ON SOME LOCALLY CONVEX FK SPACES

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Abstract. We provide necessary and/or sufficient conditions on vector spaces V of real sequences to be a Fréchet space such that each coordinate map is continuous, that is, to be a locally convex FK space.

In particular, we show that if $c_{00}(\mathcal{I}) \subseteq V \subseteq \ell_\infty(\mathcal{I})$ for some ideal $\mathcal{I}$ on ω, then V is a locally convex FK space if and only if there exists an infinite set $S \subseteq \omega$ for which every infinite subset does not belong to $\mathcal{I}$.

1. Introduction

Let $\mathbf{R}^\omega$ be the vector space of real sequences, endowed with the topology of pointwise convergence; hereafter, ω denotes the set of nonnegative integers. A topological vector space (V, τ) is said to be an *FK space* if $V \subseteq \mathbf{R}^\omega$, τ is completely metrizable, and the inclusion map $\iota : V \to \mathbf{R}^\omega$ is continuous. If, in addition, V admits neighborhood basis at 0 consisting of convex sets, then V is a *locally convex FK space*; equivalently, a locally convex FK space is a Fréchet vector subspace $V \subseteq \mathbf{R}^\omega$ for which the inclusion map ι is continuous, see e.g. [28, 29]. We refer the reader to [29, Chapter 4] for motivations on the study of locally convex FK spaces and their relation with summability theory.

Let $\mathcal{I}$ be an ideal on ω, that is, a proper hereditary collection of subsets of ω which is closed under finite unions. Unless otherwise stated, we assume that $\mathcal{I}$ contains the family of finite sets $\mathrm{Fin} := [\omega]^{<\omega}$. Among the most important examples, we find the ideal $\mathcal{Z}$ of asymptotic density zero sets, that is,

$$\mathcal{Z} := \left\{ S \subseteq \omega : \lim_{n\to\infty} \frac{|S \cap [0,n]|}{n+1} = 0 \right\}.$$

We write $\mathcal{I}^\star := \{S \subseteq \omega : S^c \in \mathcal{I}\}$ for its dual filter. If $A = (a_{n,k})$ is a nonnegative regular matrix, we denote its induced ideal by

$$\mathcal{I}_A := \left\{ S \subseteq \omega : \lim_n \sum_{k \in S} a_{n,k} = 0 \right\}. \quad (1)$$

Note that $\mathcal{Z} = \mathcal{I}_{C_1}$, where C_1 is usual Cesàro matrix. Ideals are regarded as subset of the Cantor space $\{0,1\}^\omega$.

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A real sequence $x = (x(n) : n \in \omega) \in \mathbf{R}^\omega$ is said to be $\mathcal{I}$-*convergent* to $\eta \in \mathbf{R}$, shortened as $\mathcal{I}\text{-}\lim x = \eta$, if $\{n \in \omega : |x(n) - \eta| > \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$; hence Fin-convergence coincides with ordinary convergence, and $\mathcal{Z}$-convergence with *statistical convergence*, see e.g. [8, 17]. Let us write $c(\mathcal{I})$ [$c_0(\mathcal{I})$, resp.] for the vector space of $\mathcal{I}$-convergent sequences [sequences $\mathcal{I}$-convergent to 0, resp.] and

$$c_{00}(\mathcal{I}) := \{x \in \mathbf{R}^\omega : \operatorname{supp} x \in \mathcal{I}\},$$

where $\operatorname{supp} x := \{n \in \omega : x(n) \neq 0\}$. Lastly, we let $\ell_\infty(\mathcal{I})$ be the vector space of $\mathcal{I}$-bounded sequences $x \in \mathbf{R}^\omega$, so that $\{n \in \omega : |x(n)| > M\} \in \mathcal{I}$ for some $M \in \mathbf{R}$.

