High-sensitivity imaging with multi-mode twin beams

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Twin entangled beams produced by single-pass parametric down-conversion (PDC) offer the opportunity to detect weak amount of absorption with an improved sensitivity with respect to standard techniques which make use of classical light sources. We propose a differential measurement scheme which exploits the spatial quantum correlation of type-II PDC to image a weak amplitude object with a sensitivity beyond the standard quantum limit imposed by shot noise. By applying a quantitative numerical model to a specific PDC system (a type-II beta-barium borate crystal), we are able to identify conditions in parameter space suitable for a substantial enhancement of the imaging sensitivity, and we investigate the robustness of this enhancement against experimental imperfections.

I. INTRODUCTION

The field of quantum imaging aims to exploit the quantum nature of light and the natural parallelism of optical signals to devise novel techniques for optical imaging and for parallel information processing at the quantum level. To cite a few examples, multi-mode quantum correlations in macroscopic twin beams have been used to measure small laser beam displacements beyond the Rayleigh limit [1] and for the noiseless amplification of optical images [2]. Meanwhile, a number of new applications that use multi-mode twin beams in the low gain regime have been proposed, such as e.g., quantum lithography [3], entangled photon microscopy [4], and dispersion canceled quantum optical coherence tomography [5]. The state of the art in this field can be appreciated by reading [6]. An old review is given by [7], while a new review has recently been published [8].

In this paper we focus our attention on a specific application, namely the possibility of improving the sensitivity in the measurements of very weak images, i.e., the intensity distribution transmitted by objects with a small absorption coefficient. The detection of a weak amount of absorption (but not of its spatial distribution) with a sensitivity beyond the standard quantum limit (SQL) was demonstrated in the past by using twin beams produced by cw optical parametric oscillators (OPOs) [9–12]. In those experiments the OPO operated above threshold and the generated twin beams were single spatial modes selected both by the optical resonator and the three-wave mixing interaction process. Their intensity fluctuations were correlated in the temporal domain well below the shot-noise level and this feature was exploited in order to perform differential intensity measurements of very slight amounts of absorption. For example, in [12] this technique was used in order to measure small intensity absorptions from an electro-optical amplitude modulator, achieving 7 dB (80%) of noise reduction with respect to the SQL of a classical differential scheme. In [11] a spectroscopic measurement of a two-photon transition was implemented with an improvement of 1.9 dB (35%) with respect to the shot-noise limited experiment.

However, twin beams that are single-mode in the spatial domain, as those used in the experiments [11,12], cannot be exploited to retrieve information on the spatial distribution of the transmitted field since the correlation vanishes as soon as one detects small portions of the two beams instead of the whole beams. Being interested in measuring an image, we are forced to consider a source with many spatial modes, which is able to display quantum correlation also in the spatial domain. Recently, our group demonstrated the existence of such kind of correlations in the high gain regime of single-pass parametric down-conversion (PDC) [7,13–15]. In the experiment [14,15] the subshot-noise correlation between symmetrical points of the PDC far field was observed and was interpreted as a manifestation at the macroscopic level (i.e., in the large photon number regime) of the transverse momentum conservation of photons. The aim of this paper is to show that this spatial twin beam effect existing over the several phase conjugate signal and idler mode pairs offers the opportunity to retrieve the full two-dimensional (2D) spatial distribution of a weak object in parallel, with an improved signal-to-noise ratio with respect to standard techniques. This opportunity is interesting, for example, in the case of biological samples or whenever there is the need for illuminating the object at low intensities.

The paper is organized in the following manner. In Sec. II we briefly review the principle of operation of differential measurements used to detect faint amounts of absorption, showing with a simple two-mode model that the use of a source displaying quantum correlations can provide a higher sensitivity than a classical source. In Sec. III we propose an imaging scheme that can be used to detect the spatial distribution of amplitude objects with a PDC source. The model equations that describe the multi-mode PDC process are illustrated in Sec. IV. In the last sections of the paper (Secs V and VII) we investigate the robustness of the spatial quantum correlation necessary to implement the high sensitivity imaging technique, considering experimental imperfections that are difficult to avoid in a real experiment. In particular in Sec. VII we present the results obtained through a numerical stochastic model that simulates the imaging experiment with realistic parameters.

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two beams at the plane before the object, and with the ratio between the variance of $N_1$ and $N_2$ is determined by

\[ \Delta N = N_2 - N_1 \]

In its classical version [Fig. 1(a)] a laser beam (not necessarily shot-noise limited) is separated by a 50:50 beam splitter into two classical “twin” beams, a “test” beam which illuminates the object under examination, and a “reference” beam which does not interact with the object. In the ideal case in which the two beams are perfectly balanced, they can be considered as two classical copies, one of each other, meaning that their intensity fluctuations display the strongest level of correlation allowed by a classical source (corresponding to the shot-noise level of the incident beams). The subtraction of the detected intensities allows one to eliminate the classical excess noise contained in the source field and to retrieve information about the object with a far better signal-to-noise ratio (SNR) than through direct illumination. However, when a classical source is used, this technique is limited intrinsically by the level of shot noise.

We shall now illustrate how, under appropriate conditions, quantum correlations allow one to achieve a better SNR of the object than classical correlations. In the quantum scheme [see Fig. 1(b)], the reference and the test beams are replaced by the signal and idler fields generated by a PDC source, for example, the single-mode twin beams generated by an OPO above threshold.

