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Loops in de Sitter space

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Abstract: We discuss general one and two-loops banana diagrams with arbitrary masses on the de Sitter spacetime by using direct methods of dS quantum field theory in the dimensional regularization approach. In the one-loop case we also compute the effective potential for an $O(N)$ model in $d = 4$ dimension as an explicit function of the cosmological constant Λ, both exactly and perturbatively up to order Λ. For the two-loop case we show that the calculation is made easy thanks to a remarkable Källén-Lehmann formula that has been in the literature for a while. We discuss the divergent cases at $d = 3$ using a contiguity formula for generalized hypergeometric functions and we extract the dominant term at $d = 4$ proving a general formula to deal with a divergent hypergeometric series.

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1 Introduction

General Relativity and the Standard Model of Particle Physics are the greatest successes of theoretical physics and are among the highest achievements of mankind. Starting from the researches of the pre-Socratic philosophers of the Greek world of the VI century B.C., it took almost three millennia to reach this level of comprehension of Nature [1].

On its side, General Relativity offers the natural framework for gravity and cosmology, not only by providing equations that plausibly describe the evolution of the universe and of the objects that populate it, but also as a source of ideas and technical tools for improving and sometimes changing the way we observe and measure the sky; examples are the already mature gravitational lensing and microlensing techniques and the brand new multi-messenger astronomy. On the other side, the Standard Model gives a deep understanding of the microscopic world and there is no need to recall its great successes here.

The mysterious fact remains that all attempts to construct a single coherent theory that includes both of them have consistently failed. Nevertheless, cosmology, astrophysics and the physics of elementary particles are nowadays inextricably interconnected and even an incomplete understanding of the history and the dynamics of the universe requires some form of coexistence of the two theories. At the moment, the best thing that is available is Quantum Field Theory (QFT) on a curved background, possibly including a quantum linearized gravitational field and backreaction.

QFT in an inertial frame of flat Minkowski space is quite well understood, mainly at the perturbative level but non-perturbative methods have also been developed over the years. The necessity of improving our computational skills for a deeper understanding of the Standard Model has recently generated a great flurry of activity; new strategies for computing Feynman-like integrals in QFT and GR (especially for gravitational waves) are abundant in contemporary literature. Here are a few examples:

- the proposed reformulation of perturbative QFT in terms of positive Grassmannian geometry in a complexified momentum space, leading to the notion of Amplituhedron and its generalizations [27]–[30];
- the traditional method of integration by parts to uncover relations among different Feynman integrals in order to reduce their calculation to a subset of Master Integrals; the system of differential equations they have to satisfy [31]–[32];
- the cohomological techniques based on the interpretation of Feynman integrals as periods of generators of a suitable twisted-cohomology [33]–[53];
- viewing Feynman integrals as a linear space with the intersection product of the twisted cohomology used scalar product to determine bases of Master Integrals, Picard Fuchs-equations and so on [54]–[63];
- other methods that consist in investigating the (algebraic) geometry underlying Feynman integrals relating them to suitable Calabi-Yau varieties [64]–[70].

The crucial common ingredient of all the above approaches is the Fourier momentum-space representation of Feynman integrals. Unfortunately, such representation is not available on curved backgrounds because translation invariance, which is the foundation of Fourier analysis, pertains only to flat space;[1] the main road for calculating amplitudes in curved spacetimes is

[1]Even in flat space, a non trivial topology may limit the effectiveness of the momentum space formulation.

to do it in position space. This approach is surprisingly efficient also in Minkowski space: in [2] we presented a study of one and two-loop diagrams in position space, improving by a margin the existing literature and also providing new insights into the method of partial integration.

In this paper we perform the same loop calculations in the de Sitter universe working in position space at Euclidean times. The status of the de Sitter spacetime is however rather exceptional. Even in the absence of a linear momentum space, a complete harmonic analysis has been available for some time for de Sitter quantum field theory [3–6] but this possibility has not yet been fully exploited in applications (see however [7, 8]). We intend to fill this gap and show how this bunch of rigorous methods and results is also very effective to compute exact expressions for loop integrals which may in turn prove to be relevant for cosmology.

In particular, we consider one and two-loop diagrams with no external legs, which are the crucial ones for computing the effective potential for the Standard Model in presence of a positive cosmological constant (see [9] for the case of zero cosmological constant); calculations are performed without relying on particular choices of coordinates on the de Sitter manifold; the results are explicit exact formulae for loop integrals with two and, respectively, three independent scalar fields; three different masses enter in the loop, no conformal invariance is supposed, no room for bootstrapping anything.

Banana integrals are of great interest for several reasons in the recent literature since, through various kind of dualities, they are related to several equivalent mathematical/physical problems. However, the true reason we are interested in them is indeed more direct physically: they represent particular kinds of vacuum fluctuations. The knowledge of the associated integrals allow us to determine the effective potential for the given field, $O(N)$ in the present case. At a given loop, one should know all the vacuum integrals, but at 1 and 2 loops the banana integrals are essentially all one needs. The results from the $O(N)$ case can then be adapted to the Standard Model fields in order to compute the effective potential of the whole Standard Model of Particles as been computed in [9] for the case of a flat space time, with the aim of apply it to early time cosmology. However, such an application requires for the inclusion of the correction due to the presence of a cosmological constant, which at early times must be considered nonperturbatively. This is why later we concentrate on the one and two loop integrals.

Our results generalize to the de Sitter spacetime the state of the art of the knowledge available in flat space [2, 11]. We expect that the methods and the tools exposed in this papers will be useful to face other calculations involving loops on the (complex) de Sitter manifold.

We remark that he computations in de Sitter and anti de Sitter are quite different and are not related by an analytic continuation. This fact has been known for a long time (see e.g. [3–5]), but it seems to be often forgotten so that several authors underestimate it. A clear way to see it is to look at the complexification of the starting manifolds, connecting to the euclidean manifolds on which the quantization of fields is based. These manifolds are different in the two cases and cannot analytically deformed into each other. The KL formulas we used for the calculations are very different. This is reflected in the fact that the 1-loop effective potentials in the two cases are indeed not related by an analytic continuation: while the effective potential in de Sitter is a function of Λ and $\log \Lambda$, the effective potential in anti de Sitter is a function of $\sqrt{-\Lambda}$ and $\log(-\Lambda)$. As we explain in [20], "This is due to the symmetry

$\nu \to \nu$ of the Wightman function in the de Sitter case (that is a symmetry of Legendre functions of the first kind) a symmetry that anti de Sitter quantum fields do not share."

The paper is organized as follows: in section 2, after recalling some generalities about de Sitter QFT, we write multiloop (banana) diagrams with no external legs in a form suitable for their computation in the rest of the paper. Section 3 is devoted to the one-loop diagram with two independent masses.

The proof of the main formula (3.4) is performed in position space and crucially relies on the identification of the two-point function as a Legendre function of the first kind (a Ferrers function of the first kind for the Euclidean propagator) as opposed to an equivalent but otherwise less specific hypergeometric function ${}_2F_1$.

In section 4 we discuss an application: we compute the one-loop effective potential for the $O(N)$ model on the de Sitter manifold in dimension $d = 4$, exact at one-loop. Computations of effective potentials on the de Sitter background have sporadically appeared in the literature [10]–[13] but there exists to date no rigorous deduction based on a formulation sufficiently general to allow for systematic strategies as is the case in flat spacetime.

In section 6 we compute the flat limit of the effective potential which exactly reproduces the well-known result in Minkowski space. It is worthwhile to underline already here that this is indeed a non-trivial fact, as the flat limit relates quantities that are the outcome of integrals of different functions over different manifolds and is not just the asymptotic value of some functions close to a given event. We express the exact one-loop potential as a function of the cosmological constant and show how to compute it perturbatively at any desired order, when the cosmological constant is small.

In section 7 we focus on the two-loop diagram with no external legs with three arbitrary masses; we study the diagram in dimensional regularization for an arbitrary complex dimension of the de Sitter manifold and produce exact formulae for it in eqs. (7.4), (7.14), (7.17), (7.22) and (7.24). The crucial ingredient allowing for a full solution of the problem is the Källén-Lehmann representation of the product of two propagators that was explicitly constructed a few years ago [7] and later independently rediscovered in [8].

In the remaining sections we use our explicit formulae to study the diagram at integer spacetime dimension. We start with a discussion of odd negative spacetime-dimensions which also provides a check for our formulae; then we produce a detailed study of the non-trivial cases $d = 2$ and $d = 3$.

Extracting the finite part in $d = 4$ can be done using the contiguity relations discussed in appendix (C) precisely as it is done for the $d = 3$ case in appendix D. In $d = 4$ the situation is complicated by the presence of a double pole; formulae for the finite part of the diagrams are too long to be reproduced here. We limit the discussion to the residues that we compute by using the Erdélyi-Tricomi theorem.

2 Banana integrals on the Euclidean sphere

2.1 Geometry

The easiest way to look at either the real or the complex d-dimensional de Sitter manifolds is to visualize them as subsets of the complex Minkowski spacetime with one spacelike dimension more. This viewpoint allows for a natural description of the fundamental tubular domains encoding the spectral condition of de Sitter quantum field theory [3–5].

Let therefore M_{d+1} be the real $(d+1)$-dimensional Minkowski space-time and $M^{(c)}_{d+1}$ be its complexification. In a chosen Lorentz frame the scalar product of two (complex) events is

$$z_1 \cdot z_2 = z_1^0 z_2^0 - z_1^1 z_2^1 - \ldots - z_1^d z_2^d. \tag{2.1}$$

The future cone V_+ and the future and past tubes $T_\pm$ of the (complex) Minkowski spacetime are defined as follows:

$$V_+ = \{x \in M_{d+1} \;:\; x \cdot x > 0, \quad x^0 > 0\}, \tag{2.2}$$
$$T_\pm = \{x + iy \in M^{(c)}_{d+1} \;:\; y \in \pm V_+\}\ . \tag{2.3}$$

The tubes $T_\pm$ are the geometrical sets corresponding to the spectral condition which requires the positivity of the spectrum of the energy operator in every Lorentz frame [14]. This is the very characteristic property at the heart of QFT at zero temperature. All the well-known features of QFT, and above all the Euclidean formulation and renormalization, depend on it.

The real de Sitter universe may be represented as the one-sheeted hyperboloid immersed in M_{d+1}:

$$dS_d = \{x \in M_{d+1} \;:\; x \cdot x = -R^2 = -1\}\ ; \tag{2.4}$$

the same definition, *mutatis mutandis*, holds for its complexification:

$$dS^{(c)}_d = \{z \in M^{(c)}_{d+1} \;:\; z \cdot z = -R^2 = -1\}\ . \tag{2.5}$$

The de Sitter invariant complex variable ζ is the scalar product in the ambient spacetime of two complex events $z_1, z_2 \in dS^{(c)}_d$:

$$\zeta = z_1 \cdot z_2. \tag{2.6}$$

Two real events x_1 and x_2 in dS_d are timelike separated if and only if

$$(x_1 - x_2)^2 = -2 - 2x_1 \cdot x_2 > 0. \tag{2.7}$$

The future and past tuboids $\mathcal{T}_\pm$ are the intersections of the ambient tubes $T_\pm$ with the complex de Sitter manifold:

$$\mathcal{T}_\pm = \{x + iy \in X^{(c)}_d \;:\; y \in \pm V_+\}\ . \tag{2.8}$$

Harmonic analysis. A natural basis of plane-wave solutions of the de Sitter Klein-Gordon equation

$$\Box\psi(z) + m^2\psi(z) = 0, \tag{2.9}$$

is parameterized by the choice of a lightlike vector $\xi \in C^+ = \partial V^+$ and a complex number λ which is in turn parametrized by the spacetime dimension and another complex number ν as follows:

$$\psi_\lambda(z,\xi) = (z \cdot \xi)^\lambda, \qquad \lambda = -\frac{d-1}{2} + i\nu. \tag{2.10}$$

The parameters λ and ν are related to the complex mass squared:

$$m^2 = -\lambda(\lambda + d - 1) = \frac{(d-1)^2}{4} + \nu^2. \tag{2.11}$$

Plane waves are well-defined and analytic in each of the tubes $\mathcal{T}^+$ and $\mathcal{T}^-$ [3, 4]. Of course their squared mass is real and positive only when:
a) ν is real; this correspond in a group-theoretical language to the principal series of unitary representations of the Lorentz group;
b) ν is purely imaginary such the $|\nu| < \frac{d-1}{2}$; this correspond to the complementary series of unitary representations of the Lorentz group.

In de Sitter spacetime there is no global timelike Killing vector. A spectral condition may however be formulated as the following requirement of *normal analyticity* [4]: the two-point distributions are boundary values of functions analytic in the domain $\mathcal{T}_- \times \mathcal{T}_+$; this condition, together with de Sitter invariance and the Canonical Commutation Relations, selects a unique two-point function[2] for any de Sitter Klein-Gordon field [4]:

Main result: [3, 4] *the normally analytic canonical Wightman function of a de Sitter Klein-Gordon field in spacetime dimension d whose complex mass squared $m^2 = \frac{(d-1)^2}{4} + \nu^2$ is parametrized by a complex parameter ν, has the following spectral representation*[3] *in plane waves and is holomorphic for $z_1 \in \mathcal{T}_-$ and $z_2 \in \mathcal{T}_+$:*

$$W_\nu^d(z_1, z_2) = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)e^{\pi\nu}}{2^{d+1}\pi^d} \int_\gamma (\xi \cdot z_1)^{-\frac{d-1}{2} - i\nu} (\xi \cdot z_2)^{-\frac{d-1}{2} + i\nu}\, \alpha(\xi) \tag{2.12}$$

$$= w_\nu^d(\zeta) = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{2(2\pi)^{d/2}} (\zeta^2 - 1)^{-\frac{d-2}{4}}\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta) \tag{2.13}$$

$$= \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\, {}_2F_1\left(\frac{d-1}{2} + i\nu, \frac{d-1}{2} - i\nu;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right). \tag{2.14}$$

In standard coordinates, the $(d-1)$-form $\alpha(\xi)$ in eq. (2.12) is written

$$\alpha(\xi) = (\xi^0)^{-1} \sum_{j=1}^{d} (-1)^{j+1} \xi^j\, d\xi^1 \ldots \widehat{d\xi^j} \ldots d\xi^d\,. \tag{2.15}$$

γ denotes any $(d-1)$-cycle in the forward light-cone C^+. (2.12) does not depend on the choice of γ being the integral of a closed differential form. In particular we may choose the unit sphere $\mathbf{S}_{d-1}$ (equipped with its canonical orientation):

$$\gamma_0 = \mathbf{S}_{d-1} = C^+ \cap \{\xi\ :\ \xi^0 = 1\} = \{\xi \in C^+ : {\xi^1}^2 + \ldots + {\xi^d}^2 = 1\}. \tag{2.16}$$

[2] It is called — for a strange historical habit — Bunch-Davis vacuum, but W. Thirring has been the first to find it.

[3] Remark: what in the recent literature has been called the "split representation" of the de Sitter propagator is nothing but a special application of the general analytic formula (2.12). We believe that the name we gave to it in 1994 of "Fourier-like" or "plane-waves representation" of the two-point function.

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With this choice $\alpha(\xi)$ coincides with the rotation invariant measure $d\xi$ on $\mathbf{S}_{d-1}$ normalized as follows:

$$\omega_d = \int_{\gamma_0} d\xi = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{2.17}$$

By chosing any two points in the respective tubes one can show the validity of eq. (2.13). A crucial fact is that the Legendre function [15] of the first kind in that formula

$$P^{\mu}_{\nu}(\zeta) = \frac{1}{\Gamma(1-\mu)} \left(\frac{\zeta+1}{\zeta-1}\right)^{\frac{\mu}{2}} F\left(-\nu,\ 1+\nu\ ;\ 1-\mu\ ;\ \frac{1-\zeta}{2}\right) \tag{2.18}$$

is holomorphic in the cut-plane $\mathbf{C} \setminus (-\infty,\ 1]$ but the reduced two-point function $w^d_\nu(\zeta)$ is holomorphic in the larger domain[4]

$$\zeta = z_1 \cdot z_2 \in \Delta = \{\mathbf{C} \setminus (-\infty,\ -1]\}, \tag{2.19}$$

i.e. everywhere except on the causality cut (2.7); this is the *maximal analyticity property.*

When ν is either real or is purely imaginary and such that $|\nu| < \frac{d-1}{2}$ the corresponding two-point function (2.12) is *positive-definite* and admits a direct quantum probabilistic interpretation.

When $\frac{d-1}{2} + i\nu = -n$, where n is zero or a positive integer, a more involved but yet acceptable quantum interpretation of the above formula is also possible; on the de Sitter universe there exist *tachyonic fields* having no counterpart in flat space [16, 17].

Finally, the Schwinger function (in short: the propagator) is the restriction of the maximally analytic two-point function to the Euclidean sphere. It can be obtained as follows: in eq. (2.12) choose the two points as follows

$$z_1 = \begin{pmatrix} \mathrm{sh}(-i\epsilon) \\ 0 \\ \vdots \\ 0 \\ \mathrm{ch}(-i\epsilon) \end{pmatrix}, \qquad z_2(s) = \begin{pmatrix} \mathrm{sh}(is) \\ 0 \\ \vdots \\ 0 \\ \mathrm{ch}(is) \end{pmatrix}, \qquad 0 < s < \pi, \tag{2.20}$$

so that $z_1 \cdot z_2(s) = -\cos(s - i\epsilon)$; we get the following expression for the propagator

$$G_\nu(-\cos s) = \frac{\Gamma(\frac{d-1}{2} + i\nu)\Gamma(\frac{d-1}{2} - i\nu)}{2(2\pi)^{d/2}} (\sin s)^{-\frac{d-2}{2}} \mathbf{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(-\cos s) \tag{2.21}$$

where $\mathbf{P}^{\mu}_{\rho}(z)$ is the so called "Legendre function on the cut" or Ferrers function of the first kind [15] (see appendix A). It is important to keep in mind that Ferrers functions $\mathbf{P}^{\alpha}_{\beta}(z)$ and Legendre functions $P^{\alpha}_{\beta}(z)$ are holomorphic in different cut-planes; as regards Ferrers function this is

$$\Delta_2 = \mathbf{C} \setminus \{(-\infty, -1] \cup [1, \infty)]\}. \tag{2.22}$$

[4]The prefactor $(\zeta^2-1)^{-\frac{d-2}{4}}$ exactly compensates the singularity of $P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta)$ at $\zeta = 1$, making the reduced two-point function regular there.