With these premises, Connor proved in [6, Theorem 3.3] that:

Theorem 1.1. *$c(\mathcal{Z})$ cannot be endowed with a locally convex FK topology.*

Building on his methods, Kline [21, Theorem 1] extended it by showing that:

Theorem 1.2. *Let $A = (a_{n,k})$ be a nonnegative regular matrix with the property that $\lim_n \max_k a_{n,k} = 0$. Then $c(\mathcal{I}_A)$ does not admit a locally convex FK topology.*

An alternative proof of Theorem 1.2 has been given by Demirci and Orhan in [10, Theorem 3]. An analogue of Theorem 1.2 has also been proved for lacunary statistical convergence in [10]:

Theorem 1.3. *Let $\theta := (I_n)$ be an increasing sequence of nonempty consecutive finite intervals of ω such that $\lim_n |I_n| = \infty$ and define*

$$\mathcal{I}_\theta := \left\{S \subseteq \omega : \lim_{n \to \infty} \mu_n(S) = 0\right\}, \quad \textit{where} \quad \mu_n(S) := |I_n \cap S|/|I_n|. \tag{2}$$

Then $c(\mathcal{I}_\theta)$ does not admit a locally convex FK topology.

Notice that the ideal $\mathcal{I}_\theta$ defined in (2) is a special type of *generalized density ideal* in the sense of Farah, namely, an ideal of the type

$$\mathcal{I}_\varphi = \left\{S \subseteq \omega : \lim_{n \to \infty} \varphi_n(S) = 0\right\}, \tag{3}$$

where $\varphi = (\varphi_n)$ is a sequence of submeasures $\varphi_n : \mathcal{P}(\omega) \to [0, \infty]$ (that is, monotone subadditive maps with $\varphi_n(\emptyset) = 0$ and $\varphi_n(\{k\}) < \infty$ for all $n, k \in \omega$) supported on finite pairwise disjoint sets, see [14, Section 2.10] and [15]; as observed in [3, 4, 22], the theory of representability of certain analytic P-ideals may have some potential for the study of the geometry of Banach spaces.

Lastly, Connor and Temiszu [9] recently proved the following variant:

Theorem 1.4. *Let $\mathcal{I}$ be an ideal on ω with the property that there exists a partition (I_n) of ω into consecutive finite intervals such that, if $|S \cap I_n| \le 1$ for all $n \in \omega$, then $S \in \mathcal{I}$. Then both $c_0(\mathcal{I})$ and $\ell_\infty(\mathcal{I})$ do not admit a locally convex FK topology.*

A combinatorial characterization (with respect to the Katětov–Blass order) of ideals $\mathcal{I}$ which fulfill the above property can be found in [19].

2. Main results

The aim of this work is to close the above line of results introduced in Section 1 by proving, in particular, that the corresponding (negative) analogues hold if and only if $\mathcal{I}$ is tall (recall that an ideal $\mathcal{I}$ is said to be tall if every infinite set of nonnegative integers contains an infinite subset which is in $\mathcal{I}$), see Theorem 2.3 below. The proofs of all results follow in Section 3.

On the negative side, we have the following:

Theorem 2.1. *Let $\mathcal{I}$ be a tall ideal on ω and V be a proper vector subspace of $\mathbf{R}^\omega$ which contains $c_{00}(\mathcal{I})$. Then V does not admit a locally convex FK topology.*

On the positive side, however, we have:

Theorem 2.2. *Let $\mathcal{I}$ be a nontall ideal on ω and V be a dense vector subspace of $\mathbf{R}^\omega$ contained in $\ell_\infty(\mathcal{I})$. Then V admits a locally convex FK topology.*

As a consequence, thanks to Theorem 2.1 and Theorem 2.2, we have the claimed characterization which extends all the results given in Section 1:

Theorem 2.3. *Let $\mathcal{I}$ be an ideal on ω and V be a vector space such that*

$$c_{00}(\mathcal{I}) \subseteq V \subseteq \ell_\infty(\mathcal{I}).$$

Then V admits a locally convex FK topology if and only if $\mathcal{I}$ is not tall.

As we are going to see, our main results have a number of consequences.

2.1. Special spaces.
To start with, we get immediately by Theorem 2.3:

Corollary 2.4. *Let $\mathcal{I}$ be an ideal on ω. Then $c(\mathcal{I})$ admits a locally convex FK topology if and only if $\mathcal{I}$ is not tall.*

In particular, this proves that $c(\emptyset \times \mathrm{Fin})$ is a locally convex FK space, which is the first nontrivial positive example of a $c(\mathcal{I})$ space admitting such well-behaved topology. Here, we recall that the Fubini product $\emptyset \times \mathrm{Fin}$ is the ideal on ω^2 which is isomorphic,[1] e.g., to the ideal $\{S \subseteq \omega : \forall k \in \omega, \{n \in S : \nu_2(n) = k\} \in \mathrm{Fin}\}$ on ω, where $\nu_2(n)$ stands for the 2-adic valutation of a positive integer n and $\nu_2(0) := 0$, cf. [13, Section 1.2].

