a \in A} 1/a < \infty \right\}.$$

We refer to [16] for a survey on ideals and associated filters. Finally, a normalized capacity $\nu : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ is said to be $\mathcal{I}$-*invariant* provided that $\nu(A) = \nu(B)$ whenever the symmetric difference $A \bigtriangleup B$ belongs to the ideal $\mathcal{I}$.

2010 *Mathematics Subject Classification.* Primary: 28A12, 40A35, 46A40. Secondary: 28A25, 46B45, 54D35.

*Key words and phrases.* Archimedean Riesz space, Stone–Čech compactification, Choquet averages, normalized capacities, submeasures, ideal convergence.

**Choquet integrals.** The idea of normalized capacity naturally calls for the notion of (nonlinear) integral. Consider a bounded $\Sigma$-measurable function $x : S \to \mathbf{R}$ and a normalized capacity $\nu : \Sigma \to \mathbf{R}$. The *Choquet integral* of $x$ with respect to $\nu$ is the quantity

$$\int_S x \, \mathrm{d}\nu := \int_0^\infty \nu(x \ge t) \, \mathrm{d}t + \int_{-\infty}^0 [\nu(x \ge t) - \nu(S)] \, \mathrm{d}t,$$

where the integrals on the right hand side are meant to be improper Riemann integrals. This naturally generalizes the standard notion of integral since the two coincide when $\nu$ is finitely additive. From a functional point of view, the Choquet integral fails to be linear, but it is characterized by comonotonic additivity (see the seminal paper of Schmeidler [27]). In terms of definitions, note that, to compute the Choquet integral, one only needs a monotone set function defined over the upper level sets of the integrand and not necessarily over the entire $\sigma$-algebra $\Sigma$ (this is something we will exploit in our proofs). Nonadditive set functions and their integrals are widely used in applications: for applications in economics, see [24]; for applications in probability and statistics, see [6, 23, 30].

**Topology.** Given the set $\mathbf{N}$ of positive integers, we consider the Stone–Čech compactification $\beta\mathbf{N}$. Recall that $\beta\mathbf{N}$ is homeomorphic to the space of ultrafilters $\mathcal{F}$ on $\mathbf{N}$, which we still denote by $\beta\mathbf{N}$ and is topologised by the base of clopen subsets $\{\{\mathcal{F} \in \beta\mathbf{N} : A \in \mathcal{F}\} : A \subseteq \mathbf{N}\}$. By $\mathrm{Ult}(\mathcal{I})$ we denote the compact subspace of free ultrafilters which contain the dual filter of a given ideal $\mathcal{I}$, that is,

$$\mathrm{Ult}(\mathcal{I}) := \{\mathcal{F} \in \beta\mathbf{N} : \mathcal{I}^\star \subseteq \mathcal{F}\}.$$

We endow $\mathrm{Ult}(\mathcal{I})$ with its relative topology. Note that the subspace $\mathrm{Ult}(\mathcal{I})$, sometimes called "support set," has been introduced in Henriksen [15] and further studied in [3, 8, 10, 25] in the context of ideals generated by nonnegative regular summability matrices,[1] cf. also [4, 18]. Given an ultrafilter $\mathcal{F} \in \mathrm{Ult}(\mathcal{I})$, there is a corresponding $\{0,1\}$-valued finitely additive probability measure $\mu_{\mathcal{F}}$ defined by

$$\mu_{\mathcal{F}}(A) = 1 \quad \text{if and only if} \quad A \in \mathcal{F}.$$

Finally, given a topological space $X$, denote by $\mathscr{B}(X)$ its Borel $\sigma$-algebra (recall that if $Y \subseteq X$ is endowed with its relative topology then $\mathscr{B}(Y) = \{A \cap Y : A \in \mathscr{B}(X)\}$).

[1] However, it is known [4, Proposition 13] that, if we regard ideals as subsets of the Cantor space $\{0,1\}^{\mathbf{N}}$ endowed with the product topology, then the latter sets are necessarily $F_{\sigma\delta}$ (hence, the notion of support set was applied only to a restricted class of ideals).

**Results.** We are now ready to state our main result.

**Theorem 1.1.** *Let $\nu : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ be a normalized capacity and $\mathcal{I}$ an ideal on $\mathbf{N}$. The following statements are equivalent:*

(i) *$\nu$ is $\mathcal{I}$-invariant;*

(ii) *There exists a normalized capacity $\rho : \mathscr{B}(\mathrm{Ult}(\mathcal{I})) \to \mathbf{R}$ such that*

$$\forall x \in \ell_\infty, \quad \int_{\mathbf{N}} x \, \mathrm{d}\nu = \int_{\mathrm{Ult}(\mathcal{I})} \left( \int_{\mathbf{N}} x \, \mathrm{d}\mu_{\mathcal{F}} \right) \mathrm{d}\rho(\mathcal{F});$$

(iii) *There exists a normalized capacity $\rho : \mathscr{B}(\mathrm{Ult}(\mathcal{I})) \to \mathbf{R}$ such that*

$$\forall A \subseteq \mathbf{N}, \quad \nu(A) = \int_{\mathrm{Ult}(\mathcal{I})} \mu_{\mathcal{F}}(A) \, \mathrm{d}\rho(\mathcal{F}).$$

It is easy to see that if $\nu : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ is an *abstract upper density* [11, 14], that is, a normalized capacity which is also diffuse (i.e., $\nu(A) = 0$ for all $A \in \mathrm{Fin}$) and subadditive (namely, $\nu(A \cup B) \le \nu(A) + \nu(B)$ for all $A, B \subseteq \mathbf{N}$), then

$$\mathcal{Z}_\nu := \{A \subseteq \mathbf{N} : \nu(A) = 0\} \tag{1}$$

is an ideal which contains Fin (in particular, $\mathcal{Z} = \mathcal{Z}_{\mathsf{d}^\star}$) and $\nu$ is $\mathcal{Z}_\nu$-invariant. This gives us the following easy corollary:

**Corollary 1.2.** *If $\nu : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ is an abstract upper density, then there exists a normalized capacity $\rho : \mathscr{B}(\mathrm{Ult}(\mathcal{Z}_\nu)) \to \mathbf{R}$ such that*

$$\forall A \subseteq \mathbf{N}, \quad \nu(A) = \int_{\mathrm{Ult}(\mathcal{Z}_\nu)} \mu_{\mathcal{F}}(A) \, \mathrm{d}\rho(\mathcal{F}),$$

*where $\mathcal{Z}_\nu$ is the ideal defined in* (1).

