Mapping the geometry of the E 6 group
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In this paper, we present a construction for the compact form of the exceptional Lie
group $E_6$ by exponentiating the corresponding Lie algebra $e_6$, which we realize as
the sum of $f_4$, the derivations of the exceptional Jordan algebra $J_3$ of dimension 3
with octonionic entries, and the right multiplication by the elements of $J_3$ with
vanishing trace. Our parametrization is a generalization of the Euler angles for
$SU_2$ and it is based on the fibration of $E_6$ via an $F_4$ subgroup as the fiber. It makes
use of a similar construction we have performed in a previous article for $F_4$. An
interesting first application of these results lies in the fact that we are able to
determine an explicit expression for the Haar invariant measure on the $E_6$ group

I. INTRODUCTION

The standard model (SM) provides a very good description of elementary particle physics. However, despite its success, there are reasons to go beyond it, for example, the recent discovery of neutrino oscillations, the fine tuning of the mixing matrices, the hierarchy problem, the difficulty in including gravity, and so on.

A starting point could be the fact that the renormalization flow of the coupling constants suggests the unification of gauge interactions at energies of the order of $10^{15}$ GeV, which can be improved (fine tuned) by supersymmetry. It is natural to expect the gauge group $G$ of the grand unification theory (GUT) theory to be a simple group. Obviously, at low energies, the GUT model must reproduce the SM physics, so that not only $G$ should contain $SU(3)_C \times SU(2)_L \times U(1)_Y$ (the SM gauge group) but it should also predict the correct spectra after spontaneous symmetry breaking. The increasing accuracy in the analysis of particle spectra imposes even more restrictions in the possible choices for $G$. For example, it is already known that the current estimate for the lower bound of the proton lifetime rules out some of the GUT model candidates. Recently, the particular structure of the neutrino mixing matrix seems to suggest that a good candidate for a GUT could be based on the semidirect product between the exceptional group $E_6$ and the discrete group $S_4$.

In general, while the local properties of group $G$, which for a Lie group are all encoded in the corresponding Lie algebra, are enough to perform a perturbative analysis, in order to obtain nonperturbative results, such as instantonic calculations or lattice simulations, a knowledge of the

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entire group structure is required, in particular, of the invariant measure on the group in a suitable
global parametrization. There are many ways to give such an explicit expression for the Haar
measure on a Lie group. However, for large dimensions, it becomes quite hard to find a realization,
which is able at the same time to provide both a reasonably simple form for the measure and an
explicit determination for the range of the angles. In this paper, we solve this problem for the
exceptional Lie group \(E_6\), by constructing a generalization of the Euler parametrization, with a
technique we have introduced in Ref. 2 and fully developed in Ref. 1. In Sec. II, we explain how
the \(e_6\) algebra can be represented using a theorem due to Chevalley and Schafer. The construction
of the group and the determination of the corresponding Haar measure is made in Sec. III.

II. THE CONSTRUCTION OF THE \(e_6\) ALGEBRA

Our starting point for the construction of the exceptional algebra \(e_6\) is a theorem due to
Chevalley and Schafer\(^5,7\) which we rewrite here for convenience.

**Theorem 2.1:** The exceptional simple Lie algebra \(f_4\) of dimension 52 and rank 4 over \(K\) is the
derivation algebra \(D\) of the exceptional Jordan algebra \(J\) of dimension 27 over \(K\). The
exceptional simple Lie algebra \(e_6\) of dimension 78 and rank 6 over \(K\) is the Lie algebra

\[ D \{ R_Y \}, \quad \text{Tr} \ Y = 0, \quad (2.1) \]

spanned by the derivations of \(J\) and the right multiplications of elements \(Y\) of trace 0.

We will refer to Ref. 8 for the notations. Then, the exceptional Jordan algebra is the algebra
\(J_3\). For our purposes, \(K\) will be \(R\) or \(C\). The right multiplication is \(R_Y(X) = Y \circ X\) and the trace is the
sum of diagonal elements. The exceptional Jordan algebra is 27 dimensional, just like the principal
fundamental representation of \(e_6\).

In our previous paper,\(^1\) we determined the algebra of derivations \(D\) and used it to obtain an
irreducible representation of the exceptional algebra \(f_4\). To obtain a representation of \(e_6\), we only
need to determine the matrix representation of \(R_Y\). This is a very simple task and we solved it in
\(R^{27}\) with the product inherited from \(J_3\) by means of the linear isomorphism

\[
\Phi: J_3 \rightarrow R^{27}, \quad A \mapsto \Phi(A),
\]

where \(A\) is the Jordan matrix,

\[
A = \begin{pmatrix}
a_1 & o_1 & a_2 \\
o_1^* & a_2 & o_2 \\
o_2^* & o_3 & a_3
\end{pmatrix},
\]

and \(\rho\) is the linear isomorphism between the octonions \(O\) and \(R^8\) given by\(^1\)

\[
\rho: O \rightarrow R, \quad o = o^0 + \sum_{i=1}^7 o^i j_i \mapsto \rho(o),
\]

\(^1\)See Ref. 1 for more details.
After choosing a 26 dimensional base of traceless Jordan matrices, we used the Mathematica program in Appendix A to find the matrices which complete the base for \( f_4 \) to a base for the whole \( e_6 \) algebra. In particular, if we work in the real case, we obtain the split form of \( e_6 \), with signature \((52, 26)\). However, multiplying the 26 added generators by \( i \), we find that the algebra remains real, but the Killing form becomes the compact one.

### III. THE CONSTRUCTION OF THE GROUP \( E_6 \)

#### A. The generalized Euler parametrization for \( E_6 \)

To give an Euler parametrization for \( E_6 \), we start by choosing its maximal subgroup. It is \( H=F_4 \), the group generated by the first 52 matrices. Let \( \mathcal{P} \) be the linear complement of \( f_4 \) in \( e_6 \). We then search for a minimal linear subset \( V \) of \( \mathcal{P} \), which, under the action of \( \text{Ad}(F_4) \), generates the whole \( \mathcal{P} \). Looking at the structure constants, we see that \( V \) can be chosen as the linear space generated by \( c_{53}, c_{70} \). Note that they commute.