Recall also that an ideal $\mathcal{I}$ on ω is said to be a P-ideal if for every sequence $(S_n) \in \mathcal{I}^\omega$ there exists $S \in \mathcal{I}$ such that $S_n \setminus S$ is finite for all $n \in \omega$; in addition, for each $E \notin \mathcal{I}$, we write

$$\mathcal{I} \upharpoonright E := \{S \cap E : S \in \mathcal{I}\}$$

for the restriction of $\mathcal{I}$ on E. Accordingly, we have the following:

[1]Given ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on two countably infinite sets Ω_1 and Ω_2, respectively, we say that $\mathcal{I}_1$ is isomorphic to $\mathcal{I}_2$ if there exists a bijection $f : \Omega_1 \to \Omega_2$ such that $f[S] \in \mathcal{I}_2$ if and only if $S \in \mathcal{I}_1$.

Corollary 2.5. *Let $\mathcal{I}$ be an analytic P-ideal on ω which is not isomorphic to* Fin *or* $\emptyset \times \mathrm{Fin}$. *Then $c(\mathcal{I} \upharpoonright E)$ does not admit a locally convex FK topology for some $E \notin \mathcal{I}$.*

Corollary 2.6. *Let $\mathcal{I}$ be a nonmeager ideal on ω. Then $c(\mathcal{I})$ does not admit a locally convex FK topology.*

In addition, we can characterize the family of nonnegative regular matrices A for which $c(\mathcal{I}_A)$ has this property, where $\mathcal{I}_A$ has been defined in (1). Recall that a double real sequence $(a_{n,k} : n, k \in \omega)$ has Pringsheim limit $\eta \in \mathbf{R}$, shortened as $\mathrm{P}\text{-}\lim a_{n,k} = \eta$ if for each $\varepsilon > 0$ there exists $n_0 \in \omega$ such that $|a_{n,k} - \eta| < \varepsilon$ for all $n, k \geq n_0$, see e.g. [26, Section 4.2]. In particular, if $A = (a_{n,k})$ is a nonnegative regular matrix (so that, in particular, $\lim_n a_{n,k} = 0$ for all $k \in \omega$), the property $\lim_n \max_k a_{n,k} = 0$ of Theorem 1.2 can be rewritten equivalently as $\mathrm{P}\text{-}\lim a_{n,k} = 0$.

Corollary 2.7. *Let A be a nonnegative regular matrix. Then $c(\mathcal{I}_A)$ admits a locally convex FK topology if and only if* $\limsup_n \max_k a_{n,k} > 0$.

Corollary 2.7 may be considered as an improvement of Kline's result [21] and Demirci and Orhan's result [10] in the sense that it provides necessary and sufficient conditions, though the papers [10, 21] give just sufficient conditions for the space $c(\mathcal{I}_A)$ not to have a locally convex FK topology.

Similarly, we extend Theorem 1.3 for the class of generalized density ideals $\mathcal{I}_\varphi$ in the sense of Farah with an explicit condition:

Corollary 2.8. *Let $\mathcal{I}_\varphi$ be a generalized density ideal as in (3). Then $c(\mathcal{I}_\varphi)$ admits a locally convex FK topology if and only if* $\limsup_n \max_k \varphi_n(\{k\}) > 0$.

An analogous characterization holds for summable ideals, that is, ideals on ω of the type

$$\mathcal{I}_f := \{S \subseteq \omega : \sum\nolimits_{n \in S} f(n) < \infty\},$$

where $f : \omega \to [0, \infty)$ is a function such that $\sum_n f(n) = \infty$:

Corollary 2.9. *Let $\mathcal{I}_f$ be a summable ideal on ω. Then $c(\mathcal{I}_f)$ admits a locally convex FK topology if and only if* $\limsup_n f(n) > 0$.