2.2 Banana integrals

We are now ready to write the n-loop banana integrals on the sphere with $n+1$ propagators:

$$I_n(\nu_1, \ldots, \nu_{n+1}, d) = \int G^d_{\nu_1}(x_0 \cdot x) G^d_{\nu_2}(x_0 \cdot x) \ldots G^d_{\nu_{n+1}}(x_0 \cdot x) \sqrt{g}\, dx; \tag{2.23}$$

here x varies on the de Sitter sphere, x_0 is a fixed reference point over there and $\sqrt{g}\, dx$ is the rotation invariant measure. By integrating over the angles we get

$$I_n(\nu_1, \ldots, \nu_{n+1}, d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\pi G^d_{\nu_1}(-\cos s) G^d_{\nu_2}(-\cos s) \ldots G^d_{\nu_{n+1}}(-\cos s)(\sin s)^{d-1} ds. \tag{2.24}$$

The aim of this paper is to actually compute the two and three-lines banana integrals. Here we add some information on the three lines case: this is explicitly written as the integral of the product of three Ferrers function on the interval $(-1, 1)$ which is the projection on the plane Θ of the Euclidean sphere:

$$I_3(\nu_1, \nu_2, \nu_3, d) = K_3(\nu_1, \nu_2, \nu_3, d) \int_{-1}^1 \mathbf{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_1}(u) \mathbf{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_2}(u) \mathbf{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_3}(u)(1-u^2)^{-\frac{d-2}{4}} du, \tag{2.25}$$

with

$$K_3(\nu_1, \nu_2, \nu_3, d) = \frac{\prod_{j=1}^3 \Gamma(\frac{d-1}{2} - i\nu_j)\Gamma(\frac{d-1}{2} + i\nu_j))}{2^{2+\frac{3d}{2}} \pi^d \Gamma\left(\frac{d}{2}\right)}. \tag{2.26}$$

In a previous paper [7] another integral of three Legendre functions has been computed:

$$\begin{aligned} h_d(\lambda, \nu, \kappa) &= \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, (u^2-1)^{-\frac{d-2}{4}}\, du \\ &= \frac{2^{\frac{d}{2}}}{(4\pi)^{\frac{3}{2}} \Gamma\left(\frac{d-1}{2}\right)} \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\lambda + i\epsilon'\nu + i\epsilon''\kappa}{2}\right)}{\prod_{\epsilon,\epsilon'\epsilon''=\pm1} \Gamma\left(\frac{d-1}{2} + i\epsilon\lambda\right) \Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right) \Gamma\left(\frac{d-1}{2} + i\epsilon''\kappa\right)}. \end{aligned} \tag{2.27}$$

It was far from obvious and made possible by a mix of geometrical ideas with analytical and probabilistic tools. The steps involved in the proof-computation also gave rise to many interesting quantities having possibly geometrical interpretations that are not yet fully explored in their mathematical and physical consequences.

An output of the above result is an explicit Källén-Lehmann representation of the product of two Wightman two-point functions

$$w^d_\lambda(\zeta) w^d_\nu(\zeta) = \int_{-\infty}^\infty \rho_d(\lambda, \nu, \kappa) w^d_\kappa(\zeta) \kappa d\kappa, \tag{2.28}$$

where

$$\rho_d(\lambda, \nu, \kappa) = \frac{1}{2^d \pi^{\frac{d-1}{2}} \kappa \Gamma\left(\frac{d-1}{2}\right)} \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\lambda + i\epsilon'\nu + i\epsilon''\kappa}{2}\right)}{\prod_{\epsilon=\pm1} \Gamma\left(\frac{i\epsilon\kappa}{2}\right) \Gamma\left(\frac{1}{2} + \frac{i\epsilon\kappa}{2}\right) \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa}{2}\right) \Gamma\left(\frac{d+1}{4} + \frac{i\epsilon\kappa}{2}\right)}. \tag{2.29}$$

This formula was later rediscovered in [8]. Contrary to what happens in flat space, the Källén-Lehmann weight is here a meromorphic function of the three mass variables; this fact in

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particular implies that particle decays that in flat space are forbidden by mass subadditivity, can take place in the de Sitter universe [7].

It turns out that the direct evaluation of (2.25) is extremely difficult; the geometry of the complex de Sitter universe is less helpful here than it was in the evaluation of the previous integral (2.27). The knowledge of the Källén-Lehmann representation (2.28), which is deeply rooted in the harmonic analysis that we have recalled above, offers to us one opportunity to solve the problem; we are going to describe that construction later, in section 7.

3 1-loop: the bubble

Let us start with the already nontrivial two-line case, i.e. the bubble:

$$I(\lambda,\nu,d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\pi G_\lambda(-\cos s)G_\nu(-\cos s)(\sin s)^{d-1}ds \tag{3.1}$$

$$= K(\lambda,\nu,d)\int_{-1}^{1}\mathbf{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)\mathbf{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u)du, \tag{3.2}$$

$$K_d(\lambda,\nu) = \frac{\Gamma(\frac{d-1}{2}-i\lambda)\Gamma(\frac{d-1}{2}+i\lambda)\Gamma(\frac{d-1}{2}-i\nu)\Gamma(\frac{d-1}{2}+i\nu)}{2\left(2\sqrt{\pi}\right)^d\Gamma\left(\frac{d}{2}\right).} \tag{3.3}$$

To the best of our knowledge, as simple as it may look, the integral at the r.h.s. of eq. (3.2) is not listed anywhere in the literature. We will compute it in three different ways.

Using the Wronskian. The method that gives the cleanest result is based on the properties of the Wronskian of two Legendre functions. We summarize this result in the following

Formula 1

$$I(\lambda,\nu,d) = \frac{\Gamma\left(1-\frac{d}{2}\right)}{2^d\pi^{\frac{d}{2}}(\lambda^2-\nu^2)}\left(\frac{\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)} - \frac{\Gamma\left(\frac{d-1}{2}-i\lambda\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}-i\lambda\right)\Gamma\left(\frac{1}{2}+i\lambda\right)}\right). \tag{3.4}$$

To give a proof of the above formula let us consider two solutions u^μ_ν and v^μ_σ of the Legendre equation, with $u^\mu_\nu(z)$ standing for either $\mathbf{P}^\mu_\nu(z)$ or $\mathrm{Q}^\mu_\nu(z)$ and, independently, $v^\mu_\sigma(z)$ standing for either $\mathbf{P}^\mu_\sigma(z)$ or $\mathrm{Q}^\mu_\sigma(z)$. Then

$$\int_a^b u^\mu_\nu(z)v^\mu_\sigma(z)(\nu-\sigma)(\sigma+\nu+1)dz = \Big[(1-z^2)^{\frac{1}{2}}(\sigma+\mu)(\sigma-\mu+1)u^\mu_\nu(z)v^{\mu-1}_\sigma(z)$$
$$-(1-z^2)^{\frac{1}{2}}(\nu+\mu)(\nu-\mu+1)u^{\mu-1}_\nu(z)v^\mu_\sigma(z)\Big]_a^b. \tag{3.5}$$

This is the main Wronskian equation. Its derivation and more details are given in appendix A (see also [15, 3.12 (1) p. 169]). If $a=-1$ and $b=1$, the integral on the l.h.s. converges when $-1 < \mathrm{Re}\,\mu < 1$.

In this section we will examine the case when $u^\mu_\nu = \mathbf{P}^\mu_\nu$, $v^\mu_\sigma = \mathbf{P}^\mu_\sigma$ which is relevant to the subject of this paper. The cases when $u^\mu_\nu = \mathbf{P}^\mu_\nu$, $v^\mu_\sigma = \mathrm{Q}^\mu_\sigma$, and when $u^\mu_\nu = \mathrm{Q}^\mu_\nu$, $v^\mu_\sigma = \mathrm{Q}^\mu_\sigma$ are discussed in appendix A.

Let us thus substitute $u^\mu_\nu = \mathbf{P}^\mu_\nu$ and $v^\mu_\sigma = \mathbf{P}^\mu_\sigma$ in (3.5):

$$(\nu-\sigma)(\nu+\sigma+1)\int_a^b \mathbf{P}^\mu_\nu(z)\mathbf{P}^\mu_\sigma(z)\,dz = \Big[(\sigma+\mu)(\sigma-\mu+1)(1-z^2)^{\frac12}\mathbf{P}^\mu_\nu(z)\mathbf{P}^{\mu-1}_\sigma(z) \\ -(\nu+\mu)(\nu-\mu+1)(1-z^2)^{\frac12}\mathbf{P}^{\mu-1}_\nu(z)\mathbf{P}^\mu_\sigma(z)\Big]_a^b \tag{3.6}$$

and evaluate the r.h.s. of this equation under the conditions $a=-1$, $b=1$ and $0<\mu<1$. To do this it suffices to evaluate the first term in the r.h.s. of (3.6), since the second term is obtained by exchanging ν and σ and by a global change of sign.

Since $0<\mu<1$, as $z\in(0,\ 1)$, $z\to 1$, eq. (A.8) shows that

$$(\sigma+\mu)(\sigma-\mu+1)(1-z^2)^{\frac12}\mathbf{P}^\mu_\nu(z)\mathbf{P}^{\mu-1}_\sigma(z)\sim C(1-z)^{\frac12-\frac{\mu}{2}-\frac{\mu}{2}+\frac12}=C(1-z)^{1-\mu}\to 0\ . \tag{3.7}$$

As $z\in(-1,\ 1)$, $z\to -1$ we use eqs. (A.11) and (A.12) to get

$$(\sigma+\mu)(\sigma-\mu+1)(1-z^2)^{\frac12}\mathbf{P}^\mu_\nu(z)\mathbf{P}^{\mu-1}_\sigma(z)\to\frac{2\pi^{-1}\sin(\pi\nu)\Gamma(\mu)\Gamma(1-\mu)}{\Gamma(1+\sigma-\mu)\Gamma(-\sigma-\mu)}\ . \tag{3.8}$$

Putting together the contributions of the two terms in the r.h.s. of (3.6) gives

$$\int_{-1}^1 \mathbf{P}^\mu_\nu(z)\mathbf{P}^\mu_\sigma(z)\,dz = 2\pi^{-1}\Gamma(\mu)\Gamma(1-\mu)\Phi_{\mathbf{P}}(\nu,\sigma,\mu), \tag{3.9}$$

where

$$\Phi_{\mathbf{P}}(\nu,\sigma,\mu)=\frac{\frac{\sin(\pi\sigma)}{\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)}-\frac{\sin(\pi\nu)}{\Gamma(1+\sigma-\mu)\Gamma(-\sigma-\mu)}}{(\nu+\sigma+1)(\nu-\sigma)}. \tag{3.10}$$

$\Phi_{\mathbf{P}}(\nu,\sigma,\mu)$ extends to an entire function of all its arguments, it is symmetric in ν and σ, invariant under the involution $\nu\to-\nu-1$ and vanishes at $\mu=0$. Equation (3.9) remains valid, by analytic continuation, for $-1<\operatorname{Re}\mu<1$. Combining (3.9)–(3.10) with (3.3) we obtain the beautiful formula 3.4 for the bubble.

Here are a few consequences of the above result:

1. The bubble is regular at $d=2$:

$$I(\lambda,\nu,2)=\frac{\left(\psi\left(\frac12-i\lambda\right)+\psi\left(\frac12+i\lambda\right)-\psi\left(\frac12-i\nu\right)-\psi\left(\frac12+i\nu\right)\right)}{4\pi\left(\lambda^2-\nu^2\right)}. \tag{3.11}$$

2. In odd spacetime dimension the formula becomes very simple; for instance at $d=3$

$$I(\lambda,\nu,3)=\frac{\lambda\coth(\pi\lambda)-\nu\coth(\pi\nu)}{4\pi\lambda^2-4\pi\nu^2}. \tag{3.12}$$

3. At $d=4$ we encounter the first divergence. The Laurent expansion of the formula near $d=4$ gives

$$I(\lambda,\nu,4)\simeq-\frac{1}{8\pi^2(d-4)}+\frac{1-\gamma+\log(4\pi)}{16\pi^2} \\ -\frac{\left(4\lambda^2+1\right)\left(\psi\left(\frac32-i\lambda\right)+\psi\left(\frac32+i\lambda\right)\right)-\left(4\nu^2+1\right)\left(\psi\left(\frac32-i\nu\right)+\psi\left(\frac32+i\nu\right)\right)}{64\pi^2\left(\lambda^2-\nu^2\right)} \\ +\mathrm{O}(d-4). \tag{3.13}$$

4 The O(N) model: a summary

Let $E \cong \mathbb{R}^N$ be the N dimensional Euclidean vector space with the standard scalar product denoted by $\langle \cdot \mid \cdot \rangle$ and $O(N)$ the corresponding orthogonal group. Let us consider a scalar multi-component field

$$\phi : S_d \longrightarrow E, \tag{4.1}$$

where S_d is the (Euclidean) de Sitter sphere of radius R. The radius R and the cosmological constant

$$\Lambda = \frac{(d-1)(d-2)}{2R^2} \tag{4.2}$$

will reappear when necessary (see section (6)). For the moment we take $R = 1$.

The action for ϕ is the quartic $O(N)$-invariant action

$$S[\phi] = \int_{S_d} \left[\Lambda_0 + \frac{1}{2}\langle \partial_\mu \phi \mid \partial^\mu \phi \rangle + \frac{m_0^2}{2}\langle \phi \mid \phi \rangle + \frac{c_0}{4}\langle \phi \mid \phi \rangle^2 \right] \sqrt{g}\, d^d x, \tag{4.3}$$

where m_0 and c_0 are respectively the bare mass and bare self-coupling constant; x^μ are local coordinates on the sphere. We included also an extra cosmological constant Λ_0 for convenience when renormalizing.

We will compute the effective potential at $d = 4$ for the constant configuration

$$\bar{\phi} \equiv \varphi\, e_0, \tag{4.4}$$

where φ is a real constant and e_0 is a given vector of norm 1 in E. Of course a nonzero expectation value of ϕ breaks the symmetry down to $O(N-1)$. After choosing any orthonormal basis $\{e_j\}_{j=0}^{N-1}$ of E whose first element is e_0, we may write

$$\phi = (\varphi + \psi_0)e_0 + \sum_{j=1}^{N-1} \psi_j e_j, \tag{4.5}$$

so that

$$S[\phi] = \int_{S_d} \left[\Lambda_0 + \frac{m_0^2}{2}\varphi^2 + \frac{c_0}{4}\varphi^4 \right] \sqrt{g}\, d^d x + \sum_{a=1}^{4} S_a[\psi; \varphi], \tag{4.6}$$

where

$$S_1[\psi; \varphi] = \int_{S_d} \left[m_0^2 \varphi \psi_0 + c_0 \varphi^3 \psi_0 \right] \sqrt{g}\, d^d x, \tag{4.7}$$

$$S_2[\psi; \varphi] = \frac{1}{2}\int_{S_d} \Big[\sum_{j=0}^{N-1} \partial_\mu \psi_j \partial^\mu \psi_j + (m_0^2 + 3c_0\varphi^2)\psi_0^2 + (m_0^2 + c_0\varphi^2) \sum_{j=1}^{N-1} \psi_j^2 \Big] \sqrt{g}\, d^d x, \tag{4.8}$$

$$S_3[\psi; \varphi] = \int_{S_d} \left[c_0 \varphi \psi_0 \sum_{j=0}^{N-1} \psi_j^2 \right] d^d x, \tag{4.9}$$

$$S_4[\psi; \varphi] = \frac{c_0}{4}\int_{S_d} \sqrt{g} \Big[\sum_{j=0}^{N-1} \psi_j^2 \Big]^2 \sqrt{g}\, d^d x. \tag{4.10}$$

The effective potential $\mathcal{V}(\varphi)$ is defined as follows

$$\exp(-\Omega_d \mathcal{V}(\varphi)) = \int [\prod_j D\psi_j] \exp(-S[\phi]); \tag{4.11}$$

here Ω_d is the volume of the Euclidean de Sitter spacetime (which is finite) and $[\prod_j D\psi_j]$ is the formal path integral measure. By construction, S_1 does not contribute to the effective potential; at 1-loop, we get

$$\mathcal{V}(\varphi) = \Lambda_0 + \frac{m_0^2}{2}\varphi^2 + \frac{c_0}{4}\varphi^4 - \frac{1}{\Omega_d}\log \int [\prod_j D\psi_j] \exp(-S_2[\phi]), \tag{4.12}$$

which we rewrite in the standard form:

$$\mathcal{V}(\varphi) = \Lambda_0 + \frac{m_0^2}{2}\varphi^2 + \frac{c_0}{4}\varphi^4 + \mathcal{V}_0(\varphi) + (N-1)\mathcal{V}_1(\varphi), \tag{4.13}$$

where, for $a = 0, 1$,

$$\mathcal{V}_a(\varphi) = -\frac{1}{\Omega_d}\log \int [D\Phi] \exp\left(-\frac{1}{2}\int_{M_d} \sqrt{g}\Big[\partial_\mu \Phi \partial^\mu \Phi + M_a^2(\varphi)\Phi^2\Big] d^d x\right), \tag{4.14}$$

$$M_0^2(\varphi) = m_0^2 + 3c_0\varphi^2, \tag{4.15}$$

$$M_1^2(\varphi) = m_0^2 + c_0\varphi^2. \tag{4.16}$$

Several techniques can be used to compute $\mathcal{V}_a(\varphi)$. Instead of looking at it as a function of φ, it is useful to see it as a function of $M_a^2 \equiv z_a$. By taking the first derivative w.r.t. z we get the tadpole integral which leads to the Lee-Sciaccaluga equation for determining the effective potential [18].

When we differentiate twice, we get the one mass bubble integral; this is the strategy we want to adopt, since it will allow us to develop methods that will be helpful to compute the two-loop contribution (a research that will be considered elsewhere). More precisely, if $J(m_1^2, m_2^2)$ is the two-masses bubble, one has

$$\frac{\partial^2}{\partial z^2}\mathcal{V}(\varphi) = -\frac{1}{2}J(z,z), \tag{4.17}$$

from which we can recover the effective potential. As we will review in section 6, in the flat case and $d = 4 - 2\epsilon$, we have

$$J(z,z) = \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} - \log\frac{z}{4\pi e^{\gamma}}\right) + O(\epsilon). \tag{4.18}$$

After integrating twice in z and renormalising, we get the standard result [11, 19]

$$\begin{aligned}\mathcal{V}(\varphi) = \frac{m^2}{2}\varphi^2 + \frac{c_g}{4}\varphi^4 + \frac{(m^2 + 3c_g\varphi^2)^2}{64\pi^2}\log\frac{m^2 + 3c_g\varphi^2}{\mu^2} \\ + (N-1)\frac{(m^2 + c_g\varphi^2)^2}{64\pi^2}\log\frac{m^2 + c_g\varphi^2}{\mu^2},\end{aligned} \tag{4.19}$$

where m and c_g are the renormalised mass and coupling constant, and μ^2 is a scale related to the precise renormalisation conditions. We will now extend this calculation to the de Sitter case.