Let us indicate with $a_1$ and $a_2$ the field operators of the two beams at the plane before the object, and with $N_j = a_j^\dagger a_j$, $j=1,2$, the corresponding photon fluxes. The fields $a_1$ and $a_2$ obey the usual canonical commutation relations

\[ [a_i, a_j^\dagger] = \delta_{ij}, \quad (i,j = 1,2). \]

The degree of correlation of the two beams is determined by the ratio between the variance of $N = N_2 - N_1$ and the corresponding level of shot noise, i.e.,

\[ \sigma = \frac{\langle \Delta N^2 \rangle}{\langle N \rangle + \langle N \rangle}. \]

It is equal to unity in the case of the classical copies obtained with a well-balanced 50:50 beam splitter, while it can go above unity when the PDC source is employed (it vanishes in the ideal limit of perfect quantum intensity correlations). We assume that the two beams are perfectly balanced and have symmetric statistical properties, so that in particular

\[ \langle N \rangle = \langle N \rangle, \]

\[ \langle \Delta N^2 \rangle = \langle \Delta N^2 \rangle. \]

Indicating with $t_{obj}$ the transmission coefficient of the object, the field in the test arm undergoes the unitary transformation

\[ a_1' = t_{obj} a_1 + i \sqrt{1-t_{obj}^2} \nu, \]

where $\nu$ denotes here a vacuum field operator. The measured quantity is the difference of the photon fluxes collected on the two detectors after the test beam has passed through the object, associated with the operator $N' = a_2^\dagger a_2 - a_1^\dagger a_1'$. Recalling the condition $\langle N_1 \rangle = \langle N_2 \rangle$, one immediately sees that the mean value of $N'$ is proportional to the object absorption coefficient $\alpha = 1 - |t_{obj}|^2$:

\[ \langle N \rangle = a_2^\dagger a_2 - |t_{obj}|^2 (a_1^\dagger a_1) = \langle N \rangle \alpha. \]

The noise of the measurement is determined by the variance of $N'$. By using relations (1) and (4) together with the symmetry conditions (3), we can express $\langle \Delta N'^2 \rangle$ in terms of the unprimed quantities $\langle N \rangle$, $\langle \Delta N^2 \rangle$, and $\langle \Delta N'^2 \rangle$, which depend on the photon statistics and correlation of the two beams in the absence of the object:

\[ \langle \Delta N'^2 \rangle = a_2^2 \langle \Delta N^2 \rangle + (1 - \alpha) \langle \Delta N^2 \rangle + \alpha (1 - \alpha) \langle N \rangle. \]

Using Eq. (6) together with definition (2) we obtain the SNR of the measurement

\[ \text{SNR}_{q} = \frac{\langle N \rangle}{\sqrt{\langle \Delta N'^2 \rangle}} = \frac{\alpha \sqrt{\langle N \rangle}}{\sqrt{\alpha E_n + 2 \alpha (1 - \alpha) + \alpha}}, \]

where we have introduced the quantity

\[ E_n = \frac{\langle \Delta N'^2 \rangle - \langle N \rangle}{\langle N \rangle}, \]

which characterizes the signal (idler) noise in excess with respect to that of a coherent beam with the same intensity (it is the so-called Mandel Q factor, used when dealing with sub-Poissonian statistics for which $Q$ takes negative values).

In the classical scheme of Fig. 1(a), provided the two beams are perfectly balanced and satisfy conditions (3), it can be verified that $\sigma = 1$ (i.e., $\langle \Delta N'^2 \rangle \approx 2 \langle N \rangle$) whatever the amount of excess noise. Thus we find

\[ \text{SNR}_{class} = \frac{\alpha \sqrt{\langle N \rangle}}{\sqrt{\alpha E_n + 2 - \alpha}}. \]

The standard quantum limit is obtained from this expression by considering the case of a coherent beam, for which $E_n = 0$ and
Noting that for a weak object the condition $E_n \ll 1/\alpha^2$ is easy to achieve, the SNR obtained with the differential measurement (9) usually approaches the SQL value (10) even if the classical source is not shot-noise limited but presents some excess noise.

It is worth noting that the SQL could also be defined by considering the SNR of a direct measurement scheme, which makes use of a single coherent beam with the same mean photon number $\langle N_i \rangle$; this would give $\text{SNR}_{\text{SQL}} = \alpha \sqrt{\langle N_i \rangle}$, an improvement by a factor of $\sqrt{2}$ with respect to the SQL of the differential measurement (10). However, in the following we will use definition (10) since the detection of weak amount absorption is most commonly obtained through the differential measurement technique that allows one to approach the SQL much more easily than direct measurements.

Let us now investigate the conditions under which the SQL can be beaten by using photon number correlated beams. To this end we introduce the ratio

$$R = \frac{\text{SNR}_{\text{SNR}}}{\text{SNR}_{\text{SQL}}} = \sqrt{\frac{2 - \alpha}{\alpha^2 E_n + 2\sigma(1-\alpha) + \alpha}}, \tag{11}$$

which measures the improvement of the SNR achieved by using the PDC source with respect to the SQL. In the limit of a weak object we have

$$R \approx \frac{1}{\sqrt{\sigma}} \quad \text{for} \quad \alpha \ll 1, \quad \alpha^2 E_n \ll 2\sigma. \tag{12}$$

Thus, provided the excess noise is not too large, the SNR is improved by a factor of $\sim 1/\sqrt{\sigma}$ with respect to the SQL. More generally, it is easily seen that $R > 1$ provided that

$$\sigma < \sigma_{\text{MAX}} = 1 - \frac{\alpha^2 E_n}{2(1-\alpha)}. \tag{13}$$

$\sigma_{\text{MAX}}$ represents the maximum value of $\sigma$ for which we have an improvement in the SNR with respect to the SQL. Clearly we have always $\sigma_{\text{MAX}} < 1$ (since $E_n \geq 1$ and $0 < \alpha < 1$). This is an expected result: the twin beams of the PDC source need to be correlated below the shot-noise level (i.e., $\sigma < 1$) in order to beat the classical source configuration. It can be seen that condition (13) is easily verified when $\alpha \to 0$ since $\sigma_{\text{MAX}}$ is close to unity in this limit. However, the goal is to obtain a substantial improvement of the SNR and Eq. (12) shows that the condition on $\sigma$ is rather severe.