Corollary 2.4 studies whether the sequence space $c(\mathcal{I}) = \{x \in \mathbf{R}^\omega : Ix \in c(\mathcal{I})\}$ is a locally convex FK space, where I is the infinite identity matrix. On the same lines of [24], given an infinite matrix $A = (a_{n,k})$, one could ask whether the same results hold for the space

$$c_A(\mathcal{I}) := \{x \in d_A : Ax \in c(\mathcal{I})\},$$

where

$$d_A := \{x \in \mathbf{R}^\omega : Ax \text{ is well defined}\}.$$

Here, Ax stands for the sequence whose n-th term is the series $\sum_k a_{n,k}x_k$, provided that it converges. Note that $c_A := c_A(\mathrm{Fin})$ is usually called *convergence domain* of A, see e.g. [29, p. 3].

Based on several properties of FK-spaces and matrix maps, we obtain the following extension of Corollary 2.4:

Proposition 2.10. *Let A be an infinite matrix and $\mathcal{I}$ be an ideal on ω. Then:*

(i) *$c_A(\mathcal{I})$ admits a locally convex FK topology, provided that $\mathcal{I}$ is not tall;*

(ii) *$c_A(\mathcal{I})$ does not admit a locally convex FK topology, provided that there exists a tall ideal $\mathcal{J}$ on ω such that $c_{00}(\mathcal{J}) \subseteq c_A(\mathcal{I}) \subsetneq \mathbf{R}^\omega$.*

2.2. **Further results.** Quite interestingly, we have the following non-inclusion:

Corollary 2.11. *Let $\mathcal{I}$ be a tall ideal on ω and $\mathcal{J}$ be a nontall ideal on ω. Then*

$$c_{00}(\mathcal{I}) \setminus \ell_\infty(\mathcal{J}) \neq \emptyset,$$

namely, there exists a real sequence supported on $\mathcal{I}$ which is not $\mathcal{J}$-bounded.

It has been conjectured by DeVos [11] that, if V is a locally convex FK space, then either $\ell_\infty \subseteq V$ or $V \cap \{0,1\}^\omega$ is a meager set in $\{0,1\}^\omega$. The case where $\{0,1\}^\omega \subseteq V$ has been proved by Bennett and Kalton in [1]. The case where $V = c_A := \{x \in \mathbf{R}^\omega : Ax \in c\}$, for some summability matrix A such that $c_{00} \subseteq c_A$, has been shown in [18, 20]. For related results, see also [24, 27]. Here, we provide some additional positive examples (notice that $c(\mathcal{I})$ is nonseparable whenever $\mathcal{I} \neq \mathrm{Fin}$ by [25, Lemma 2.1]):

Corollary 2.12. *Let $\mathcal{I}$ be a nontall ideal on ω. Then $c(\mathcal{I})$ is a locally convex FK space and $c(\mathcal{I}) \cap \{0,1\}^\omega$ is meager in $\{0,1\}^\omega$.*

To conclude, we show that spaces $c_0(\mathcal{I})$ do not have stronger properties than being FK spaces unless $\mathcal{I} = \mathrm{Fin}$.

For instance, recall that a locally convex FK space V is said to be an *AK space* if $c_{00} \subseteq V$ and (e_n) is a Schauder basis for V, where $e_n(n) = 1$ and $e_n(k) = 0$ for all distinct $n, k \in \omega$, see [29, Definition 4.2.13].

Proposition 2.13. *Let $\mathcal{I}$ be an ideal on ω. Then $c_0(\mathcal{I})$ is an AK space if and only if $\mathcal{I} = \mathrm{Fin}$.*

On a similar note, recall that a locally convex FK space V is said to be a *BK space* if its metric is induced by a norm, or equivalently, if V is also a Banach space, see [29, p. 55] or [7].

Proposition 2.14. *Let $\mathcal{I}$ be an ideal on ω. Then $c_0(\mathcal{I})$ is included in some BK space if and only if $\mathcal{I} = \mathrm{Fin}$.*

It is worth to remark that the key feature inside the proof of Proposition 2.14 has been incidentally noted in [8, Section 1].