5 The effective potential

To compute the effective potential at $d = 4$, we set $d = 4 - 2\epsilon$. Since $d\nu^2 = dm^2$, we have to integrate twice $-I(\nu, \nu, 4)/2$ in $d\nu^2 = 2\nu d\nu$. From (3.13), we have

$$\begin{aligned} I(\nu, \nu, 4 - 2\epsilon) = & \frac{1}{16\pi^2\epsilon} + \frac{1 - \gamma + \log(4\pi)}{16\pi^2} \\ & - \frac{1}{128\pi^2\nu} \frac{d}{d\nu} \left[\left(4\nu^2 + 1\right) \left(\psi\left(\frac{3}{2} - i\nu\right) + \psi\left(\frac{3}{2} + i\nu\right)\right)\right] \\ & + \ \mathrm{O}(\epsilon). \end{aligned} \tag{5.1}$$

The nontrivial part comes from the second line. A first integration gives:

$$\begin{aligned} & \int \frac{1}{8\nu} \frac{d}{d\nu} \left[\left(4\nu^2 + 1\right) \left(\psi\left(\frac{3}{2} - i\nu\right) + \psi\left(\frac{3}{2} + i\nu\right)\right)\right] 2\nu d\nu = \\ & \quad = \left(\nu^2 + \frac{1}{4}\right) \left(\psi\left(-\frac{1}{2} - i\nu\right) + \psi\left(-\frac{1}{2} + i\nu\right)\right) + C, \end{aligned} \tag{5.2}$$

which, integrated once again becomes

$$C\nu^2 + \int \left(\nu^2 + \frac{1}{4}\right) \left(\psi\left(-\frac{1}{2} - i\nu\right) + \psi\left(-\frac{1}{2} + i\nu\right)\right) d\nu^2. \tag{5.3}$$

An integration by parts then gives

$$\begin{aligned} & \int \left(\nu^2 + \frac{1}{4}\right) \left(\psi\left(-\frac{1}{2} - i\nu\right) + \psi\left(-\frac{1}{2} + i\nu\right)\right) d\nu^2 = \\ & = \frac{1}{2} \left(\nu^2 + \frac{1}{4}\right)^2 \left(\psi\left(-\frac{1}{2} - i\nu\right) + \psi\left(-\frac{1}{2} + i\nu\right)\right) \\ & \quad - \frac{i}{2} \int d\nu \left(\nu^2 + \frac{1}{4}\right)^2 \left(\psi'\left(-\frac{1}{2} - i\nu\right) - \psi'\left(-\frac{1}{2} + i\nu\right)\right). \end{aligned} \tag{5.4}$$

Notice that

$$\psi'(z) = \zeta(2, z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}, \tag{5.5}$$

is a Hurwitz zeta function [23]. After setting $y := \nu^2 + \frac{1}{4}$, we can introduce the function

$$\begin{aligned} B(y) \equiv & \frac{y^2}{4} - \frac{i}{2} \int d\nu \left(\nu^2 + \frac{1}{4}\right)^2 \left(\psi'\left(-\frac{1}{2} - i\nu\right) - \psi'\left(-\frac{1}{2} + i\nu\right)\right) \\ = & -\frac{1}{2} \int dy\, y^2 \left(\sum_{n=0}^{\infty} \frac{2n - 1}{(n(n-1) + y)^2} - \frac{1}{y}\right), \end{aligned} \tag{5.6}$$

where in the second line we have used the series expansion (5.5). By using the standard Abel-Plana's formula [24] we can write

$$\sum_{n=0}^{\infty} \frac{2n-1}{(n(n-1)+y)^2} = -\frac{1}{2y^2} - 4\int_0^{\infty} \frac{t}{e^{2\pi t} - 1} \frac{(t^2 - y)^2 - y}{[(t^2 - y)^2 + t^2]^2}, \tag{5.7}$$

so that

$$B(y) = \frac{y}{3} + \int_0^\infty \frac{2t^4 + t^2 - 1 + y(4 - 2t^2)}{(t^2 - y)^2 + t^2} \frac{t^3 dt}{e^{2\pi t} - 1} + 6 \int_0^\infty \arctan\left(t - \frac{y}{t}\right) \frac{t^2 dt}{e^{2\pi t} - 1}$$
$$+ 2 \int_0^\infty \left(t^2 - \frac{1}{2}\right) \log[t^2 + (t^2 - y)^2] \frac{t dt}{e^{2\pi t} - 1}, \tag{5.8}$$

where we have conveniently set to zero an integration constant. With these notations, the (renormalized) effective potential is

$$\mathcal{V}_R = \frac{m_R^2}{2}\varphi_R^2 + \frac{c_g}{4}\varphi_R^4 + \frac{1}{64\pi^2}(m_R^2 + 3c_g\varphi_R^2 - 2)^2 \left(\psi\left(-\frac{1}{2} - i\nu_0\right) + \psi\left(-\frac{1}{2} + i\nu_0\right)\right)$$
$$+ \frac{1}{32\pi^2} B(m_R^2 + 3c_g\varphi_R^2 - 2) + \frac{N-1}{32\pi^2} B(m_R^2 + c_g\varphi_R^2 - 2)$$
$$+ \frac{N-1}{64\pi^2}(m_R^2 + c_g\varphi_R^2 - 2)^2 \left(\psi\left(-\frac{1}{2} - i\nu_1\right) + \psi\left(-\frac{1}{2} + i\nu_1\right)\right)$$
$$- \frac{1}{64\pi^2}\left[(m_R^2 + 3c_g\varphi_R^2 - 2)^2 + (N-1)(m_R^2 + c_g\varphi_R^2 - 2)^2\right] \log \mu_R^2. \tag{5.9}$$

The last line is just the introduction of a reference energy scale, related to the renormalization, while

$$\nu_0 = \sqrt{m_R^2 + 3c_g\varphi_R^2 - \frac{9}{4}}, \tag{5.10}$$
$$\nu_1 = \sqrt{m_R^2 + c_g\varphi_R^2 - \frac{9}{4}}. \tag{5.11}$$

R refers to the de Sitter radius and appears through the formula $m_R = mR$, $\mu_R = \mu R$, and $\varphi_R = R\varphi$, m_R, μ_R and φ_R being adimensional quantities.
We can find an alternative expression for the function B by directly integrating (5.6):

$$B(y) = -\sum_{n=1}^{\infty} \left[(2n+1)y + \frac{n(n+1)(2n+1)y}{n(n+1) + y}\right.$$
$$\left. - 2(2n+1)(n+1)n \log\left(1 + \frac{y}{n(n+1)}\right)\right] + \frac{y^2}{4} + D, \tag{5.12}$$

where $D = B(0)$ is a constant that can be determined from (5.8) as

$$D = -\frac{1}{72} + \frac{\gamma - \log(2\pi)}{60} - \frac{3\zeta(3)}{4\pi^2} + 4\log(A_3), \tag{5.13}$$

where A_3 is the third generalized Glaisher-Kinkelin constant (see appendix B).
This formula allows us to write the effective potential at any finite value of ν, but it is not suitable for taking the flat limit, for which it is convenient to use (5.8).

6 Flat limit

Before discussing the flat limit let us recall that in flat space the Schwinger function is proportional to a MacDonald function:

$$G_m(x) = \frac{1}{(2\pi)^d} \int \frac{e^{-ipx}}{p^2 + m^2} dp = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{r}{m}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(mr), \quad r = \sqrt{x^2}. \tag{6.1}$$

An easy computation in x-space [2] gives the textbook answer for the bubble:

$$\int G_{m_1}(x)G_{m_2}(x)dx = \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\frac{m_2^{d-2}-m_1^{d-2}}{m_1^2-m_2^2}. \tag{6.2}$$

which, when $d = 2$, reduces to

$$I(m_1, m_2, 2) = \frac{\log(m_1)-\log(m_2)}{2\pi(m_1^2-m_2^2)}. \tag{6.3}$$

To study the flat limit of eq. (3.4), we have to restore a generic de Sitter radius R and rescale the masses accordingly. When $m > 0$ and $R \to +\infty$, by using the Stirling formula

$$\left|\frac{\Gamma\left(\frac{d-1}{2}+iRm\right)}{\Gamma\left(\frac{1}{2}+iRm\right)}\right|^2 \sim (Rm)^{d-2}\ . \tag{6.4}$$

Hence,

$$I(Rm_1, Rm_2, d) \sim -R^{d-4}\frac{\Gamma\left(1-\frac{d}{2}\right)(m_1^{d-2}-m_2^{d-2})}{2^d\pi^{\frac{d}{2}}(m_2^2-m_1^2)} \quad (R\to+\infty)\ . \tag{6.5}$$

This fits exactly with (6.2).

Once more, we would like to draw the attention of the reader on the non triviality of this result as it expresses the flat limit of an integrated quantity as opposed to the asymptotic properties of the correlation functions which are merely local properties.

Even more interestingly, the same nontrivial result holds true for the effective potential. To show it, we need to compute the behaviour of $B(\nu)$ for $\nu \to \infty$. This is obtained from (5.8) by noticing that the integral in the first line goes to zero and using that

$$\int_0^\infty \frac{t^k dt}{e^{2\pi t}-1} = \frac{k!}{(2\pi)^{k+1}}\zeta(k+1). \tag{6.6}$$

Therefore,

$$B\left(\nu^2+\frac{1}{4}\right) = \frac{1}{3}\nu^2 - \frac{1}{15}\log\left(\nu^2+\frac{1}{4}\right)+\frac{1}{12}-\frac{3\zeta(3)}{4\pi^3}+\ldots, \tag{6.7}$$

where the dots stay for terms vanishing for $\nu \to \infty$.

On the other hand, for $|\arg(x)| \leq \pi - \delta$, $\delta > 0$, one has

$$\psi(x) = \log x - \frac{1}{2x} - \sum_{j=1}^{m}\frac{B_{2j}}{2j}\frac{1}{x^{2j}} + O\left(\frac{1}{x^{2m+2}}\right), \tag{6.8}$$

where B_j are the Bernoulli numbers. Since $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$, it follows that

$$\frac{1}{2}\left(\nu^2+\frac{1}{4}\right)^2\left(\psi\left(-\frac{1}{2}-i\nu\right)+\psi\left(-\frac{1}{2}+i\nu\right)\right) = \frac{1}{2}\left(\nu^2+\frac{1}{4}\right)^2\log\left(\nu^2+\frac{1}{4}\right)+\frac{1}{3}\nu^2+\frac{1}{20}+\ldots. \tag{6.9}$$

In this limit we have

$$
\begin{aligned}
\mathcal{V}_R = \frac{m_R^2}{2}\varphi_R^2 + \frac{c_g}{4}\varphi_R^4 + \frac{1}{32\pi^2}\Bigg[&\frac{(m_R^2+3c_g\varphi_R^2-2)^2}{2}\log\frac{m_R^2+3c_g\varphi_R^2-2}{\mu_R^2} \\
&+\frac{2}{3}(m_R^2+3c_g\varphi_R^2-2)-\frac{1}{15}\log(m_R^2+3c_g\varphi_R^2-2)-\frac{1}{30}-\frac{3\zeta(3)}{4\pi^3}+\dots\Bigg] \\
&+\frac{N-1}{32\pi^2}\Bigg[\frac{(m_R^2+c_g\varphi_R^2-2)^2}{2}\log\frac{m_R^2+c_g\varphi_R^2-2}{\mu_R^2} \\
&+\frac{2}{3}(m_R^2+c_g\varphi_R^2-2)-\frac{1}{15}\log(m_R^2+c_g\varphi_R^2-2)-\frac{1}{30}-\frac{3\zeta(3)}{4\pi^3}+\dots\Bigg]. \qquad (6.10)
\end{aligned}
$$

The potential in the flat limit is defined by $\mathcal{V} = \lim_{R\to\infty} R^{-4}\mathcal{V}_R$ and gives exactly (4.19). For very large but finite R, we can write

$$
\begin{aligned}
\mathcal{V}_\Lambda = \frac{m^2}{2}\varphi^2 + \frac{c_g}{4}\varphi^4 + \frac{1}{64\pi^2}\Bigg[&(m^2+3c_g\varphi^2)^2\log\frac{m^2+3c_g\varphi^2}{\mu^2} \\
&+(N-1)(m^2+c_g\varphi^2)^2\log\frac{m^2+c_g\varphi^2}{\mu^2}\Bigg] \\
&-\frac{\Lambda}{48\pi^2}\Bigg[(m^2+3c_g\varphi^2)\left(\log\frac{m^2+3c_g\varphi^2}{\mu^2}+\frac{1}{6}\right) \\
&\qquad+(N-1)(m^2+c_g\varphi^2)\left(\log\frac{m^2+c_g\varphi^2}{\mu^2}+\frac{1}{6}\right)\Bigg] \\
&+\frac{\Lambda^2}{144\pi^2}\Bigg[-\frac{1}{30}\log\frac{3(m^2+3c_g\varphi^2)}{\Lambda}+\log\frac{m^2+3c_g\varphi^2}{\mu^2}+\frac{49}{60}-\frac{3\zeta(3)}{4\pi^3} \\
&+(N-1)\left(-\frac{1}{30}\log\frac{3(m^2+c_g\varphi^2)}{\Lambda}+\log\frac{m^2+c_g\varphi^2}{\mu^2}+\frac{49}{60}-\frac{3\zeta(3)}{4\pi^3}\right)\Bigg] \\
&+O(\Lambda^3), \qquad (6.11)
\end{aligned}
$$

where $\Lambda = 3R^{-2}$ is the cosmological constant. The first two lines are the standard Coleman-Weinberg effective potential [11, 19], while the remaining terms provide the perturbative corrections in the cosmological constant, up to order two. The exact expression is given by (5.9), which in terms of the full dimensional quantities is

$$
\begin{aligned}
\mathcal{V}_\Lambda = \frac{m^2}{2}\varphi^2 + \frac{c_g}{4}\varphi^4 &+ \frac{1}{64\pi^2}(m^2+3c_g\varphi^2-\frac{2}{3}\Lambda)^2\left(\psi\left(-\frac{1}{2}-i\nu_0\right)+\psi\left(-\frac{1}{2}+i\nu_0\right)\right) \\
&+\frac{\Lambda^2}{288\pi^2}B\left(3\frac{m^2}{\Lambda}+9c_g\frac{\varphi^2}{\Lambda}-2\right)+\frac{(N-1)\Lambda^2}{288\pi^2}B\left(3\frac{m^2}{\Lambda}+3c_g\frac{\varphi^2}{\Lambda}-2\right) \\
&+\frac{N-1}{64\pi^2}(m^2+c_g\varphi^2-\frac{2}{3}\Lambda)^2\left(\psi\left(-\frac{1}{2}-i\nu_1\right)+\psi\left(-\frac{1}{2}+i\nu_1\right)\right) \\
&-\frac{1}{64\pi^2}\left[(m^2+3c_g\varphi^2-\frac{2}{3}\Lambda)^2+(N-1)(m^2+c_g\varphi^2-\frac{2}{3}\Lambda)^2\right]\log\frac{3\mu^2}{\Lambda}, \qquad (6.12)
\end{aligned}
$$

with

$$\nu_0 = 3\sqrt{\frac{m^2}{3\Lambda} + c_g\frac{\varphi^2}{\Lambda} - \frac{1}{4}}, \tag{6.13}$$

$$\nu_1 = 3\sqrt{\frac{m^2}{3\Lambda} + c_g\frac{\varphi^2}{3\Lambda} - \frac{1}{4}}. \tag{6.14}$$

It is interesting to compare this result with the one for anti de Sitter, [20]. In that case, the perturbative expansion in the cosmological constant Λ of the effective potential contains also terms of order $\sqrt{|\Lambda|}$. This is strictly related to the invariance of the Legendre functions of the first kind $P^{\mu}_{-\frac{1}{2}-\nu}$ under the $\nu \to -\nu$, which is not true for the functions of the second kind.

7 2-loop: the watermelon

For the sake of comparison let us recall at first the flat space formula for the watermelon:

$$\begin{aligned} I_3(m_1,m_2,m_3,d) = &-2^{1-2d}\pi^{1-d}\Gamma(2-d)(-S(m_1,m_2,m_3))^{\frac{d-3}{2}} \\ &+\frac{(m_1m_2)^{d-4}\left(m_1^2+m_2^2-m_3^2\right)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};M_{123}^2\right)}{4^d\pi^{d-2}(\cos(\pi d)-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \\ &+\frac{(m_2m_3)^{d-4}\left(-m_1^2+m_2^2+m_3^2\right)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};M_{231}^2\right)}{4^d\pi^{d-2}(\cos(\pi d)-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \\ &+\frac{(m_1m_3)^{d-4}\left(m_1^2-m_2^2+m_3^2\right)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};M_{312}^2\right)}{4^d\pi^{d-2}(\cos(\pi d)-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)}. \end{aligned} \tag{7.1}$$

where

$$M_{ijk} = \left(\frac{m_i^2+m_j^2-m_k^2}{2m_im_j}\right) \tag{7.2}$$

and

$$S(m_1,m_2,m_3) = m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2 \tag{7.3}$$

is the Symanzik polynomial. The above formula is valid when no mass is bigger than the sum of the other two; this happens if and only if the Symanzik polynomial is negative. It may be obtained by solving an appropriate differential equation for the diagram [11] or else by a direct calculation in position space which is indeed the easier and shorter way to get it [2]. A similar formula holds for $S > 0$ [2].

As regards de Sitter sphere the situation is trickier. At the moment there is no substitute for the method of differential equations which seems to be well-adapted only to flat space but useless otherwise. We are left with x-space calculations.

Faced directly, the integral at the r.h.s. of (2.25) is significantly more challenging than the already difficult (2.27) and at the moment the only way we found to evaluate it makes indeed use of eq. (2.27): by singling out one of the masses, say ν_3, we get an integral representation of the two-loop watermelon as a superposition of one-loop diagrams:

$$\begin{aligned} I_3(\nu_1,\nu_2,\nu_3,d) &= \int G_{\nu_1}(x_0\cdot x)G_{\nu_2}(x_0\cdot x)G_{\nu_3}(x_0\cdot x)\sqrt{g}dx = \\ &= \int \rho(\nu_1,\nu_2,\kappa)I_2(\kappa,\nu_3,d)\kappa d\kappa = I_3^{(1)}(\nu_1,\nu_2,\nu_3,d) - I_3^{(2)}(\nu_1,\nu_2,\nu_3,d). \end{aligned} \tag{7.4}$$

where

$$I_3^{(1)}(\nu_1,\nu_2,\nu_3,d) = \frac{\left(\frac{\Gamma\left(\frac{d-1}{2}-i\nu_3\right)\Gamma\left(\frac{d-1}{2}+i\nu_3\right)}{\Gamma\left(\frac{1}{2}-i\nu_3\right)\Gamma\left(\frac{1}{2}+i\nu_3\right)}\right)}{2^d\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)} \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\mathcal{A}(\nu_1,\nu_2,\nu_3,d) \tag{7.5}$$

$$\mathcal{A}(\nu_1,\nu_2,\nu_3,d) = \int \frac{\prod_{\epsilon,\epsilon',\epsilon''}\Gamma\left(\frac{d-1}{4}+\frac{i\epsilon'\nu_1+i\epsilon'\nu_2+i\epsilon''\kappa}{2}\right)}{\prod_\epsilon \Gamma\left(\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{1}{2}+\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{d-1}{4}+\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{d+1}{4}+\frac{i\epsilon\kappa}{2}\right)} \frac{d\kappa}{(\kappa^2-\nu_3^2)} \tag{7.6}$$

and

$$I_3^{(2)}(\nu_1,\nu_2,\nu_3,d) = \frac{1}{2^d\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)} \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\mathcal{B}(\nu_1,\nu_2,\nu_3) \tag{7.7}$$

$$\mathcal{B}(\nu_1,\nu_2,\nu_3) = \int \frac{\frac{\Gamma\left(\frac{d-1}{2}-i\kappa\right)\Gamma\left(\frac{d-1}{2}+i\kappa\right)}{\Gamma\left(\frac{1}{2}-i\kappa\right)\Gamma\left(\frac{1}{2}+i\kappa\right)}\prod_{\epsilon,\epsilon',\epsilon''}\Gamma\left(\frac{d-1}{4}+\frac{i\epsilon'\nu_1+i\epsilon'\nu_2+i\epsilon''\kappa}{2}\right)}{\prod_\epsilon \Gamma\left(\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{1}{2}+\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{d-1}{4}+\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{d+1}{4}+\frac{i\epsilon\kappa}{2}\right)} \frac{d\kappa}{(\kappa^2-\nu_3^2)}. \tag{7.8}$$

$I_3^{(1)}(\nu_1,\nu_2,\nu_3,d)$ and $I_3^{(2)}(\nu_1,\nu_2,\nu_3,d)$ are symmetric only in the exchange of ν_1 and ν_2. To work with symmetric expressions we might replace them by their totally symmetric counterparts but this is not really helpful.