If we then write the general element \( g \) of \( E_6 \) in the form

\[
g = \exp(\tilde{h}) \exp(v) \exp(h), \quad h, \tilde{h} \in f_4, \quad v \in V,
\]

we have a redundancy of dimension 28. As argued in Ref. 1, we expect to find a 28 dimensional subgroup of \( F_4 \), which, acting by adjunction, defines an automorphism of \( V \). This can be done by noticing that \( V \) commutes with the first 28 matrices \( c_i, i=1, \ldots, 28 \), which generate an \( \text{SO}(8) \) subgroup of \( F_4 \).

We found convenient to introduce the change of base

\[
\tilde{c}_{53} := \frac{1}{2} c_{53} + \frac{3}{2} c_{70},
\]

\[
\tilde{c}_{70} := -\frac{3}{2} c_{53} + \frac{1}{2} c_{70}.
\]

Thus, we have

\[
g[x_1, \ldots, x_{78}] = B[x_1, \ldots, x_{24}] e^{x_2 \tilde{c}_5} e^{x_3 \tilde{c}_16} e^{x_4 \tilde{c}_10} e^{x_5 \tilde{c}_14} e^{x_6 \tilde{c}_5} e^{x_7 \tilde{c}_6}, \quad v \in V,
\]

where \( B=F_4/\text{SO}(8) \). We chose for \( F_4 \) the Euler parametrization given in Ref. 1 so that we found for \( B \),

\[
B[x_1, \ldots, x_{24}] = B_{F_4}[x_1, \ldots, x_{16}] B_{B}[x_{17}, \ldots, x_{23}] e^{x_{24} \tilde{c}_4},
\]

where

\[
B_{F_4}[x_1, \ldots, x_{15}] = B_{B}[x_1, \ldots, x_7] e^{x_8 \tilde{c}_4} B_{B}[x_9, \ldots, x_{14}] e^{x_9 \tilde{c}_{10} e^{x_{15} \tilde{c}_{16}}} e^{x_{16} \tilde{c}_5} e^{x_{17} \tilde{c}_6} e^{x_{18} \tilde{c}_{22}},
\]

\[
B_{B}[x_1, \ldots, x_7] = e^{x_1 \tilde{c}_4} e^{x_2 \tilde{c}_{10} e^{x_3 \tilde{c}_{16}}} e^{x_4 \tilde{c}_{14} e^{x_5 \tilde{c}_{5}} e^{x_6 \tilde{c}_5} e^{x_7 \tilde{c}_6}},
\]
\[ B_{8[x_1, \ldots, x_6]} = e^{y_1^{25}y_2^{26}y_3^{27}y_4^{28}y_5^{29}y_6^{30}y_7^{31}}, \]  
and the tilded matrices are the ones introduced in Ref. 1.

**B. Determination of the range for the parameters**

To determine the range of the parameters, we will use the topological method we have developed in Ref. 2. Let us first determine the volume of \( E_6 \) by means of the Macdonald formula.

1. The volume of \( E_6 \)

The rational homology of the exceptional Lie group \( E_6 \) is that of a product of odd dimensional spheres, \( H_*(E_6) = H_*(\Pi_{i=1}^6 S^{d_i}) \), with

\[ d_1 = 3, \quad d_2 = 9, \quad d_3 = 11, \quad d_4 = 15, \quad d_5 = 17, \quad d_6 = 23. \]  

The simple roots of \( E_6 \) are

\[ r_1 = L_1 + L_2, \]  
\[ r_2 = L_2 - L_1, \]  
\[ r_3 = L_3 - L_2, \]  
\[ r_4 = L_4 - L_3, \]  
\[ r_5 = L_5 - L_4, \]  
\[ r_6 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2}, \]  
where \( L_i, i=1, \ldots, 6, \) is an orthonormal base for the Cartan algebra. The volume of the fundamental region is then

\[ \text{Vol}(f_{E_6}) = \frac{2}{L}. \]  

Indeed, we computed the 36 positive roots, all having length \( \sqrt{2} \). They have exactly the structure given in Ref. 8, with \( L_i = e_i \), the canonical base of \( \mathbb{R}^6 \). The Macdonald formula\(^9,11\) gives for the volume of the compact form of \( E_6 \),

\[ \text{Vol}(E_6) = \frac{\sqrt{3} \cdot 2^{17} \cdot \pi^{42}}{3^{10} \cdot 5^5 \cdot 7^3 \cdot 11}. \]  

2. The invariant measure on \( E_6 \)

With the chosen generalized Euler parametrization, the invariant measure on \( E_6 \) decomposes into the product of the measure of \( F_4 \) and the one on \( M = E_6/F_4 \). The invariant measure on \( F_4 \) was computed in Ref. 1 so that we need to compute here only the induced measure on \( M \).

Using the notation of Ref. 1, let us define

\[ J_M := \pi_p(e^{-y_1^{25}y_2^{26}y_3^{27}y_4^{28}y_5^{29}y_6^{30}y_7^{31}}B[x_1, \ldots, x_{24}]^{-1}d(B[x_1, \ldots, x_{24}]e^{y_1^{25}y_2^{26}y_3^{27}y_4^{28}y_5^{29}y_6^{30}y_7^{31}})), \]  
where \( \pi_p \) is the projection on the subspace generated by \( c_j, j=53, \ldots, 78 \). The metric induced on \( M \) by the bi-invariant metric on \( E_6 \) is then
and the invariant measure on $E_6$ is then

$$d\mu_{E_6} = |\det(J_{M_i})|d\mu_{P_i}\prod_{i=1}^{26} dx_i,$$

where $J_{M_i}$ is the $26 \times 26$ matrix defined by

$$J_M = \sum_{i,j=1}^{26} J_{M_i}c_{ij} dx^i,$$

with $c_{ij}$ a base $\{\tilde{e}_{53}, \tilde{e}_{70}, c_{54}, \ldots, \tilde{e}_{69}, c_{71}, \ldots, c_{78}\}$ of $P$.