### III. HIGH-SENSITIVITY IMAGING SCHEME

The previous treatment did not consider the spatial aspects of the detection process. Since we are interested in developing an imaging system, parallel multi-mode operation is required. We focus here on the nonclassical source case, being interested in determining the conditions where the SQL can be beaten. The process of spontaneous parametric down-conversion is naturally broadband in the spatial as well as in the temporal domains and is therefore well-suited for this task. In particular, we can exploit the subshot-noise photon-number correlation arising between pairs of symmetrical regions of the far field, theoretically predicted in [13,16] and experimentally demonstrated in [14,15]. Close to the degenerate frequency, photon pairs are indeed emitted almost symmetrically with respect to the pump axis because of the conservation of the transverse momentum. The finite waist $w_p$ of the pump beam introduces a spread in the relative angular directions of the twin photons on the order of the pump angular bandwidth $\Delta \omega / w_p$, $\Delta \omega$ denoting the signal or the idler wavelength. Assuming the far field is observed in the focal plane of a lens with focal length $f$ (see Fig. 2) this angular spread corresponds to the coherence length of the field in this plane $x_{\text{coh}} = \lambda / w_p$. In [16] it was shown that the fluctuations in the photon number difference detected in two symmetrical regions go well below shot noise (ideally to zero when losses are negligible), provided the detection areas are large compared to $x_{\text{coh}}$.

In accordance with this result, we consider the imaging scheme illustrated in Fig. 2. The pump field which illuminates the $\chi^{(2)}$ nonlinear crystal is a coherent pump pulse with a large beam waist $w_p$. The object, characterized by an absorption coefficient $a(x)$ depending on the transverse plane position $x$, is placed in the far field of the source. The lens shown in the figure is used to perform the Fourier transformation of the fields from the source into its second focal plane, which is taken as the detection plane. Both the signal (test) and the idler (reference) photons are collected by two arrays of detectors, typically the pixels of a high quantum efficiency charge-coupled device (CCD) camera [17]. The detectors are either placed immediately after the object plane as in the figure, or more realistically, a telescopic system is used to image the object plane into the detection plane. The far-field coherence length $x_{\text{coh}} = \lambda / w_p$ is assumed to be on the same order of magnitude of the pixel detector size or smaller, in order to fulfill the sub-shot-noise correlation condition and compatibly with the requirement of large photon number operation. In this way, the fluctuations of the reference and the test beams display strong spatial (point to point) correlation for each symmetrical pair of detectors of the CCD array. Sub-shot-noise correlation for each of these pairs of detection channels is the goal to achieve in order to beat the SQL in a differential measurement.

### IV. NUMERICAL MODEL

With the aim of simulating the high-sensitivity imaging experiment proposed in the previous section, we used a fully
three-dimensional (3D) numerical model which was developed in [16] and successfully employed to model, also quantitatively, the experiment on PDC spatial correlation [15]. It takes into account realistic features of the PDC process, such as the phase-matching conditions inside the crystal as well as the finite size of the pump beam, both in the temporal and the spatial domains. The model is based on a set of equations describing the propagation of the slowly varying envelope operators of the signal and idler fields, which we denote by \( a_1(z, \vec{x}, t) \) and \( a_2(z, \vec{x}, t) \), and satisfy bosonlike commutation rules at equal \( z \), i.e., 
\[
[a_1(z, \vec{x}, t), a_2^\dagger(z, \vec{x}, t')] = \delta_{t-t'} \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (i,j = 1,2)
\]
(see [16] for more details). As we are considering the parametric regime, it is assumed that the pump field \( A_0(z, \vec{x}, t) \) remains almost undepleted and can be treated as a known classical field. For definiteness, we shall assume it has a Gaussian profile with a plane-wave front at \( z = 0 \), with a beam waist \( w_p \) and duration \( \tau_p \):
\[
A_0(z = 0, \vec{x}, t) = (2\pi)^{-3/2} A_0 e^{-(\vec{x}^2 r^2)/w_p^2} e^{-i t r^2/\tau_p^2}.
\]
(14) The propagation equation for the signal field \( a_1 \) can then be written in Fourier space as [16,18]
\[
\frac{\partial}{\partial z} a_1(z, \vec{q}, \Omega) = i [k_j(\vec{q}, \Omega) - k_j] a_1(z, \vec{q}, \Omega) + \frac{g}{L_c} e^{-i k_0 \vec{q} \cdot \vec{r}} \int d\Omega' \nonumber
\]
\[
\times a_1(z, \vec{q} - \vec{q}', \Omega - \Omega') a_2^\dagger(z, -\vec{q}', -\Omega'),
\]
(15) the equation for the idler wave \( a_2 \) can be obtained by exchanging the subscript \( 1 \rightarrow 2 \). The operators \( a_j(z, \vec{q}, \Omega) \) denote the Fourier transformation of the fields \( a_j(z, \vec{x}, t) \):
\[
a_j(z, \vec{q}, \Omega) = \int \frac{d\vec{x}}{2\pi} \int \frac{dt}{\sqrt{2\pi}} a_j(z, \vec{x}, t) e^{-i \vec{q} \cdot \vec{x} + \Omega t} \quad (j = 1,2),
\]
(16) where \( \Omega \) denotes the temporal frequency relative to the carrier frequency \( \omega_c \) of field \( j \), \( \vec{q} \) the transverse vector-component; and \( k_j(\vec{q}, \Omega) = \sqrt{k_j(\vec{q}, \Omega) - \vec{q}^2} \) is the corresponding longitudinal component along the propagation direction \( z \). The wave-number function \( k_j(\vec{q}, \Omega) \) determines the linear propagation properties of field \( j \); in particular its linear and quadratic dependence on \( \Omega \) is responsible for temporal walk-off and second-order temporal dispersion, while its linear and quadratic dependence on \( \vec{q} \) leads to spatial walk-off and diffraction. The parametric gain \( g \) is a dimensionless constant proportional to the strength of the nonlinear interaction, to the crystal length \( L_c \), and to the peak value of the pump field \( A_p \); its value determines the number of photons that are generated in the down-conversion process in mode pairs that are well-phase-matched. \( \Delta_0 = k_1 + k_2 - k_0 \) indicates the collinear phase-mismatch parameter of the three carrier waves, \( k_1, k_2, \) and \( k_0 \) being the corresponding wave number. The normalized function \( \alpha_0(\vec{q}, \Omega) = 1/(\pi^{3/2} \delta q^{2} \delta \omega_p) e^{-\vec{q}^2/\delta \vec{q}^2 - \omega^2/\omega_p^2} \) is the Fourier transform of \( A_0(\vec{x}, t)/A_p \), with \( \delta q_p = 2/w_p \) and \( \delta \omega_p = 2/\tau_p \), denoting the pump spatial and temporal bandwidths, respectively.