First term. Let us consider the first term $I_3^{(1)}(\nu_1,\nu_2,\nu_3,d)$; the change of variables[5]

$$s = \frac{i\kappa}{2}, \quad \delta = \frac{d-1}{4}, \quad x = \frac{i\nu_1}{2}, \quad y = \frac{i\nu_2}{2}, \quad w = \frac{i\nu_3}{2}, \quad u = x+y \tag{7.9}$$

allows to rewrite the integral as follows:

$$\mathcal{A}(x,y,w,d) = \int_{-i\infty}^{i\infty}\frac{ds}{2i}\prod_{\epsilon,\epsilon',\epsilon''=\pm}\frac{\Gamma\left(\epsilon s+\delta+\epsilon' x+i\epsilon'' y\right)\Gamma\left(\epsilon s+w\right)}{\Gamma\left(\epsilon s\right)\Gamma\left(\frac{1}{2}+\epsilon s\right)\Gamma\left(\delta+\epsilon s\right)\Gamma\left(\delta+\frac{1}{2}+\epsilon s\right)\Gamma\left(\epsilon s+w+1\right)}. \tag{7.10}$$

There are poles on the integration path at

$$s = -\delta + \frac{\pm i\nu_1 \pm i\nu_2}{2} - n, \qquad s = -\frac{i\nu_3}{2} - n, \tag{7.11}$$

$$s = -\frac{i\nu_3}{2} + n, \qquad s = \delta + \frac{\pm i\nu \pm i\lambda}{2} + n. \tag{7.12}$$

[5]In the following we will use either (ν_1,ν_2,ν_3) or (x,y,w) and either d or $\delta(d)$ interchangeably.

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By integrating along the imaginary axis (with suitable indentations) the result may be expressed as a linear combination of four hypergeometric functions

$$I_3^{(1)}(x,y,z,d) = \sum_{\epsilon,\epsilon'=\pm1} A_d(\epsilon x,\epsilon' y,w) \tag{7.13}$$

where

$$A_d(x,y,w) = a_d(x,y,w)\times \\ {}_9F_8\left(\begin{matrix}2\delta, u+\frac{1}{2}, u+1, \delta+u+1, 2\delta+2x, 2\delta+2y, 2\delta+2u, \delta-w+u, \delta+w+u \\ 2x+1, 2y+1, 2u+1, \delta+u, 2\delta+u, 2\delta+u+\frac{1}{2}, \delta-w+u+1, \delta+w+u+1\end{matrix};1\right), \tag{7.14}$$

$$a_d(x,y,w) = -\frac{4^{-2\delta-3}\pi^{-4\delta-\frac{1}{2}}\Gamma\left(\frac{1}{2}-2\delta\right)\cos(2\pi w)\Gamma(2\delta-2w)\Gamma(2w+2\delta)\Gamma(2x+2\delta)\Gamma(2y+2\delta)}{\sin(2\pi x)\sin(2\pi y)\Gamma(2x+1)\Gamma(2y+1)\left(w^2-(\delta+u)^2\right)\Gamma(-2u-2\delta)\Gamma(2u+4\delta)}. \tag{7.15}$$

The hypergeometric series in (7.14) converges absolutely for $3-4\delta = 4-d > 0$.

It is possible to proceed to a simplification of the above expressions by observing that in each term of the hypergeometric series (7.14) the following product of Pochhammers reduces nicely to a rational function:

$$\frac{(u+\delta+1)_n(u-w+\delta)_n(u+w+\delta)_n}{(u+\delta)_n(u-w+\delta+1)_n(u+w+\delta+1)_n} = \frac{((\delta+u)^2-w^2)\left(\frac{1}{\delta+n+u+w}+\frac{1}{\delta+n+u-w}\right)}{2(\delta+u)}. \tag{7.16}$$

We obtain in this way a formula consisting of eight terms ${}_7F_6$:

$$I_3^{(1)}(x,y,z,d) = \sum_{\epsilon,\epsilon'=\pm1} A'_d(\epsilon x,\epsilon' y,\epsilon'' w) \tag{7.17}$$

where

$$A'_d(x,y,w) = a'_d(x,y,w)\times \\ {}_7F_6\left(\begin{matrix}2\delta, u+\frac{1}{2}, u+1, \delta+u-w, 2\delta+2u, 2\delta+2x, 2\delta+2y \\ 2u+1, 2x+1, 2y+1, \delta+u-w+1, 2\delta+u, 2\delta+u+\frac{1}{2}\end{matrix};1\right) \tag{7.18}$$

$$a'_d(x,y,w) = \\ = \frac{2^{-4\delta-7}\pi^{-4\delta-\frac{1}{2}}\Gamma\left(\frac{1}{2}-2\delta\right)\cos(2\pi w)\Gamma(2\delta-2w)\Gamma(2w+2\delta)\Gamma(2x+2\delta)\Gamma(2y+2\delta)}{\sin(2\pi x)\sin(2\pi y)(\delta+u)\Gamma(2x+1)\Gamma(2y+1)\Gamma(-2u-2\delta)\Gamma(2u+4\delta)(\delta+u-w)}. \tag{7.19}$$

This simplification has not worsened the convergence: (7.18) converges absolutely under the same conditions: $4\delta-3 = -d+4 > 0$.

Second term. As regards the second term, we have

$$\mathcal{B}(x,y,w,d) = \int \frac{ds}{2i} \prod_{\epsilon,\epsilon',\epsilon''=\pm1} \frac{2^{d-2}\Gamma(\epsilon s+w)\Gamma(\epsilon s+\delta+\epsilon' x+\epsilon'' y)}{\Gamma(\epsilon s)\Gamma\left(\epsilon s+\frac{1}{4}\right)\Gamma\left(\epsilon s+\frac{1}{2}\right)\Gamma\left(\epsilon s+\frac{3}{4}\right)\Gamma(\epsilon s+w+1)} \tag{7.20}$$

Again, the result may be expressed as a linear combination of four hypergeometric functions:

$$I_3^{(2)} = \sum_{\epsilon,\epsilon'=\pm 1} B_d(\epsilon x, \epsilon' y, w) \tag{7.21}$$

where

$$B_d(x,y,w) = b_d(x,y,w) \times {}_7F_6\left(\begin{matrix} 2\delta, \delta+u+1, \delta-w+u, \delta+w+u, 2\delta+2x, 2\delta+2y, 2\delta+2u \\ 2x+1, 2y+1, 2x+2y+1, \delta+u, \delta-w+u+1, \delta+w+u+1 \end{matrix}; 1\right) \tag{7.22}$$

$$b_d(x,y,w) = \frac{2^{-8\delta-4u-7}\pi^{1-4\delta}\Gamma\left(\frac{1}{2}-2\delta\right)\csc(2\pi x)\csc(2\pi y)\csc(2\pi u)\Gamma(2x+2\delta)\Gamma(2y+2\delta)}{((\delta+u)^2-w^2)\,\Gamma(2x+1)\Gamma(2y+1)\Gamma(2u+1)\Gamma(-4u-4\delta)\Gamma\left(2u+2\delta+\frac{1}{2}\right)}. \tag{7.23}$$

The hypergeometric series in (7.22) converges absolutely for $4-8\delta = 2(3-d) > 0$. Proceeding exactly as before we may find a simplification of the above formula:

$$I_3^{(2)}(x,y,z,d) = \sum_{\epsilon,\epsilon',\epsilon''=\pm 1} B'_d(\epsilon x, \epsilon' y, \epsilon'' w) \tag{7.24}$$

where

$$B'_d(x,y,w) = b'_d(x,y,w)\ {}_5F_4\left(\begin{matrix} 2\delta, \delta+u+w, 2\delta+2u, 2\delta+2x, 2\delta+2y \\ 2u+1, 2x+1, 2y+1, \delta+u+w+1 \end{matrix}; 1\right) \tag{7.25}$$

$$b'_d(x,y,w) = -\frac{2^{-6-4u-8\delta}\pi^{1-4\delta}\csc(2\pi x)\csc(2\pi y)\csc(2\pi u)\Gamma\left(\frac{1}{2}-2\delta\right)\Gamma(2x+2\delta)\Gamma(2y+2\delta)}{(w+u+\delta)\Gamma(1+2x)\Gamma(1+2y)\Gamma(1+2u)\Gamma(1-4u-4\delta)\Gamma\left(\frac{1}{2}+2u+2\delta\right)}. \tag{7.26}$$

Again, the simplification has not worsened the convergence: (7.25) converges absolutely for $4-8\delta = 2(3-d) > 0$.

In conclusion, $I_3^{(1)}$ is the most regular of the two terms; it starts diverging at $d=4$. The second term diverges already at $d=3$. This means that no compensation among the two terms may be expected. The only exception is at $d=2$: in that case the watermelon is finite; the two terms have to compensate each other to render the divergence in the coefficient $\Gamma\left(1-\frac{d}{2}\right)$ harmless.

8 Odd dimensions $d \le 1$

In odd integer dimension $d \le 1$ the result simplifies a great deal because the hypergeometric series reduce to finite sums. Let us examine the simplest case $d=1$; our general formulae give

$$\begin{aligned} I_3^{(1)}(\nu_1,\nu_2,\nu_3,1) = &\frac{\coth(\pi\nu_1)\coth(\pi\nu_3)\left(\nu_2^2-\nu_1^2-\nu_3^2\right)}{4\nu_1\nu_3(\nu_1-\nu_2-\nu_3)(\nu_1+\nu_2-\nu_3)(\nu_1-\nu_2+\nu_3)(\nu_1+\nu_2+\nu_3)}+ \\ &+\frac{\coth(\pi\nu_2)\coth(\pi\nu_3)\left(\nu_1^2-\nu_2^2-\nu_3^2\right)}{4\nu_2\nu_3(\nu_1-\nu_2-\nu_3)(\nu_1+\nu_2-\nu_3)(\nu_1-\nu_2+\nu_3)(\nu_1+\nu_2+\nu_3)} \end{aligned} \tag{8.1}$$

and

$$I_3^{(2)}(\nu_1,\nu_2,\nu_3,1) = \frac{-\coth(\pi\nu_1)\coth(\pi\nu_2)\left(\nu_3^2-\nu_1^2-\nu_2^2\right)}{4\nu_1\nu_2(\nu_1-\nu_2-\nu_3)(\nu_1+\nu_2-\nu_3)(\nu_1-\nu_2+\nu_3)(\nu_1+\nu_2+\nu_3)} - \frac{1}{2(\nu_1-\nu_2-\nu_3)(\nu_1+\nu_2-\nu_3)(\nu_1-\nu_2+\nu_3)(\nu_1+\nu_2+\nu_3)}. \tag{8.2}$$

$I_3^{(2)}(\nu_1,\nu_2,\nu_3,1)$ contains a term which is totally symmetric w.r.t. the mass parameters and a term which is symmetric only w.r.t. the exchange of ν_1 and ν_2.

Subtracting $I_3^{(2)}$ from $I_3^{(1)}$ reestablishes the global symmetry of the diagram w.r.t. the three mass parameters. Since $I_3^{(1)}$ do not include a totally symmetric term it can be fully deduced from the knowledge of $I_3^{(2)}$.

The above situation is generic in odd negative dimension: the first addendum contains the terms proportional to $\coth \pi\nu_3$ and the second term contains a totally symmetric contribution $S(\nu_1,\nu_2,\nu_3,d)$ which is just a product of poles.

Both the symmetric and the non symmetric contributions admit expansions in partial fractions. The totally symmetric term has indeed a very simple expression:

$$S(\nu_1,\nu_2,\nu_3,d) = \frac{\cos(2\pi i\delta)\pi^{-4\delta}\Gamma(1-4\delta)}{32} \prod_{\epsilon,\epsilon'=\pm} \frac{\Gamma\left(\delta+\frac{i\nu_1}{2}+\frac{i\epsilon\nu_2}{2}+\frac{i\epsilon'\nu_3}{2}\right)}{\Gamma\left(1-\delta+\frac{i\nu_1}{2}+\frac{i\epsilon\nu_2}{2}+\frac{i\epsilon'\nu_3}{2}\right)}; \tag{8.3}$$

it amounts to a product of simple poles. The non-symmetric term is also proportional to a product of simple poles but the polynomial at the numerator becomes more and more cumbersome as the dimension grows (negative).

A straightforward derivation of $I_3(\nu_1,\nu_2,\nu_3,1)$ by direct integration of eq. (7.4) points towards a fully symmetric expression: in spacetime dimension $d=1$ the Schwinger function is simply

$$G^1_\lambda(-\cos s) = \frac{\Gamma(-i\nu)\Gamma(i\nu)\,\mathrm{ch}\left(\lambda(\pi-s)\right)}{2\pi} \tag{8.4}$$

and therefore

$$\begin{aligned} I(\nu_1,\nu_2,\nu_3,1) &= 2\int_0^\pi G^1_{\nu_1}(-\cos s)G^1_{\nu_2}(-\cos s)G^1_{\nu_3}(-\cos s)ds \\ &= \frac{\frac{\mathrm{sh}(\pi(\nu_1+\nu_2+\nu_3))}{\nu_1+\nu_2+\nu_3}+\frac{\mathrm{sh}(\pi(\nu_1+\nu_2-\nu_3))}{\nu_1+\nu_2-\nu_3}+\frac{\mathrm{sh}(\pi(\nu_1-\nu_2+\nu_3))}{\nu_1-\nu_2+\nu_3}+\frac{\mathrm{sh}(\pi(\nu_1-\nu_2-\nu_3))}{\nu_1-\nu_2-\nu_3}}{16\,\nu_1\nu_2\nu_3\,\mathrm{sh}(\pi\nu_1)\,\mathrm{sh}(\pi\nu_2)\,\mathrm{sh}(\pi\nu_3)}. \end{aligned} \tag{8.5}$$

In the special case when all the masses are equal the formula reduces to

$$I(\nu,\nu,\nu,1) = \frac{5+\mathrm{ch}(2\pi\nu)}{24\nu^4\,\mathrm{sh}(\pi\nu)^2}. \tag{8.6}$$

Studying the case of three particles with the same mass *ab initio* is not easier; it only hides the underlying beautiful structure.

Rescaling the masses $\nu \to \nu R$ and taking the limit $R\to\infty$ gives the correct result in flat space, as obtained by taking $d=1$ in eq. (7.1):

$$I(m_1,m_2,m_3,1) = \frac{1}{4m_1m_2m_3(m_1+m_2+m_3)}; \tag{8.7}$$

we stress again that this result is non trivial, being the flat limit of an integrated quantity and not just the asymptotic behaviour of a correlation function close to a given point.

Also the above symmetric structure in four terms is generic in odd dimension $d \leq 1$:

$$I(\nu_1, \nu_2, \nu_3, d) = \sum_{\epsilon, \epsilon' = \pm} \frac{\operatorname{sh}(\pi(\nu_1 + \epsilon\nu_2 + \epsilon'\nu_3)) P_d(\nu_1, \epsilon\nu_2, \epsilon'\nu_3)}{\nu_1\nu_2\nu_3 \operatorname{sh}(\pi\nu_1) \operatorname{sh}(\pi\nu_2) \operatorname{sh}(\pi\nu_3) Q_d(\nu_1, \epsilon\nu_2, \epsilon'\nu_3)} \tag{8.8}$$

where P_d and Q_d are polynomials. The polynomial at the denominator is always a product of monomials and the result admits a simple partial fraction expansion. We will not elabotrate further on this point.

9 Spacetime dimension $d = 2$

In dimension $d = 2$ the watermelon is finite. However, as for the 1-loop diagram (see eq. (3.11)) the two terms in eq. (7.4) are multiplied by the coefficient $\Gamma\left(1 - \frac{d}{2}\right)$ that becomes singular at $d = 2$ and a limit procedure is needed. Only in this particular two-dimensional case $I_3^{(1)}$ and $I_3^{(2)}$ compensate each other.

To extract the finite result we make use of the elementary identity

$$\begin{aligned} I_3(x, y, z, 2) &= \frac{\partial}{\partial d} \left((d-2) I_3(x, y, z, d)\right)\Big|_{d=2} = \\ &= \partial_d \left((d-2) \sum_{\epsilon, \epsilon', \epsilon'' = \pm} \left(A'_d(\epsilon x, \epsilon' y, \epsilon'' w) - B'_d(\epsilon x, \epsilon' y, \epsilon'' w)\right) \right)\Bigg|_{d=2} . \end{aligned} \tag{9.1}$$

It is useful to rewrite the typical term in the above expression as follows:

$$A'_d(x, y, w) - B'_d(x, y, w) = a'_d(x, y, w) \left({}_7F_6\left(\ldots; 1\right) - \frac{b'_d(x, y, w)}{a'_d(x, y, w)}\, {}_5F_4\left(\ldots; 1\right) \right). \tag{9.2}$$

The term in parentheses at the r.h.s. tends to zero when $d \to 2$. This is precisely the compensation we were expecting. Taking the limit we get

$$\begin{aligned} T(x, y, w) &= \partial_d \left((d-2)(A'_d(x, y, w) - B'_d(x, y, w))\right)\big|_{d=2} \\ &= c_2(x, y, w) \left(\frac{\partial}{\partial d}\, {}_7F_6\left(\ldots; 1\right) - \frac{\partial}{\partial d}\, {}_5F_4\left(\ldots; 1\right) - {}_5F_4\left(\ldots; 1\right) \frac{\partial}{\partial d} \frac{b'_d(x, y, w)}{a'_d(x, y, w)} \right)\Bigg|_{d=2} \end{aligned} \tag{9.3}$$

where $c_2(x, y, w) = \lim_{d \to 2} (d-2) a'_d(x, y, w)$. All the terms together provide the following finite expression:

$$I_3(x, y, z, 2) = \sum_{\epsilon, \epsilon', \epsilon'' = \pm} T(\epsilon x, \epsilon' y, \epsilon'' w) \tag{9.4}$$

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where

$$T(x,y,w)=\frac{\left(\pi\cot(2\pi(x+y))-\psi\left(\frac{1}{2}-2w\right)-\psi\left(\frac{1}{2}+2w\right)+2\psi(1+2x+2y)\right)}{32\sqrt{\pi}\sin(2\pi x)\sin(2\pi y)\Gamma\left(\frac{1}{2}-2x-2y\right)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}+2x\right)\Gamma\left(\frac{1}{2}+2y\right)\,{}_5F_4\left(\begin{matrix}\frac{1}{2}+2x,\frac{1}{4}+w+x+y,\frac{1}{2}+2y,\frac{1}{2}+2x+2y,\frac{1}{2}\\ 1+2x,\frac{5}{4}+w+x+y,1+2y,1+2x+2y\end{matrix};1\right)}{(1+4w+4x+4y)\Gamma(1+2x)\Gamma(1+2y)\Gamma(1+2x+2y)}+$$
$$-\frac{\Gamma(-2x)\Gamma\left(\frac{1}{2}+2x\right)\Gamma(-2y)\Gamma\left(\frac{1}{2}+2y\right)}{32\pi^{5/2}(1+4w+4x+4y)\Gamma\left(\frac{1}{2}-2x-2y\right)\Gamma(1+2x+2y)}\times$$
$$\times\left({}_6\ddot{F}_5\left(\begin{matrix}\frac{1}{2}+x+y,\frac{1}{2}+2x,\frac{1}{4}+w+x+y,\frac{1}{2}+2y,\frac{1}{2}+2x+2y,\frac{1}{2}\\ \frac{1}{2}+x+y,1+2x,\frac{5}{4}+w+x+y,1+2y,1+2x+2y\end{matrix};1\right)+\right.$$
$$\left.+\;{}_6\dot{F}_5\left(\begin{matrix}1+x+y,\frac{1}{2}+2x,\frac{1}{4}+w+x+y,\frac{1}{2}+2y,\frac{1}{2}+2x+2y,\frac{1}{2}\\ 1+x+y,1+2x,\frac{5}{4}+w+x+y,1+2y,1+2x+2y\end{matrix};1\right)\right) \tag{9.5}$$

where we introduced the notation

$${}_p\dot{F}_q\left(\begin{matrix}a,\cdots\\ b,\cdots\end{matrix};z\right)=\frac{\partial}{\partial a}{}_pF_q\left(\begin{matrix}a,\cdots\\ b,\cdots\end{matrix};z\right),\qquad {}_p\ddot{F}_q\left(\begin{matrix}a,\cdots\\ b,\cdots\end{matrix};z\right)=\frac{\partial}{\partial b}{}_pF_q\left(\begin{matrix}a,\cdots\\ b,\cdots\end{matrix};z\right). \tag{9.6}$$

While the above result is perfectly finite and usable, there should exist an underlying simplification, as eqs. (9.4)–(9.5) should be a one-parameter[6] deformation of the flat space formula which may simply be written in terms of dilogarithms [2, 73]. Actually, also in flat space a very natural construction in position space gives a formula containing derivatives of the hypergeometric function ${}_2F_1$ w.r.t. the parameters [2]. In that case it is been possible to go further and proceed to dilogarithms which, by the way, are not more explicit the hypergeometrics. Here the task is much more difficult and at the moment we are happy with eq. (9.5).