In order to compute $|\det(J_{M_i})|$, it is convenient to introduce the notations

$$\omega[x,y,z] = e^{x45e^{75}e^{-70}},$$

$$J_\omega := \omega^{-1}d\omega,$$

$$J_9 := B_9^{-1}dB_9,$$

$$J_{F_4} := B_4^{-1}dB_{F_4},$$

so that

$$J_M[x_1, \ldots, x_{26}] = \omega[x_{24}, x_{25}, x_{26}]^{-1}B_9[x_{17}, \ldots, x_{23}]^{-1}J_{F_4}[x_1, \ldots, x_{16}]B_9[x_{17}, \ldots, x_{23}]\omega[x_{24}, x_{25}, x_{26}]
+ \omega[x_{24}, x_{25}, x_{26}]^{-1}J_9[x_1, \ldots, x_{23}]\omega[x_{24}, x_{25}, x_{26}] + J_\omega[x_{24}, x_{25}, x_{26}].$$

Some remarks are in order now.

1. The following relations are true:

$$e^{-\alpha_{53}c_L}e^{53} = \cos ac_L + \sin ac_L e^{53},$$

if $L = 45, \ldots, 52$. Moreover, $\tilde{e}_{70}$ commutes with $c_p$, $I = 45, \ldots, 52, 71, \ldots, 78$, and with $\tilde{e}_{53}$.

2. $\tilde{e}_{53}$ and $\tilde{e}_{70}$ commute with the so(8) algebra generated by the matrices $c_p$, $I = 1, \ldots, 21, 30, \ldots, 36$.

3. The adjoint action of $e^{245}$ on the above so(8) algebra generates in addition the matrices $c_p$, $J = 46, \ldots, 52$.

4. From $J_{F_4} \in F_4$ and $B_9 \subset F_4$, it follows that

$$\tilde{J}_{F_4} := B_9^{-1}J_{F_4}B_9 \in F_4.$$

5. The adjoint action of $\omega$ on the $F_4$ matrices generates all the remaining matrices of $c_p$. In particular, the projection of $\omega^{-1}c_{p,0}L = 1, \ldots, 52$, on $c_p$, $J = 54, \ldots, 69$, is different from zero only if $L = 22, \ldots, 29, 37, \ldots, 44$.

6. The adjoint action of the SO(8) group, corresponding to the above so(8) algebra, gives a rotation both on the indices $I = {22, \ldots, 29}$ and $J = {37, \ldots, 44}$. More precisely,

$$\text{SO}(8)^{-1}c_p\text{SO}(8) = R^c_Ic_L,$$

where $R^c_I$ is the 26-dimensional representation of $SO(8)$, with $I = 1, \ldots, 26$.
$$\text{SO}(8)^{-1}c_j\text{SO}(8) = \tilde{R}^K_jc_K,$$
where $L$ runs from 22 to 29 and $K$ runs from 37 to 44, and $R, \tilde{R}$ are both orthogonal matrices.

To verify these, it suffices to note that

$$e^{-xc_J}c_e^{-x} = \cos \left( \frac{x}{2}c_J \right) \pm \sin \left( \frac{x}{2}c_J \right),$$

$$e^{-xc_A}c_e^{-x} = \cos \left( \frac{x}{2}c_A \right) \pm \sin \left( \frac{x}{2}c_A \right),$$

where $A=1,\ldots,21,30,\ldots,36$, $I, J_A \in \{22, \ldots, 29\}$, $J, J_A \in \{37, \ldots, 44\}$. In particular, $\det(R \otimes \tilde{R}) = 1$.

From the first three points, we see that the matrix $J_M$ takes the form

$$M = \begin{pmatrix}
A & 0 & 0 \\
* & C & 0 \\
* & * & D
\end{pmatrix},$$

where on the rows we indicate the coefficients of $dx_I$ with $I=1, \ldots, 26$ starting from the bottom and on the columns the projections on $c \tilde{c}_A$, $c_L$, following the order $A=53,70$, $L=71,\ldots,78,54,\ldots,69$. The asterisks indicate the elements which do not contribute to the determinant.

$$\det J_M = \det A \det C \det D.$$ 

In particular, $A$ is a $3 \times 3$ block obtained by projecting $J_{\omega}$ on $c \tilde{c}_{53}, c \tilde{c}_{70}, c \tilde{c}_{71}$. Point 1 implies

$$A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \sin x_{25}
\end{pmatrix},$$

so that

$$\det A = \sin x_{25}.$$ 

The $C$ block is a $7 \times 7$ matrix obtained by projecting $\omega^{-1}J_{\omega}$ on $c_K$, $K=72,\ldots,78$. From point 1, it follows that it can be obtained by multiplying the projection of $e^{-x\tilde{c}_A}J_{\omega}e^{x\tilde{c}_A}$ on $c_L$, $L=46,\ldots,52$, by $\sin x_{25}$. If we call $\tilde{C}$ the matrix corresponding to such a projection, we find

$$\det C = \sin^7 x_{25} \det \tilde{C}.$$ 

The determinant of $\tilde{C}$ can be computed directly and gives

$$\det C = \sin x_{20} \cos x_{21} \cos x_{22} \sin^2 x_{22} \sin^2 x_{23} \cos^4 x_{23} \sin^7 x_{24} \sin^7 x_{25}.$$ 