Abstract upper densities (and, more generally, submeasures) have been commonly used in number theory, functional analysis, and economics. Examples include the upper asymptotic, upper analytic, upper Pólya, and upper Banach densities [21], exhaustive submeasures [17, 29], nonpathological and lower semicontinuous submeasures [12, 28], duals of exact games [24, 26], and, of course, finitely additive and $\sigma$-additive measures.

A remarkable example is suggested by the well-known characterization of analytic P-ideals $\mathcal{I}$ (i.e., analytic ideals such that if $(A_n)$ is a sequence in $\mathcal{I}$ then there exists $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathrm{Fin}$ for all $n \in \mathbf{N}$). Indeed, a result due to Solecki states that an ideal $\mathcal{I}$ is an analytic P-ideal if and only if there exists a lower semicontinuous submeasure $\varphi : \mathcal{P}(\mathbf{N}) \to [0, \infty]$ such that

$$\mathcal{I} = \mathrm{Exh}(\varphi) := \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\} \tag{2}$$

with $\|\mathbf{N}\|_\varphi < \infty$, where $\|A\|_\varphi := \lim_n \varphi(A \setminus [1, n])$ represents the "mass at infinity" of each $A \subseteq \mathbf{N}$ (see, e.g., [12, Theorem 1.2.5(b)]). Replacing $\|\cdot\|_\varphi$ with $\|\cdot\|_\varphi / \|\mathbf{N}\|_\varphi$ if necessary, we can assume without loss of generality that $\|\mathbf{N}\|_\varphi = 1$. Now, it is easy to see that $\|\cdot\|_\varphi$ is an abstract upper density and, in particular, it is $\mathcal{I}$-invariant. Therefore, thanks to Corollary 1.2, we obtain:

**Corollary 1.3.** *If $\mathcal{I}=\mathrm{Exh}(\varphi)$ is an analytic P-ideal as in (2), then there exists a normalized capacity $\rho:\mathscr{B}(\mathrm{Ult}(\mathcal{I}))\to\mathbf{R}$ such that*

$$\forall A\subseteq\mathbf{N},\quad \|A\|_\varphi=\int_{\mathrm{Ult}(\mathcal{I})}\mu_{\mathcal{F}}(A)\,\mathrm{d}\rho(\mathcal{F}).$$

On the other hand, it is also clear that every ideal $\mathcal{I}$ coincides with $\mathcal{Z}_\nu$, where $\nu:\mathcal{P}(\mathbf{N})\to\mathbf{R}$ is the $\{0,1\}$-valued abstract upper density such that $\nu(A)=1$ if and only if $A\notin\mathcal{I}$. Lastly, note that Theorem 1.1 holds also for normalized capacities $\nu$ which are not subadditive, e.g., the lower asymptotic density $\mathsf{d}_\star:\mathcal{P}(\mathbf{N})\to\mathbf{R}$ defined by $\mathsf{d}_\star(A):=\liminf_n|A\cap[1,n]|/n$ for all $A\subseteq\mathbf{N}$: in this case, the set $\{A\subseteq\mathbf{N}:\mathsf{d}_\star(A)=0\}$ is, of course, not closed under finite unions, however the normalized capacity $\mathsf{d}_\star$ is $\mathcal{Z}$-invariant.

The proof of Theorem 1.1 is given in Section 3. An abstract version in the context of Riesz spaces will be given in Theorem 2.3.

## 2. An abstract version on Riesz spaces

Recall that an *ordered vector space* is a real vector space $X$ with a compatible partial order $\le$, that is, $\alpha x\le\alpha y$ and $x+z\le y+z$ for all scalars $\alpha\ge 0$ and all vectors $x,y,z\in X$ with $x\le y$. An ordered vector space $X$ is a *Riesz space* when $X$ is also a lattice. In addition, $X$ is said to be *Archimedean* if $0\le nx\le y$ for some $x,y\in X$ and all $n\in\mathbf{N}$ implies $x=0$. Finally, a nonzero vector $e\in X$ is called a (*strong order*) *unit* if for each $x\in X$ there exists $\lambda\ge 0$ such that $-\lambda e\le x\le\lambda e$. In what follows, $X$ will be assumed to be an Archimedean Riesz space with unit $e$.

A vector subspace $N$ of $X$ is a *Riesz subspace* if $x\vee y\in N$ whenever $x,y\in N$. A vector subspace $N$ of $X$ is said to be an *order ideal* if it is solid, i.e., $x\in N$ whenever there exists $y\in N$ such that $|x|\le|y|$, where $|z|:=z\vee(-z)$. Since an order ideal $N$ contains the absolute values of its elements and $x\vee y=\frac{1}{2}(x+y+|x-y|)$ for all $x,y\in X$, an order ideal is automatically a Riesz subspace. We denote its positive cone by $N_+:=\{x\in N:x\ge 0\}$. Finally, an order ideal $N$ is said to be *uniformly closed* if $x\in N$ whenever there exist a sequence of vectors $(x_n)$ in $N$, a vector $y>0$, and a decreasing sequence of positive reals $(\varepsilon_n)$ such that $\lim_n\varepsilon_n=0$ and $|x_n-x|\le\varepsilon_n y$ for all $n\in\mathbf{N}$. A detailed theory of Riesz spaces can be found in [2].

We endow $X$ with the norm $\|\cdot\|:X\to[0,\infty)$ defined by

$$\forall x\in X,\quad \|x\|:=\inf\{\lambda\ge 0:|x|\le\lambda e\}. \tag{3}$$

Note that $\|\cdot\|$ is indeed a norm of $X$ which is also a Riesz norm, that is, $\|x\|\le\|y\|$ whenever $|x|\le|y|$. In particular, $(X,\|\cdot\|)$ is an $M$-space and its norm dual, discussed below, is an $AL$-space (see [2, pp. 93–98]).