In the plane-wave pump (PWP) limit where \( \delta q_p \) and \( \delta \omega_p \) are much smaller than the spatial and temporal bandwidths of phase-matching, \( \alpha_0(\vec{q}, \Omega) \rightarrow \delta(\vec{q}) \delta(\Omega) \) and the model equations can be solved analytically. The solution of Eq. (15) can be written as an input-output relation in the form of a two-mode squeezing transformation:
\[
a_1^{out}(\vec{q}, \Omega) = U_1(\vec{q}, \Omega) a_1^{in}(\vec{q}, \Omega) + V_1(\vec{q}, \Omega) a_2^{in}(\vec{q}, \Omega),
\]
\[
a_2^{out}(\vec{q}, \Omega) = U_2(\vec{q}, \Omega) a_2^{in}(\vec{q}, \Omega) + V_2(\vec{q}, \Omega) a_1^{in}(\vec{q}, \Omega).
\]
(17) The gain coefficients \( U_j \) and \( V_j \) are functions of the parametric gain \( g \) and of the phase-mismatch function \( \Delta(\vec{q}, \Omega) = k_1(\vec{q}, \Omega) + k_2(\vec{q}, \Omega) - k_0 \). Their explicit expressions can be found in [16,18]. One finds that the mean number of photons generated in mode \( (\vec{q}, \Omega) \) of beam \( j \) is on the order of \( \sin^2 g \) for well phase-matched modes, for which \( \Delta(\vec{q}, \Omega) \), is small compared to unity. For what concerns correlations, we observe that relation (17) links only phase-conjugate modes \( (\vec{q}, \Omega) \) and \( (-\vec{q}, -\Omega) \) of the signal and idler fields. This feature reflects the perfect correlation of the transverse momenta and energies of the emitted twin photons which occurs in the PWP limit. In particular, the correlation function of the signal and idler field Fourier transforms,
\[
\langle a_1^{out}(\vec{q}, \Omega) a_2^{out}(\vec{q}', \Omega') \rangle = \delta(\vec{q} + \vec{q}') \delta(\Omega + \Omega') U_1(\vec{q}, \Omega) \times V_2(-\vec{q}, -\Omega),
\]
(18) displays a deltalike correlation peak in correspondence to opposite transverse momenta. As discussed in details in [16], this leads to subshot correlations in photon number between symmetrical positions in the far field.

As already mentioned in the previous section, the finite pump bandwidth \( \delta q_p \) in the spatial domain introduces an indeterminacy in the correlation between the angular directions of the emitted twin photons. This indeterminacy gives rise to a spread of the far-field signal-idler correlation function, which acquires a finite width on the order of the far-field coherence length \( x_{coh} \approx \lambda f/w_p \). Thus as a consequence of the finite transverse size of the pump beam, perfect spatial correlations between two symmetrical ideal detectors can be achieved only if the linear size of the detectors is large compared to \( x_{coh} \); in this ideal limit we would have \( \sigma = 0 \). Conversely, if this condition is not fulfilled the fluctuations of \( N \) just lie below or close to the shot-noise level, i.e., we have \( 0 < \sigma < 1 \). When the finite quantum efficiency of the detectors \( \eta < 1 \) is taken into account, the highest level of correlation that can be reached is \( \sigma = 1 - \eta \) (see, e.g., [13]).

In the simulations we shall consider an optical setup close to that of the experiment performed by Jedrkiewicz et al. [14,15] to measure subshot-noise far-field correlations in the high gain regime of PDC: the nonlinear crystal is a 4 mm long beta-barium borate (BBO) crystal cut for type II phase-matching and PDC is observed around the degenerate wave-length \( \lambda = 704 \) nm. The far field is measured in the focal plane of a lens of focal length \( f = 5 \) cm as shown in Fig. 2. The role of the \( f - f \) lens system is to perform the mapping.
between the spatial frequency plane and the focal plane of
the lens which is used as the detection plane. The pixel
detectors of the CCD are characterized by a $20 \times 20$ $\mu$m$^2$
area and a high quantum efficiency $\eta$ approaching 0.9
at the signal wavelength. The effective quantum efficiency
which takes into account the losses of all optical elements
was evaluated to be around 0.75 [14].

All the numerical results presented in the following sec-
tions have been obtained by considering a stochastic model
based on the Wigner representation equivalent to Eq. (15).
All the expectation values of the field operator moments nec-
essary to evaluate the observables of interest were evaluated
by integrating numerically classical-looking propagation
equations for the $c$-number stochastic fields [formally iden-
tical to Eq. (15)] and by performing ensemble averages on
the output fields (see, e.g., [19] for a detailed treatment).
Taking into account that the Wigner representation provides
only symmetrically ordered operator moments, we had to
apply appropriate corrections to obtain the desired ordering.
A detailed description of the model can be found in [16,19].