The $D$ block requires some further discussion. It is a $16 \times 16$ matrix obtained by projecting $\omega^{-1}J_{\tilde{R}_8} \omega$ on $c_P$, $I=54,\ldots,69$. First, from points (4) and (5) of our remarks, we see that only the $c_L$ with $L=22,\ldots,29,37,\ldots,44$ in $\tilde{J}_{F_4}$ contribute to the determinant. Let us define the $16 \times 16$ matrix $U$ with

$$U_A^{\chi} = -\frac{1}{6} \text{Tr}(\omega^{-1}c_A \omega c_B),$$

$$A = 22,\ldots,29,37,\ldots,44,$$
Moreover, let $\tilde{D}$ be the matrix obtained by projecting $J_{F_4}$ on $c_L$ with $L=22,\ldots,29,37,\ldots,44$, and

$$Q := \begin{pmatrix} R & 0 \\ 0 & \tilde{R} \end{pmatrix},$$

where $R$ and $\tilde{R}$ are the rotation matrices defined at point (6). Then, we can deduce from points (4)–(6) that

$$\det D = \det(U \cdot Q \cdot \tilde{D}) = \det U \det \tilde{D}. \quad (3.39)$$

The matrix $U$ and its determinant are easily computed. Indeed, we have

$$\omega^{-1} c_{22} \omega = \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{61} + \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{69} + \cdots,$$

$$\omega^{-1} c_{23} \omega = -\cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{57} - \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{65} + \cdots,$$

$$\omega^{-1} c_{24} \omega = -\cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{60} - \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{68} + \cdots,$$

$$\omega^{-1} c_{25} \omega = \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{55} + \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{63} + \cdots,$$

$$\omega^{-1} c_{26} \omega = -\cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{59} - \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{67} + \cdots,$$

$$\omega^{-1} c_{27} \omega = \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{58} + \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{66} + \cdots,$$

$$\omega^{-1} c_{28} \omega = \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{56} + \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{64} + \cdots,$$

$$\omega^{-1} c_{29} \omega = -\cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{54} - \sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{62} + \cdots,$$

$$\omega^{-1} c_{37} \omega = -\sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{44} + \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{63} + \cdots,$$

$$\omega^{-1} c_{38} \omega = -\sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{45} + \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{63} + \cdots,$$

$$\omega^{-1} c_{39} \omega = -\sin \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) c_{62} + \cos \left( \frac{x_{24}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) c_{64} + \cdots.$$
\[
\omega^{-1} c_{40} \omega = - \sin \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} + x_{25}}{2} \right) c_{57} + \cos \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} - x_{25}}{2} \right) c_{65} + \cdots,
\]

\[
\omega^{-1} c_{41} \omega = - \sin \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} + x_{25}}{2} \right) c_{58} + \cos \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} - x_{25}}{2} \right) c_{66} + \cdots,
\]

\[
\omega^{-1} c_{42} \omega = - \sin \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} + x_{25}}{2} \right) c_{59} + \cos \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} - x_{25}}{2} \right) c_{67} + \cdots,
\]

\[
\omega^{-1} c_{43} \omega = - \sin \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} + x_{25}}{2} \right) c_{60} + \cos \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} - x_{25}}{2} \right) c_{68} + \cdots,
\]

\[
\omega^{-1} c_{44} \omega = - \sin \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} + x_{25}}{2} \right) c_{61} + \cos \left( \frac{\sqrt{3} x_{24}}{2} \right) \sin \left( \frac{\sqrt{3} x_{26} - x_{25}}{2} \right) c_{69} + \cdots,
\]

where ellipses stay for terms which vanish when projected on \( c_B \), \( B=54,\ldots,61,70,\ldots,78 \). From these, it follows

\[
\det U = \sin \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) \sin \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right).
\]  

(3.40)

At this point, we need to compute \( \det \tilde{D} \). This can be done by noticing that \( J_{F_4} \) coincides exactly with the current \( J_M \) of \( F_4 \) in Ref. 1. We then find

\[
\det \tilde{D} = 2^7 \sin^{15} \frac{x_{16}}{2} \cos^7 \frac{x_{16}}{2} \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8
\]

\[
\times \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15}.
\]

(3.41)

We can finally write down the invariant measure on the base space,

\[
d\mu_M = 2^7 \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8
\]

\[
\times \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15} \sin^{15} \frac{x_{16}}{2} \cos^7 \frac{x_{16}}{2}
\]

\[
\times \sin x_{20} \cos x_{21} \cos x_{22} \sin^2 x_{22} \cos^2 x_{23} \sin^4 x_{23} \sin^7 x_{24}
\]

\[
\times \sin^8 x_{25} \sin^5 \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) \sin^8 \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right).
\]

(3.42)

Note that the periods of the variables are \( 4\pi \) so that one should take the range \( x_i = [0, 4\pi] \) for \( i = 1, 2, 3, \ldots, 9, 10, 11, 17, 18, 19 \). However, as in Ref. 1, it is easy to show directly from the parameterization that they can all be restricted to \( [0, 2\pi] \). The range of \( x_i \) is then

\[
x_1 \in [0, 2\pi], \quad x_2 \in [0, 2\pi], \quad x_3 \in [0, 2\pi], \quad x_4 \in [0, \pi],
\]

\[
x_5 \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad x_6 \in [0, \frac{\pi}{2}], \quad x_7 \in [0, \frac{\pi}{2}], \quad x_8 \in [0, \pi],
\]

\[
x_9 \in [0, 2\pi], \quad x_{10} \in [0, 2\pi], \quad x_{11} \in [0, 2\pi], \quad x_{12} \in [0, \pi].
\]
\[
\begin{align*}
    x_{13} & \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \quad x_{14} \in \left[ 0, \frac{\pi}{2} \right], \quad x_{15} \in \left[ 0, \frac{\pi}{2} \right], \quad x_{16} \in [0, \pi], \\
    x_{17} & \in [0, 2\pi], \quad x_{18} \in [0, 2\pi], \quad x_{19} \in [0, 2\pi], \quad x_{20} \in [0, \pi], \\
    x_{21} & \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \quad x_{22} \in \left[ 0, \frac{\pi}{2} \right], \quad x_{23} \in \left[ 0, \frac{\pi}{2} \right], \quad x_{24} \in [0, \pi], \\
    x_{25} & \in \left[ 0, \frac{\pi}{2} \right], \quad -\frac{x_{25}}{\sqrt{3}} \leq x_{26} \leq \frac{x_{25}}{\sqrt{3}}.
\end{align*}
\]