**Remark 2.1.** Since $(X, \|\cdot\|)$ is a normed Riesz space, the positive cone $X_+$ is closed (see, e.g., [2, Theorem 2.21]). This immediately yields that $\{\lambda \geq 0 : |x| \leq \lambda e\}$ is closed. By the definition of $\|\cdot\|$, the infimum in (3) is thus attained. In particular, we have that $|x| \leq \|x\|e$ for all $x$ in $X$. Finally, since $\|\cdot\|$ is a Riesz norm, this latter fact implies that an order ideal is uniformly closed if and only if it is norm closed.

Let $X^\star$ be the norm dual of $X$. We endow $X^\star$ and any of its subsets with the weak$^\star$ topology and with the canonical ordering induced by the positive cone of $X$, i.e.,

$$X_+^\star := \{\xi \in X^\star : \langle x, \xi \rangle \geq 0 \text{ for all } x \in X_+\},$$

where $\langle x, \xi \rangle$ stands for the dual pairing. Finally, we denote the positive unit sphere by

$$\Delta := \{\xi \in X_+^\star : \langle e, \xi \rangle = 1\}.$$

By the Banach-Alaoglu Theorem, $\Delta$ is compact. Moreover, the set of extreme points of $\Delta$ is nonempty and compact as well.

Following [7], we say that a functional $V : X \to \mathbf{R}$ is:

(i) *normalized* if $V(\lambda e) = \lambda$ for all $\lambda \in \mathbf{R}$;
(ii) *monotone* if $V(x) \leq V(y)$ for all vectors $x \leq y$ in $X$;
(iii) *unit-additive* if $V(x + \lambda e) = V(x) + V(\lambda e)$ for all $x \in X$ and all $\lambda \geq 0$;
(iv) *unit-modular* if $V(x \vee \lambda e) + V(x \wedge \lambda e) = V(x) + V(\lambda e)$ for all $x \in X$ and all $\lambda \in \mathbf{R}$.

**Remark 2.2.** If a functional $V : X \to \mathbf{R}$ satisfies properties (i)–(iii), then $V$ is Lipschitz continuous of order 1: in fact,

$$V(x) = V(y + (x - y)) \leq V(y + \|x - y\|e) = V(y) + \|x - y\|$$

for all $x, y \in X$ so that, by symmetry, $|V(x) - V(y)| \leq \|x - y\|$.

If $N \subseteq X$ is a proper uniformly closed order ideal, we say that a functional $V : X \to \mathbf{R}$ is

(v) *$N$-invariant* if $V(x) = V(y)$ whenever $x - y \in N$.

An important class of $N$-invariant functionals are those that are also linear. In particular, we denote by $\Delta_N$ the subset of functionals in the positive unit sphere $\Delta$ that annihilate $N$ and we denote by $\mathcal{E}_N$ the set of extreme points of $\Delta_N$, that is,

$$\Delta_N := \Delta \cap N^\perp \quad \text{and} \quad \mathcal{E}_N := \text{ext}(\Delta_N), \tag{4}$$

where $N^\perp$ is the annihilator of $N$. Note that $\Delta_N$ is a nonempty, convex, and compact subset of $X^\star$ (cf. Proposition 2.9). By the Krein-Milman Theorem, also $\mathcal{E}_N$ is nonempty.

In what follows, we are going to show that a functional $V : X \to \mathbf{R}$ which satisfies properties (i)–(v) has to be necessarily a Choquet average of the continuous linear functionals in $\mathcal{E}_N$.

**Theorem 2.3.** *Let $X$ be an Archimedean Riesz space with unit $e$ and let $N$ be a proper uniformly closed order ideal of $X$. If $V : X \to \mathbf{R}$ is a functional which satisfies properties (i)–(v), then there exists a normalized capacity $\nu : \mathscr{B}(\mathcal{E}_N) \to \mathbf{R}$ such that*

$$\forall x \in X, \quad V(x) = \int_{\mathcal{E}_N} \langle x, \xi \rangle \, \mathrm{d}\nu(\xi). \tag{5}$$

In order to prove this result, we need four ancillary lemmas. From now on, let $X$ and $N$ be as in the statement of Theorem 2.3. We denote by $\dot{X}$ the quotient space $X/N$. For each $x \in X$ denote by $\dot{x} := \pi(x)$ the corresponding vector in $\dot{X}$, where $\pi : X \to \dot{X}$ is the canonical projection. Recall that $\pi$ is an onto lattice homomorphism (see, e.g., [2, Theorem 1.34]). If a functional $V : X \to \mathbf{R}$ is $N$-invariant, then we denote by $\dot{V} : \dot{X} \to \mathbf{R}$ the induced functional on the quotient space, that is, $\dot{V}(\dot{x}) = V(x)$, where $\dot{x} = \pi(x)$ for some $x \in X$. Since $V$ is $N$-invariant, the functional $\dot{V}$ is well defined. For, if $\dot{x} = \dot{y}$ for some $x, y \in X$, then $\pi(x-y) = 0$, that is $x - y \in N$, and we can conclude that $\dot{V}(\dot{x}) = V(x) = V(y) = \dot{V}(\dot{y})$.

**Lemma 2.4.** *$\dot{X}$ is an Archimedean Riesz space with unit $\dot{e}$.*

*Proof.* By [22, Theorem 5.1], $\dot{X}$ is an Archimedean Riesz space. We are only left to show that $\dot{e} = \pi(e)$ is a unit of $\dot{X}$. Consider $a \in \dot{X}$. It follows that $a = \pi(x)$ for some $x \in X$. Since $e$ is a unit of $X$, there exists $\lambda \geq 0$ such that $x \in [-\lambda e, \lambda e]$. Since $\pi$ is a lattice homomorphism, $\pi$ is a positive linear operator, yielding that $a \in [-\lambda \dot{e}, \lambda \dot{e}]$. Finally, $\dot{e}$ is nonzero. By contradiction, assume that $\pi(e) = \dot{e} = \dot{0}$. This would imply that $\lambda e \in N$ for all $\lambda \geq 0$. Since $e$ is a unit and $N$ an order ideal, we would have that for each $x \in X$ there exists $\lambda \geq 0$ such that $|x| \leq \lambda e$, yielding that $x \in N$ and, in particular, $X = N$, a contradiction with $N$ being proper. □