10 Spacetime dimension $d = 3$

Flat space recapitulation. Let us review at first the three-dimensional case in flat space. Only the first line in formula (7.1) has a pole at $d = 3$, namely the simple pole in the Gamma function. A Laurent expansion near $d \sim 3$ gives

$$I_3(m_1,m_2,m_3,d)=-\frac{1}{32\pi^2(d-3)}+\frac{2-2\gamma+\log\left(16\pi^2\right)}{64\pi^2}+$$
$$-\frac{\operatorname{th}^{-1}\left(\frac{2m_1m_2}{m_1^2+m_2^2-m_3^2}\right)+\operatorname{th}^{-1}\left(\frac{2m_1m_3}{m_1^2-m_2^2+m_3^2}\right)+\operatorname{th}^{-1}\left(\frac{2m_2m_3}{-m_1^2+m_2^2+m_3^2}\right)}{32\pi^2}+$$
$$-\frac{\log\left(-m_1^4-m_2^4-m_3^4+2m_1^2m_2^2+2m_1^2m_3^2+2m_2^2m_3^2\right)}{64\pi^2}+\mathrm{O}(d-3). \tag{10.1}$$

[6]The parameter is the de Sitter radius R, which in the formula is set to 1.

By using the identity $\mathrm{th}^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ the result simplifies to

$$I_3(m_1, m_2, m_3, d) = -\frac{1}{32\pi^2(d-3)} - \frac{1}{16\pi^2}\log(m_1+m_2+m_3) + \frac{1-\gamma+\log(4\pi)}{32\pi^2} + \mathrm{O}(d-3). \tag{10.2}$$

Alternatively, we may proceed by an ultraviolet cutoff of the integral defining the watermelon; in position space we cut a little sphere surrounding the origin $x = 0$:

$$\begin{aligned} I_{3,\Lambda}(m_1, m_2, m_3, 3) &= \frac{(m_1 m_2 m_3)^{\frac{1}{2}}}{(2\pi)^{\frac{5}{2}}\pi}\int_{\Lambda^2}^{\infty} r^{\frac{1}{2}} K_{\frac{1}{2}}(m_1 r)\, K_{\frac{1}{2}}(m_2 r)\, K_{\frac{1}{2}}(m_3 r)\, dr \\ &= \int_{\Lambda^2}^{\infty} \frac{e^{-r(m_1+m_2+m_3)}}{16\pi^2 r} dr = \frac{\Gamma(0, (m_1+m_2+m_3)\Lambda^2)}{16\pi^2}. \end{aligned} \tag{10.3}$$

Expansion in Λ now gives

$$I_{3,\Lambda}(m_1, m_2, m_3, 3) = -\frac{\log(\Lambda^4)}{32\pi^2} - \frac{1}{16\pi^2}\log(m_1+m_2+m_3) - \frac{\gamma}{16\pi^2}. \tag{10.4}$$

We see that the mass dependent finite parts coincide in the two expression. On the other hand the additive constants depend on the renormalization scheme.

de Sitter: UV cutoff. At $d = 3$ the Schwinger function is elementary:

$$G^3_\nu(-\cos s) = \frac{\mathrm{sh}(\nu(\pi - s))}{4\pi\,\mathrm{sh}(\pi\nu)\sin s}. \tag{10.5}$$

We may cutoff the integral as in flat space

$$\begin{aligned} &I_{3,K}(\nu_1, \nu_2, \nu_3, d) = 4\pi\int_K^\pi G^3_{\nu_1}(-\cos s) G^3_{\nu_2}(-\cos s) G^3_{\nu_3}(-\cos s) \sin^2 s ds = \\ &= \int_K^\pi \frac{\mathrm{sh}((\pi-s)(\nu_1-\nu_2-\nu_3)) - \mathrm{sh}((\pi-s)(\nu_1+\nu_2-\nu_3)) - \mathrm{sh}((\pi-s)(\nu_1-\nu_2+\nu_3)) + \mathrm{sh}((\pi-s)(\nu_1+\nu_2+\nu_3))}{64\pi^2\,\mathrm{sh}(\pi\nu_1)\,\mathrm{sh}(\pi\nu_2)\,\mathrm{sh}(\pi\nu_3)\sin s} ds. \end{aligned} \tag{10.6}$$

This leads us to consider the following indefinite integral

$$\begin{aligned} F_a(s) = \int \frac{\mathrm{sh}(a(\pi - s))}{\sin s} ds &= \frac{1}{2} e^{as - \pi a + is} \Phi\left(e^{2is}, 1, \frac{1}{2} - \frac{ia}{2}\right) - \frac{1}{2}\psi\left(\frac{1}{2} - \frac{ia}{2}\right) + \\ &- \frac{1}{2} e^{-as + \pi a + is} \Phi\left(e^{2is}, 1, \frac{1}{2} + \frac{ia}{2}\right) + \frac{1}{2}\psi\left(\frac{1}{2} + \frac{ia}{2}\right). \end{aligned} \tag{10.7}$$

Here Φ is the Lerch transcendent function and we adjusted the primitive $F_a(s)$ so that at the upper bound it vanishes at $s = \pi$. It follows that

$$\begin{aligned} \int_K^\pi \frac{\mathrm{sh}(a(\pi - s))}{\sin s} ds = -F_a(K) &\simeq -\mathrm{sh}(\pi a)(\log(2K) + \gamma) + \\ &- \frac{1}{2}\mathrm{sh}(\pi a)\left(\psi\left(\frac{1}{2} - \frac{ia}{2}\right) + \psi\left(\frac{1}{2} + \frac{ia}{2}\right)\right) + \mathrm{O}(K) \end{aligned} \tag{10.8}$$

and therefore

$$\int_K^\pi G^3_{\nu_1}(-\cos s)G^3_{\nu_2}(-\cos s)G^3_{\nu_3}(-\cos s)\sin^2 s ds = -\frac{\log(2K)+\gamma}{16\pi^2}+$$
$$-\frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1-\nu_2-\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1-\nu_2-\nu_3)}{2}\right)}{128\pi^2\,\mathrm{sh}(\pi\nu_1)\,\mathrm{sh}(\pi\nu_2)\,\mathrm{sh}(\pi\nu_3)}\mathrm{sh}(\pi(\nu_1-\nu_2-\nu_3))+$$
$$-\frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1+\nu_2+\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1+\nu_2+\nu_3)}{2}\right)}{128\pi^2\,\mathrm{sh}(\pi\nu_1)\,\mathrm{sh}(\pi\nu_2)\,\mathrm{sh}(\pi\nu_3)}\mathrm{sh}(\pi(\nu_1+\nu_2+\nu_3))+$$
$$+\frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1-\nu_2+\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1-\nu_2+\nu_3)}{2}\right)}{128\pi^2\,\mathrm{sh}(\pi\nu_1)\,\mathrm{sh}(\pi\nu_2)\,\mathrm{sh}(\pi\nu_3)}\mathrm{sh}(\pi(\nu_1-\nu_2+\nu_3))+$$
$$+\frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1+\nu_2-\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1+\nu_2-\nu_3)}{2}\right)}{128\pi^2\,\mathrm{sh}(\pi\nu_1)\,\mathrm{sh}(\pi\nu_2)\,\mathrm{sh}(\pi\nu_3)}\mathrm{sh}(\pi(\nu_1+\nu_2-\nu_3))+\mathrm{O}(K). \tag{10.9}$$

Dimensional regularization. The task is now to extract the finite part of the watermelon at $d=3$ from our general formulae. We accomplish this task in full detail, as in this example we explain the algorithm to extract information from our general formula for integer dimensions. The hypergeometric series at the r.h.s. of eq. (7.14) converges for $d<4$ and therefore the first term $I_3^{(1)}$ is finite. A direct calculation shows that

$$A_3(x,y,w)=\frac{\cot(2\pi w)(\cot(2\pi x)+\cot(2\pi y))\left(\psi\left(\frac{1}{2}+w+x+y\right)-\psi\left(\frac{1}{2}-w+x+y\right)\right)}{128\pi^2}. \tag{10.10}$$

The full $I_3^{(1)}(x,y,w,3)$ follows from (7.13). On the other hand $I_3^{(2)}$ contains the divergent part of the diagram, here a simple pole:

$$I_3^{(2)}(x,y,w,d)\simeq\frac{R(x,y,w,3)}{d-3}+\widetilde{I}_3(x,y,w,3). \tag{10.11}$$

Both the residue and the finite term may be computed with the help of the contiguity relations explained in appendix C. Details are provided in appendix D. In particular the residue of $B_d(x,y,w)$ at $d=3$ is given by

$$R_B(x,y,w)=b_3(x,y,w)f_3(x,y,w)=\frac{1-\cot(2\pi x)\cot(2\pi y)}{128\pi^2} \tag{10.12}$$

and does not depend on w (see eqs. (7.22) and (D.6)). In turn, the residue of $I_3^{(2)}$ at $d=3$ is the sum of four terms:

$$R(x,y,w)=R_B(x,y,w)+R_B(x,-y,w)+R_B(-x,y,w)+R_B(-x,-y,w)=\frac{1}{32\pi^2}. \tag{10.13}$$

We get the same result as in flat space. This fact is indeed nontrivial: the calculation in flat space and the calculation on the Euclidean sphere have essentially nothing to do with each other; the latter expresses a global quantity resulting from integrating on the curved sphere (here the radius of the sphere has been set to 1, but the result does not depend on the curvature) the de Sitter propagators which are completely different from the propagators

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in flat space in the large; they are similar to each other only in a small region as compared with the curvature and for lage value of the masses.

As regards the finite term, calculations are cumbersome; some details may found in appendix D. In the special case where $w = 0$ (i.e. $\nu_3 = 0$), eqs. (D.2), (D.6), (D.7) and (D.13) together give

$$\widetilde{B}_3(x,y,0) = \frac{(\cot(2\pi x)\cot(2\pi y)-1)\left(1-\gamma-\pi\cot(4\pi(x+y))+\log(\pi)-2\psi\left(\frac{1}{2}+x+y\right)\right)}{128\pi^2} \tag{10.14}$$

which allows for the calculation of $\widetilde{I}_3(x,y,0,3)$. We may actually solve for $\widetilde{I}_3(x,y,w,3)$. in full generality by observing that the functions

$$\Delta I_3^{(2)} = I_3^{(2)}(x,y,w,d) - I_3^{(2)}(x,y,0,d), \tag{10.15}$$

$$\Delta B_d = B_d(x,y,w) - B_d(x,y,0), \tag{10.16}$$

are regular at $d = 3$ because the residues (10.12) and (10.13) do not depend on w. In particular, by using eq. (7.4) and the identity

$$\frac{1}{(\kappa^2-\nu^2)} - \frac{1}{\kappa^2} = \frac{\nu^2\Gamma\left(-\frac{i\kappa}{2}\right)\Gamma\left(\frac{i\kappa}{2}\right)}{4(\kappa^2-\nu^2)\Gamma\left(1-\frac{i\kappa}{2}\right)\Gamma\left(\frac{i\kappa}{2}+1\right)}$$

we get

$$\Delta I_3^{(2)} = -\frac{w^2}{2\pi i}\int \prod_{\epsilon,\epsilon',\epsilon''=\pm1} \frac{2^{d-2}\pi\Gamma(\epsilon s+w)\Gamma(\epsilon s+\delta+x+y)}{\Gamma(1+\epsilon s)\Gamma\left(\epsilon s+\frac{1}{4}\right)\Gamma\left(\epsilon s+\frac{1}{2}\right)\Gamma\left(\epsilon s+\frac{3}{4}\right)\Gamma(\epsilon s+w+1)}. \tag{10.17}$$

By expressing $\Delta I_3^{(2)}$ in terms of hypergeometrics as in eq. (7.22), after some work, we obtain that

$$\begin{aligned}\Delta B_3 &= \lim_{d\to3}(B_d(x,y,w)-B_d(x,y,0))\\ &= -\frac{w^2(\cot(2\pi x)\cot(2\pi y)-1)}{8\pi^2(2x+2y+1)(4w^2-(2x+2y+1)^2)}\times\\ &\quad\times{}_4F_3\left(\begin{matrix}1,x+y+\frac{1}{2},w+x+y+\frac{1}{2},-w+x+y+\frac{1}{2}\\ x+y+\frac{3}{2},w+x+y+\frac{3}{2},-w+x+y+\frac{3}{2}\end{matrix};1\right)\\ &= \frac{(\cot(2\pi x)\cot(2\pi y)-1)\left(2\psi\left(\frac{1}{2}+x+y\right)\right)-\psi\left(\frac{1}{2}-w+x+y\right)-\psi\left(\frac{1}{2}+w+x+y\right)}{128\pi^2}.\end{aligned} \tag{10.18}$$

Combining eqs. (10.14) and (10.18) we get

$$\begin{aligned}\widetilde{B}_3(x,y,w) &= \frac{(1-\cot(2\pi x)\cot(2\pi y))(\pi\cot(4\pi(x+y))-1+\gamma-\log(\pi))}{128\pi^2}\\ &\quad+\frac{(1-\cot(2\pi x)\cot(2\pi y))\left(\psi\left(\frac{1}{2}-w+x+y\right)+\psi\left(\frac{1}{2}+w+x+y\right)\right)}{128\pi^2}\end{aligned} \tag{10.19}$$

and $\widetilde{I}_2(x,y,w,3)$ is computed by (8.2).

Collecting all the contributions the final result at $d \sim 3$ is given by

$$I(x,y,w,d\sim 3) = -\frac{1}{32\pi^2(d-3)} + \frac{1-\gamma+\log(\pi)}{32\pi^2} +$$
$$\sum_{\epsilon,\epsilon'=\pm} \frac{\left(\psi\left(\frac{1}{2}-w-\epsilon x-\epsilon' y\right)+\psi\left(\frac{1}{2}+w+\epsilon x+\epsilon' y\right)\right)\sin\left(2\pi\left(w+\epsilon x+\epsilon' y\right)\right)}{128\pi^2\sin(2\pi w)\sin(2\pi\epsilon x)\sin\left(2\pi\epsilon' y\right)} \tag{10.20}$$

to be compared with (10.9): the mass-dependent finite part is the same as in eq. (10.9).

11 Spacetime dimension $d = 4$

In the four dimensional case both $I_3^{(1)}$ and $I_3^{(2)}$ diverge. The situation is complicated even more by the presence of a double pole. While the algorithm remains the same, the whole procedure of extracting the residues and the finite parts from our general formulae through the contiguity scheme is quite heavy and the formulae cumbersome.

We will refrain to reproduce those formulae here. Instead we proceed to identify the residues of the poles but yet another method which has a quite general domain of applicability. We describe the general construction first.

Convergence, poles and residues. In section 7 we considered integrals of the form:

$$\mathbf{J} = \int_{\eta+i\mathbf{R}} J((a_p),(c_p),s)ds\,,$$
$$J((a_p),(c_p),s) = \frac{\prod_{j=1}^{p-1}\Gamma(a_j+s)\Gamma(a_j-s)}{(a_p+s)(a_p-s)\prod_{j=1}^{p-1}\Gamma(c_j+s)\Gamma(c_j-s)} = \frac{\prod_{j=1}^{p}\Gamma(a_j+s)\Gamma(a_j-s)}{\prod_{j=1}^{p}\Gamma(c_j+s)\Gamma(c_j-s)}\,, \tag{11.1}$$

where $c_p = a_p + 1$. Note that $J((a_p),(c_p),s) = J((a_p),(c_p),-s)$. It is assumed that $\mathrm{Re}\, a_j > 0$ for $1 \leq j \leq p-1$, a_p is pure imaginary, $\eta > 0$ is small, so that all the poles of J (as a function of s) on the right of the contour are those located at $s = a_j + n$, $n \geq 0$ integer, $1 \leq j \leq p-1$. We also assume that outside of $\{|s| \leq K\}$ the contour is deformed so that it lies on the imaginary axis $i\mathbf{R}$. Let

$$Z = 2\sum_{j=1}^{p}(a_j - c_j)\ . \tag{11.2}$$

Since $|\arg s| \leq \frac{\pi}{2}$ we may use the Erdélyi-Tricomi Theorem [23, eq. 5.11.13], [82, pp. 118 ff]: as s tends to $\pm i\infty$

$$\frac{\Gamma(a_j+s)}{\Gamma(c_j+s)} \sim s^{a_j-c_j}\sum_{k=0}^{\infty} G_k(a_j,\ c_j)\, s^{-k} \tag{11.3}$$

where the first three terms are given by

$$G_0 = 1, \qquad G_1(a,b) = \frac{1}{2}(a-b)(a+b-1),$$
$$G_2(a,b) = \frac{1}{12}\binom{a-b}{2}\left(3(a+b-1)^2-(a-b+1)\right)\ . \tag{11.4}$$

In general, $G_n(a,b)$ is a polynomial with real rational coefficients in a and b.

It follows that for a fixed integer $N \geq 1$, there is a $L > 0$ and a bounded analytic function $h_N((a_p),(c_p),s)$ such that, for $s = it$, $t \geq L$,

$$J((a_p),(c_p),s) = t^Z \left(\sum_{k=0}^{N} \frac{u_k}{t^k} \right) + t^{Z-N-1} h_N((a_p),(c_p),s) \; . \tag{11.5}$$

Here $u_0 = 1$ and the other u_k are polynomial expressions of the $G_n(a_j,c_j)$, $0 \leq n \leq N$, $1 \leq j \leq p$. Since $J((a_p),(c_p),s)$ is an even function of s, $u_k = 0$ for odd k.

Suppose that $\operatorname{Re} Z < -1$: integrating $J((a_p),(c_p),s)$ over the whole integration contour gives

$$2i \int_L^\infty t^Z \left(\sum_{k=0}^{N} \frac{u_k}{t^k} \right) dt = \sum_{k=0}^{N} \frac{-2iu_k L^{Z-k+1}}{Z-k+1} \tag{11.6}$$

plus a bounded analytic function of the parameters.