We considered in the simulations a fairly temporal bandwidth,
corresponding to the PDC field that a 10 nm
interference filter centered at the degenerate wavelength $\lambda$
=704 nm would transmit. At such a wavelength, collinear
type II phase-matching occurs when the pump axis forms an
angle $\theta_p=49.05^\circ$ with the crystal axis. As $\theta_p$ is close to this
value, the phase-matching mechanism leads to the generation
of two PDC radiation cones, one for the signal and one for
the idler field, that have tilted directions one with respect to
the other because of spatial walk-off (see, e.g., [16] for more
details). The distribution of the mean intensity in the far field
is therefore characterized by a pair of nonconcentric rings
whose radius varies with the orientation of the crystal. Figure
3 displays the behavior of those rings for decreasing values of
$\theta_p$ starting from case (a) where collinear phase-matching is
achieved and the rings are tangents in the origin (for $\theta_p$
=49.05°, $\Delta_0=0$), and ending up to case (c) where the radius
of the rings shrinks to zero (for $\theta_p=48.18^\circ$, $\Delta_0=-186$ $\text{cm}^{-1}$).
In the three cases the angular separation of the PDC emission
cones is close to 6.6°, while the width of the rings increases
as the radius decreases. In case (c), the rings reduce to two
disk-shaped areas with a diameter of about 2.2 mm (corre-
sponding to an angular range of 2.5°). This particular phase-
matching configuration suits well our imaging purpose, as
the intensity distribution is nearly uniform on a broad area
which covers a huge number of coherence areas (as can be
seen from the intensity profile shown in the figure). In the
numerical simulations we exploited this translational invari-
ance of the signal and idler far field to perform spatial aver-
age (in addition to ensemble averages) that speedup the con-
vergence rate substantially. In fact, our choice of a 10 nm
temporal bandwidth corresponds to a good compromise be-
tween staying close to frequency degeneracy and having a
reasonably wide range of angular emission. Moreover, the
two disks in Fig. 3(c) are spatially well-separated. For this
reason, in principle there is no need to divide the two beams
with a polarization beam splitter into two different detection
arms (as, e.g., it was done in experiment [14,15], where
nearly collinear phase-match was considered). This latter
feature represents an advantage with respect to other con-
figurations, as the use of additional optical elements in the
arms of the differential imaging scheme is a potential source
of imbalance and can lead to correlation losses (see discus-
sion in Sec. VI). For the modeling of the imaging experiment
we shall therefore consider the phase-matching conditions of
case (c). The boxes shown in Fig. 3 indicate the square areas
considered in the simulations: all the quantitative results that
follow have been obtained with a numerical grid corresponding
to these two areas of the signal and idler beams. It
should, however, be stressed that other system configurations
can in principle be exploited.

V. BEHAVIOR OF THE SIGNAL-IDLER-FAR-FIELD
CORRELATION FUNCTION

In this section we briefly discuss some aspects of the far-
field signal-idler intensity correlation that are relevant for the
proposed imaging scheme. The spread of this function in-
deed determines the minimum size of the detection area for
which it is possible to observe subshot-noise correlation and
sets therefore a lower limit for the imaging resolution. To be
more precise, let us consider the normalized correlation func-
tion for the signal and idler intensity fluctuations, defined as

$$
\Gamma_{12}(\vec{x}_1, \vec{x}_2) = \frac{\langle \delta N_{1}(\vec{x}_1) \delta N_{2}(\vec{x}_2) \rangle}{\sqrt{\langle \delta N_{1}(\vec{x}_1)^2 \rangle \langle \delta N_{2}(\vec{x}_2)^2 \rangle}},
$$

where $\delta N_j(\vec{x}_i) = N_j(\vec{x}_i) - \langle N_j(\vec{x}_i) \rangle$ ($j=1,2$) denotes the photon number fluctuation density operator in position $\vec{x}_i$ of the de-

FIG. 3. Far-field mean intensity distribution for three different
crystal orientations: (a) $\theta_p=49.05^\circ$, $\Delta_0=0$, (b) $\theta_p=48.5^\circ$, $\Delta_0=$
$-117.6$ $\text{cm}^{-1}$, and (c) $\theta_p=48.18^\circ$, $\Delta_0=-186$ $\text{cm}^{-1}$. The millimeter
scale corresponds to that obtained with the chosen lens focal length
$f=5$ cm. The temporal bandwidth is of 10 nm. The radial profile of
the signal and idler rings is shown above each pattern. The boxes in
(c) indicate the $1280 \times 1280$ $\mu$m$^2$ square areas of uniform intensity
chosen to model the imaging experiment.
As can be seen from the figure legend, the smallest value of $g=1$ yields a FWHM of $\sim 11 \, \mu m$, close to the low gain limit result (21). However, by increasing the parametric gain we observe that the correlation function becomes broader and broader: its FWHM is more than doubled with respect to the low gain value as $g$ is raised to 4.5. This stronger spread occurring at high gains can be explained with the following intuitive arguments. Inside the crystal, the cascading effect which causes the exponential growth of the number of generated photon pairs is enhanced in the regions where the pump field takes its highest values. Thus in a regime of very high gain, most of the photon pairs are produced where the pump field is close to its peak value, i.e., close to the center of the beam. As a result the effective region of amplification inside the crystal becomes narrower than the pump beam profile (see also [15,21]), and this effect produces a broadening of the signal and idler intensity correlation function in the far field.

Finally, it clearly appears from Fig. 4 that the peak value of $\Gamma_{12}$ of the discretized correlation function increases with the parametric gain. We can explain this feature by noticing that $\Gamma_{12}$ is by definition equal to $\langle \delta N_1 \delta N_2 \rangle / \langle \delta N_1 \rangle \langle \delta N_2 \rangle$, where $\delta N_1$ and $\delta N_2$ denote the photon number fluctuations on a particular symmetrical pixel pair of the numerical grid. A straightforward calculation based on definitions (2) and (8) and symmetry condition (3) leads to the relation $\Gamma_{12}^{\text{max}} = 1 - \sigma / (E_n + 1)$. As we shall see in the next section [see discussion, Eq. (23)], for PDC beams the excess noise $E_n$ is on the order of the mean number of photons per mode $\sim \sinh^2 g$. This explains the behavior of $\Gamma_{12}^{\text{max}}$, which approaches unity as $g$ (and hence $E_n$) is increased, although the degree of correlation $\sigma$ usually deteriorates at high gains, as we shall show in the next section. In this particular case, it is found that $\sigma$ is only slightly below unity, since the size of the pixels (5 $\mu m$) on which the photons are collected is rather small compared to the coherence area.