**Lemma 2.5.** *Let $V : X \to \mathbf{R}$ be a $N$-invariant functional. If $V$ satisfies properties (i)–(iv), so does $\dot{V}$.*

*Proof.* By Lemma 2.4, $\dot{e}$ is a unit of $\dot{X}$. By definition of $\dot{V}$, we have that

$$\forall \lambda \in \mathbf{R}, \quad \dot{V}(\lambda \dot{e}) = \dot{V}(\lambda \pi(e)) = \dot{V}(\pi(\lambda e)) = V(\lambda e) = \lambda,$$

proving that $\dot{V}$ is normalized. Fix $a, b \in \dot{X}$ such that $a \leq b$. By [2, p.17], there exist $x, y \in X$ such that $\pi(x) = a$, $\pi(y) = b$, and $x \leq y$. By the monotonicity of $V$ and the definition of $\dot{V}$, we obtain that $\dot{V}(a) = V(x) \leq V(y) = \dot{V}(b)$, proving that $\dot{V}$ is monotone. Since $\pi$ is linear and a lattice homomorphism, the remaining properties are straightforward to check. Fix $a \in \dot{X}$, so that $a = \pi(x)$ for some $x \in X$. It follows that for each $\lambda \geq 0$

$$\begin{aligned} \dot{V}(a + \lambda \dot{e}) &= \dot{V}(\pi(x + \lambda e)) = V(x + \lambda e) \\ &= V(x) + V(\lambda e) = \dot{V}(\pi(x)) + \dot{V}(\pi(\lambda e)) = \dot{V}(a) + \dot{V}(\lambda \dot{e}). \end{aligned}$$

Similarly, we have that for each $\lambda \in \mathbf{R}$

$$\begin{aligned}\dot{V}(a \vee \lambda\dot{e}) + \dot{V}(a \wedge \lambda\dot{e}) &= \dot{V}(\pi(x) \vee \pi(\lambda e)) + \dot{V}(\pi(x) \wedge \pi(\lambda e)) \\ &= \dot{V}(\pi(x \vee \lambda e)) + \dot{V}(\pi(x \wedge \lambda e)) \\ &= V(x \vee \lambda e) + V(x \wedge \lambda e) \\ &= V(x) + V(\lambda e) = \dot{V}(a) + \dot{V}(\lambda\dot{e}).\end{aligned}$$

We can conclude that $\dot{V}$ is unit-additive and unit-modular. □

**Lemma 2.6.** *If $V \in \Delta$ is $N$-invariant, then*

$$\dot{V} \in \dot{\Delta} := \{\dot{\xi} \in \dot{X}^\star : \langle \dot{x}, \dot{\xi}\rangle \geq 0 \text{ for all } \dot{x} \in \dot{X}_+ \text{ and } \langle \dot{e}, \dot{\xi}\rangle = 1\}. \quad (6)$$

*Proof.* Since $V \in \Delta$, it satisfies properties (i)–(iv). Since $V$ is $N$-invariant, $\dot{V}$ is well defined and clearly linear. By Lemma 2.5, $\dot{V}$ is normalized (thus, $\langle \dot{e}, \dot{V}\rangle = \dot{V}(\dot{e}) = 1$), monotone (thus, $\langle \dot{x}, \dot{V}\rangle = \dot{V}(\dot{x}) \geq \dot{V}(\dot{0}) = 0$ for all $\dot{x} \in \dot{X}_+$), and unit-additive. By Remark 2.2, $\dot{V}$ is continuous, proving the statement. □

In what follows, the set $\dot{\Delta}$ defined in (6) will be endowed with the weak$^\star$ topology $\sigma\left(\dot{\Delta}, \dot{X}\right)$. Recall that $\Delta_N$ is the set of linear functionals $\xi \in \Delta$ that annihilate $N$, as defined in (4).

**Lemma 2.7.** *The map $\Pi : \dot{\Delta} \to \Delta_N$ defined by*

$$\zeta \mapsto \zeta \circ \pi$$

*is an affine homeomorphism.*

*Proof.* First, we prove that $\Pi$ is well defined. Fix $\zeta \in \dot{\Delta}$ and define $\xi := \Pi(\zeta)$, that is, $\xi(x) = \zeta(\pi(x))$ for all $x \in X$. Since $\zeta$ and $\pi$ are positive linear operators, it follows that $\xi$ is linear and positive, that is, $\xi(x) = \zeta(\pi(x)) \geq \zeta(\dot{0}) = 0$ for all $x \in X_+$ as well as $\xi(e) = \zeta(\pi(e)) = \zeta(\dot{e}) = 1$. By Remark 2.2, $\xi$ is also continuous, proving that $\xi \in \Delta$. Since $\pi(x) = \dot{0}$ for all $x \in N$, we have that $\xi(x) = \zeta(\pi(x)) = \zeta(\dot{0}) = 0$ for all $x \in N$, yielding that $\xi \in \Delta_N$.

We now show that $\Pi$ is bijective. To this aim, fix $\zeta_1, \zeta_2 \in \dot{\Delta}$ and suppose that $\Pi(\zeta_1) = \Pi(\zeta_2)$, that is, $\zeta_1(\dot{x}) = \zeta_2(\dot{x})$ for all $x \in X$. Since $\pi$ is surjective, it follows that $\zeta_1 = \zeta_2$, hence $\Pi$ is injective. Next, fix $\xi \in \Delta_N$ and $x, y \in X$ such that $x - y \in N$. Since $N$ is an order ideal, it follows that $|x-y| \in N$ and $\xi(|x-y|) = 0$. Since $\xi \in X_+^\star$, this implies that $\xi(x) \leq \xi(y + |x-y|) = \xi(y)$ and similarly $\xi(y) \leq \xi(x)$, proving that $\xi$ is $N$-invariant. By Lemma 2.6, we can conclude that $\dot{\xi} \in \dot{\Delta}$. By definition of $\Pi$ and $\dot{\xi}$, we have that $\Pi(\dot{\xi})(x) = \dot{\xi}(\pi(x)) = \xi(x)$ for all $x \in X$, i.e., $\Pi(\dot{\xi}) = \xi$, proving that $\Pi$ is surjective.