The result has a meromorphic continuation for $\operatorname{Re} Z < N$, with poles given by (11.6). The residue at $Z = k-1$ is $-2iu_k$. In particular the residue at $Z = -1$ is $-2i$ and is independent of the parameters. Recall that $u_{2n+1} = 0$ for all n. For $N = 2$ we can write

$$J((a_p),(c_p),s) = t^Z \left(\Big(1 + s^{-2} \sum_{j=1}^{p} (2G_2(a_j,c_j) - G_1(a_j,c_j)^2) \Big) + O(s^{-3}) \right) . \tag{11.7}$$

Since $s^2 = -t^2$, we get

$$u_2 = \sum_{j=1}^{p} (G_1(a_j,c_j)^2 - 2G_2(a_j,c_j)) = \sum_{j=1}^{p} (f_2(a_j) - f_2(c_j)), \tag{11.8}$$

where

$$f_2(a) = \frac{a(a-1)(2a-1)}{6} \; . \tag{11.9}$$

Summary. $\mathbf{J}$ extends to a meromorphic function of the parameters having poles at

$$Z = -1 + 2n, \quad n \geq 0 \;\; \text{integer}. \tag{11.10}$$

Near $Z = -1 + 2n$,

$$\mathbf{J} \sim \frac{-2iu_{2n}}{\zeta + 1 - 2n} \; . \tag{11.11}$$

Application at $d = 3$ and $d = 4$. Let us consider at first the integral (7.10). It fits with the definition (11.1) with $Z = 4\delta - 4 = d - 5$. Here we trade the variable Z for the spacetime dimension d and conclude that there are poles $d = 4 + 2n$. The residue of $\mathcal{A}$ at $d = 4$ is thus simply:

$$\text{residue of } \mathcal{A} \text{ at } d = 4 : \;\; \frac{-2iu_0}{2i} = -1 \; . \tag{11.12}$$

Also the integral (10.17) fits with the definition (11.1); now $Z = 8\delta - 5 = 2d - 7$. There are poles at $d = 3 + n$ with residues (in the variable d) $-2^{d-2} i u_{2n}/(2i) = -2^{d-3} u_{2n}$; in particular

$$\begin{aligned}
&\text{residue of } \mathcal{B} \text{ at } d = 3: && -u_0 = -1\,, \\
&\text{residue of } \mathcal{B} \text{ at } d = 4: && -2u_2 = 2w^2 - 2x^2 - 2y^2 + \frac{1}{8}\,.
\end{aligned} \tag{11.13}$$

Multiplying by the right normalizations as in eqs. (7.5) and (7.7) and restoring the mass variables we obtain the leading singularities as follows:

$$I_3 \text{ at } d = 3: \quad -\frac{1}{32\pi^2(d-3)}, \tag{11.14}$$

$$I_3 \text{ at } d = 4: \quad -\frac{4\nu_1^2 + 4\nu_2^2 + 4\nu_3^2 + 3}{512\pi^4(d-4)^2}. \tag{11.15}$$

Once more, we notice that the dominant divergence in $d = 3$ is exactly the same as in the flat case. (11.15) reproduces the flat dominant divergence in the limit $R \to \infty$.

12 Conclusions

In a recent paper we have shown that calculating loop integrals in position space may be advantageous also in flat Minkowski space from several viewpoints. While we believe that that possibility in flat space deserves attention and is not just a luxury or a mathematical ornament, in curved space performing loop calculation in position space is compulsory. Here we have started this program in de Sitter space. Calculations are significantly more intricate than in flat space but can be performed successfully till the end. In a companion paper we present a study of the same diagrams in anti de Sitter space. We have found that in the AdS case things are a little simpler than in the present de Sitterian study because of the presence of a true spectral condition. The two papers together open a new way for precision calculations of QFT in the presence of a cosmological constant.

A Details about section 3

A.1 The functions $\mathbf{P}_\nu^\mu$ and $\mathbf{Q}_\nu^\mu$

These functions are solutions of the Legendre equation [15, 3.2 (1) p. 121]:

$$\begin{aligned}
&\frac{d}{dz}(1 - z^2)w'(z) + C_{\nu,\mu}(z)w(z) = 0\,, \\
&\qquad\qquad C_{\nu,\mu}(z) = \nu(\nu+1) - \mu^2(1-z^2)^{-1}\,.
\end{aligned} \tag{A.1}$$

They are called "Legendre functions on the cut" in [15] or Ferrers functions [23, 14.3.1, 14.3.2]. They are holomorphic in the domain

$$\Delta_2 = \{z \in \mathbf{C} \;:\; \operatorname{Im} z \neq 0 \;\text{ or }\; -1 < z < 1\} \tag{A.2}$$

and given there by:

$$\begin{aligned}
\mathbf{P}_\beta^\alpha(z) &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{1+z}{1-z}\right)^{\frac{\alpha}{2}} F\left(-\beta,\ 1+\beta\ ;\ 1-\alpha\ ;\ \frac{1-z}{2}\right), \qquad (A.3)\\
&= \frac{\Gamma(-\alpha)}{\Gamma(1+\beta-\alpha)\Gamma(-\beta-\alpha)}\left(\frac{1+z}{1-z}\right)^{\frac{\alpha}{2}} F\left(-\beta,\ 1+\beta\ ;\ 1+\alpha\ ;\ \frac{1+z}{2}\right)\\
&\quad -\pi^{-1}\sin(\pi\beta)\Gamma(\alpha)\left(\frac{1-z}{1+z}\right)^{\frac{\alpha}{2}} F\left(-\beta,\ 1+\beta\ ;\ 1-\alpha\ ;\ \frac{1+z}{2}\right) \qquad (A.4)\\
&= \frac{\mathrm{tg}(\pi\alpha)+\mathrm{tg}(\pi\beta)}{\pi}\mathrm{Q}_\beta^\alpha(z) + \frac{\mathrm{tg}(\pi\alpha)-\mathrm{tg}(\pi\beta)}{\pi}\mathrm{Q}_{-\beta-1}^\alpha(z), \qquad (A.5)\\
\mathrm{Q}_\beta^\alpha(z) &= \frac{\pi\cos(\pi\alpha)}{2\sin(\pi\alpha)}\mathbf{P}_\beta^\alpha(z) - \frac{\pi\Gamma(\beta+\alpha+1)}{2\sin(\pi\alpha)\Gamma(\beta-\alpha+1)}\mathbf{P}_\beta^{-\alpha}(z) \qquad (A.6)\\
&= \frac{1}{2}\Gamma(\alpha)\cos(\alpha\pi)\left(\frac{1+z}{1-z}\right)^{\frac{\alpha}{2}} F\left(-\beta,\ \beta+1\ ;\ 1-\alpha\ ;\ \frac{1-z}{2}\right)\\
&\quad + \frac{\Gamma(1+\beta+\alpha)\Gamma(-\alpha)}{2\Gamma(1+\beta-\alpha)}\left(\frac{1-z}{1+z}\right)^{\frac{\alpha}{2}} F\left(-\beta,\ \beta+1\ ;\ \alpha+1\ ;\ \frac{1-z}{2}\right). \qquad (A.7)
\end{aligned}$$

(See [15, 3.4 (6) p. 143, 3.2 (15) pp 124-125, 3.4 (13), (10) p. 144]). From this it follows that:
1) as $z \to 1$

$$\mathbf{P}_\beta^\alpha(z) \sim \frac{1}{\Gamma(1-\alpha)}\left(\frac{1-z}{2}\right)^{-\frac{\alpha}{2}}, \qquad (A.8)$$

$$\mathrm{Q}_\beta^\alpha(z) \sim \frac{1}{2}\Gamma(\alpha)\cos(\pi\alpha)\left(\frac{1-z}{2}\right)^{-\frac{\alpha}{2}} \quad \text{if} \quad \mathrm{Re}\,\alpha > 0\,, \qquad (A.9)$$

$$\mathrm{Q}_\beta^\alpha(z) \sim \frac{\Gamma(1+\beta+\alpha)\Gamma(-\alpha)}{2\Gamma(1+\beta-\alpha)}\left(\frac{1-z}{2}\right)^{\frac{\alpha}{2}} \quad \text{if} \quad \mathrm{Re}\,\alpha < 0\,; \qquad (A.10)$$

2) as $z \to -1$

$$\mathbf{P}_\beta^\alpha(z) \sim -\pi^{-1}\sin(\pi\beta)\Gamma(\alpha)\left(\frac{1+z}{2}\right)^{-\frac{\alpha}{2}} \quad \text{if} \quad \mathrm{Re}\,\alpha > 0\ , \qquad (A.11)$$

$$\mathbf{P}_\beta^\alpha(z) \sim \frac{\Gamma(-\alpha)}{\Gamma(1+\beta-\alpha)\Gamma(-\beta-\alpha)}\left(\frac{1+z}{2}\right)^{\frac{\alpha}{2}} \quad \text{if} \quad \mathrm{Re}\ \alpha < 0\ , \qquad (A.12)$$

$$\mathrm{Q}_\beta^\alpha(z) \sim -\frac{\Gamma(\alpha)\cos(\pi\beta)}{2}\left(\frac{1+z}{2}\right)^{-\frac{\alpha}{2}} \quad \text{if} \quad \mathrm{Re}\,\alpha > 0\,, \qquad (A.13)$$

$$\begin{aligned}
\mathrm{Q}_\beta^\alpha(z) &\sim \frac{\Gamma(-\alpha)\Gamma(1+\beta+\alpha)}{2\sin(\pi\alpha)\Gamma(1+\beta-\alpha)}\Big[\sin(\pi\beta)-\cos(\pi\alpha)\sin\pi(\alpha+\beta)\Big]\left(\frac{1+z}{2}\right)^{\frac{\alpha}{2}}\\
&= -\frac{\Gamma(-\alpha)\Gamma(1+\beta+\alpha)\cos\pi(\alpha+\beta)}{2\Gamma(1+\beta-\alpha)}\left(\frac{1+z}{2}\right)^{\frac{\alpha}{2}} \quad \text{if} \quad \mathrm{Re}\,\alpha < 0\,.
\end{aligned} \qquad (A.14)$$

A.2 Derivation of the main Wronskian equation (3.5)

Let u_1, u_2 be solutions of

$$\frac{d}{dz}(1-z^2)u_j'(z) + B_j(z)u_j(z) = 0\,, \quad j = 1,\ 2, \qquad (A.15)$$

where B_j is analytic in the domain we consider. Let $D(z)$ be the determinant

$$D(z) = \begin{vmatrix} u_1(z) & u_2(z) \\ (1-z^2)u_1'(z) & (1-z^2)u_2'(z) \end{vmatrix} . \tag{A.16}$$

Then

$$\frac{d}{dz}D(z) = \begin{vmatrix} u_1(z) & u_2(z) \\ -B_1(z)u_1(z) & -B_2(z)u_2(z) \end{vmatrix} = u_1(z)u_2(z)\Big[B_1(z) - B_2(z)\Big] . \tag{A.17}$$

Thus

$$\begin{aligned}\int_a^b u_1(z)u_2(z)\Big[B_1(z) - B_2(z)\Big]dz &= \Big[D(z)\Big]_a^b = \\ &= \Big[(1-z^2)[u_1(z)u_2'(z) - u_2(z)u_1'(z)]\Big]_a^b . \end{aligned} \tag{A.18}$$

We now take $u_1 = u_\nu^\mu$, $u_2 = v_\sigma^\rho$ with the understanding that u_ν^μ stands for $\mathbf{P}_\nu^\mu$ or Q_ν^μ, and independently v_σ^ρ stands for $\mathbf{P}_\sigma^\rho$ or Q_σ^ρ. Thus $B_1(z) = C_{\nu,\mu}(z)$ and $B_2(z) = C_{\sigma,\rho}(z)$, and

$$B_1(z) - B_2(z) = (\nu - \sigma)(\nu + \sigma + 1) + (\rho^2 - \mu^2)(1-z^2)^{-1} . \tag{A.19}$$

We recall the formulae [15, 3.8 (19),(15) p.161]

$$(1-z^2)\frac{d\mathbf{P}_\nu^\mu(z)}{dz} = -\nu z\mathbf{P}_\nu^\mu(z) + (\nu+\mu)\mathbf{P}_{\nu-1}^\mu(z) , \tag{A.20}$$

$$\mathbf{P}_{\nu-1}^\mu(z) = z\mathbf{P}_\nu^\mu(z) + (\nu - \mu + 1)(1-z^2)^{\frac{1}{2}}\mathbf{P}_\nu^{\mu-1}(z) . \tag{A.21}$$

Hence

$$(1-z^2)\frac{d\mathbf{P}_\nu^\mu(z)}{dz} = \mu z\mathbf{P}_\nu^\mu(z) + (\nu+\mu)(\nu-\mu+1)(1-z^2)^{\frac{1}{2}}\mathbf{P}_\nu^{\mu-1}(z)] . \tag{A.22}$$

As stated in [15] the formulae (A.20)–(A.22) remain valid when $\mathbf{P}$ is replaced by Q. Therefore, setting $\rho = \mu$, we get:

$$\begin{aligned}&\int_a^b u_\nu^\mu(z)v_\sigma^\mu(z)(\nu-\sigma)(\sigma+\nu+1)dz = \\ &= \Big[(1-z^2)^{\frac{1}{2}}(\sigma+\mu)(\sigma-\mu+1)u_\nu^\mu(z)v_\sigma^{\mu-1}(z) - (1-z^2)^{\frac{1}{2}}(\nu+\mu)(\nu-\mu+1)u_\nu^{\mu-1}(z)v_\sigma^\mu(z)\Big]_a^b . \end{aligned} \tag{A.23}$$

This is eq. (3.5). Recall that here the integration is over an arc (with extremities a and b) contained in the domain $\Delta_2 = \{z \in \mathbf{C} \ : \ \mathrm{Im}\, z \neq 0 \ \text{ or } \ |z| < 1\}$, and that $z \mapsto (1-z^2)^{\frac{1}{2}}$ is the function holomorphic in this domain and equal to $|1-z^2|^{\frac{1}{2}}$ when $z \in (-1,\ 1)$.

A.2.1 Case of the functions P_ν^μ and Q_ν^μ

Some details have to be modified if we now suppose that u_ν^μ stands for P_ν^μ or Q_ν^μ, and independently v_σ^ρ stands for P_σ^ρ or Q_σ^ρ. These functions[7] will be considered as holomorphic

[7]We use the notations of [15, 3.2 (3),(5) p. 122]. Note that Q_ν^μ is not defined where $\Gamma(\nu+\mu+1)$ has a pole.

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in the domain $\Delta_1 = \{z \in \mathbf{C} \ : \ \operatorname{Im} z \neq 0 \quad \text{or} \quad z > 1\}$. Then eqs. (A.20)–(A.22) have to be replaced by

$$(1-z^2)\frac{dP^\mu_\nu(z)}{dz} = -\nu z P^\mu_\nu(z) + (\nu+\mu)P^\mu_{\nu-1}(z)\,, \tag{A.24}$$

$$P^\mu_{\nu-1}(z) = zP^\mu_\nu(z) - (\nu-\mu+1)(z^2-1)^{\frac{1}{2}}P^{\mu-1}_\nu(z)\,, \tag{A.25}$$

(see [15, 3.8 (10), (5) p. 161])

$$(1-z^2)\frac{dP^\mu_\nu(z)}{dz} = \mu z P^\mu_\nu(z) - (\nu+\mu)(\nu-\mu+1)(z^2-1)^{\frac{1}{2}}P^{\mu-1}_\nu(z)]\,. \tag{A.26}$$

Here $z \mapsto (z^2-1)^{\frac{1}{2}}$ is the function holomorphic in Δ_1 equal to $|z^2-1|^{\frac{1}{2}}$ when $z>1$. Again as stated in [15] eqs. (A.24)–(A.26) remain valid if P is replaced by Q. Therefore, setting $\rho = \mu$, we get from (A.18):

$$\begin{aligned}&\int_a^b u^\mu_\nu(z)v^\mu_\sigma(z)(\nu-\sigma)(\sigma+\nu+1)dz = \\ &= \Big[-(z^2-1)^{\frac{1}{2}}(\sigma+\mu)(\sigma-\mu+1)u^\mu_\nu(z)v^{\mu-1}_\sigma(z) + (z^2-1)^{\frac{1}{2}}(\nu+\mu)(\nu-\mu+1)u^{\mu-1}_\nu(z)v^\mu_\sigma(z)\Big]_a^b\,.\end{aligned} \tag{A.27}$$

Here the integration is over an arc contained in Δ_1.

A.3 The basic case $u^\mu_\nu = \mathrm{Q}^\mu_\nu$, $v^\mu_\sigma = \mathrm{Q}^\mu_\sigma$

This case may be considered as basic since by using (A.5) it is possible to obtain the other cases from it. The formula (3.5) becomes in this case:

$$\begin{aligned}&(\nu-\sigma)(\nu+\sigma+1)\int_a^b \mathrm{Q}^\mu_\nu(z)\mathrm{Q}^\mu_\sigma(z)\,dz = \\ &= \Big[(\sigma+\mu)(\sigma-\mu+1)(1-z^2)^{\frac{1}{2}}\mathrm{Q}^\mu_\nu(z)\mathrm{Q}^{\mu-1}_\sigma(z) - (\nu+\mu)(\nu-\mu+1)(1-z^2)^{\frac{1}{2}}\mathrm{Q}^{\mu-1}_\nu(z)\mathrm{Q}^\mu_\sigma(z)\Big]_a^b.\end{aligned} \tag{A.28}$$

As in section 3 it is sufficient to evaluate the contribution of the first term in the r.h.s. of (A.28) since the contribution of the second can be obtained by exchanging ν and σ and a global change of sign.