VI. FRAGILITY OF QUANTUM CORRELATION TOWARD IMPERFECT DETECTION

This section aims at identifying regimes suitable for high sensitivity image detection, which requires quantum correlation and a large number of photons per pixel at the same time. The latter condition is necessary because of the unavoidable presence of detection noise. If we denote by $\langle \delta N_1^2 \rangle_{\text{bkg}}$ the variance of the difference of background noise from symmetrical pixels of the CCD, we need to achieve $\langle \delta N_1^2 \rangle_{\text{bkg}} / (\langle N_1 + N_2 \rangle) \ll 1$. For example, referring to the experiment performed by Jedrkiewicz et al. [14,15], the CCD device was affected by a background photocurrent giving $\langle \delta N_1^2 \rangle_{\text{bkg}} = 100$. As a result, for that CCD one should maintain subshot-noise correlations for $\langle N_1 + N_2 \rangle \gg 100$. As it was clearly illustrated in [14,15], the task to achieve strong local subshot-noise correlation with a large number of detected photons is experimentally demanding. We wish to investigate under which conditions the SNR obtained with the PDC source is able to beat the SQL defined in Eq. (10). In particular we shall emphasize the negative role of the excess noise present in the signal and idler beams,
showing that the quantum nature of the correlation becomes fragile against unavoidable imperfections in the detection procedure. Because of this feature, the conditions of high parametric gain and high excess noise in the experiment described in [14] would not suit our purpose. Indeed, the experimental data displayed a transition from quantum (subshot noise) to classical (above shot noise) correlation as the number of PDC photons per pixel was raised above 15–20, a regime where the intrinsic noise of the CCD is not negligible.

It is well-known that the signal and idler fields taken separately display a thermal-like statistics. The variance of the photon number intercepted by the pixel detectors can therefore be written in the form [22]

$$\langle \delta N_2^2 \rangle = \langle N_2 \rangle + \frac{\langle N_2 \rangle^2}{M},$$

(22)

where $M$ denotes the degeneracy factor representing the number of spatial and temporal modes collected by the detectors. (Accurate measurements of the photon statistics in the high gain regime of PDC are described, e.g., in [23].) According to definition (8), it follows therefore that the excess noise coincides with the mean number of photons per mode:

$$E_n = \frac{\langle N_2 \rangle}{M}.$$  

(23)

In the large pump beam waist conditions we consider here, this quantity depends only on the parametric gain and is on the order of $\sinh^2 g$.

A large amount of excess noise has a detrimental effect on quantum correlation as one takes into account small unbalances between the test and the reference arms of the imaging system, which in real experiments can never be completely suppressed. Let assume, for example, that the signal and the idler fields are not perfectly balanced, but rather undergo different losses in the test and the reference arms, either during propagation, or in the detection process. Indicating with $\eta_1$ and $\eta_2$ the different effective quantum efficiencies in the two detection channels, it is found that

$$\sigma = 1 - \eta + \frac{(\eta_1 - \eta_2)^2}{2 \bar{\eta}} \left( E_n + \frac{1}{2} \right),$$

(24)

where $\bar{\eta} = (\eta_1 + \eta_2)/2$. Thus we see from relation (24) that a necessary condition to have subshot-noise correlation is that $E_n \ll 2 \bar{\eta} / (\eta_1 - \eta_2)^2$. Because $(\eta_1 - \eta_2)^2$ is usually small compared to unity, the condition can be generally fulfilled, e.g., by applying a careful compensation of the losses in the two detection channels [10,12].

However, when dealing with an imaging scheme as in our case, another source of unbalance which is usually more difficult to control experimentally derives from the inaccuracy in the determination of the center of symmetry of the far-field pattern in the detection plane. Typically, the distance $x_{\text{shift}}$ between the selected and the exact center of symmetry is on the order of one-fourth the size of a CCD pixel (20 $\mu$m in the experiment [14]) and is small compared with the far-field coherence length $x_{\text{coh}}$ (a few tens of microns in [14]) but not negligible. In the following, all the quantities of interest will refer to a pair of pixel detectors located at symmetrical positions with respect to the optical axis as the one shown in Fig. 2. In particular $N_1$ and $N_2$ will indicate the operators associated with the number of photons measured by such a pair of detectors.

Using the numerical model illustrated in Sec. IV, we investigated the behavior of the correlation factor $\sigma$ toward errors in the center of symmetry $x_{\text{shift}}$. Considering different values of the parametric gain $g$ in order to vary the amount of excess noise. The far-field measurement scheme considered in the simulation is identical to the one illustrated in Fig. 2, except we do not include the absorbing object, as we are presently interested only in evaluating the correlation factor. The results are illustrated in Fig. 5. We considered decreasing values of the parametric gain $g$ in order to reduce progressively the amount of excess noise (which behaves as...
sins^2 g) starting from rather high values ($E_n \sim 100$ for $g = 4.15$) down to a small amount ($E_n \sim 0.5$ for $g = 1.45$). Our aim is to compare the SNR obtained in the imaging experiment at different parametric gains for a fixed number of detected photons (see Sec. VII). For this reason, as $g$ is decreased the pump pulse duration $\tau_p$ is increased as indicated in the figure legend, so that the number of photons ($N_1$) detected by the pixel detectors remains unchanged, about 3500 photons/pixel. In such conditions, the excess noise decreases as the inverse of the pump pulse duration $\tau_p$. The two plots correspond to two different sizes of the pixel detectors: 20 $\mu m$ in case (a) and 40 $\mu m$ in case (b). Referring to the 20 $\mu m$ pixels of the CCD used in experiment [14], the 40 $\mu m$ pixel size can be obtained by performing a 2 $\times$ 2 pixel binning. With our choice of the lens focal length ($f = 5$ cm) and the pump beam waist ($w_p = 1500$ $\mu m$), the typical coherence length in the detection plane is of the same order of magnitude of the pixel size or slightly smaller. Except for the case $\tau_p = 5$ ps, the long duration of the pump pulse with respect to the PDC coherence time ($\tau_{coh} \sim 1$ ps) did not allow us to simulate PDC propagation using a unique numerical array. For this reason we divided the pump temporal profile in a sequence of 5 ps long intervals over which the generated PDC field can be considered as nearly stationary. We performed simulations for each of these time intervals separately, summing up their independent contributions in order to evaluate the overall photon statistics.