3. PROOFS

On the same lines of [6, 9, 21], we will need the following characterization:

Theorem 3.1. *Let V be a dense proper subspace of $\mathbf{R}^\omega$. Then V admits a locally convex FK topology if and only if there exist $B \subseteq c_{00}$ and $y \in \mathbf{R}^\omega$ such that*

$$\sup_{x \in B} |x \cdot y| = \infty \quad \text{and} \quad \sup_{x \in B} |x \cdot v| < \infty \text{ for all } v \in V.$$

where $x \cdot y := \sum_n x(n)y(n)$.

Proof. It follows by [28, Theorems 15.1.1 and 15.2.7] and [2, Proposition 1]. □

Also the following property of locally convex FK spaces will be useful:

Lemma 3.2. *Let X and Y be locally convex FK spaces, with topologies generated by families of seminorms $\mathcal{A}$ and $\mathcal{B}$, respectively. Let also $T : X \to \mathbf{R}^\omega$ be a continuous linear map and define*

$$\hat{X} := T^{-1}[Y].$$

Then $\hat{X}$ is a locally convex FK space with topology generated by the family of seminorms $\mathcal{A} \cup \{b \circ T : b \in \mathcal{B}\}$.

Proof. See [28, Lemma 5.5.10]. □

We are ready for the proofs of our main results.

Proof of Theorem 2.1. Note that V contains c_{00}, hence it is a dense proper vector subspace of $\mathbf{R}^\omega$. It follows by Theorem 3.1 that V does not admit a locally convex FK topology if and only if

$$(\forall v \in V, \sup_{x \in B} |x \cdot v| < \infty) \implies (\forall y \in \mathbf{R}^\omega, \sup_{x \in B} |x \cdot y| < \infty)$$

for all $B \subseteq c_{00}$; note that this is always well defined since $x \in c_{00}$. To this aim, fix a subset $B \subseteq c_{00}$ and suppose that $y \in \mathbf{R}^\omega$ is fixed and $\sup_{x \in B} |x \cdot y| = \infty$. Hence, it is enough to show that there exists $v \in V$ such that $\sup_{x \in B} |x \cdot v| = \infty$.

For, define

$$\forall n \in \omega, \quad \kappa(n) := \sup_{x \in B} |x(n)y(n)|.$$

FIRST CASE: $\kappa(n) = \infty$ FOR SOME $n \in \omega$. For each $k \in \omega$, let e_k be the k-th unit vector of ℓ_∞, namely, the sequence $(e_k(n) : n \in \omega)$ such that $e_k(k) = 1$ and $e_k(n) = 0$ otherwise. Then $\sup_{x \in B} |x(n)e_n(n)| = \infty$. Since $e_n \in c_{00} \subseteq V$, we conclude that $\sup_{x \in B} |x \cdot v| = \infty$ for some $v \in V$.

SECOND CASE: $\kappa(n) < \infty$ FOR ALL $n \in \omega$. In this case, there exists a sequence $(x_n : n \in \omega)$ of elements of B such that $|x_n \cdot y| \geq 2^n$ for all $n \in \omega$. Now let us define three increasing sequences (k_n), (s_n), and (m_n) in ω as it follows: set $m_0 := 0$,

$s_0 := \max \operatorname{supp} x_0$ and let k_0 be an integer such that $x_0(k_0)y(k_0) \neq 0$; then, for each positive integer n, define

$$m_n := \left\lfloor \sum\nolimits_{i \le \tilde{s}_{n-1}} \kappa(i) \right\rfloor + n, \quad s_n := \max \bigcup\nolimits_{i \le n} \operatorname{supp} x_{m_i},$$

where by convenience $\tilde{s}_n := \max\{s_n, k_n\}$, and

$$k_n := \max\left(\operatorname{supp} x_{m_n} \cap \operatorname{supp} y\right).$$

Note that k_n is well defined and it is greater than $\tilde{s}_{n-1}$ because

$$\begin{aligned} \left| \sum\nolimits_{i > \tilde{s}_{n-1}} x_{m_n}(i) y(i) \right| &\ge |x_{m_n} \cdot y| - \sum\nolimits_{i \le \tilde{s}_{n-1}} \kappa(i) \\ &\ge 2^{m_n} - \sum\nolimits_{i \le \tilde{s}_{n-1}} \kappa(i) \ge 2^{m_n} - m_n > 0. \end{aligned}$$

Hence, we have by construction $x_{m_n}(k_n)y(k_n) \neq 0$ and $s_n \ge k_n > s_{n-1}$ for all $n \ge 1$; in particular, $\max \operatorname{supp} x_{m_n} = s_n$.