We again fix $\mu \in (0,\ 1)$. Using the formulae (A.9) and (A.10) we find that as $z \to 1$ the first term in the r.h.s. of (A.28) tends to

$$\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)\cos(\pi\mu)}{2\Gamma(1+\sigma-\mu)}\,. \tag{A.29}$$

As $z \to -1$ (with a minus sign due to dealing with the lower bound a) this first term contributes:

$$\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)\cos(\pi\nu)\cos\pi(\sigma+\mu)}{2\Gamma(1+\sigma-\mu)}\,. \tag{A.30}$$

The r.h.s. of (A.28) is thus given by

$$\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)[\cos(\pi\mu)+\cos(\pi\nu)\cos\pi(\sigma+\mu)]}{2\Gamma(1+\sigma-\mu)} \\ -\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\nu+\mu)[\cos(\pi\mu)+\cos(\pi\sigma)\cos\pi(\nu+\mu)]}{2\Gamma(1+\nu-\mu)}. \tag{A.31}$$

In this expression let us set $\nu=-\sigma-1$. The first term becomes

$$\begin{aligned}&\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)[\cos(\pi\mu)-\cos(\pi\sigma)\cos\pi(\sigma+\mu)]}{2\Gamma(1+\sigma-\mu)}\\ &=\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)\sin\pi(\sigma+\mu)\sin(\pi\sigma)}{2\Gamma(1+\sigma-\mu)}\\ &=-\frac{\pi\Gamma(\mu)\Gamma(1-\mu)\sin(\pi\sigma)}{2\Gamma(1+\sigma-\mu)\Gamma(-\sigma-\mu)}\ .\end{aligned} \tag{A.32}$$

The second term becomes

$$\begin{aligned}&-\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(\mu-\sigma)[\cos(\pi\mu)-\cos(\pi\sigma)\cos\pi(\mu-\sigma)]}{2\Gamma(-\sigma-\mu)}\\ &=\frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(\mu-\sigma)\sin\pi(\mu-\sigma)\sin(\pi\sigma)}{2\Gamma(-\sigma-\mu)}\\ &=\frac{\pi\Gamma(\mu)\Gamma(1-\mu)\sin(\pi\sigma)}{2\Gamma(1+\sigma-\mu)\Gamma(-\sigma-\mu)}\ .\end{aligned} \tag{A.33}$$

Thus the expression (A.31) vanishes if we set $\nu=-\sigma-1$. It also obviously vanishes if we set $\sigma=\nu$. We obtain

$$\int_{-1}^{1}\mathrm{Q}_\nu^\mu(z)\mathrm{Q}_\sigma^\mu(z)\,dz=\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\nu+\mu)\Gamma(1+\sigma+\mu)\Phi_{\mathrm{QQ}}(\nu,\ \sigma,\ \mu)\,, \tag{A.34}$$

$$\Phi_{\mathrm{QQ}}(\nu,\ \sigma,\ \mu)=\frac{1}{(\nu-\sigma)(\nu+\sigma+1)}\times \\ \left[\frac{\cos(\pi\mu)+\cos(\pi\nu)\cos\pi(\sigma+\mu)}{2\Gamma(1+\nu+\mu)\Gamma(1+\sigma-\mu)}-\frac{\cos(\pi\mu)+\cos(\pi\sigma)\cos\pi(\nu+\mu)}{2\Gamma(1+\sigma+\mu)\Gamma(1+\nu-\mu)}\right]. \tag{A.35}$$

$\Phi_{\mathrm{QQ}}(\nu,\ \sigma,\ \mu)$ extends to an entire function of all its arguments. It is symmetric in ν and σ and vanishes when $\mu=0$. The equality (A.34) extends by analytic continuation to values of μ with $-1<\mathrm{Re}\,\mu<1$.

A.4 The case $u_\nu^\mu=\mathbf{P}_\nu^\mu$, $v_\sigma^\mu=\mathrm{Q}_\sigma^\mu$

This case may be dealt with by two different methods: the first is to use the Wronskian equation as in the two preceding cases. The second is to use eq. (A.5) and the result of subsection A.3. The formula (3.5) becomes in this case:

$$(\nu-\sigma)(\nu+\sigma+1)\int_a^b\mathbf{P}_\nu^\mu(z)\mathrm{Q}_\sigma^\mu(z)\,dz= \\ =\Big[(\sigma+\mu)(\sigma-\mu+1)(1-z^2)^{\frac12}\mathbf{P}_\nu^\mu(z)\mathrm{Q}_\sigma^{\mu-1}(z)-(\nu+\mu)(\nu-\mu+1)(1-z^2)^{\frac12}\mathbf{P}_\nu^{\mu-1}(z)\mathrm{Q}_\sigma^\mu(z)\Big]_a^b. \tag{A.36}$$

We fix $\mu \in (0,\ 1)$. To evaluate the r.h.s. of this formula as $z \to 1$ we use the formulae (A.8)–(A.10): as $z \to 1$ the first term in the r.h.s. of (A.36) tends to

$$\frac{\Gamma(1+\sigma+\mu)}{\Gamma(1+\sigma-\mu)}\,. \tag{A.37}$$

The second term in the r.h.s. of (A.36) behaves like Const. $(1-z)^{1-\mu}$ and tends to 0. To evaluate the r.h.s. of the formula (A.36) as $z \to -1$ we use the formula (A.11)–(A.14). In the end we find that, for $a = -1$ and $b = 1$ the r.h.s. of (A.36) is equal to

$$\frac{\Gamma(1+\sigma+\mu)}{\Gamma(1+\sigma-\mu)} + \frac{\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)\sin(\pi\nu)\cos\pi(\mu+\sigma)}{\pi\Gamma(1+\sigma-\mu)} + \frac{\Gamma(\mu)\Gamma(1-\mu)\cos(\pi\sigma)}{\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)}. \tag{A.38}$$

If we set $\nu = \sigma$ in this expression, it becomes

$$\begin{aligned} &\frac{\Gamma(1+\sigma+\mu)}{\Gamma(1+\sigma-\mu)} - \frac{\Gamma(1+\sigma+\mu)\Gamma(\mu)\Gamma(1-\mu)}{\pi\Gamma(1+\sigma-\mu)}\times \\ &\left[-\sin(\pi\sigma)\cos\pi(\mu+\sigma) - \frac{\pi\cos(\pi\sigma)}{\Gamma(1+\sigma+\mu)\Gamma(-\sigma-\mu)}\right] = 0\,, \end{aligned} \tag{A.39}$$

as it must be. We can therefore subtract from (A.38) the same expression with $\nu = \sigma$ and we obtain, for $0 < \mu < 1$,

$$\int_{-1}^{1} \mathbf{P}_\nu^\mu(z)\mathrm{Q}_\sigma^\mu(z)\,dz = \Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)\Phi_{\mathbf{PQ}}(\nu,\ \sigma,\ \mu)\,, \tag{A.40}$$

$$\begin{aligned} \Phi_{\mathbf{PQ}}(\nu,\ \sigma,\ \mu) = \frac{1}{(\nu-\sigma)(\nu+\sigma+1)} \times \Bigg[&\frac{[\sin(\pi\nu)-\sin(\pi\sigma)]\cos\pi(\mu+\sigma)}{\pi\Gamma(1+\sigma-\mu)} \\ &+ \frac{\cos(\pi\sigma)}{\Gamma(1+\sigma+\mu)}\left[\frac{1}{\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)} - \frac{1}{\Gamma(1+\sigma-\mu)\Gamma(-\sigma-\mu)}\right]\Bigg]. \end{aligned} \tag{A.41}$$

The function $\Phi_{\mathbf{PQ}}(\nu,\ \sigma,\ \mu)$ is entire in all its variables. It vanishes at $\mu = 0$ and it is invariant under the change $\nu \to -1-\nu$. The equality (A.40) extends by analytic continuation to $-1 < \operatorname{Re}\mu < 1$.

The second method is to use (A.5) to write

$$\begin{aligned} &\int_{-1}^{1} \mathbf{P}_\nu^\mu(z)\mathrm{Q}_\sigma^\mu(z)\,dz = \\ &= \frac{\mathrm{tg}(\pi\mu)+\mathrm{tg}(\pi\nu)}{\pi}\int_{-1}^{1}\mathrm{Q}_\nu^\mu(z)\mathrm{Q}_\sigma^\mu(z)\,dz + \frac{\mathrm{tg}(\pi\mu)-\mathrm{tg}(\pi\nu)}{\pi}\int_{-1}^{1}\mathrm{Q}_{-\nu-1}^\mu(z)\mathrm{Q}_\sigma^\mu(z)\,dz = \\ &= \frac{1}{(\nu-\sigma)(\nu+\sigma+1)}\Bigg(\frac{\Gamma(\mu+\sigma+1)(\sec(\pi\mu)\cos(\pi\nu)\cos(\pi(\mu+\sigma))+1)}{\Gamma(-\mu+\sigma+1)} + \\ &\quad + \frac{\pi\nu\sec(\pi\mu)\,\mathrm{tg}(\pi\nu)\cos(\pi\sigma) - \pi\mu\csc(\pi\mu)(\sec(\pi\nu)+\cos(\pi\sigma))}{\Gamma(-\mu-\nu+1)\Gamma(-\mu+\nu+1)}\Bigg)\,. \end{aligned} \tag{A.42}$$

A.5 Other integrals

Using the formulae [15, 3.4 (14), (15), (17), (18) p. 144]

$$\mathbf{P}^{\alpha}_{\beta}(-z) = \cos\pi(\beta+\alpha)\mathbf{P}^{\alpha}_{\beta}(z) - \frac{2\sin\pi(\beta+\alpha)}{\pi}\mathrm{Q}^{\alpha}_{\beta}(z)\,, \tag{A.43}$$

$$\mathrm{Q}^{\alpha}_{\beta}(-z) = -\cos\pi(\beta+\alpha)\mathrm{Q}^{\alpha}_{\beta}(z) < -\frac{\sin\pi(\beta+\alpha)}{2\pi}\mathbf{P}^{\alpha}_{\beta}(z)\,, \tag{A.44}$$

$$\Gamma(1+\beta+\alpha)\mathbf{P}^{-\alpha}_{\beta}(z) = \Gamma(1+\beta-\alpha)\left[\mathbf{P}^{\alpha}_{\beta}(z)\cos(\pi\alpha) - \frac{2}{\pi}\sin(\pi\alpha)\mathrm{Q}^{\alpha}_{\beta}(z)\right]\,, \tag{A.45}$$

$$\Gamma(1+\beta+\alpha)\mathrm{Q}^{-\alpha}_{\beta}(z) = \Gamma(1+\beta-\alpha)\left[\mathrm{Q}^{\alpha}_{\beta}(z)\cos(\pi\alpha) + \frac{\pi}{2}\sin(\pi\alpha)\mathbf{P}^{\alpha}_{\beta}(z)\right]\,, \tag{A.46}$$

we can immediately obtain $\int_{-1}^{1}\mathbf{P}^{\pm\mu}_{\nu}(\pm z)\mathbf{P}^{\pm\mu}_{\sigma}(\pm z)dz$, $\int_{-1}^{1}\mathbf{P}^{\pm\mu}_{\nu}(\pm z)\mathrm{Q}^{\pm\mu}_{\sigma}(\pm z)dz$, $\int_{-1}^{1}\mathrm{Q}^{\pm\mu}_{\nu}(\pm z)\mathrm{Q}^{\pm\mu}_{\sigma}(\pm z)dz$, for instance:

$$\begin{aligned}\int_{-1}^{1}\mathbf{P}^{\mu}_{\nu}(z)\mathbf{P}^{\mu}_{\sigma}(-z)\,dz &= 2\pi^{-1}\cos\pi(\sigma+\mu)\Gamma(\mu)\Gamma(1-\mu)\Phi_{\mathbf{PP}}(\nu,\ \sigma,\ \mu)\\ &\quad - \frac{2}{\pi}\sin\pi(\sigma+\mu)\Gamma(\mu)\Gamma(1-\mu)\Gamma(1+\sigma+\mu)\Phi_{\mathbf{PQ}}(\nu,\ \sigma,\ \mu)\ .\end{aligned} \tag{A.47}$$

Finally let us use subsubsection A.2.1 to evaluate the integral

$$\int_{1}^{\infty} Q^{\mu}_{\nu}(z)Q^{\mu}_{\sigma}(z)dz\ . \tag{A.48}$$

It follows from [15, 3.9.2 (21), (5), (6) pp 163-164] that:

$$\text{as}\quad z\to+\infty,\qquad Q^{\mu}_{\nu}(z)\sim \text{const.}\ z^{-\nu-1}\ ; \tag{A.49}$$

$$\text{if}\ \operatorname{Re}\mu>0,\quad \text{as}\quad z\to 1,\qquad Q^{\mu}_{\nu}(z)\sim e^{i\pi\mu}2^{\frac{\mu}{2}-1}\Gamma(\mu)(z-1)^{-\frac{\mu}{2}}\ ; \tag{A.50}$$

$$\text{if}\ \operatorname{Re}\mu<0,\quad \text{as}\quad z\to 1,\qquad Q^{\mu}_{\nu}(z)\sim \frac{e^{i\pi\mu}2^{-\frac{\mu}{2}-1}\Gamma(-\mu)\Gamma(\nu+\mu+1)(z-1)^{\frac{\mu}{2}}}{\Gamma(\nu-\mu+1)}\ . \tag{A.51}$$

Note also ([15, 3.2 (2) p. 140])

$$e^{i\pi\mu}\Gamma(\nu+\mu+1)Q^{-\mu}_{\nu}(z) = e^{-i\pi\mu}\Gamma(\nu-\mu+1)Q^{\mu}_{\nu}(z)\ . \tag{A.52}$$

It follows that the integral (A.48) converges if $\operatorname{Re}(\nu+\sigma)>-1$ and $|\operatorname{Re}\mu|<1$. For our evaluation we suppose $\operatorname{Re}(\nu+\sigma)>-1$ and $0<\mu<1$. We then set $u^{\mu}_{\nu}(z)=Q^{\mu}_{\nu}(z)$ and $v^{\mu}_{\sigma}=Q^{\mu}_{\sigma}$ in (A.27) and let b tend to infinity and a tend to 1. The result is that, under our assumptions,

$$\int_{1}^{\infty} Q^{\mu}_{\nu}(z)Q^{\mu}_{\sigma}(z)\,dz = \frac{e^{2i\pi\mu}\Gamma(\mu)\Gamma(1-\mu)}{2(\nu-\sigma)(\sigma+\nu+1)}\left[\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} - \frac{\Gamma(\sigma+\mu+1)}{\Gamma(\sigma-\mu+1)}\right]\ . \tag{A.53}$$

This equation remains valid, by analytic continuation, when $|\operatorname{Re}\mu|<1$ (and $\operatorname{Re}(\nu+\sigma)>-1$) (note that the bracket vanishes for $\mu=0$). The formula is compatible with (A.52). Letting ν tend to σ we find

$$\int_{1}^{\infty} Q^{\mu}_{\sigma}(z)Q^{\mu}_{\sigma}(z)\,dz = \frac{e^{2i\pi\mu}\Gamma(\mu)\Gamma(1-\mu)\Gamma(\sigma+\mu+1)}{2(2\sigma+1)\Gamma(\sigma-\mu+1)}\Big[\psi(\sigma+\mu+1)-\psi(\sigma-\mu+1)\Big]\ . \tag{A.54}$$

Again this equation is valid when $|\operatorname{Re}\mu|<1$ and $\operatorname{Re}2\sigma>-1$, but its r.h.s. can be continued outside of this region.

B The constant D

We have

$$D = B(0) = \int_0^\infty \frac{2t^4 + t^2 - 1+}{t^4 + t^2} \frac{t^3 dt}{e^{2\pi t} - 1} + 6 \int_0^\infty \arctan(t) \frac{t^2 dt}{e^{2\pi t} - 1}$$
$$+ 2 \int_0^\infty \left(t^2 - \frac{1}{2}\right) \log[t^2 + t^4] \frac{t dt}{e^{2\pi t} - 1}. \tag{B.1}$$

The first line gives

$$\int_0^\infty \left(2 - \frac{1}{t^2}\right) \frac{t^3 dt}{e^{2\pi t} - 1} = 2\zeta(4) \frac{\Gamma(4)}{(2\pi)^4} - \frac{\zeta(2)}{(2\pi)^2} = -\frac{1}{30}. \tag{B.2}$$

For the second line we use

$$\int_0^\infty \arctan(t) \frac{t^2 dt}{e^{2\pi t} - 1} = -\frac{2}{9} + \frac{\zeta(3)}{8\pi^2} + \frac{1}{4} \log(2\pi) - \log A, \tag{B.3}$$

where A is the Glaisher-Kinkelin constant related to the first derivative of the Riemann ζ function by [25]

$$\log A = \frac{1}{12} - \zeta'(-1). \tag{B.4}$$

For the last line we use

$$\int_0^\infty (2t^2 - 1) \log\left(1 + t^2\right) \frac{t^2 dt}{e^{2\pi t} - 1} = \frac{11}{8} - \frac{3}{2} \left(\log(2\pi) + \frac{\zeta(3)}{\pi^2}\right) - 2\zeta'(-3) + 5 \log A, \tag{B.5}$$

$$\int_0^\infty (2t^2 - 1) \log t^2 \frac{t^2 dt}{e^{2\pi t} - 1} = \frac{1}{60} \left(-\frac{19}{6} + \gamma - \log(2\pi)\right) + \frac{3\zeta'(4)}{2\pi^4} + \log A. \tag{B.6}$$

The standard functional relation for $\zeta(z)$ gives

$$\zeta'(-3) = -\frac{4!}{2(2\pi)^4} \zeta'(4) + \frac{B_4}{4} (H_3 - \gamma - \log(2\pi)). \tag{B.7}$$

On the other hand, we can use the Adamchik formula [26]

$$\zeta'(-3) = \frac{B_4}{4} H_3 - \log A_3, \tag{B.8}$$

where A_3 is the third generalized Glaisher-Kinkelin number (or third Bendersky's number) defined by [25]

$$\log A_3 = \lim_{n \to \infty} \left[\sum_{k=1}^n k^3 \log k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \log n + \frac{n^4}{16} - \frac{n^2}{12}\right]$$
$$\simeq -0.02065635, \tag{B.9}$$

H_n are the harmonic numbers, and B_n the Bernoulli numbers. Summing up, we get

$$D = -\frac{1}{72} + \frac{\gamma - \log(2\pi)}{60} - \frac{3\zeta(3)}{4\pi^2} + 4 \log(A_3) \approx -0.2088707. \tag{B.10}$$

C Poles from contiguity in hypergeometrics

Recall that

$$ {}_{q+1}F_q\left(\begin{matrix} a_1, & \dots, & a_{q+1} \\ b_1, & \dots, & b_q \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_{q+1})_n \, z^n}{n!\,(b_1)_n \dots (b_q)_n} \; . \tag{C.1} $$

Define

$$ Z = a_1 + \cdots + a_{q+1} - b_1 - \cdots - b_q \, , \tag{C.2} $$

then

$$ \begin{aligned} \text{the series is absolutely convergent for } & |z| < 1 \\ \text{absolutely convergent for } & |z| = 1 \qquad\quad \text{if} \quad \operatorname{Re} Z < 0 \, , \\ \text{conditionally convergent for } & |z| = 1, \;\; z \neq 1 \text{ if } \;\; 0 \leq \operatorname{Re} Z < 1 \; . \end{aligned} \tag{C.3} $$

A general fact for ${}_{q+1}F_q$ is provided by the following formula [83, (37) p. 441](after a small correction), [84, (19) p. 719]

$$ \begin{aligned} & \left[\sigma + z \sum_{j=1}^{q} (a_j - b_j)\right] {}_{q+1}F_q\left(\begin{matrix} (a_q), \ \sigma \\ (b_q) \end{matrix}\middle| z\right) + \\ & + z \sum_{j=1}^{q} \frac{(b_j - \sigma) \prod_{k=1}^{q} (b_j - a_k)}{b_j \prod_{\substack{k=1 \\ k \neq j}}^{q} (b_j - b_k)} {}_{q+1}F_q\left(\begin{matrix} (a_q), \ \sigma \\ b_1, \dots, b_{j-1}, b_j + 1, b_{j+1}, \dots, b_q \end{matrix}\middle| z\right) \\ & = \sigma(1 - z) \, {}_{q+1}F_q\left(\begin{matrix} (a_q), \ \sigma + 1 \\ (b_q) \end{matrix}\middle| z\right) . \end{aligned} \tag{C.4} $$

This requires $b_j \neq b_k$ for all $j \neq k$. Denote

$$ Z = \sigma + \sum_{j=1}^{q} (a_j - b_j) \; . \tag{C.5} $$

First take the values of the parameters such that

$$ \begin{aligned} & \operatorname{Re} Z < -1 \, , \\ & b_j - b_k \neq 0 \qquad \forall \quad j \neq k \, , \qquad -b_j \notin \mathbf{Z}_+ . \end{aligned} \tag{C.6} $$

In this case it is legitimate to let z tend to 1 everywhere and we get

$$ \begin{aligned} & (Z) \, {}_{q+1}F_q\left(\begin{matrix} (a_q), \ \sigma \\ (b_q) \end{matrix}\middle| 1\right) = \\ & - \sum_{j=1}^{q} \frac{(b_j - \sigma) \prod_{k=1}^{q} (b_j - a_k)}{b_j \prod_{\substack{k=1 \\ k \neq j}}^{q} (b_j - b_k)} {}_{q+1}F_q\left(\begin{matrix} (a_q), \ \sigma \\ b_1, \dots, b_{j-1}, b_j + 1, b_{j+1}, \dots, b_q \end{matrix}\middle| 1\right) \; . \end{aligned} \tag{C.7} $$

The r.h.s. continues to be regular provided

$$ \begin{aligned} & \operatorname{Re}(1 - Z) = \sum_{j=1}^{q} \operatorname{Re} b_j - \sum_{j=1}^{q} \operatorname{Re} a_j - \operatorname{Re} \sigma + 1 > 0 \, , \\ & b_j - b_k \neq 0 \qquad \forall \quad j \neq k \, , \qquad -b_j \notin \mathbf{Z}_+ \; . \end{aligned} \tag{C.8} $$

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so that, in this region,

$$ {}_{q+1}F_q \left(\begin{matrix} (a_q), & \sigma \\ (b_q) & \end{matrix} \middle| 1 \right) = \frac{\Phi_0((a_q),\ \sigma,\ (b_q))}{Z} \,, \tag{C.9} $$

where $\Phi_0((a_q),\ \sigma,\ (b_q))$ is analytic in (C.8).