The remaining parameters \( x_j, j=27, \ldots, 78 \), will run over the range for \( F_4 \), as given in Ref. 1. The volume of the whole closed cycle \( V \) so obtained is then

\[
\text{Vol}(V) = \text{Vol}(F_4) \int_R d\mu_M = \frac{\sqrt{3} \cdot 2^{17} \cdot \pi^{42}}{3^{10} \cdot 5^2 \cdot 7^3 \cdot 11},
\]

where \( R \) is the range of parameters \( x_i, i=1, \ldots, 26 \). This is the volume of \( E_6 \), so that we cover the group exactly once.\(^2\)

**IV. CONCLUSIONS**

In this paper, we have performed an explicit construction for the compact form of the simple Lie group \( E_6 \). This is particularly interesting because recently it has been argued that this group could be the most promising for unification in GUT theories.\(^3,6\) To parameterize the group, we have used the generalized Euler angle method, a technique we introduced in Refs. 2 and 1 to give the most simple expression for the invariant measure on the group, while at the same time still being able to provide an explicit expression for the range of the parameters. Both these requirements are necessary in order to minimize the computation power needed for computer simulations, for example, of lattice models.

Our results can be easily extended to the GUT group \( E_6^4 \times S_6 \). Also, a modified Euler parametrization could be applied to evidentiate the subgroups related to the correct symmetry breaking, see Ref. 3. Finally, our parametrization could be used for a straightforward geometrical analysis of the exceptional Lie group \( E_6 \) and its quotients.

**ACKNOWLEDGMENTS**

S.L.C. is grateful to Giuseppe Berrino and Stefano Pigola for helpful comments. B.L.C. would like to thank O. Ganor for useful discussions. This work has been supported in part by the Spanish Ministry of Education and Science (Grant No. AP2005-5201) and in part by the Director, Office of Science, Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

**APPENDIX A: THE \( E_6 \) MATRICES**

The matrices we found using MATHEMATICA, and orthonormalized with respect to the scalar product \( \langle a, b \rangle := -\frac{1}{6} \text{Tr}(ab) \), was computed by means of the followings programs.

**The MATHEMATICA program**

\[
\begin{align*}
% & % & % & % \text{ the octonionic products}
\end{align*}
\]

\(^2\)Obviously, there is a subset of vanishing measure which is multiply covered.
\[ QQ[1, 1] = e[1]; \quad QQ[1, 2] = e[2]; \quad QQ[1, 3] = e[3]; \]
\[ QQ[1, 4] = e[4]; \quad QQ[1, 5] = e[5]; \quad QQ[1, 6] = e[6]; \]
\[ QQ[1, 7] = e[7]; \quad QQ[1, 8] = e[8]; \quad QQ[2, 1] = e[2]; \]
\[ QQ[2, 2] = -e[1]; \quad QQ[2, 3] = e[5]; \quad QQ[2, 4] = e[8]; \]
\[ QQ[2, 5] = -e[3]; \quad QQ[2, 6] = e[7]; \quad QQ[2, 7] = -e[6]; \]
\[ QQ[2, 8] = -e[4]; \quad QQ[3, 1] = e[3]; \quad QQ[3, 2] = -e[5]; \]
\[ QQ[3, 3] = -e[1]; \quad QQ[3, 4] = e[6]; \quad QQ[3, 5] = e[2]; \]
\[ QQ[3, 6] = -e[4]; \quad QQ[3, 7] = e[8]; \quad QQ[3, 8] = -e[7]; \]
\[ QQ[4, 1] = e[4]; \quad QQ[4, 2] = -e[8]; \quad QQ[4, 3] = -e[6]; \]
\[ QQ[4, 4] = -e[1]; \quad QQ[4, 5] = e[7]; \quad QQ[4, 6] = e[3]; \]
\[ QQ[4, 7] = -e[5]; \quad QQ[4, 8] = e[2]; \quad QQ[5, 1] = e[5]; \]
\[ QQ[5, 2] = e[3]; \quad QQ[5, 3] = -e[2]; \quad QQ[5, 4] = -e[7]; \]
\[ QQ[5, 5] = -e[1]; \quad QQ[5, 6] = e[8]; \quad QQ[5, 7] = e[4]; \]
\[ QQ[5, 8] = -e[6]; \quad QQ[6, 1] = e[6]; \quad QQ[6, 2] = -e[7]; \]
\[ QQ[6, 3] = e[4]; \quad QQ[6, 4] = -e[3]; \quad QQ[6, 5] = -e[8]; \]
\[ QQ[6, 6] = -e[1]; \quad QQ[6, 7] = e[2]; \quad QQ[6, 8] = e[5]; \]
\[ QQ[7, 1] = e[7]; \quad QQ[7, 2] = e[6]; \quad QQ[7, 3] = -e[8]; \]
\[ QQ[7, 4] = e[5]; \quad QQ[7, 5] = -e[4]; \quad QQ[7, 6] = -e[2]; \]
\[ QQ[7, 7] = -e[1]; \quad QQ[7, 8] = e[3]; \quad QQ[8, 1] = e[8]; \]
\[ QQ[8, 2] = e[4]; \quad QQ[8, 3] = e[7]; \quad QQ[8, 4] = -e[2]; \]
\[ QQ[8, 5] = e[6]; \quad QQ[8, 6] = -e[5]; \quad QQ[8, 7] = -e[3]; \]
\[ QQ[8, 8] = -e[1]; \]