As can be inferred by comparing Figs. 5(a) and 5(b), under the same gain and pulse duration conditions the 2 $\times$ 2 pixel binning [Fig. 5(b)] allows one to enhance the correlation substantially (as long as $x_{shift} \ll x_{coh}$) since the detection area covers a larger portion of the coherence area. It is here worth noting that the relevant parameter that determines the amount of correlation is the ratio between detection area and the coherence area: the same improvement can therefore be obtained by increasing the pump beam waist $w_p$ as $1/x_{coh}$.

Another relevant issue which can be inferred from both plots is the deterioration of the correlation that occurs at high parametric gains even for $x_{shift} = 0$. This effect is related to the broadening of the signal-idler correlation function $\Gamma_{12}$ which occurs at high parametric gains as discussed in Sec. V. It becomes particularly relevant for the 5 ps pump with pulse $g = 4.15$ and the 20 $\mu m$ pixel detectors [black squares in Fig. 5(a)]: the correlation factor $\sigma$ goes rather close to unity, i.e., close to the shot-noise level, since for such a high value of $g$ the FWHM of the $\Gamma_{12}$, $\Delta x_{FWHM} \sim 23$ $\mu m$, becomes larger than the size of the pixel detectors. On the other side, considering the 40 $\mu m$ pixel detectors, we see that $\sigma$ approaches the optimal value $1 - \eta = 0.1$ as the pump pulse duration is increased and the parametric gain is decreased.

From the same plots we see that $\sigma$ starts from its minimum value taken at $x_{shift} = 0$ and increases with $x_{shift}$ until it saturates to the value $\sigma_{sat} = 1 + E_n$ as $x_{shift}$ becomes larger than $x_{coh}$. It can be easily verified that the saturation value $\sigma_{sat}$ corresponds to the limit in which the photon number fluctuations revealed on the two detectors becomes completely uncorrelated, as the error in the symmetry center becomes larger than the width of the signal-idler correlation function (which can be identified with $x_{coh}$). Considering the worst case $\tau_p = 5$ ps, the large parametric gain makes the excess noise extremely high ($E_n \sim 100$). As a consequence, we see that the slightest inaccuracy in the position of the symmetry center leads to a transition of the degree of correlation from subshot noise to strongly above shot noise (black square in Fig. 5). As the parametric gain is decreased $\sigma$ saturates slightly faster (since $x_{coh}$ diminishes) but to lower and lower values which tend to approach the shot-noise level as $E_n$ goes to zero (see stars of the 1000 ps pump pulse case). As $M \tau_p$ becomes larger than $\langle N_1 \rangle$ the photon statistics on the two detectors becomes indeed nearly Poissonian, so that $\langle \delta N_2^2 \rangle = \langle N_1 \rangle$ [see Eq. (22)] and $\sigma$ is worse on the order of unity, the value corresponding to the standard quantum limit.

If we approximate the curves of Fig. 5 with straight lines that go from the minimum value $\sim 1 - \eta$ (at $x_{shift} = 0$) to the saturation value $\sigma_{sat} = 1 + E_n$ (at $x_{shift} = x_{coh}$), we can obtain a rough estimate of the error shift value for which the transition from subshot-noise to above shot-noise correlations occurs, i.e.,

$$\sigma > 1 \iff \frac{x_{shift}}{x_{coh}} > \frac{1}{1 + E_n/\eta}.$$  

A more precise analytical derivation of the dependence of $\sigma$ on $x_{shift}$ will be given in [24]; however, we can infer from the behavior illustrated in Fig. 5 and from expression (25) that the precision in the determination of the symmetry center becomes an extremely relevant issue from the experimental point of view whenever the excess noise is a large quantity, i.e., for short pump pulses and high parametric gains. Moreover, relation (25) does not take into account the broadening of the correlation function occurring with increasing $g$ and therefore the fragility of the quantum correlation is underestimated. Clearly, a way to compensate for this negative effect would be to increase the detection area. On the contrary, when the excess noise is lowered by decreasing the parametric gain, the quantum correlation becomes more robust.

The previous results clearly suggests that subshot-noise correlation can be maintained at higher photon number values if the parametric gain is reduced and the pump pulse duration is increased, typically in the nanosecond range. The simulation shown in Fig. 6 confirms this hypothesis: it plots $\sigma$ as a function of the mean photon number per pixel ($\langle N_1 \rangle$) considering a fixed error in the determination of the symmetry center, $x_{shift} = 4$ $\mu m$, and for increasing values of the pump pulse duration, starting from 5 ps up to 1 ns. Each point of the curves is obtained for different values of the parametric gain $g$ in the range between 1.2 and 3.5, while the solid lines correspond to a linear fit of the numerical data. As a general behavior, we see from the figure that $\sigma$ increases linearly with $\langle N_1 \rangle$ with a slope which decreases as the pump pulse duration is increased. We verified that the angular coefficient of the linear fits (solid lines) scales as the inverse of $M \sim \tau_p / \tau_{coh}$. For the 1 ns pulse one can therefore reach subshot-noise correlation at much higher photon number values than for shorter pulses.

VII. SIMULATION OF THE HIGH-SENSITIVITY IMAGING EXPERIMENT

From the previous results we have seen that in the presence of excess noise a careful balance of the two detection
channels and a precise determination of the symmetry center are necessary in order to approach the (ideal) minimum value $\sigma = 1 - \eta$. On the other hand, we see both from Eq. (24) and relation (25) that this requirement becomes less stringent as the excess noise is reduced by lowering the parametric gain, so that $M$ becomes on the same order of magnitude of $\langle N_f \rangle$ or larger. We can take advantage of this feature by considering long pump pulses, typically in the nanoseconds range, so that $\tau_p \gg \tau_{\text{coh}}$ and a large number of temporal modes are amplified at a low parametric gain, keeping large the total number of photons collected by the pixel detectors.