At this point, let $S \subseteq \{k_n : n \in \omega\}$ be an infinite set which belongs to $\mathcal{I}$, which exists because $\mathcal{I}$ is tall, and denote its increasing enumeration by $(k_{t_n} : n \in \omega)$. Lastly, let v be the real sequence supported on S such that $v(k_{t_0}) := 0$ and, recursively,

$$v(k_{t_n}) = \frac{1}{x_{m_{t_n}}(k_{t_n})} \left(n + \sum\nolimits_{i<n} x_{m_{t_i}}(k_{t_i}) v(k_{t_i}) \right)$$

for all positive integer n. It follows by construction that $\operatorname{supp} v \subseteq S$, so that $v \in c_{00}(\mathcal{I}) \subseteq V$, and, for all $n \ge 1$,

$$x_{m_{t_n}} \cdot v = \sum\nolimits_{i \le s_{t_n}} x_{m_{t_n}}(i) v(i) = \sum\nolimits_{i \le n} x_{m_{t_n}}(k_{t_i}) v(k_{t_i}) = n$$

Therefore $\sup_{x \in B} |x \cdot v| \ge \sup_{n \in \omega} (x_{m_{t_n}} \cdot v) = \infty$, which completes the proof. □

Proof of Theorem 2.2. By hypothesis there exists an infinite subset $S \subseteq \omega$ such that $W \notin \mathcal{I}$ whenever $W \subseteq S$ is an infinite subset. Denote its increasing enumeration by $(s_n : n \in \omega)$ and define

$$B := \left\{ \sum\nolimits_{i \le n} 2^{-i} e_{s_i} : n \in \omega \right\}.$$

Let also $y \in \mathbf{R}^\omega$ such that $y(n) = 2^n$ for all $n \in \omega$. Hence, it follows that $B \subseteq c_{00}$ and $\sup_{x \in B} |x \cdot y| = \infty$. On the other hand, if v is a sequence in V (and, in particular, $v \in \ell_\infty(\mathcal{I})$) then there exists $M \in \mathbf{R}$ such that $A := \{n \in \omega : |v(n)| >$

$M\} \in \mathcal{I}$. By the definition of S, we have that $F := A \cap S \in \mathrm{Fin}$, hence

$$\begin{aligned}\sup\nolimits_{x\in B} |x \cdot v| &= \sup\nolimits_{n\in\omega} \left|\sum\nolimits_{i\le n} 2^{-i} e_{s_i} \cdot v\right| \\ &\le \sup\nolimits_{n\in\omega} \left|\sum_{s_i \in \operatorname{supp} v \cap F, i\le n} 2^{-i} v(s_i)\right| + \sup\nolimits_{n\in\omega} \left|\sum_{s_i \in \operatorname{supp} v \setminus F, i\le n} 2^{-i} v(s_i)\right| \\ &\le \sum\nolimits_{i\in F} |v(i)| + M \cdot \sup\nolimits_{n\in\omega} \sum_{s_i \in \operatorname{supp} v \setminus F, i\le n} 2^{-i} \\ &\le \sum\nolimits_{i\in F} |v(i)| + 2M < \infty.\end{aligned}$$

It follows by Theorem 3.1 that V admits a locally convex FK topology. □

Proof of Corollary 2.5. Thanks to [13, Corollary 1.2.11], there exists $E \notin \mathcal{I}$ such that $\mathcal{I} \upharpoonright E$ is tall (cf. also [23, Remark 2.6]; accordingly, with the notation of [13, p. 10], observe that $(\mathrm{Fin}\times\emptyset)^\perp = \emptyset\times\mathrm{Fin}$, $(\mathrm{Fin}\oplus\mathcal{P}(\omega))^\perp = \mathrm{Fin}$, and $\mathrm{Fin}^\perp = \mathcal{P}(\omega)$). The claim follows by Corollary 2.4. □