% % % % the Jordan algebra product
\[ Qm[x, y_] := \text{Sum}[\text{Sum}[x[i]y[j]QQ[i, j], \{i, 8\}], \{j, 8\}] \]

\[ QP[x, y_] := \{\text{Coefficient}[Qm[x, y], e[1]], \text{Coefficient}[Qm[x, y], e[2]], \text{Coefficient}[Qm[x, y], e[3]], \text{Coefficient}[Qm[x, y], e[4]], \text{Coefficient}[Qm[x, y], e[5]], [Qm[x, y], e[6]], \text{Coefficient}[Qm[x, y], e[7]], \text{Coefficient}[Qm[x, y], e[8]]\} \]

\[ \text{OctP[a_, b_]} := \{\{\text{Sum}[QP[Part[Part[a, 1], i], Part[Part[b, i], 1]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 1], i], Part[Part[b, i], 2]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 1], i], Part[Part[b, i], 3]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 2], i], Part[Part[b, i], 1]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 2], i], Part[Part[b, i], 2]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 2], i], Part[Part[b, i], 3]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 3], i], Part[Part[b, i], 1]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 3], i], Part[Part[b, i], 2]], \{i, 3\}], \text{Sum}[QP[Part[Part[a, 3], i], Part[Part[b, i], 3]], \{i, 3\}]\} \]

\[ \text{OctPS[a_, b_]} := 1/2(\text{OctP[a, b]} + \text{OctP[b, a]} \]

% % % % correspondence between the Jordan algebra and \(\mathbb{R}^{27}\)

\[ \text{FF[AA_] := \{Part[Part[Part[Part[AA, 1], 1], 1], 1], \text{Part[Part[Part[Part[AA, 1], 2], 1], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 2], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 3], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 4], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 5], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 6], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 7], 1]}, \text{Part[Part[Part[Part[AA, 1], 2], 8], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 1], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 2], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 3], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 4], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 5], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 6], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 7], 1]}, \text{Part[Part[Part[Part[AA, 1], 3], 8], 1]}, \text{Part[Part[Part[Part[AA, 2], 1], 1], 1]}, \text{Part[Part[Part[Part[AA, 2], 2], 1], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 1], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 2], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 3], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 4], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 5], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 6], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 7], 1]}, \text{Part[Part[Part[Part[AA, 2], 3], 8], 1]}, \text{Part[Part[Part[Part[AA, 3], 1], 1], 1]} \} \]
FF[i_] := {{Part[vv, 1], 0, 0, 0, 0, 0, 0}, {Part[vv, 2], Part[vv, 3], Part[vv, 4]}, Part[vv, 5], Part[vv, 6], Part[vv, 7], Part[vv, 8], Part[vv, 9]}, {Part[vv, 10], Part[vv, 11], Part[vv, 12], Part[vv, 13], Part[vv, 14], Part[vv, 15], Part[vv, 16], Part[vv, 17]], {{Part[vv, 2], − Part[vv, 3], − Part[vv, 4], − Part[vv, 5], − Part[vv, 6], − Part[vv, 7], − Part[vv, 8], − Part[vv, 9]}, {Part[vv, 18], 0, 0, 0, 0, 0, 0}, {Part[vv, 19], Part[vv, 20], Part[vv, 21], Part[vv, 22], Part[vv, 23], Part[vv, 24], Part[vv, 25], Part[vv, 26]}, {{Part[vv, 10], − Part[vv, 11], − Part[vv, 12], − Part[vv, 13], − Part[vv, 14], − Part[vv, 15], − Part[vv, 16], − Part[vv, 17]}, {Part[vv, 19], − Part[vv, 20], − Part[vv, 21], − Part[vv, 22], − Part[vv, 23], − Part[vv, 24], − Part[vv, 25], − Part[vv, 26]}, {Part[vv, 27], 0, 0, 0, 0, 0, 0}}}

% % construction of the matrices

MT = {{mt[1], 0, 0, 0, 0, 0, 0, 0}, {mt[2], mt[3], mt[4], mt[5], mt[6], mt[7], mt[8], mt[9]}, {mt[10], mt[11], mt[12], mt[13], mt[14], mt[15], mt[16], mt[17]}, {{mt[2], − mt[3], − mt[4], − mt[5], − mt[6], − mt[7], − mt[8], − mt[9]}, {mt[18], 0, 0, 0, 0, 0, 0, 0}, {mt[19], mt[20], mt[21], mt[22], mt[23], mt[24], mt[25], mt[26]}, {{mt[10], − mt[11], − mt[12], − mt[13], − mt[14], − mt[15], − mt[16], − mt[17]}, {mt[19], − mt[20], − mt[21], − mt[22], − mt[23], − mt[24], − mt[25], − mt[26]}, {− mt[1] − mt[18], 0, 0, 0, 0, 0, 0}}};

h = Array[hh, 27];

M = Array[mm, {27, 27}];

imm = FF[OctPS[MT, FF[i]]];

Do[Do[mm[i, j] = Coefficient[Part[imm, i], Part[h, j]], {i, 27}], {j, 27}]

Do[econi[j] = D[M, mt[j]], {j, 26}];

Mno[1] = econi[1];

Do[Do[AA[i, j] = 0, {i, 26}], {j, 26}]


Do[Mno[i] = − \sqrt{6}Mno[i]/sqrt[Tr[Mno[i] . Mno[i]]], {i, 26}];
Mapping the geometry of the $E_8$ group

APPENDIX B: STRUCTURE CONSTANTS

The matrices $cc[i]$, $i=53, \ldots, 78$, so obtained, must be added to the 52 matrices $cc[i]$ determined in Ref. 1 to give a base of the $e_8$ algebra.