Finally, by construction, $\Pi$ is affine and continuous. Since $\dot{\Delta}$ and $\Delta_N$ are Hausdorff and compact, this implies that $\Pi$ is a homeomorphism. □

We can now prove Theorem 2.3.

*Proof of Theorem 2.3.* By Lemma 2.4, $\dot{X}$ is an Archimedean Riesz space with unit $\dot{e}$. By Lemma 2.5, $\dot{V}$ satisfies properties (i)–(iv). We define

$$\mathscr{U}(\mathrm{ext}(\dot{\Delta})) := \{\{\dot{\xi} \in \mathrm{ext}(\dot{\Delta}) : f(\dot{\xi}) \geq t\} : f \in C(\mathrm{ext}(\dot{\Delta})),\, t \in \mathbf{R}\}$$

the lattice of upper level sets generating the Baire $\sigma$-algebra of $\mathrm{ext}(\dot{\Delta})$. By [7, Theorem 6], there exists a (unique outer continuous) capacity $\mu : \mathscr{U}(\mathrm{ext}(\dot{\Delta})) \to \mathbf{R}$ such that

$$\forall \dot{x} \in \dot{X}, \quad \dot{V}(\dot{x}) = \int_{\mathrm{ext}(\dot{\Delta})} \dot{\xi}(\dot{x})\, \mathrm{d}\mu(\dot{\xi}).$$

In addition, since $\dot{V}$ is normalized, we have $\mu(\mathrm{ext}(\dot{\Delta})) = 1$. By Lemma 2.7, $\Pi : \dot{\Delta} \to \Delta_N$ is an affine homeomorphism. This implies that

$$\Pi[\,\mathrm{ext}(\dot{\Delta})] = \mathrm{ext}(\Delta_N) =: \mathcal{E}_N.$$

If we define

$$\mathscr{U}(\mathcal{E}_N) := \{\{\xi \in \mathcal{E}_N : f(\xi) \geq t\} : f \in C(\mathcal{E}_N),\, t \in \mathbf{R}\},$$

then it is routine to check that

$$\mathscr{U}(\mathrm{ext}(\dot{\Delta})) = \{\Pi^{-1}(E) : E \in \mathscr{U}(\mathcal{E}_N)\}.$$

Define $\nu_0 : \mathscr{U}(\mathcal{E}_N) \to \mathbf{R}$ by $\nu_0(E) := \mu(\Pi^{-1}(E))$ for all $E \in \mathscr{U}(\mathcal{E}_N)$. Clearly, $\nu_0$ is a normalized capacity. Note also that for each $x \in X$ and for each $t \in \mathbf{R}$

$$\begin{aligned}\nu_0(\{\xi \in \mathcal{E}_N : \langle x, \xi\rangle \geq t\}) &= \mu\left(\{\dot{\xi} \in \mathrm{ext}(\dot{\Delta}) : \langle x, \Pi(\dot{\xi})\rangle \geq t\}\right)\\ &= \mu\left(\{\dot{\xi} \in \mathrm{ext}(\dot{\Delta}) : \langle \dot{x}, \dot{\xi}\rangle \geq t\}\right).\end{aligned}$$

By the definition of Choquet integral, this implies that

$$\forall x \in X, \quad V(x) = \dot{V}(\dot{x}) = \int_{\mathrm{ext}(\dot{\Delta})} \dot{\xi}(\dot{x})\, \mathrm{d}\mu(\dot{\xi}) = \int_{\mathcal{E}_N} \xi(x)\, \mathrm{d}\nu_0(\xi). \tag{7}$$

Since $\mathscr{U}(\mathcal{E}_N) \subseteq \mathscr{B}(\mathcal{E}_N)$, the claim (5) follows by defining the normalized capacity $\nu : \mathscr{B}(\mathcal{E}_N) \to \mathbf{R}$ as, for example,

$$\forall B \in \mathscr{B}(\mathcal{E}_N), \quad \nu(B) := \sup\{\nu_0(A) : A \in \mathscr{U}(\mathcal{E}_N), A \subseteq B\}.$$

By definition of Choquet integral and since $\nu$ extends $\nu_0$, the statement follows. □

We conclude the section with an example and a result. The example exhibits a class of proper uniformly closed order ideals. The result proves that these sets are the building blocks of any proper uniformly closed order ideal.

**Remark 2.8.** For each $\xi \in \Delta$, the set

$$N_\xi := \{x \in X : \xi(|x|) = 0\} \tag{8}$$

is a proper subset of $X$ (since $e \notin N_\xi$) with the following properties:

(a) $N_\xi$ is a vector subspace: if $x, y \in N_\xi$ and $\alpha, \beta \in \mathbf{R}$, then
$$0 \leq \xi(|\alpha x + \beta y|) \leq |\alpha|\xi(|x|) + |\beta|\xi(|y|) = 0, \quad \text{hence } \alpha x + \beta y \in N_\xi;$$

(b) $N_\xi$ is solid (and, in particular, a Riesz subspace): if $x \in X$ and $y \in N_\xi$ are such that $|x| \leq |y|$, then
$$0 \leq \xi(|x|) \leq \xi(|y|) = 0, \quad \text{hence } x \in N_\xi;$$

(c) $N_\xi$ is uniformly closed: by Remark 2.1, it is enough to check that $N_\xi$ is norm closed. Since $\|\cdot\|$ is a Riesz norm and $||x| - |y|| \leq |x - y|$ for all $x, y \in X$, we have that if $(x_n)$ is a sequence in $N_\xi$ which converges in norm to $x$, then $(|x_n|)$ converges in norm to $|x|$. Since $\xi$ is continuous, we can conclude that $\xi(|x|) = \lim_n \xi(|x_n|) = 0$, that is, $x \in N_\xi$.

Therefore $N_\xi$ is a proper uniformly closed order ideal of $X$.