Proof of Corollary 2.6. It is well known that if $\mathcal{I}$ is a nonmeager ideal then it is tall, see e.g. [5, Proposition 2.4]. The claim follows by Corollary 2.4. □

Proof of Corollary 2.7. Thanks to [12, Proposition 7.2], cf. also [16, Proposition 2.34], the ideal $\mathcal{I}_A$ is tall if and only if $\lim_n \max_k a_{n,k} = 0$. The claim follows by Corollary 2.4. □

Proof of Corollary 2.8. As in the previous proof, it is enough to observe that $\mathcal{I}_\varphi$ is tall if and only if $\lim_n \max_k \varphi_n(\{k\}) > 0$, and use Corollary 2.4. □

Proof of Corollary 2.9. It is easy to check that $\mathcal{I}_f$ is tall if and only if $\lim_n f(n) = 0$, and use Corollary 2.4 as in the previous proofs. □

Proof of Proposition 2.10. (i) Thanks to [29, Theorem 4.3.8] and Corollary 2.4, both d_A and $c(\mathcal{I})$ are locally convex FK spaces. Hence there exist nonempty families $\mathcal{A}$ and $\mathcal{B}$ of seminorms on d_A and $c(\mathcal{I})$, respectively, which generate their topologies, with $|\mathcal{A}| \le \omega$ and $|\mathcal{B}| \le \omega$; here, one may want to recall that we can choose $\mathcal{A} = \bigcup_n \{p_n, q_n\}$, where $p_n(x) := |x_n|$ and $q_n(x) := \sup_m \left|\sum_{k\le m} a_{n,k} x_k\right|$ for all $n \in \omega$ and $x \in d_A$. At this point, let

$$T : d_A \to \mathbf{R}^\omega$$

be the function defined by $Tx := Ax$ for all $x \in d_A$. Since T is a linear map between two locally convex FK spaces, then T is continuous, thanks to [29, Theorem 4.2.8]. It follows by Lemma 3.2 that $T^{-1}[c(\mathcal{I})]$, that is, $c_A(\mathcal{I})$, is a locally convex FK space whose topology is generated by the family of seminorms

$$\mathcal{A} \cup \{b \circ T : b \in \mathcal{B}\}.$$

(ii) It follows by Theorem 2.1 □

Proof of Corollary 2.11. Note that $V := c_{00}(\mathcal{I})$ is a dense proper vector subspace of $\mathbf{R}^\omega$. Suppose that $V \subseteq \ell_\infty(\mathcal{J})$, so that V admits a locally convex FK topology by Theorem 2.2. However, this would be in contradiction with Theorem 2.1. □

Proof of Corollary 2.12. The first part follows by Corollary 2.4. If $\mathcal{I}$ is not tall then it is meager, cf. Corollary 2.6. Therefore $c(\mathcal{I}) \cap \{0,1\}^\omega = \{\mathbf{1}_A : A \in \mathcal{I} \cup \mathcal{I}^\star\}$, which is a meager subset of $\{0,1\}^\omega$. □

Proof of Proposition 2.13. It is known that $c_0 = c_0(\mathrm{Fin})$ is a Banach space for which (e_n) is a Schauder basis. Suppose now that $\mathcal{I} \neq \mathrm{Fin}$. Then $c_0(\mathcal{I})$ is non-separable by [25, Lemma 2.1]. However, every AK space is necessarily separable, completing the proof. □

Proof of Proposition 2.14. Thanks to [29, Theorem 4.2.11], a sequence space V is included in some BK space if and only if there exists a sequence $y \in \mathbf{R}^\omega$ such that

$$\forall x \in V, \quad x(n) = O(y(n)) \text{ as } n \to \infty.$$

If $\mathcal{I} = \mathrm{Fin}$ choose $y = (1, 1, \ldots)$. If $\mathcal{I} \neq \mathrm{Fin}$ such sequence y cannot exist. Indeed, let us suppose for the sake of contradiction that we can find such y, and fix an infinite set $S \in \mathcal{I}$. At this point, define the sequence $x \in \mathbf{R}^\omega$ by $x(n) = n(|y_n|+1)$ if $n \in S$ and $x(n) = 0$ otherwise. Then $x \in c_0(\mathcal{I})$ and, on the other hand, $x(n) \neq O(y(n))$ as $n \to \infty$. □

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