APPENDIX B: STRUCTURE CONSTANTS

The structure constants $s^K_{ij}$ are defined by $[C_i, C_j] = \sum_{K=1}^{78} s^K_{ij} C_K$. We checked that the coefficients $s^K_{ij}$ are completely antisymmetric in the indices. We write only the nonvanishing terms, up to symmetries, which are not yet written in Ref. 1.
$s_{1,54,55} = \frac{1}{2}, \quad s_{1,56,58} = -\frac{1}{2}, \quad s_{1,57,61} = -\frac{1}{2}, \quad s_{1,59,60} = -\frac{1}{2},$

$s_{1,62,63} = -\frac{1}{2}, \quad s_{1,64,66} = \frac{1}{2}, \quad s_{1,65,69} = \frac{1}{2}, \quad s_{1,67,68} = \frac{1}{2},$

$s_{1,71,72} = -1, \quad s_{2,54,56} = \frac{1}{2}, \quad s_{2,55,58} = \frac{1}{2}, \quad s_{2,57,59} = -\frac{1}{2},$

$s_{2,60,61} = -\frac{1}{2}, \quad s_{2,62,64} = -\frac{1}{2}, \quad s_{2,63,66} = -\frac{1}{2}, \quad s_{2,65,67} = \frac{1}{2},$

$s_{2,68,69} = \frac{1}{2}, \quad s_{2,71,73} = -1, \quad s_{3,54,58} = \frac{1}{2}, \quad s_{3,55,56} = -\frac{1}{2},$

$s_{3,57,60} = -\frac{1}{2}, \quad s_{3,59,61} = \frac{1}{2}, \quad s_{3,62,66} = \frac{1}{2}, \quad s_{3,63,64} = -\frac{1}{2},$

$s_{3,65,68} = -\frac{1}{2}, \quad s_{3,67,69} = \frac{1}{2}, \quad s_{3,72,73} = -1, \quad s_{4,54,57} = \frac{1}{2},$

$s_{4,55,61} = \frac{1}{2}, \quad s_{4,56,59} = \frac{1}{2}, \quad s_{4,58,60} = -\frac{1}{2}, \quad s_{4,62,65} = -\frac{1}{2},$

$s_{4,63,69} = -\frac{1}{2}, \quad s_{4,64,67} = -\frac{1}{2}, \quad s_{4,66,68} = \frac{1}{2}, \quad s_{4,71,74} = -1,$

$s_{5,54,60} = 1, \quad s_{5,55,59} = 1, \quad s_{5,56,61} = -\frac{1}{2}, \quad s_{5,58,59} = \frac{1}{2},$

$s_{5,62,63} = \frac{1}{2}, \quad s_{5,64,66} = \frac{1}{2}, \quad s_{5,65,67} = \frac{1}{2}, \quad s_{5,66,68} = \frac{1}{2},$

$s_{5,72,74} = -1, \quad s_{6,54,59} = \frac{1}{2}, \quad s_{6,55,60} = -\frac{1}{2}, \quad s_{6,56,57} = -\frac{1}{2},$

$s_{6,58,61} = -\frac{1}{2}, \quad s_{6,62,67} = \frac{1}{2}, \quad s_{6,63,68} = -\frac{1}{2}, \quad s_{6,64,65} = -\frac{1}{2},$

$s_{6,66,69} = -\frac{1}{2}, \quad s_{6,73,74} = -1, \quad s_{7,54,58} = \frac{1}{2}, \quad s_{7,55,56} = -\frac{1}{2},$

$s_{7,57,60} = \frac{1}{2}, \quad s_{7,59,61} = -\frac{1}{2}, \quad s_{7,62,66} = -\frac{1}{2}, \quad s_{7,63,64} = \frac{1}{2},$

$s_{7,65,68} = -\frac{1}{2}, \quad s_{7,67,69} = \frac{1}{2}, \quad s_{7,71,75} = -1, \quad s_{8,54,56} = -\frac{1}{2},$

$s_{8,55,68} = -\frac{1}{2}, \quad s_{8,57,59} = -\frac{1}{2}, \quad s_{8,60,61} = -\frac{1}{2}, \quad s_{8,62,64} = -\frac{1}{2},$

$s_{8,63,66} = -\frac{1}{2}, \quad s_{8,65,67} = -\frac{1}{2}, \quad s_{8,68,69} = -\frac{1}{2}, \quad s_{8,72,75} = -1,$

$s_{9,54,55} = \frac{1}{2}, \quad s_{9,56,58} = -\frac{1}{2}, \quad s_{9,57,61} = \frac{1}{2}, \quad s_{9,59,60} = \frac{1}{2},$

$s_{9,62,63} = \frac{1}{2}, \quad s_{9,64,66} = -\frac{1}{2}, \quad s_{9,65,69} = \frac{1}{2}, \quad s_{9,67,68} = \frac{1}{2},$

$s_{9,73,75} = -1, \quad s_{10,54,60} = \frac{1}{2}, \quad s_{10,55,59} = \frac{1}{2}, \quad s_{10,56,61} = -\frac{1}{2},$

$s_{10,57,58} = -\frac{1}{2}, \quad s_{10,62,68} = \frac{1}{2}, \quad s_{10,63,67} = \frac{1}{2}, \quad s_{10,64,69} = -\frac{1}{2},$
<table>
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<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_{10,65,66} )</td>
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</tr>
<tr>
<td>( s_{10,74,75} )</td>
<td>(- 1)</td>
</tr>
<tr>
<td>( s_{11,54,59} )</td>
<td>( \frac{1}{2})</td>
</tr>
<tr>
<td>( s_{11,55,60} )</td>
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<tr>
<td>( s_{11,56,57} )</td>
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<td>( s_{12,54,57} )</td>
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<td>( s_{12,55,59} )</td>
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<tr>
<td>( s_{12,64,69} )</td>
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\( s^{39,59,74} = \frac{1}{2}, \quad s^{39,60,78} = -\frac{1}{2}, \quad s^{39,61,77} = \frac{1}{2}, \quad s^{40,53,65} = -1, \)