**Proposition 2.9.** *If $N$ is a proper uniformly closed order ideal of an Archimedean Riesz space $X$ with unit $e$, then*
$$N = \bigcap\nolimits_{\xi \in \Delta_N} N_\xi,$$
*where $\Delta_N$ and $N_\xi$ are defined as in (4) and (8), respectively.*

*Proof.* By Remark 2.1 and since $N$ is a proper uniformly closed order ideal of $X$, then $N$ is norm closed and convex. Consider a point $z \in X \backslash N$. By the Hahn–Banach Theorem, there exists $\xi \in X^\star$ such that $\xi(z) > \xi(x)$ for all $x \in N$. We next show that $\xi(x) = 0$ for all $x \in N$ and $\xi$ can be chosen to be in $\Delta$. Since $N$ is a vector space, we have that $\xi(x) = 0$ for all $x \in N$ and, in particular, $\xi(z) \neq 0$. Since $N$ is a Riesz subspace, we have that the positive and negative part of each vector $x$ in $N$, that is $x^+$ and $x^-$, belong to $N$. By a Riesz–Kantorovich formula and since $X^\star$ is an $AL$-space, we have that $\xi = \xi^+ - \xi^-$, where $\xi^+, \xi^- \in X^\star_+$ and for each $x \geq 0$
$$\xi^+(x) = \sup\{\xi(y) : 0 \leq y \leq x\}.$$
Since $N$ is an order ideal, for each $x \in N$ we have that if $0 \leq y \leq x$, then $y \in N$. Since $\xi(N) = \{0\}$, this implies that $\xi^+(x) = 0$ for all $x \in N_+$, proving that $\xi^+(x^+) = \xi^+(x^-) = 0$ for all $x \in N$. We can conclude that $\xi^+(x) = \xi^+(x^+ - x^-) = \xi^+(x^+) - \xi^+(x^-) = 0$ for all $x \in N$. Since $\xi = \xi^+ - \xi^-$ and $\xi(N) = \{0\}$, it follows that $\xi^-(x) = 0$ for all $x \in N$. Note that either $\xi^+(z) \neq 0$ or $\xi^-(z) \neq 0$. Otherwise, since $\xi = \xi^+ - \xi^-$, we would have that $0 = \xi^+(z) - \xi^-(z) = \xi(z) \neq 0$, a contradiction. Assume $\xi^+(z) \neq 0$ (resp. $\xi^-(z) \neq 0$). In this case, we have that $\xi^+(e) \neq 0$ (resp. $\xi^-(e) \neq 0$). Otherwise, since $\xi^+ \in X^\star_+$ (resp. $\xi^- \in X^\star_+$) and $e$ is a unit, we would have that $\xi^+ = 0$ (resp. $\xi^- = 0$), a contradiction with $\xi^+(z) \neq 0$ (resp. $\xi^-(z) \neq 0$). In the first case, define $\bar{\xi} = \xi^+/\xi^+(e)$. In the second case, define $\bar{\xi} = \xi^-/\xi^-(e)$. Since $\xi^+, \xi^- \in X^\star_+$, we have that $\bar{\xi} \in \Delta$. Moreover, since $\xi^+(N) = \xi^-(N) = \{0\}$, we can conclude that $\bar{\xi}(z) \neq 0 = \bar{\xi}(x)$ for all $x \in N$ and, in particular, $\bar{\xi} \in \Delta_N$.

By Remarks 2.1 and 2.8 and since $N$ is an order ideal, $N_\xi$ is a proper norm closed order ideal which contains $N$ for all $\xi \in \Delta_N$, yielding that $M := \bigcap_{\xi \in \Delta_N} N_\xi$ has the same properties. By contradiction, assume that there exists $z \in M \setminus N$. By the initial part of the proof, there exists $\bar{\xi} \in \Delta_N$ such that $\bar{\xi}(z) \neq 0 = \bar{\xi}(x)$ for all $x \in N$. Since $z \in M$, we have that $z \in N_{\bar{\xi}}$. Since $\bar{\xi} \in \Delta$ and $\bar{\xi}(|z|) = 0$, we have that $0 \le \bar{\xi}(z^+), \bar{\xi}(z^-) \le \bar{\xi}(|z|) = 0$, yielding that $\bar{\xi}(z) = \bar{\xi}(z^+) - \bar{\xi}(z^-) = 0$, a contradiction. □

## 3. Proof of Theorem 1.1

Before we proceed with the proof of Theorem 1.1, we need some preliminary results related to ideal convergence. Given an ideal $\mathcal{I}$ on $\mathbf{N}$, a real-valued sequence $x = (x_n)$ is said to be $\mathcal{I}$-convergent to $\eta \in \mathbf{R}$, shortened as $\mathcal{I}\text{-}\lim x = \eta$, if $\{n \in \mathbf{N} : |x_n - \eta| \ge \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$. Accordingly, we define

$$c_0(\mathcal{I}) := \{x \in \mathbf{R}^{\mathbf{N}} : \mathcal{I}\text{-}\lim x = 0\},$$

that is, $c_0(\mathcal{I})$ is the set of real-valued sequences which are $\mathcal{I}$-convergent to 0. As usual, we denote by $\ell_\infty$ the space of real-valued bounded sequences. Recall that $\ell_\infty$ is an Archimedean Riesz space with unit $e = (1, 1, \ldots)$ (see, e.g., [1, p. 538]). Note that if $\mathcal{I} \neq \mathrm{Fin}$, then $c_0(\mathcal{I})$ is not contained in $\ell_\infty$ (choose, for example, $(x_n)$ such that $x_n = n$ if $n \in S$ and $x_n = 0$ otherwise, with $S \in \mathcal{I} \setminus \mathrm{Fin}$). The set of bounded $\mathcal{I}$-convergent sequences, $c_0(\mathcal{I}) \cap \ell_\infty$, has been studied in several works (see, e.g., [5, 9, 13, 18, 19, 20]).

In what follows, given a set $A \subseteq \mathbf{N}$, let $x = \mathbf{1}_A$ be the characteristic function of $A$, so that $x_n = 1$ if $n \in A$ and $x_n = 0$ otherwise (hence, $e = \mathbf{1}_{\mathbf{N}}$).