This process can be iterated, i.e. it can be applied to each of the hypergeometic functions appearing in the r.h.s. of (C.7), and it can be iterated further, so that, for any integer $n \geq 0$, there is a function $\Phi_n((a_q),\ \sigma,\ (b_q))$, analytic in

$$ \begin{aligned} &\mathrm{Re}(n+1-Z) > 0 \,, \\ &b_j - b_k \notin \mathbf{Z} \qquad \forall \quad j \neq k \,, \qquad -b_j \notin \mathbf{Z}_+ \,, \end{aligned} \tag{C.10} $$

such that, in this region,

$$ {}_{q+1}F_q \left(\begin{matrix} (a_q), & \sigma \\ (b_q) & \end{matrix} \middle| 1 \right) = \frac{\Phi_n((a_q),\ \sigma,\ (b_q))}{Z(Z-1)\ldots(Z-n)} \,. \tag{C.11} $$

Of course some of the poles appearing in this formula may not actually occur, i.e. their residues might be zero.

Remark C.1 Let f be a function meromorphic on a domain $D \subset \mathbf{C}$ that has at $a \in D$ a Laurent expansion

$$ f(z) = \frac{f_{-1}}{z-a} + f_0 + f_1(z-a) + \ldots + f_n(z-a)^n + \ldots \tag{C.12} $$

Then we have

$$ f_{-1} = (z-a)f(z)\Big|_{z=a} \,, \quad f_0 = \left(\frac{d}{dz}\right)(z-a)f(z)\Big|_{z=a} \,, \quad f_1 = \frac{1}{2}\left(\frac{d}{dz}\right)^2 (z-a)f(z)\Big|_{z=a} \,, \quad \text{etc.} \tag{C.13} $$

In case e.g. f is a hypergeometric function that, as a function of Z, as above, has a pole at $Z = a$, the contiguity formula (possibly iterated) may provide an expression for $(Z-a)f(Z)$, as a sum of convergent hypergeometric series, allowing a computation of the first terms of the Laurent series of f at a. If

$$ f(z) = \frac{f_{-p}}{(z-a)^p} + \cdots + \frac{f_{-1}}{z-a} + f_0 + f_1(z-a) + \ldots + f_n(z-a)^n + \ldots \tag{C.14} $$

then

$$ f_n = \frac{1}{(n+p)!}\left(\frac{d}{dz}\right)^{n+p} (z-a)^p f(z)\Big|_{z=a} \,, \quad n \geq -p \,. \tag{C.15} $$

D Details about the derivation of the finite term in the three-dimensional case

The hypergeometric series in (7.22) converges absolutely for $4 - 8\delta = 2(3-d) > 0$. The residue and the finite term at $d = 3$ are computed as follows:

$$ R_B(x,y,w,3) = \lim_{d\to 3} \left((d-3) B_d(x,y,w) \right) , \tag{D.1} $$

$$ \begin{aligned} \widetilde{B}_3(x,y,w) &= \lim_{d\to 3}\ \partial_d \left((d-3) B_d(x,y,w) \right) \\ &= b_3(x,y,w) F_3(x,y,w) + f_3(x,y,w) \partial_d b_d(x,y,w) \end{aligned} \tag{D.2} $$

where

$$f_3(x,y,w) = \lim_{d\to 3} \;(d-3)\; {}_7F_6\left(\begin{matrix} 2\delta,\dots,2\delta+2u \\ 2x+1,\dots,\delta+w+u+1 \end{matrix};1\right), \tag{D.3}$$

$$F_3(x,y,w) = \lim_{d\to 3} \frac{\partial}{\partial d}\;(d-3)\,{}_7F_6\left(\begin{matrix} 2\delta,\dots,2\delta+2u \\ 2x+1,\dots,\delta+w+u+1 \end{matrix};1\right). \tag{D.4}$$

To compute the above quantities the first step is to rewrite the hypergeometric functions at the r.h.s. of eq. (7.22) by using the contiguity relation

$$\begin{aligned}
&(d-3)\,{}_7F_6\left(\begin{matrix} 2\delta, 1+u+\delta, u-w+\delta, u+w+\delta, 2u+2\delta, 2x+2\delta, 2y+2\delta \\ 1+2x, 1+2y, 1+2u, u+\delta, 1+u-w+\delta, 1+u+w+\delta \end{matrix};1\right) = \\
&= \frac{(2\delta-1)(1+2x-2\delta)(1-2y-2\delta)(1+2x-2y-2\delta)(x-y-\delta)(1-w+x-y-\delta)(1+w+x-y-\delta)\,{}_7F_6(2\delta,\dots;2+2x,\dots;1)}{8(1+2x)(x-y)y(1+x-y-\delta)(w+x-y-\delta)(w-x+y+\delta)} \\
&\quad + \frac{(2\delta-1)(1+2y-2\delta)(x-y+\delta)(1+w-x+y-\delta)(1-w-x+y-\delta)(1-2x-2\delta)(1-2x+2y-2\delta)\,{}_7F_6(2\delta,\dots;2+2y,\dots;1)}{8x(x-y)(1+2y)(1-x+y-\delta)(w-x+y-\delta)(w+x-y+\delta)} \\
&\quad - \frac{(2\delta-1)(1+2x-2\delta)(1+2y-2\delta)(1+2x+2y-2\delta)(x+y-\delta)(1-w+x+y-\delta)(1+w+x+y-\delta)\,{}_7F_6(2\delta,\dots;2+2x+2y,\dots;1)}{8xy(1+2x+2y)(1+x+y-\delta)(w-x-y+\delta)(w+x+y-\delta)} \\
&\quad + \frac{w^2(x-y-\delta)(x+y-\delta)(x-y+\delta)\,{}_7F_6(2\delta,\dots;1+x+y+\delta,\dots;1)}{2(1+x-y-\delta)(1-x+y-\delta)(1+x+y-\delta)(1-w)(1+w)} \\
&\quad - \frac{(1+2w)(1+w-x-y-\delta)(1+w+x-y-\delta)(1+w-x+y-\delta)(1+w+x+y-\delta)\,{}_7F_6(2\delta,\dots;2+w+x+y+\delta,\dots;1)}{4(1+w)(w-x-y+\delta)(w+x-y+\delta)(w-x+y+\delta)(1+w+x+y+\delta)} \\
&\quad + \frac{(1-2w)(1-w-x-y-\delta)(1-w+x-y-\delta)(1-w-x+y-\delta)(1-w+x+y-\delta)\,{}_7F_6(2\delta,\dots;2-w+x+y+\delta,\dots;1)}{4(1-w)(w+x+y-\delta)(w-x+y-\delta)(w+x-y-\delta)(1-w+x+y+\delta)}.
\end{aligned} \tag{D.5}$$

At the r.h.s., we have only specified the parameters of the hypergeometric functions which are shifted by 1; the hypergeometric functions at the r.h.s. converge for $d < 7/2$. Eq. (D.5) gives us immediately the residue at $d = 3$ of the hypergeometric function at the l.h.s.:

$$f_3(x,y,w) = \frac{4w^2-(2x+2y+1)^2}{8x+8y+4}. \tag{D.6}$$

Calculating the finite part is more difficult. Let us start with the easiest:

$$\begin{aligned}
f_3(x,y,w)\,\partial_d b_d(x,y,w)\big|_{d=3} &= -\frac{(2x+2y+1)(\cot(2\pi x)\cot(2\pi y)-1)}{128\pi^2\,(4w^2-(2x+2y+1)^2)} + \\
&- \frac{(\cot(2\pi x)\cot(2\pi y)-1)\left(\psi(2x+1)+\psi(2y+1)-\gamma+2+\log\left(4\pi^2\right)\right)}{256\pi^2} + \\
&+ \frac{(\cot(2\pi x)\cot(2\pi y)-1)\left(\psi\left(2x+2y+\frac{3}{2}\right)-2\psi(-4x-4y-2)\right)}{256\pi^2}.
\end{aligned} \tag{D.7}$$

On the other hand computing $F_3(x,y,w)$ is quite involved. The first three lines at the r.h.s. of (D.5) may be grouped into a single expression that we denote $(d-3)R_{123}$; their contribution to $F_3(x,y,z)$ is therefore

$$\begin{aligned}
&R_{123}(x,y,w) = \\
&= \frac{x(2x-2y-1)\left((1+2x+2y)^2-4w^2\right)\left((2+4x-4y)\psi(1+2x)-(1-2w+2x-2y)\psi\left(\frac{1}{2}-w+x+y\right)-(1+2w+2x-2y)\psi\left(\frac{1}{2}+w+x+y\right)\right)}{4\left((1+2x)^2-4y^2\right)\left((1-2x+2y)^2-4w^2\right)} \\
&+ \frac{y(2y-2x-1)\left((1+2x+2y)^2-4w^2\right)\left((2-4x+4y)\psi(1+2y)-(1-2w-2x+2y)\psi\left(\frac{1}{2}-w+x+y\right)-(1+2w-2x+2y)\psi\left(\frac{1}{2}+w+x+y\right)\right)}{4\left((1+2y)^2-4x^2\right)\left((1+2x-2y)^2-4w^2\right)} \\
&+ \frac{(x+y)(2x+2y-1)\left((1+2x+2y)^2-4w^2\right)\left((1-2w+2x+2y)\psi\left(\frac{1}{2}-w+x+y\right)+(1+2w+2x+2y)\psi\left(\frac{1}{2}+w+x+y\right)-2(1+2x+2y)\psi(1+2x+2y)\right)}{4(1+2x+2y)^2\left(4w^2-(1-2x-2y)^2\right)}.
\end{aligned} \tag{D.8}$$

The other terms are trickier as they involve derivatives of the hypergeometric functions. Let us consider the fourth line. After some work we get the following expression:

$$\begin{aligned}
R_4 = &\frac{w\big((1-2x-2y)\big((1+2x+2y)^2-4w^2\big)\big)\Big(\Big(H_{-\frac{1}{2}-w+x+y}\Big)^2-\Big(H_{-\frac{1}{2}+w+x+y}\Big)^2\Big)}{32\,(w^2-1)\,(1+2x+2y)} \\
&+\frac{w\big(2w^2\big(-3+12x^2+8xy+12y^2\big)+(1+2x+2y)^2\big(2-x-8x^2+4x^3-y-4x^2y-8y^2-4xy^2+4y^3\big)\big)\psi\big(\frac{1}{2}+w+x+y\big)}{4\,(-1+w^2)\,(1+2x+2y)^2\,(-1+4x^2-8xy+4y^2)} \\
&-\frac{w\big(2w^2\big(-3+12x^2+8xy+12y^2\big)+(1+2x+2y)^2\big(2+4x^3-y-8y^2+4y^3-4x^2(2+y)-x\big(1+4y^2\big)\big)\big)\psi\big(\frac{1}{2}-w+x+y\big)}{4\,(-1+w^2)\,(1+2x+2y)^2\,(-1+4x^2-8xy+4y^2)} \\
&+\frac{2w^2\big(-\frac{1}{2}+x+y\big)\big({}_4\dot{F}_3\big(\big\{1+2x,1,\frac{1}{2}-w+x+y,\frac{1}{2}+w+x+y\big\},\big\{1+2x,\frac{3}{2}-w+x+y,\frac{3}{2}+w+x+y\big\},1\big)\big)}{4(-1+w)(1+w)\left(\frac{1}{2}+x+y\right)} \\
&+\frac{2w^2\big(-\frac{1}{2}+x+y\big)\big({}_4\dot{F}_3\big(\big\{1+2y,1,\frac{1}{2}-w+x+y,\frac{1}{2}+w+x+y\big\},\big\{1+2y,\frac{3}{2}-w+x+y,\frac{3}{2}+w+x+y\big\},1\big)\big)}{4(-1+w)(1+w)\left(\frac{1}{2}+x+y\right)} \\
&+\frac{2w^2\big(-\frac{1}{2}+x+y\big)\big({}_4\dot{F}_3\big(\big\{1+2x+2y,1,\frac{1}{2}-w+x+y,\frac{1}{2}+w+x+y\big\},\big\{1+2x+2y,\frac{3}{2}-w+x+y,\frac{3}{2}+w+x+y\big\},1\big)\big)}{4(-1+w)(1+w)\left(\frac{1}{2}+x+y\right)}.
\end{aligned} \tag{D.9}$$

As regards the fifth and the sixth line it is useful to compute at first the following auxiliary expression

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix} s,\,\frac{3}{2}+u,\,\frac{1}{2}+u-w,\,\frac{1}{2}+u+w \\ \frac{1}{2}+u,\,\frac{3}{2}+u-w,\,\frac{5}{2}+u+w \end{matrix};1\right) = \\
&=\frac{\Gamma(1-s)\left(\frac{(1+2u+2w)(3+2u+2w)\Gamma\left(\frac{3}{2}+u-w\right)}{\Gamma\left(\frac{3}{2}-s+u-w\right)}-\frac{(1+2u-2w)(-1+2u-2w+s(2+4w))\Gamma\left(\frac{5}{2}+u+w\right)}{\Gamma\left(\frac{5}{2}-s+u+w\right)}\right)}{4(1+2u)(1+2w)}
\end{aligned} \tag{D.10}$$

which allows to compute ${}_4F_3\left(\begin{matrix} 1,\,\frac{3}{2}+u,\,\frac{1}{2}+u-w,\,\frac{1}{2}+u+w \\ \frac{1}{2}+u,\,\frac{3}{2}+u-w,\,\frac{5}{2}+u+w \end{matrix};1\right)$, its derivative w.r.t. u and ${}_4\dot{F}_3\left(\begin{matrix} 1,\,\frac{3}{2}+u,\,\frac{1}{2}+u-w,\,\frac{1}{2}+u+w \\ \frac{1}{2}+u,\,\frac{3}{2}+u-w,\,\frac{5}{2}+u+w \end{matrix};1\right)$.
We get in this way an expression for R_5 where, as before, there remain three terms expressed as derivatives of hypergeometric functions:

$$\begin{aligned}
R_5(x,y,w) = &\frac{(1+2w+2x+2y)\big(4w^2+4w^3+(1+2x+2y)^2(w+2(1+x+y))\big)}{32(1+w)(1+2x+2y)^2(3+2w+2x+2y)}\big(H_{-\frac{1}{2}-w+x+y}-H_{\frac{1}{2}+w+x+y}\big) \\
&+\frac{\gamma\left(4w^2-(1+2x+2y)^2\right)\left(2+4w+(1+2w+2x+2y)\left(H_{\frac{1}{2}+w+x+y}-H_{-\frac{1}{2}-w+x+y}\right)\right)}{64(1+w)(1+2x+2y)} \\
&+\frac{(1-2w+2x+2y)(1+2w+2x+2y)^2\left(\psi\left(\frac{1}{2}-w+x+y\right)^2-\psi\left(\frac{3}{2}+w+x+y\right)^2\right)}{128(1+w)(1+2x+2y)}+ \\
&+\frac{\left(5+8w-\frac{8(1+w)(1+2w)}{1+2w-2x-2y}-\frac{8(1+w+x)(1+2w+2x)}{1+2w+2x-2y}-\frac{8x(1+2x)}{1+2w-2x+2y}\right)}{16(1+w)(1+2x+2y)(6+4w+4x+4y)}\left((1+2x+2y)^2-4w^2\right)\times \\
&\times\left(\psi\left(\frac{1}{2}-w+x+y\right)+(1+2w)\psi\left(\frac{1}{2}+w+x+y\right)-2(1+w)\psi\left(\frac{3}{2}+w+x+y\right)\right)
\end{aligned}$$

$$+\frac{(1+2w)(1+2w+2x+2y)\left(4w(1+w)+(1+2x+2y)^2-(1-2w+2x+2y)(1+2x+2y)(3+2w+2x+2y)\left(H_{\frac{1}{2}+w+x+y}-\gamma\right)\right)}{32(1+w)(1+2x+2y)^2(3+2w+2x+2y)}$$
$$-\frac{(1+2w)(1+2w+2x+2y)\ {}_5\dot{F}_4\left(\left\{1+2x,\frac{3}{2}+x+y,\frac{1}{2}-w+x+y,\frac{1}{2}+w+x+y,1\right\},\left\{1+2x,\frac{1}{2}+x+y,\frac{3}{2}-w+x+y,\frac{5}{2}+w+x+y\right\},1\right)}{8(1+w)(3+2w+2x+2y)}$$
$$-\frac{(1+2w)(1+2w+2x+2y)\ \left({}_5\dot{F}_4(\{1+2y,\ldots\},\{1+2y,\ldots\},1)+{}_5\dot{F}_4(\{1+2x+2y,\ldots\},\{1+2x+2y,\ldots\},1)\right)}{8(1+w)(3+2w+2x+2y)}. \tag{D.11}$$

Finally, $R_6(x,y,w) = R_5(x,y,-w)$. Collecting all the above terms we are able to fully construct $F_3(x,y,w)$ determine $B_3(x,y,w)$ and finally $\tilde{I}_2(x,y,w)$. The result is however utterly complicated. It simplifies a lot when $w=0$ by taking into account the following identity:

$$\begin{aligned}
{}_3\dot{F}_2\left(\begin{matrix}a,b,1\\a,b+2\end{matrix};1\right) &= \frac{d}{ds}F_2\left(\begin{matrix}a+s,b,1\\a,b+2\end{matrix};1\right)\Bigg|_{s=0} = \sum_{n=1}^{\infty}\frac{b(1+b)(\psi(a+n)-\psi(a))}{(b+n)(1+b+n)}\\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{b(1+b)}{(b+n)(1+b+n)(k+a)} = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{b(1+b)}{(b+n)(1+b+n)(k+a)}\\
&= \sum_{k=0}^{\infty}\frac{b(1+b)}{(a+k)(1+b+k)} = \frac{b(1+b)(\psi(1+b)-\psi(a))}{1-a+b}.
\end{aligned} \tag{D.12}$$

Collecting everything we land on an indeed very simple result:

$$F(x,y,0) = \frac{(2x+2y+1)}{8}\left(H_{2x+2y}+H_{2x}+H_{2y}-4H_{x+y-\frac{1}{2}}\right)-\frac{1}{8}. \tag{D.13}$$

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