\( s^{40,54,74} = -\frac{1}{2}, \quad s^{40,55,78} = \frac{1}{2}, \quad s^{40,56,76} = \frac{1}{2}, \quad s^{40,57,71} = -\frac{1}{2}, \)

\( s^{40,58,77} = -\frac{1}{2}, \quad s^{40,59,73} = -\frac{1}{2}, \quad s^{40,60,72} = \frac{1}{2}, \quad s^{41,53,66} = -1, \)

\( s^{41,54,75} = -\frac{1}{2}, \quad s^{41,55,73} = -\frac{1}{2}, \quad s^{41,56,72} = \frac{1}{2}, \quad s^{41,57,77} = 1, \)

\( s^{41,58,71} = -\frac{1}{2}, \quad s^{41,59,78} = \frac{1}{2}, \quad s^{41,60,74} = -\frac{1}{2}, \quad s^{42,53,67} = -1, \)

\( s^{42,54,76} = -\frac{1}{2}, \quad s^{42,55,77} = \frac{1}{2}, \quad s^{42,56,74} = \frac{1}{2}, \quad s^{42,57,73} = -\frac{1}{2}, \)

\( s^{42,58,72} = -\frac{1}{2}, \quad s^{42,59,71} = \frac{1}{2}, \quad s^{43,53,69} = -1, \quad s^{43,54,78} = -\frac{1}{2}, \)

\( s^{43,55,76} = -\frac{1}{2}, \quad s^{43,56,78} = \frac{1}{2}, \quad s^{43,57,75} = -\frac{1}{2}, \quad s^{43,58,74} = \frac{1}{2}, \)

\( s^{43,59,72} = \frac{1}{2}, \quad s^{43,60,71} = -\frac{1}{2}, \quad s^{43,61,73} = -\frac{1}{2}, \quad s^{44,53,62} = -1, \)

\( s^{44,54,78} = \frac{1}{2}, \quad s^{44,55,74} = \frac{1}{2}, \quad s^{44,56,77} = -\frac{1}{2}, \quad s^{44,57,72} = \frac{1}{2}, \)

\( s^{44,58,76} = -\frac{1}{2}, \quad s^{44,59,75} = \frac{1}{2}, \quad s^{44,60,73} = \frac{1}{2}, \quad s^{44,61,71} = -\frac{1}{2}, \)

\( s^{45,53,71} = \frac{1}{2}, \quad s^{45,54,62} = -\frac{1}{2}, \quad s^{45,55,63} = -\frac{1}{2}, \quad s^{45,56,64} = \frac{1}{2}, \)

\( s^{45,57,65} = -\frac{1}{2}, \quad s^{45,58,66} = -\frac{1}{2}, \quad s^{45,59,67} = -\frac{1}{2}, \quad s^{45,60,68} = \frac{1}{2}, \)

\( s^{45,61,69} = -\frac{1}{2}, \quad s^{45,70,71} = -\frac{7}{2}, \quad s^{46,53,72} = -\frac{1}{2}, \quad s^{46,54,63} = \frac{1}{2}, \)

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\( s^{47,57,67} = \frac{1}{2}, \quad s^{47,58,63} = \frac{1}{2}, \quad s^{47,59,65} = -\frac{1}{2}, \quad s^{47,60,69} = \frac{1}{2}, \)

\( s^{47,61,68} = -\frac{1}{2}, \quad s^{47,70,73} = -\frac{7}{2}, \quad s^{48,53,74} = -\frac{1}{2}, \quad s^{48,54,65} = -\frac{1}{2}, \)

\( s^{48,55,69} = -\frac{1}{2}, \quad s^{48,56,67} = -\frac{1}{2}, \quad s^{48,57,62} = -\frac{1}{2}, \quad s^{48,58,68} = \frac{1}{2}, \)
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\[ S_{49,57,68} = -\frac{1}{2}, \quad S_{49,58,62} = \frac{1}{2}, \quad S_{49,59,69} = \frac{1}{2}, \quad S_{49,60,65} = \frac{1}{2}, \]
\[ S_{49,61,67} = -\frac{1}{2}, \quad S_{49,70,75} = -\frac{3}{2}, \quad S_{50,53,76} = -\frac{1}{2}, \quad S_{50,54,67} = -\frac{1}{2}, \]
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\[ S_{50,59,66} = \frac{1}{2}, \quad S_{50,60,63} = \frac{1}{2}, \quad S_{50,61,66} = \frac{1}{2}, \quad S_{50,70,76} = -\frac{3}{2}, \]
\[ S_{51,53,77} = -\frac{1}{2}, \quad S_{51,54,68} = -\frac{1}{2}, \quad S_{51,55,67} = \frac{1}{2}, \quad S_{51,56,69} = -\frac{1}{2}, \]
\[ S_{51,57,66} = \frac{1}{2}, \quad S_{51,58,65} = -\frac{1}{2}, \quad S_{51,59,63} = -\frac{1}{2}, \quad S_{51,60,62} = \frac{1}{2}, \]
\[ S_{51,61,64} = \frac{1}{2}, \quad S_{51,70,77} = -\frac{3}{2}, \quad S_{52,53,78} = -\frac{1}{2}, \quad S_{52,54,69} = -\frac{1}{2}, \]
\[ S_{52,55,65} = \frac{1}{2}, \quad S_{52,56,68} = -\frac{1}{2}, \quad S_{52,57,63} = -\frac{1}{2}, \quad S_{52,58,67} = \frac{1}{2}, \]
\[ S_{52,59,66} = -\frac{1}{2}, \quad S_{52,60,64} = -\frac{1}{2}, \quad S_{52,61,62} = \frac{1}{2}, \quad S_{52,70,78} = -\frac{3}{2}. \]