We now illustrate the numerical simulation of a high-sensitivity experiment performed considering the measurement scheme of Fig. 2. Let us first consider the case in which test and reference detection channels are perfectly balanced (in particular $x_{\text{shift}} = 0$). The object we considered took the form of a weakly absorbing mask on the numerical grid with a constant absorption coefficient $\alpha = 0.04$, as the one shown in Fig. 8(a). Figure 7 illustrates the behavior of the SNR as a function of pump pulse duration. In those simulations the parametric gain is kept constant ($g = 1.45$), while $\tau_p$ is increased up to 1000 ps with the unbinned pixels [Fig. 7(a)] and up to 250 ps with the $2 \times 2$ pixel binning [Fig. 7(b)]; in such a way $\langle N_f \rangle$ (lower horizontal scale) increases almost linearly with the pump pulse duration (upper axis) up to 3500 photons per pixel and the SNR increases with the square root of $\langle N_f \rangle$ as predicted by Eqs. (9) and (7).

The lower curves (black squares) represent the SQL obtained with the split coherent beams, while the upper curves (hollow triangles) reproduce the results obtained with the PDC source. With the chosen parameters, the coherence length $\lambda_{\text{FWHM}}$ in the detection plane was evaluated on the order of 10 $\mu$m. As discussed in Sec. IV (see Fig. 5), only the 40 $\mu$m pixels size allows us to approach the optimal degree of correlation determined by the finite quantum efficiency of the detectors, i.e., $\sigma = 1 - \eta = 0.1$. For this reason the factor of improvement $R$ obtained with the unbinned pixels [Fig. 7(a)] is only $\sim 1.9$, while with the $2 \times 2$ pixel binning [Fig. 7(b)] we are close to the optimal value $R = 2.9$ [obtained from Eq. (11) with $\sigma = 0.1, E_n = 0.5$]. From these results it clearly emerges that the size of the coherence area determined by the optical setup imposes a lower limit to the size of the object details that can be resolved with a sensitivity beyond the SQL.

The dashed lines in the figure are obtained by using the expressions of the two-mode calculations (7) and (9) and fit well the numerical results. The values of the relevant fitting parameter $\sigma$ are indicated beside each plot ($E_n$ is on the order of unity and does not affect the SNR value significantly, as can be inferred from expression (7)). It should be noticed that both $R$ and $\sigma$ are almost independent on the pump pulse duration, as long as the parametric gain $g$ is fixed.

The images of Fig. 8 allow one to appreciate visually the improvement that can be achieved with the PDC source (b) with respect to the classical source (c) with the 40 $\mu$m pixel size. The indicated SNR values are those obtained with a 125 ps pump pulse (giving $\sim 1750$ photons/pixel) corresponding to the dotted vertical line in Fig. 7.

Finally we investigated on the sensitivity of the SNR improvement with respect to inaccuracies in the determination of the center of symmetry of the PDC far field. In Fig. 9 we plot the ratio $R = \text{SNR}_{\sigma}/\text{SNR}_{\text{SQL}}$ as a function of $x_{\text{shift}}$, for different values of the pump pulse duration and for a pixel...
To conclude, our simulations show that the detrimental effects of the broadening of the coherence area added to the fragility against imperfections in the measurement process become a much less relevant issue in the low-gain, long-pulse regime.

VIII. CONCLUSIONS

We have proposed an imaging scheme that uses the multimode twin beams generated in the process of single-pass parametric down-conversion to detect the spatial distribution of faint objects through a differential measurement technique. The object is located in the PDC far field and the image is retrieved by subtracting pixel by pixel the measured signal and idler photon fluxes. In this way the detection scheme can achieve a sensitivity beyond the SQL by exploiting the local character of the quantum correlations of the PDC far field, a property which was demonstrated experimentally in [14,15] in a regime of high parametric gain.

We developed a stochastic numerical model which includes realistic features of the imaging system in order to verify to which extent the PDC source allows one to enhance the SNR of the measurement with respect to that obtained from a classical source. We pointed out the fragility of the subshot-noise correlation against imperfections of the detection system in the presence of a large amount of excess noise. In particular we analyzed the effect of small errors (on the order of a few microns) in the determination of the center of symmetry of the PDC far field, showing that they lead to a loss of correlation proportional to the excess noise of the source. Other experimental imperfections, such as unbalances due to different losses in the test and reference arms and light scattering from the background, produce a similar deterioration of the subshot-noise correlation. The quantum character of the correlation is therefore rapidly lost as the parametric gain (and therefore the excess noise) is increased.

We suggest that the use of pump pulses with duration on the order of a few nanoseconds or much, longer than the PDC coherence time, would allow one to generate the same number of photons than in the case of the picoseconds pulses used in [14] with a much lower parametric gain and a negligible excess noise. Our simulations demonstrate that this represents a clear advantage that the subshot-noise correlation becomes more robust and can be maintained up

FIG. 9. The ratio $R=\text{SNR}_a/\text{SNR}_{\text{SQL}}$ is plotted as a function of $x_{\text{shift}}$ for different values of the pump pulse duration, considering 20 $\mu$m (a) and 40 $\mu$m (b) pixel detectors. All other parameters are the same as in Fig. 5.
to gains corresponding to several thousands of PDC photons per pixel, a regime where the noise of the CCD can be neglected [25]. We used the same numerical model to simulate the imaging experiment, showing that under these conditions the use of a PDC source leads to a substantial SNR enhancement with respect to classical imaging in a domain of parameters accessible to experimental implementations.

Moreover, from our analysis it emerges that the degree of quantum correlations, and hence the improvement in the parameters accessible to experimental implementations.

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[25] Experimental results obtained by one of us (O.J.) with a type-I BBO crystal with a cw pump tend to confirm the result of our numerical model and will be the object of a future publication.