**Lemma 3.1.** *If $\mathcal{I}$ is an ideal on $\mathbf{N}$, then $c_0(\mathcal{I}) \cap \ell_\infty$ is a proper uniformly closed order ideal of $\ell_\infty$.*

*Proof.* It is easy to see that $c_0(\mathcal{I})$ is a vector subspace of $\mathbf{R}^{\mathbf{N}}$ and so is $\ell_\infty$. Since $\mathcal{I}\text{-}\lim e = 1$, we obtain that $c_0(\mathcal{I}) \cap \ell_\infty$ is a proper vector subspace of $\ell_\infty$. Consider $y \in c_0(\mathcal{I}) \cap \ell_\infty$ and $x \in \ell_\infty$ such that $|x| \le |y|$. By the definition of $\mathcal{I}$-convergence, it follows that $\mathcal{I}\text{-}\lim x = 0$, proving that $c_0(\mathcal{I}) \cap \ell_\infty$ is an order ideal. Finally, it is routine to check that $c_0(\mathcal{I}) \cap \ell_\infty$ is norm closed. By Remark 2.1, $c_0(\mathcal{I}) \cap \ell_\infty$ is uniformly closed, proving the statement. □

Denote by $ba$ the space of signed finitely additive measures $\mu : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ with finite total variation, that is, such that

$$\sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| : \{A_1, \ldots, A_n\} \text{ is a partition of } \mathbf{N} \right\} < \infty$$

(see, e.g., [1, Section 10.10]). Recall that $\ell_\infty^\star$ can be identified with $ba$ (see, e.g., [1, Theorem 14.4]) via the lattice isomorphism $T : \ell_\infty^\star \to ba$ defined by

$$\forall \xi \in \ell_\infty^\star, \forall A \subseteq \mathbf{N}, \quad T(\xi)(A) := \xi(\mathbf{1}_A).$$

We endow both $ba$ and the norm dual $\ell_\infty^\star$ with the weak$^\star$ topology. Note that $T$ is continuous and its inverse is given by

$$\forall \mu \in ba, \forall x \in \ell_\infty, \quad T^{-1}(\mu)(x) = \int_{\mathbf{N}} x \, \mathrm{d}\mu. \tag{9}$$

**Lemma 3.2.** *If $\mathcal{I}$ is an ideal on $\mathbf{N}$ and $N = c_0(\mathcal{I}) \cap \ell_\infty$, then*

$$T[\Delta_N] = \mathrm{M}(\mathcal{I}),$$

*where $\mathrm{M}(\mathcal{I})$ is the set of finitely additive probability measures $\mu$ such that $\mu(A) = 0$ for all $A \in \mathcal{I}$.*

*Proof.* Set $X = \ell_\infty$. By definition of $T$, it is immediate to see that $T(\xi)$ is a finitely additive probability measure for all $\xi \in \Delta$. Consider $\xi \in \Delta_N = \Delta \cap N^\perp$. Since $\xi \in N^\perp$, if $A \in \mathcal{I}$, then $\mathbf{1}_A \in N$, yielding that $T(\xi)(A) = \xi(\mathbf{1}_A) = 0$ for all $A \in \mathcal{I}$ and proving that $T[\Delta_N] \subseteq \mathrm{M}(\mathcal{I})$.

Conversely, fix $\mu \in \mathrm{M}(\mathcal{I})$. Given Equation (9), define $\xi := T^{-1}(\mu) \in \ell_\infty^\star$. It is straightforward to check that $\xi \in \Delta$. Consider $x \in N_+$. Since $\mathcal{I}$-$\lim x = 0$, $x \geq 0$, and $\mu \in \mathrm{M}(\mathcal{I})$, we have that $\{n \in \mathbf{N} : x_n \geq t\} \in \mathcal{I}$ and $\mu(\{n \in \mathbf{N} : x_n \geq t\}) = 0$ for all $t > 0$, proving that

$$\forall x \in N_+, \quad \xi(x) = \int_{\mathbf{N}} x \, \mathrm{d}\mu = \int_0^\infty \mu(\{n \in \mathbf{N} : x_n \geq t\}) \, \mathrm{d}t = 0. \tag{10}$$

Since $N$ is a Riesz subspace, we have that the positive and negative part of each vector $x$ in $N$, that is $x^+$ and $x^-$, belong to $N$. By (10), we can conclude that $\xi(x) = \xi(x^+) - \xi(x^-) = 0$ for all $x \in N$, proving that $\xi \in N^\perp$ and, in particular, that $\xi \in \Delta \cap N^\perp = \Delta_N$. This implies that $T(\xi) = \mu$, proving that $\mathrm{M}(\mathcal{I}) \subseteq T[\Delta_N]$. □

As a side result, we obtain a representation of $c_0(\mathcal{I}) \cap \ell_\infty$.

**Proposition 3.3.** *If $\mathcal{I}$ is an ideal on $\mathbf{N}$, then*

$$c_0(\mathcal{I}) \cap \ell_\infty = \left\{ x \in \ell_\infty : \int_{\mathbf{N}} |x| \, \mathrm{d}\mu = 0 \text{ for all } \mu \in \mathrm{M}(\mathcal{I}) \right\}.$$

*Proof.* Set $X = \ell_\infty$ and $N = c_0(\mathcal{I}) \cap \ell_\infty$. By Lemma 3.1, $N$ is a proper uniformly closed order ideal of $X$. By Proposition 2.9, Lemma 3.2, and Equation (9), we obtain that

$$\begin{aligned} N &= \bigcap\nolimits_{\xi \in \Delta_N} \{x \in X : \xi(|x|) = 0\} = \bigcap\nolimits_{\mu \in T[\Delta_N]} \{x \in X : T^{-1}(\mu)(|x|) = 0\} \\ &= \bigcap\nolimits_{\mu \in \mathrm{M}(\mathcal{I})} \{x \in X : \int_{\mathbf{N}} |x| \, \mathrm{d}\mu = 0\}, \end{aligned}$$

proving the statement. □

We are finally ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* (i) $\implies$ (ii) Let $X$ be the Archimedean Riesz space $\ell_\infty$ with unit $e = \mathbf{1}_{\boldsymbol{N}}$ and $S$ the Riesz subspace of sequences which take finitely many values. Define $N := c_0(\mathcal{I}) \cap \ell_\infty$. By Lemma 3.1, $N$ is a proper uniformly closed order ideal. Define the functional $V : X \to \mathbf{R}$ by

$$\forall x \in X, \quad V(x) := \int_{\mathbf{N}} x \, \mathrm{d}\nu.$$

Note that $V$ satisfies properties (i)–(iv) (see, e.g., [24, Proposition 4.11, Theorem 4.3, and Lemma 4.6]). We next show that it is $N$-invariant.

CLAIM 1. $V(x+z) = V(x)$ whenever $x \in S_+$ and $z \in N_+$.

*Proof.* Consider $x \in S_+$ and $z \in N_+$. Define for each $\varepsilon > 0$

$$A_\varepsilon := \{n \in \mathbf{N} : x_n + z_n \geq \varepsilon\} \setminus \{n \in \mathbf{N} : x_n \geq \varepsilon\}.$$

Fix $\varepsilon > 0$. If $K_\varepsilon := \{x_n : x_n < \varepsilon\}$ is empty, then $A_\varepsilon = \emptyset$ and $A_\varepsilon \in \mathcal{I}$. Since $x \in S$ and $\mathcal{I}$-$\lim z = 0$, if $K_\varepsilon$ is a nonempty set, then $K_\varepsilon$ is finite and

$$A_\varepsilon = \{n \in \mathbf{N} : x_n \in K_\varepsilon \text{ and } z_n \geq \varepsilon - x_n\} \subseteq \{n \in \mathbf{N} : z_n \geq \varepsilon - \max K_\varepsilon\} \in \mathcal{I},$$

proving that $A_\varepsilon \in \mathcal{I}$. Since $x, z \geq 0$, $\{n \in \mathbf{N} : x_n \geq \varepsilon\} \setminus \{n \in \mathbf{N} : x_n + z_n \geq \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$ (being empty). By the definition of Choquet integral and since $\nu$ is $\mathcal{I}$-invariant and $x, z \geq 0$, this implies that $V(x+z) = V(x)$. □

CLAIM 2. $V(x+z) \leq V(x)$ whenever $x \in S$ and $z \in N$.

*Proof.* Consider $x \in S_+$ and $z \in N$. Since $N$ is an order ideal, $|z| \in N_+$. By Claim 1 and since $V$ is monotone, it follows that $V(x+z) \leq V(x+|z|) = V(x)$. Next, consider $x \in S$. Since $e$ is a unit, there exists $\lambda > 0$ such that $x + \lambda e \geq 0$ and, clearly, $x + \lambda e \in S_+$. Since $V$ is unit-additive and normalized, it follows that

$$V(x+z) + \lambda = V(x + \lambda e + z) \leq V(x + \lambda e) = V(x) + \lambda,$$

proving the claim. □

CLAIM 3. $V$ is $N$-invariant.

*Proof.* Fix $x, y \in X$ such that $x - y \in N$. Since $S$ is dense in $X$, there exist two sequences $(x^k)$ and $(y^k)$ in $S$ which are norm convergent to $x$ and $y$, respectively. We obtain by Claim 2 that

$$\forall k \in \mathbf{N}, \quad V(y^k + x - y) \leq V(y^k) \text{ and } V(x^k + y - x) \leq V(x^k).$$

By Remark 2.2, $V$ is continuous. By passing to the limit, this implies that $V(x) \leq V(y)$ and $V(y) \leq V(x)$. □

By Theorem 2.3 and since $V$ satisfies properties (i)–(v), there exists a normalized capacity $\psi : \mathscr{B}(\mathcal{E}_N) \to \mathbf{R}$ such that

$$\forall x \in X, \quad V(x) = \int_{\mathcal{E}_N} \langle x, \xi \rangle \, \mathrm{d}\psi(\xi). \tag{11}$$

Define

$$\mathrm{F}(\mathcal{I}) := \{\mu_{\mathcal{F}} : \mathcal{F} \in \mathrm{Ult}(\mathcal{I})\}.$$

CLAIM 4. $T(\mathcal{E}_N) = \mathrm{F}(\mathcal{I})$.

*Proof.* By Lemma 3.2, the map from $\Delta_N$ to $\mathrm{M}(\mathcal{I})$, defined by $\xi \mapsto T(\xi)$, is an affine bijection. This implies that $T(\mathcal{E}_N) = \mathrm{ext}(\mathrm{M}(\mathcal{I}))$. The proof that $\mathrm{ext}(\mathrm{M}(\mathcal{I})) = \mathrm{F}(\mathcal{I})$ goes verbatim as in the case $\mathcal{I} = \{\emptyset\}$ (see, e.g., [1, p. 544]). $\square$

Define the capacity $\kappa : \mathscr{B}(\mathrm{F}(\mathcal{I})) \to \mathbf{R}$ by $\kappa(A) := \psi(T^{-1}(A))$. By the same arguments used in proving (7) and using (9), (11), and Claim 4, we obtain that

$$\forall x \in X, \quad V(x) = \int_{\mathrm{F}(\mathcal{I})} \left( \int_{\mathbf{N}} x \, \mathrm{d}\mu \right) \mathrm{d}\kappa(\mu),$$

Lastly, by a similar reasoning, since the map $\mathrm{Ult}(\mathcal{I}) \to \mathrm{F}(\mathcal{I})$ defined by $\mathcal{F} \mapsto \mu_{\mathcal{F}}$ is a homeomorphism, we get

$$\forall x \in X, \quad V(x) = \int_{\mathrm{Ult}(\mathcal{I})} \left( \int_{\mathbf{N}} x \, \mathrm{d}\mu_{\mathcal{F}} \right) \mathrm{d}\rho(\mathcal{F}),$$

which concludes the proof.

(ii) $\Longrightarrow$ (iii) Choose $x = \mathbf{1}_A$ with $A \subseteq \mathbf{N}$.

(iii) $\Longrightarrow$ (i) It follows by the fact that $\mu_{\mathcal{F}}$ is $\mathcal{I}$-invariant for all $\mathcal{F} \in \mathrm{Ult}(\mathcal{I})$. $\square$

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